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THE BOCHNER TYPE CURVATURE TENSOR OF PSEUDO CONVEX CR-STRUCTURES

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Abstract. For a pseudo-convex CR-structure on an odd dimensional manifold, we introduce a family of canonical torsion-free linear connections. Every connection in this family is uniquely determined by an almost contact structure associated with the given pseudo convex CR-structure. We study the change of the connections in this family under the gauge transformations and, accordingly, the corresponding change of the gauge tensor fields. The Bochner type curvature tensor field we get is invariant under gauge transformations.

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§0. Introduction

In [5] the authors have introduced a Bochner type curvature tensor field for the pseudo convex CR-structures by using an adapted connection with torsion considered by N.Tanaka in [6]. Some results concerning this tensor field have been obtained in [7], [8].

In this paper we use a torsion-free linear connection adapted to an almost contact structure associated with a given pseudo-convex CR-manifold in order to get a Bochner type curvature tensor field for the CR-manifold. The fundamental tensor field, the 1-form and the vector field defining the associated almost contact structure are no longer parallel with respect to this connection. The expression of the obtained curvature tensor field is very much similar to the C-projective curvature tensor field in the case of the normal almost contact manifolds [2], the H-projective curvature tensor field in the case of the complex manifolds and the usual Bochner curvature tensor field in the case of the Kaehler manifolds. At the end we establish the relation between our adapted linear connection and the Tanaka connection and get that our Bochner type curvature tensor field and that obtained in [5] are the same, only their expressions are different. However, it seems to us that our expression is simpler and we use only the Ricci tensor field in order to get it.

The structure of the paper is as follows. In the first of two sections we study the integrability problem for a CR-structure, obtaining a tensor field S of type $(1,2)$ which is related to the tensor field used in defining the normality condition for an almost contact manifold and which vanishes in the case of an integrable CR-structure. Next, we introduce the family of canonical torsion-free linear connections for a pseudo convex CR-structure and study their change under gauge transformations. From the curvature tensor fields of the connections in the obtained family and their changes under the gauge transformations, we get the Bochner type curvature tensor field.

§1. Preliminaries

Let M be a real hypersurface of a complex manifold $(\widetilde{M}, J)$ with $\dim_{\mathbf{C}}\widetilde{M} = n+1$. Denote by TM the tangent bundle of M and let $H(M) \subset TM$ be the distribution of the holomorphic tangent vectors on M, i.e.

$$H(M) = \{X \in TM | JX \in TM\};$$

$H(M)$ can be thought of as the decomplexification of the subbundle in the complexification $TM \otimes \mathbf{C} = T^cM$ of TM denoted also by $H(M)$ and defined as:

$$H(M) = \{X - iJX \mid X \in TM, JX \in TM\}$$

(see [3] for more details). Then $\mathrm{rank}_{\mathbf{R}} H(M) = 2n$.
Denote by $\Gamma(H(M))$ the $C^\infty(M)-$module of sections in $H(M)$ where $C^\infty(M)$ is the ring of smooth functions on M. If $H(M)$ is thought of as a complex vector subbundle in T^cM then its sections are the holomorphic vector fields. For the holomorphic vector fields $X - iJX$ and $Y - iJY$, $X, Y \in \Gamma(H(M))$, the condition $[X - iJX, Y - iJY] \in \Gamma(H(M))$ (which is natural if we think these vector fields as holomorphic vector fields on the complex manifold $\widetilde{M}$) is expressed by the following involutivity conditions for $H(M)$

$$(1.1)\quad \begin{cases} [X,Y] - [JX,JY] \in \Gamma(H(M)), \\ \\ [JX,JY] - [X,Y] - J[JX,Y] - J[X,JY] = 0. \end{cases}$$

So, taking into account the previous formulas, $(M, H(M))$ defines a CR-structure on M [4], [5].

Further we shall consider that the CR-structure of M is a pseudo-convex structure i.e. its Levi form is nondegenerate; then, if we denote by η the local 1-form on M which defines, locally, the 1-codimensional distribution $H(M) \subset TM$

(1.2) $H(M) = \{X \in TM \mid \eta(X) = 0\}$,

η is a contact form on M i.e. $\eta \wedge (d\eta)^n \neq 0$ on M.

§2. CR-structures and almost contact structures

Given the pseudo-convex CR-structure $(M, H(M))$ with the contact form η, let φ and ξ be the endomorphism and the vector field on M given, respectively, by the relations

(2.1) $\varphi = J \circ h, \qquad \eta(\xi) = 1, \qquad d\eta(\xi, X) = 0$

where $X \in \mathcal{X}(M)$ and $h = I - \eta \otimes \xi$ is the projection operator on $H(M)$.

It is not difficult to show that the equation

(2.2) $\varphi^2 = -I + \eta \otimes \xi$

holds (I still denotes the identity endomorphism), and (φ, ξ, η) defines an (local) almost contact structure on M which is called associated with the pseudo-convex CR-structures $(M, H(M))$ [1],[6].

It is a natural question to study the relation between the complex involutivity conditions (1.1) of $(M, H(M))$ and the normality of the almost contact structure (φ, ξ, η) given by the vanishing of the (1,2)-tensor [1]

(2.3) $N = N_\varphi + d\eta \otimes \xi$,

where N_φ denotes the Nijenhuis tensor of the tensor φ.

So, starting from the equations (1.1), for every $X, Y \in \mathcal{X}(M)$ we can write

(2.4)
$$[J(X - \eta(X)\xi), J(Y - \eta(Y)\xi)] - [X - \eta(X)\xi, Y - \eta(Y)\xi] = J([J(X - \eta(X)\xi), Y - \eta(Y)\xi]) + J([X - \eta(X)\xi, J(Y - \eta(Y)\xi)])$$

and, after a long but straightforward computation, we obtain that the complex involutivity conditions are equivalently expressed by the equation

(2.5) $S(X, Y) = 0, \qquad X, Y \in \mathcal{X}(M)$

where

(2.6) $S(X, Y) = N_\varphi(X, Y) + d\eta(X, Y)\xi + \eta(X)\varphi(\mathcal{L}_\xi\varphi)Y - \eta(Y)\varphi(\mathcal{L}_\xi\varphi)X.$

Here $\mathcal{L}$ denotes the Lie derivative. Taking into account that, on every normal almost contact manifold $\mathcal{L}_\xi\varphi = 0$, it is clear how the normality of (φ, ξ, η) implies $S = 0$.

On the other hand, the equation (2.5) with S defined by (2.6) does not guarantee the normality of (φ, ξ, η) without the further condition $\mathcal{L}_\xi \varphi = 0$.

Moreover it will be useful to know some relations concerning the tensor S we defined; so, it is not difficult to verify the following equations

$$(2.7)\quad \begin{cases} \eta(S(X,Y)) = d\eta(X,Y) - d\eta(\varphi X, \varphi Y), \\ S(X,\xi) = 0, \\ S(\varphi X, \varphi Y) = -S(X,Y), \\ \varphi S(X, \varphi Y) = S(X,Y) - \{d\eta(X,Y) - d\eta(\varphi X, \varphi Y)\}\xi, \end{cases}$$

and, if we put

$$(2.8)\quad \psi = \frac{1}{2}\mathcal{L}_\xi \varphi,$$

we have too

$$(2.9)\quad \begin{cases} \psi\xi = 0, \qquad \eta \circ \psi = 0, \qquad \varphi\psi + \psi\varphi = 0 \\ d\eta(\psi X, Y) + d\eta(X, \psi Y) = 0. \end{cases}$$

In the following the equality $S = 0$ will be assumed.

§3. The canonical torsion-free connections of pseudo-convex CR-structures

Let (φ, ξ, η) be an almost contact structure associated with the pseudo-convex CR-structure $(M, H(M))$. Looking for a torsion-free connection ∇ on M related in a natural way to the 1-form η , we obtain the following

Theorem 3.1. *If (φ, ξ, η) is the almost contact structure associated with the pseudo-convex CR-structure $(M, H(M))$, then there exists one and only one torsion-free connection ∇ such that for every $X, Y \in \mathcal{X}(M)$*

$$(3.1)\quad \begin{cases} (\nabla_X\eta)(Y) = \dfrac{1}{2}d\eta(X,Y), \qquad \nabla_X d\eta = 0, \qquad \nabla_X\xi = 0 \\[2ex] (\nabla_X\varphi)Y = 2\eta(X)\psi Y - \dfrac{1}{2}d\eta(X,\varphi Y)\xi. \end{cases}$$

Proof. Before starting with the actual proof of the theorem, let us briefly remark that the first two conditions in (3.1) have been suggested by the well known formulas

$$(3.2)\quad \begin{cases} d\eta(X,Y) = (\nabla_X\eta)(Y) - (\nabla_Y\eta)(X), \\[1ex] 0 = d^2\eta(X,Y,Z) = (\nabla_X d\eta)(Y,Z) + (\nabla_Y d\eta)(Z,X) + (\nabla_Z d\eta)(X,Y), \end{cases}$$

relating the exterior differential with the torsion free linear connection ∇, while the third condition is simply obtained from the compatibility conditions. The point was to get the last condition in order to assure the compatibility conditions and the vanishing of the torsion of ∇ under the condition $S = 0$. To get the connection ∇ we shall compute $\eta(\nabla_X Y)$ and $d\eta(\nabla_X Y, Z)$. Taking into account the first relation in (3.1), we easily find

$$(3.3)\quad 2\eta(\nabla_X Y) = 2X(\eta(Y)) - d\eta(X,Y).$$

On the other hand, from the symmetry of ∇ and (3.1) we also have

$$(3.4)\quad X(d\eta(Y,Z)) = d\eta(\nabla_X Y, Z) + d\eta(Y, \nabla_X Z)$$

$$\begin{aligned}(3.5)\quad \varphi Z(d\eta(X,\varphi Y)) = {} & d\eta(\nabla_X Z, Y) + 2\eta(X)d\eta(\psi Z, \varphi Y) \\ & - d\eta([X,\varphi Z],\varphi Y) - 2\eta(Y)d\eta(\varphi X, \psi Z) + d\eta(\varphi X, [Y,\varphi Z]) \\ & + d\eta(\nabla_X Y, Z) - d\eta([X,Y],Z) - Y(d\eta(X,Z)),\end{aligned}$$

and finally, adding (3.4), (3.5),

$$\begin{aligned}(3.6)\quad 2d\eta(\nabla_X Y, Z) = {} & 2\eta(X)d\eta(\varphi Y, \psi Z) + 2\eta(Y)d\eta(\varphi X, \psi Z) \\ & + \varphi Z(d\eta(X,\varphi Y)) + d\eta([X,\varphi Z],\varphi Y) + d\eta([Y,\varphi Z],\varphi X) \\ & + X(d\eta(Y,Z)) + Y(d\eta(X,Z)) + d\eta([X,Y],Z).\end{aligned}$$

Recalling that $d\eta$ is not degenerate on $H(M)$, it is easy to see that (3.3) and (3.6) completely define $\nabla_X Y$.

Furthermore, if $\widetilde{\nabla}$ is another connection satisfying the hypotheses of the theorem, we obviously have $\eta(\nabla_X Y) = \eta(\widetilde{\nabla}_X Y)$ and $d\eta(\nabla_X Y, Z) = d\eta(\widetilde{\nabla}_X Y, Z)$ which imply $\nabla = \widetilde{\nabla}$. □

Remark 3.2. In [6] the author already associated a non symmetric canonical connection to strongly pseudo-convex CR-structures asking the parallelism of tensor fields φ, ξ, η together with some conditions on its nonvanishing torsion. Such a connection has been used by different authors to study curvature invariants on contact and on strongly pseudo-convex CR-manifolds (see in particular [7], [8], [5]). In the same way, we shall call our connection "torsion-free canonical connection associated to the pseudo-convex CR-structure $(M, H(M))$."

§4. Gauge transformations of almost contact structures

If, starting from the equation (1.2), we consider another 1-form $\eta' = e^f\eta, f \in C^\infty(M)$ defining the same distribution $H(M)$,then, examining the relations between the associated almost contact structures (φ, ξ, η) and (φ', ξ', η') respectively induced by η and η', we obtain

Proposition 4.1. *The two almost contact structures (φ, ξ, η) and (φ', ξ', η') are associated to the same pseudo-convex CR-structure iff for some function $f \in C^\infty(M)$ they satisfy*

$$(4.1)\quad \begin{cases} \eta' = \varepsilon e^f \eta \qquad d\eta' = \varepsilon e^f\{d\eta + df \wedge \eta\} \\ \varphi' = \varphi + \eta \otimes A \qquad \xi' = \varepsilon e^{-f}\{\xi + \varphi A\} \\ \varepsilon = \pm 1 \end{cases}$$

where, assuming $\varepsilon = 1$ (see (4.2)), the vector field A is defined by the conditions

$$(4.2)\quad \eta(A) = 0, \qquad d\eta(\varphi A, X) = df(hX)$$

and $df(X) = X(f)$.

Proof. See [4]. □

Remark 4.2. The case where $\varepsilon = -1$ is similar and the final result, concerning the invariance of the Bochner type tensor field, is the same.

Following [8], from now on, we shall call (4.1) a "gauge transformation of almost contact structures".

Proposition 4.3. *The complex involutivity of the CR-structure $(M, H(M))$ is invariant under gauge transformations.*

Proof. Let S and S' be the two (1,2)-tensor fields defined by (2.6), obtained from the two almost contact structures (φ, ξ, η) and (φ', ξ', η'); we have to prove that $S = 0$ iff $S' = 0$.

Taking into account the equations (3.1) we get the following relation

(4.3) $S'(X, Y) = S(X, Y) - \eta(X)S(\varphi A, Y) - \eta(Y)S(X, \varphi A)$

for every $X, Y \in \mathfrak{X}(M)$

Obviously $S = 0$ implies $S' = 0$. On the other hand, if we assume $S' = 0$, substituting in the last equation of (3.1) ξ for Y, we have as a consequence $S(X, \varphi A) = 0$ for every $X \in \mathfrak{X}(M)$ and, immediately $S = 0$. □

Moreover, taking into account (3.1), we obtain

Theorem 4.4. *A gauge transformation between two almost contact structures (φ, ξ, η) and (φ', ξ', η') induces a transformation between their associated torsion-free canonical connections ∇ and ∇' given by*

(4.4) $\nabla'_X Y = \nabla_X Y + P(X, Y) \qquad X, Y \in \mathfrak{X}(M)$

where P is the symmetric tensor field of type (1,2) with the following expression

(4.5)
$$\begin{aligned} 2P(X, Y) = {} & df(hX)hY + df(hY)hX + d\eta(X, \varphi Y)A \\ & - df(\varphi X)\varphi Y - df(\varphi Y)\varphi X + \eta(X)\{df(hY)\xi + df(hY)\varphi A \\ & - df(A)\varphi Y - 2df(\varphi Y)A - 2\varphi\nabla_{hY}A\} + \eta(Y)\{df(hX)\xi \\ & + df(hX)\varphi A - df(A)\varphi X - 2df(\varphi X)A - 2\varphi\nabla_{hX}A\} \\ & + 2\eta(X)\eta(Y)\{df(\xi)\xi + df(\xi)\varphi A - [\xi, \varphi A] - \frac{3}{2}df(A)A + \varphi\nabla_{\varphi A}A\}. \end{aligned}$$

Proof. We notice at first that, from equation (4.1)

$$(\nabla'_X \eta')(Y) = \frac{1}{2}d\eta'(X, Y) \qquad X, Y \in \mathfrak{X}(M)$$

we find

(4.6) $\eta'(\nabla'_X Y) = \frac{1}{2}e^f\{df(X)\eta(Y) + df(Y)\eta(X) - d\eta(X, Y)\},$

which easily implies

(4.7) $\eta(P(X, Y)) = \frac{1}{2}\{df(X)\eta(Y) + df(Y)\eta(X)\}.$

On the contrary, to obtain the complete expression of $P(X,Y)$ we need a very long computation. So, we shall describe here only the principal steps and formulas we used to prove the theorem.

The first idea is to introduce an auxiliary partial metric tensor g' on $H(M)$ given by

$$g'(X,Y) = d\eta'(\varphi' X, Y) \tag{4.8}$$

finding, by means of the equation

$$g'(P(X,Y),Z) + g'(Y,P(X,Z)) = (\nabla_X g')(Y,Z) - (\nabla'_X g')(Y,Z) \tag{4.9}$$

true for every $X,Y,Z \in \mathcal{X}(M)$, the formula

$$\begin{aligned} 2g'(P(X,Y),Z) &= (\nabla_X g')(Y,Z) + (\nabla_Y g')(X,Z) - (\nabla_Z g')(X,Y) \\ &\quad - (\nabla'_X g')(Y,Z) - (\nabla'_Y g')(X,Z) + (\nabla'_Z g')(X,Y) \end{aligned} \tag{4.10}$$

Then, computing the right hand side of (4.10) by using the conditions (3.1), the relations (4.1) and the formula (4.8) defining g', we get the expression of $d\eta(P(X,Y),Z)$.

By using the obvious formula

$$P(X,Y) = P(hX,hY) + \eta(X)P(\xi,hY) + \eta(Y)P(hX,\xi) + \eta(X)\eta(Y)P(\xi,\xi), \tag{4.11}$$

with $X,Y \in \mathcal{X}(M)$ we see that it is more convenient to compute the expressions of $d\eta(P(X,Y),Z)$ corresponding to the cases: (i) both X,Y are sections in $H(M)$, (ii) X is a section in $H(M)$ and $Y=\xi$, (iii) $X=Y=\xi$.

Thus, it follows

$$\left\{\begin{aligned} P(hX,hY) &= \frac{1}{2}\{df(hX)hY + df(hY)hX + d\eta(hX,\varphi Y)A \\ &\quad - df(\varphi X)\varphi Y - df(\varphi Y)\varphi X\} \\ P(hX,\xi) &= \frac{1}{2}\{df(hX)\xi + df(hX)\varphi A - df(A)\varphi X\} \\ &\quad - df(\varphi X)A - \varphi\nabla_{hX}A \\ P(\xi,\xi) &= df(\xi)\xi + df(\xi)\varphi A - [\xi,\varphi A] - \frac{3}{2}df(A)A + \varphi\nabla_{\varphi A}A. \end{aligned}\right. \tag{4.12}$$

Before we conclude this section let us indicate some useful formulas for later use.

First of all, we have the simple formulas

$$\left\{\begin{aligned} & df(\varphi A) = 0, \\ & (\nabla_X df)(Y) = (\nabla_Y df)(X) \qquad X,Y \in \mathcal{X}(M) \end{aligned}\right. \tag{4.13}$$

easily deduced from the relations $df(hX) = d\eta(\varphi A, X)$ and $d^2 f = 0$ respectively; furthemore, we remark the following complicated but useful relation between $\psi = \frac{1}{2}\mathcal{L}_\xi \varphi$ and $\psi' = \frac{1}{2}\mathcal{L}_{\xi'}\varphi'$

(4.14) $$2\psi'(X) = e^{-f}\{2\psi(X) + (df(\varphi X) + \eta(X)df(A))(\xi + \varphi A) + [\varphi A, \varphi X] \\ -\varphi[\varphi A, X] + df(hX)A + \eta(X)[\xi + \varphi A, A]\},$$

obtained for every $X \in \mathcal{X}(M)$ just applying (4.1). □

§5. Curvature of torsion free canonical connections

Given the almost contact structure (φ, ξ, η) associated to the pseudoconvex CR-structure $(M, H(M))$ and the torsion free canonical connection ∇ on M, consider the curvature tensor field R of ∇ defined by

$$R_{XY}Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \qquad X, Y, Z \in \mathcal{X}(M).$$

Since we are mainly interested in getting the curvature changes under gauge transformations, we need at first to study some general relations and properties of R, with special attention to the restriction of R on $H(M)$.

Equations (3.1) easily imply for every $X, Y, Z, W \in \mathcal{X}(M)$

(5.1) $$\begin{cases} d\eta(R_{XY}Z, W) = -d\eta(Z, R_{XY}W) \\ R_{XY}\xi = 0 \\ \eta(R_{XY}Z) = 0 \\ \varphi R_{XY}Z = R_{XY}\varphi Z - 2d\eta(X,Y)\psi Z + \eta(X)d\eta(Y, \psi Z)\xi \\ \qquad -\eta(Y)d\eta(X, \psi Z)\xi + 2\eta(X)(\nabla_Y\psi)Z - 2\eta(Y)(\nabla_X\psi)Z. \end{cases}$$

Remark that, if $X, Y, Z \in \Gamma(H(M))$, the last formula in (5.1) becomes simply

(5.2) $$\varphi R_{XY}Z = R_{XY}\varphi Z - 2d\eta(X,Y)\psi Z.$$

As before, let us introduce now the auxiliary metric g on $H(M)$

$$g(X,Y) = d\eta(\varphi X, Y) \qquad X, Y \in \Gamma(H(M))$$

to define a "generalized Riemann-Christoffel tensor" $\mathcal{R}$ for ∇; given $X, Y, Z, W \in \mathcal{X}(M)$ we put

(5.3) $$\mathcal{R}(W, Z, X, Y) = g(W, R_{XY}Z) = d\eta(\varphi W, R_{XY}Z);$$

Obviously $\mathcal{R}(W,Z,X,Y) = -\mathcal{R}(W,Z,Y,X)$; furthemore, considering the first Bianchi identity $\underset{X,Y,Z}{\mathfrak{S}} R(X,Y)Z = 0$ fulfilled by R, we also have

$$\underset{X,Y,Z}{\mathfrak{S}} \mathcal{R}(W,X,Y,Z) = 0, \tag{5.4}$$

while, using (5.1) and (5.2), for $X, Y, Z \in \Gamma(H(M))$ we get

$$\mathcal{R}(Z,Z,X,Y) = d\eta(X,Y)d\eta(Z,\psi Z)$$

which implies for $X, Y, Z, W \in \Gamma(H(M))$,

$$\mathcal{R}(W,Z,X,Y) + \mathcal{R}(Z,W,X,Y) = 2d\eta(X,Y)d\eta(W,\psi Z). \tag{5.5}$$

As a consequence of these two last equations, we obtain

$$\begin{cases} \mathcal{R}(X,Y,W,Z) + \mathcal{R}(W,X,Y,Z) \\ \qquad +\mathcal{R}(X,Z,Y,W) = 2d\eta(Y,Z)d\eta(X,\psi W), \\ \mathcal{R}(Y,Z,W,X) + \mathcal{R}(X,Y,W,Z) \\ \qquad +\mathcal{R}(W,Y,Z,X) = 2d\eta(W,Z)d\eta(X,\psi Y) + 2d\eta(Z,X)d\eta(Y,\psi W) \end{cases} \tag{5.6}$$

which, added each other give

$$\begin{aligned} &2\mathcal{R}(X,Y,W,Z) + \mathcal{R}(W,X,Y,Z) \\ &+\mathcal{R}(X,Z,Y,W) + \mathcal{R}(Y,Z,W,X) \\ &\qquad +\mathcal{R}(W,Y,Z,X) = 2\{d\eta(Y,Z)d\eta(X,\psi W) \\ &\qquad\qquad +d\eta(W,Z)d\eta(X,\psi Y) + d\eta(Z,X)d\eta(Y,\psi W)\} \end{aligned} \tag{5.7}$$

Repeating the same computations with the pairs $(X,Y),(W,Z)$ interchanged, we get

$$\begin{aligned} \mathcal{R}(W,Z,X,Y) - \mathcal{R}(X,Y,W,Z) &= d\eta(Y,W)d\eta(X,\psi Z) + d\eta(W,X)d\eta(Y,\psi Z) \\ &\quad -d\eta(W,Z)d\eta(X,\psi Y) - d\eta(Z,X)d\eta(Y,\psi W) \\ &\quad +d\eta(X,Y)d\eta(Z,\psi W) - d\eta(Y,Z)d\eta(X,\psi W). \end{aligned} \tag{5.8}$$

On the other hand, formula (5.2) becomes at once for $\mathcal{R}$

$$\mathcal{R}(\varphi W,\varphi Z,X,Y) = \mathcal{R}(W,Z,X,Y) - 2d\eta(X,Y)d\eta(Z,\psi W) \tag{5.9}$$

which, together with the previous formula (5.9), implies for $X, Y, Z, W \in \Gamma(H(M))$

(5.10) $\mathcal{R}(\varphi W, \varphi Z, \varphi X, \varphi Y) = \mathcal{R}(W, Z, X, Y) - 2d\eta(X, Y)d\eta(Z, \psi W)$

$$+d\eta(Z, X)d\eta(Y, \psi W) - -d\eta(Y, W)d\eta(X, \psi Z)$$

$$+d\eta(X, W)d\eta(Y, \psi Z) + d\eta(\varphi Y, W)d\eta(\varphi X, \psi Z)$$

$$+d\eta(X, \varphi W)d\eta(\varphi Y, \psi Z) - -d\eta(\varphi Y, Z)d\eta(\varphi X, \psi W)$$

$$-d\eta(Z, \varphi X)d\eta(\varphi Y, \psi W) + d\eta(Y, Z)d\eta(X, \psi W).$$

We are now able to find some conditions fulfilled by the Ricci tensor field $\rho(R)$ of ∇ as a consequence of the relations proved for $\mathcal{R}$. Then, if we consider the two times covariant tensor

$$\rho(R)(Y, Z) = \mathrm{trace}(X \to R_{XY}Z) \qquad X, Y, Z \in \mathcal{X}(M),$$

from (5.8), (5.10) and taking into account (5.1), for $X, Y \in \Gamma(H(M))$, we obtain respectively

$$(5.11) \begin{cases} \rho(R)(X, Y) = \rho(R)(Y, X), \\ \rho(R)(\varphi X, \varphi Y) = \rho(R)(X, Y) - 2nd\eta(\varphi X, \psi Y) \end{cases}$$

while from (5.2) we derive for $X, Y, Z \in \Gamma(H(M))$

$$\rho(R)(\varphi Y, Z) + 2d\eta(Y, \psi Z) = \mathrm{trace}(X \to \varphi R_{XY}Z).$$

§6. Changes of the curvature tensor field under gauge transformations

Let ∇ and ∇' be two torsion-free canonical connections related by (4.4). Then the curvature tensor fields R, R' of ∇, ∇' respectively are related by the well known formula

(6.1) $R'_{XY}Z = R_{XY}Z + (\nabla_X P)(Y, Z) - (\nabla_Y P)(X, Z) + P(X, P(Y, Z))$

$$-P(Y, P(X, Z)).$$

Because of the complicated and very long expression found for P (see formula (4.5)), from now on we shall do our computations for the sections in the subbundle $H(M)$. So, if P reduces to the first equation in (4.12), we get, after a straightforward computation, for $X, Y, Z \in \Gamma(H(M))$

$$(6.2)\quad R'_{XY}Z = R_{XY}Z + C(X,Z)Y - C(Y,Z)X + C(Y,\varphi Z)\varphi X$$
$$-C(X,\varphi Z)\varphi Y - \{C(X,\varphi Y) - C(Y,\varphi X)\}\varphi Z$$
$$-d\eta(X,Y)DZ - \frac{1}{2}d\eta(X,Z)DY + \frac{1}{2}d\eta(Y,Z)DX$$
$$+\frac{1}{2}d\eta(X,\varphi Z)\varphi DY - \frac{1}{2}d\eta(Y,\varphi Z)\varphi DX,$$

where, by denoting $\alpha = \frac{1}{2}df$

$$(6.3)\quad \begin{cases} C(X,Y) = (\nabla_X \alpha)(Y) - \alpha(X)\alpha(Y) + \alpha(\varphi X)\alpha(\varphi Y) - \frac{1}{4}\alpha(A)d\eta(X,\varphi Y), \\ DX = \varphi\nabla_X A + \alpha(\varphi X)A - \alpha(X)\varphi A + \frac{1}{2}\alpha(A)\varphi X + \alpha(\xi)X. \end{cases}$$

Besides the symmetry of C (see the second equation in (4.13)), it is not difficult to check that trace$D = 0$ and C, D are related by

$$(6.4)\quad \frac{1}{2}d\eta(DX,Y) = C(X,Y) \qquad X, Y \in \Gamma(H(M)).$$

From (6.2) we easily get the following relation between the Ricci tensor fields $\rho(R')$ and $\rho(R)$ of ∇' and ∇ respectively:

$$(6.5)\quad \rho(R')(X,Y) = \rho(R)(X,Y) - 2(n+1)C(X,Y) - 2C(\varphi X,\varphi Y)$$
$$-\frac{1}{2}d\eta(X,\varphi Y)\mathrm{trace}(\varphi D)$$

from which, contracting with the 2-contravariant tensor field g^{-1}, inverse of the partial metric g on $H(M)$, we find, after a brief computation,

$$(6.6)\quad \mathrm{trace}(\varphi D) = \frac{1}{2(n+1)}\{e^f\tau(R') - \tau(R)\},$$

where $\tau(R) = \mathrm{trace}(\rho(R))$ is a kind of scalar curvature of R obtained by using the partial metric g on $H(M)$. Finally we find the following expression for C

$$(6.7)\quad C(X,Y) = -\frac{n+1}{2n(n+2)}\{\rho(R')(X,Y) - \rho(R)(X,Y)\}$$
$$+\frac{1}{2n(n+2)}\{\rho(R')(\varphi' X,\varphi' Y) - \rho(R)(\varphi X,\varphi Y)\}$$
$$-\frac{1}{8(n+1)(n+2)}\{e^f\tau(R') - \tau(R)\}d\eta(X,\varphi Y).$$

So, we obtain the following

Theorem 6.1. *Let (φ, ξ, η) be an almost contact structure on M associated with the pseudo-convex CR-structure $(M, H(M))$. Then for $X, Y, Z \in \Gamma(H(M))$ the tensor field*

$$
\begin{aligned}
(6.8)\quad B(R)_{XY}Z = {} & R_{XY}Z + L(X,Z)Y - L(Y,Z)X + L(Y,\varphi Z)\varphi X \\
& - L(X,\varphi Z)\varphi Y - \{L(X,\varphi Y) - L(Y,\varphi X)\}\varphi Z \\
& - \frac{1}{2}d\eta(X,Z)KY + \frac{1}{2}d\eta(Y,Z)KX + \frac{1}{2}d\eta(X,\varphi Z)\varphi KY \\
& - d\eta(X,Y)KZ - \frac{1}{2}d\eta(Y,\varphi Z)\varphi KX
\end{aligned}
$$

where

$$
\begin{aligned}
(6.9)\quad L(X,Y) = {} & \frac{n+1}{2n(n+2)}\rho(R)(X,Y) - \frac{1}{2n(n+2)}\rho(R)(\varphi X, \varphi Y) \\
& + \frac{1}{8(n+1)(n+2)}\tau(R)d\eta(X,\varphi Y) \\
= {} & \frac{1}{2(n+2)}\{\rho(R)(X,Y) + 2d\eta(\varphi X, \psi Y)\} \\
& + \frac{1}{8(n+1)(n+2)}\tau(R)d\eta(X,\varphi Y),
\end{aligned}
$$

and

$$\frac{1}{2}d\eta(KX,Y) = L(X,Y)$$

is invariant under gauge transformations.

Proof. Substitute (6.7) in (6.2) to find $B(R') = B(R)$. □

Note that, doing the necessary computations in local coordinates we get:

$$\operatorname{trace}(B(R)_{XY}) = 0, \quad \operatorname{trace}(X \to B(R)_{XY}Z) = 0,$$

$$\operatorname{trace}(\varphi B(R)_{XY}) = 0, \quad \operatorname{trace}(X \to \varphi B(R)_{XY}Z) = 0.$$

So, it is natural now to define $B(R)$ as a "Bochner type curvature tensor" associated with the pseudo-convex CR-structure $(M, H(M))$.

§7. The relation with the Bochner tensor field obtained by using the Tanaka connection

Recall that, as we explained in Remark 3.2, different authors already studied invariant curvature tensors for strongly pseudo-convex CR-structures by means of Tanaka connection. Remark that in [5] the authors have obtained a quite complicated expression for such a tensor field. In this section we shall show that our tensor field $B(R)$ is the same as that obtained in [5]. However, its expression is more convenient , being similar to the $C-$projective curvature tensor field of the normal almost contact manifolds, [2], the $H-$projective curvature tensor field for the complex manifolds and the Bochner curvature tensor field for the Kaehler manifolds.

Denote by ∇^T the Tanaka connection of the almost contact structure (φ, ξ, η) associated with the pseudo-convex structure $(M, H(M))$ and denote by R^T the curvature tensor field of ∇^T (see [6],[4],[7]). Then, for every $X, Y, Z, W \in \mathcal{X}(M)$, ∇^T and R^T are related to our torsion free linear connection ∇ and our curvature tensor fields R respectively by the formulas

$$\nabla^T_X Y = \nabla_X Y + \eta(X)\psi\varphi Y + \frac{1}{2}d\eta(X,Y)\xi \tag{7.1}$$

$$R^T_{XY}Z = R_{XY}Z + d\eta(X,Y)\psi\varphi Z + \eta(Y)(\nabla_X\psi)\varphi Z - \eta(X)(\nabla_Y\psi)\varphi Z \\ + \frac{1}{2}\eta(Y)d\eta(\varphi X, \psi Z)\xi - \frac{1}{2}\eta(X)d\eta(\varphi Y, \psi Z)\xi. \tag{7.2}$$

So, if $X, Y, Z \in \Gamma(H(M))$, we simply have

$$R^T_{XY}Z = R_{XY}Z + d\eta(X,Y)\psi\varphi Z. \tag{7.3}$$

As a consequence, for the corresponding Ricci tensor fields $\rho(R^T), \rho(R)$ we obtain by

$$\rho(R^T)(X,Y) = \rho(R)(X,Y) - d\eta(\varphi X, \psi Y)\ , \ X, Y \in \Gamma(H(M)). \tag{7.4}$$

Now we are able to get the expressions of the tensor fields k, l, m, L, M used in [5] to express the Bochner curvature tensor field in terms of our torsion free linear connection ∇, its curvature tensor field R, its Ricci tensor field $\rho(R)$ and the tensor fields L, K. By using the expressions from [5], we get for $X, Y \in \Gamma(H(M))$ the following formulas

$$\begin{cases} k(X,Y) = \rho(R)(X,Y) - nd\eta(\varphi X, \psi Y) \\ l(X,Y) = -L(X,Y) + \frac{1}{2}d\eta(\varphi X, \psi Y) \\ m(X,Y) = L(X,\varphi Y) + \frac{1}{2}d\eta(X, \psi Y) \\ LX = \frac{1}{2}(\varphi KX + \psi X) \\ MX = \frac{1}{2}(-KX + \varphi\psi X), \end{cases} \tag{7.5}$$

and the expression for the tensor field B_0 from [5] is obtained by a straightforward computation:

$$
\begin{aligned}
(7.6)\quad B_{0,XY}Z = {} & R_{XY}Z - L(Y,Z)X + L(X,Z)Y \\
& -2L(X,\varphi Y)\varphi Z + +L(Y,\varphi Z)\varphi X - L(X,\varphi Z)\varphi Y \\
& -d\eta(X,Y)KZ + \tfrac{1}{2}d\eta(Y,Z)KX - -\tfrac{1}{2}d\eta(X,Z)KY \\
& +\tfrac{1}{2}d\eta(\varphi Y,Z)\varphi KX - \tfrac{1}{2}d\eta(\varphi X,Z)\varphi KY + \tfrac{1}{2}d\eta(\varphi Y,\psi Z)X \\
& -\tfrac{1}{2}d\eta(\varphi X,\psi Z)Y + \tfrac{1}{2}d\eta(Y,\psi Z)\varphi X - \tfrac{1}{2}d\eta(X,\psi Z)\varphi Y \\
& -\tfrac{1}{2}d\eta(Y,\varphi Z)\psi X + +\tfrac{1}{2}d\eta(X,\varphi Z)\psi Y - \tfrac{1}{2}d\eta(Y,Z)\varphi\psi X \\
& +\tfrac{1}{2}d\eta(X,Z)\varphi\psi Y - d\eta(X,\psi Y)\varphi Z.
\end{aligned}
$$

Next, the expression of B_1 is directly deduced from (7.3)

$$
\begin{aligned}
(7.7)\quad B_{1,XY}Z &= \tfrac{1}{2}\{R^T_{\varphi X\varphi Y}Z - R^T_{XY}Z\} \\
&= \tfrac{1}{2}\{R_{\varphi X\varphi Y}Z - R_{XY}Z\}
\end{aligned}
$$

with $X, Y \in \Gamma(H(M))$. Then by using (5.10) for $X, Y, Z \in \Gamma(H(M))$ we obtain

$$
\begin{aligned}
(7.8)\quad B_{1,XY}Z = {} & \tfrac{1}{2}\{-d\eta(\varphi Y,\psi Z)X + d\eta(\varphi X,\psi Z)Y - d\eta(Y,\psi Z)\varphi X \\
& +d\eta(X,\psi Z)\varphi Y + d\eta(Y,\varphi Z)\psi X - d\eta(X,\varphi Z)\psi Y \\
& +d\eta(Y,Z)\varphi\psi X - d\eta(X,Z)\varphi\psi Y
\end{aligned}
$$

and finally, the equality

$$B(R) = B_0 + B_1$$

easily follows by cancelling the terms in B_0+B_1 and using the expression (6.9) of L to get $L(X,\varphi Y) + L(Y,\varphi X)$.

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