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On certain simple cycles of the Collatz conjecture

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Abstract. The Collatz conjecture is that there exists a positive integer n which satisfies $f^n(m) = 1$ for any integer $m \geq 3$, where f is the function on the rational number field defined by $f(m) = m/2$ if the numerator of m is even and $f(m) = (3m+1)/2$ if the numerator of m is odd. Let m be a rational number such that $f^n(m) = m > 1$. Then we show that, if m has some simple sequences, then the total number of positive integer m is finite, by estimating $f(m) - m$.

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§1. Introduction

We define a function f on the set of the positive integers by

$$f(m) = \begin{cases} \dfrac{m}{2} & \text{if } m \text{ is even,} \\ \dfrac{3m+1}{2} & \text{if } m \text{ is odd.} \end{cases}$$

The Collatz conjecture is that there exists a positive integer n which satisfies $f^n(m) = (f \circ \cdots \circ f)(m) = 1$ for any integer $m \geq 3$. We call m the "starting-number" and the smallest n the "total-sequence".

This conjecture is equivalent to the next two conditions for every odd integer m:
(1) $f^n(m) \neq m$ for any $n \geq 1$. (If $f^n(m) = m$ holds, then we call m a "cycle-number".)
(2) m has total-sequence. ($f^n(m)$ dose not diverge.)

We consider (1) and assume that m is odd, since even number is mapped to an odd number by iterating f. We know only one integral cycle-number: $m = 1$. We call one the “trivial-cycle”.
Let m be a cycle-number. We define the numbers l_i $(i \geq 0)$ and m_i $(i \geq 1)$ by the following rules:
(i) We put $l_0 = 0$ and $m_1 = m$.
(ii) For $i \geq 1$, l_i is the least positive integer such that $f^{l_i}(m_i)$ is odd.
(iii) We put $m_{i+1} = f^{l_i}(m_i)$.

If $m = m_1 = m_{k+1}$, then we call k “odd-cycle-sequence”. We write

$$m_1 = \langle l_1, l_2, \cdots, l_k \rangle.$$

We can easily see that

$$m_i = \langle l_i, l_{i+1}, \cdots, l_k, l_1, \cdots, l_{i-1} \rangle. \qquad (i = 1, \cdots, k) \tag{1.1}$$

We can write trivial-cycle

$$1 = \langle 2 \rangle.$$

If m is a cycle-number, and $f^n(m) = m$, then we call n a “cycle-sequence”. We can easily see that

$$n = \sum_{i=1}^{k} l_i.$$

Theorem 1.1. *Let $m = \langle l_1, l_2, \cdots, l_k \rangle$ and $l_0 = 0$. Then we have*

$$m = \frac{\sum_{i=1}^{k} 3^{k-i} \cdot 2^{\sum_{j=0}^{i-1} l_j}}{2^n - 3^k}. \tag{1.2}$$

Theorem 1.1 was proved in [1]. The theorem shows that every cycle-number has a rational expression. So we can generalize the Collatz conjecture to rational numbers. That is, we define a function

$$f\left(\frac{a}{b}\right) = \begin{cases} \dfrac{a}{2b} & \text{if } a \text{ is even,} \\ \dfrac{3a+b}{2b} & \text{if } a \text{ is odd,} \end{cases}$$

for a/b, where a, b are positive integers such that $(a, b) = 1$. Then the rational version of the Collatz conjecture is that there exists a positive integer n which satisfies $f^n(a/b) = 1$. Except for the trivial-cycle, we know by Theorem 1.1 that there are many cycle-numbers for the rational version of the Collatz conjecture. The cycle-numbers for the original Collatz conjecture are integral cycle-numbers for the rational version of the Collatz conjecture. Therefore, the Collatz conjecture can be reduced to a problem of an exponential indeterminate equation on positive integers.

To consider the integral case, we must know when (1.2) becomes an integer. If we consider the case $2^n - 3^k = 1$ for example, we have the following:

Theorem 1.2. *The exponential indeterminate equation* $2^n - 3^k = 1$ *has only one positive integral solution* $(n, k) = (2, 1)$.

Proof. Let $n \geq 3$, then $2^n - 3^k \equiv -3^k \equiv -3$ or $-1 \not\equiv 1$ (mod.8). ∎

This solution $(n, k) = (2, 1)$ corresponds to $1 = \langle 2 \rangle$. And, the following theorem is a result in the special case, too:

Theorem 1.3. *Suppose* $m = \langle 1, 1, 1, \cdots, 1, l_k \rangle$ *is an integral cycle-number, then* $m = 1 = \langle 2 \rangle$.

Theorem 1.3 was proved in [2]. We shall prove the next two theorem in Section 3, and 4.

Theorem 1.4. *Let* m *be a cycle-number,* n *the cycle-sequence, and* k *the odd-cycle-sequence. If* $3/4 \geq 3^k/2^n$, *then* $m_1 = \langle 1, \cdots, 1, l, \cdots, l \rangle$ *is not a positive integer.*

Theorem 1.5. *Let* m *be a cycle-number,* n *the cycle-sequence, and* k *the odd-cycle-sequence. If* $1 > 3^k/2^n > 3/4$, *then the total number of positive integer of* $m_1 = \langle 1, \cdots, 1, l, \cdots, l \rangle$ *is finite.*

Combining Theorem 1.4 and Theorem 1.5, we have

Theorem 1.6. *The total number of positive integer of* $m_1 = \langle 1, \cdots, 1, l, \cdots, l \rangle$ *is finite.*

This theorem is a generalization of Theorem 1.3.

§2. Some lemmas

Lemma 2.1. *Let* $\langle l_1, l_2, \cdots, l_k \rangle = m_1$. *If* $m_i = \min\{m_1, \cdots, m_k\} > 1$, *then* $l_i = 1$.

Proof. We express m_{i+1} using m_i. If $i = k$, then let $m_{i+1} = m_1$. Then, by definition, we have

$$m_{i+1} = f^{l_i}(m_i) = \frac{3m_i + 1}{2^{l_i}}.$$

Most right side,

$$\frac{3m_i + 1}{2^{l_i}} < \frac{4m_i}{2^{l_i}} = \frac{m_i}{2^{l_i - 2}}$$

for $m_i > 1$. Let $l_i \geq 2$. Then we have

$$m_{i+1} = \frac{3m_i + 1}{2^{l_i}} < \frac{m_i}{2^{l_i - 2}} \leq m_i.$$

It is a contradiction to the assumption that m_i is the smallest. Therefore $l_i = 1$. ∎

Lemma 2.2. *$m = \langle 1, l, l, \cdots, l \rangle$ is not a positive integer.*

Proof. Let m be a positive integer, k be the odd-cycle-sequence. If $l = 1$, then the result is clear from Theorem 1.3. Assume that $l > 1$. We make $m = m_1, \cdots, m_k$ in a way similar to that of (1.1) and let $k \geq 2$. Then m_1 is the smallest. Because, if $l_i > 1$, then $m_i \neq \min\{m_1, \cdots, m_k\}$ from contraposition of Lemma 2.1. That means $\min\{m_i | l_i = 1\} = \min\{m_1, \cdots, m_k\} = m_1$.

We express m_3 using m_1. If $k = 2$, put $m_1 = m_3$. Then, by definition, we have

$$m_3 = f^{1+l}(m_1) = \frac{9m_1 + 5}{2^{l+1}}.$$

Let $l \geq 3$. Then,

$$\frac{9m_1 + 5}{2^{l+1}} \leq \frac{9m_1 + 5}{16}.$$

If $x > 5/7$, then $(9x + 5)/16 < x$. Hence we have

$$m_3 = \frac{9m_1 + 5}{2^{l+1}} \leq \frac{9m_1 + 5}{16} < m_1,$$

since m_1 is a positive integer. This contradicts the assumption that m_1 is the smallest.

Next, let $k \geq 3$, $l = 2$. And, we express m_4 using m_1. If $k = 3$, put $m_1 = m_4$. Then, we have

$$m_4 = f^{1+l+l}(m_1) = f^5(m_1) = \frac{27m_1 + 23}{32}$$

If $m > 23/5$ then $(27m + 23)/32 < m$, therefore for $m \geq 5$,

$$m_3 = \frac{27m + 23}{32} < m.$$

It is a contradiction. We know only one positive integral cycle-number if $m = 5$, i.e., $m = 1$.

Lastly, let $k = 2$, $l = 2$. Then, $m = \langle 1, 2\rangle$ and

$$m = \langle 1, 2\rangle = \frac{3+2}{2^3 - 3^2} = -5 < 0$$

It is not a positive integer. $\blacksquare$

Now, we see the case where $m_2 - m_1$ that is an integer. Because, if m_1 is an integer, then $f^{l_1}(m_1) - m_1 = m_2 - m_1$ is integral, too.

Let $m_1 = \langle 1, \cdots, 1, l, \cdots, l\rangle$, $m_2 = \langle 1, \cdots, 1, l, \cdots, l, 1\rangle$ be positive integral cycles, x be the number of one's, n be the cycle-sequence and $k \geq 2$ be the odd-cycle-sequence. Note that the number of l is $k-x$, and we get the relation $n = x + l(k - x)$. And, let $l \geq 2$, then $x \geq 2$ from Theorem 1.3 and Lemma 2.2.

By Theorem 1.1,

$$m_1 = \frac{3^{k-1} + \cdots + 2^{x-1}\cdot 3^{k-x} + 2^x \cdot 3^{k-x-1} + 2^{x+l}\cdot 3^{k-x-2} + \cdots + 2^{x+l(k-x-1)}}{2^n - 3^k}$$

and

$$m_2 = \frac{3^{k-1} + \cdots + 2^{x-1}\cdot 3^{k-x} + 2^{x-1+l}\cdot 3^{k-x-1} + 2^{x-1+2l}\cdot 3^{k-x-2} + \cdots + 2^{x-1+l(k-x)}}{2^n - 3^k}.$$

Since $m_2 > m_1$,

$$\begin{aligned} 0 < m_2 - m_1 &= \frac{(2^{x-1+l} - 2^x)\cdot 3^{k-x-1} + \cdots + (2^{x-1+l(k-x)} - 2^{x+l(k-x-1)})}{2^n - 3^k} \\ &= \frac{2^x(2^{l-1}-1)(2^{l(k-x)} - 3^{k-x})}{(2^n - 3^k)(2^l - 3)} \\ &= \frac{2^x(2^{l-1}-1)(2^{n-x} - 3^{k-x})}{(2^n - 3^k)(2^l - 3)}. \end{aligned} \tag{2.1}$$

Now, $m_2 - m_1$ is integral, $2^n - 3^k > 1$ and $2^l - 3 \geq 1$ are odd integers. It follows that

$$(2^n - 3^k)(2^l - 3) \mid (2^{l-1} - 1)(2^{n-x} - 3^{k-x}). \tag{2.2}$$

We consider the function

$$g(x) = 2^{n-x} - 3^{k-x}.$$

We have,

$$g'(x) = -2^{n-x}\log 2 + 3^{k-x}\log 3.$$

The equation $g'(x) = 0$ has only one solution

$$x = \frac{\log \frac{3^k}{2^n} + \log \frac{\log 3}{\log 2}}{\log \frac{3}{2}} = a.$$

Since $3^k/2^n < 1$,

$$a < \frac{0.461}{0.405} < 1.139.$$

Therefore, If $g(x)$ has the maximum on $x \geq 0$, then $x < 1.139$. Now, since $k \geq 2$ and $n - k > 1$,

$$g'(k) = -2^{n-k} \log 2 + \log 3 < -2 \log 2 + \log 3 = -\log 4 + \log 3 < 0.$$

So, if $a < b$, then $g(x)$ is monotone decreasing at b.

Lemma 2.3. *Let $x \geq 0$. then $g(0) \geq g(x)$ if and only if*

$$\frac{6^x - 3^x}{6^x - 2^x} \geq \frac{3^k}{2^n}.$$

Proof. By the definition of $g(x)$ and, $g(0) \geq g(x)$,

$$2^n - 3^k \geq 2^{n-x} - 3^{k-x}.$$

Thus,

$$\frac{6^x - 3^x}{6^x - 2^x} \geq \frac{3^k}{2^n}. \quad \blacksquare$$

Lemma 2.4. $3/4 \geq 3^k/2^n$ *if and only if $g(b) \geq g(a)$ for any integer a, b such that $a \geq b \geq 0$.*

Proof. $g(0) \geq g(1)$ if and only if $3/4 \geq 3^k/2^n$ by Lemma 2.3. And in this case, the equation $g'(x) = 0$ has only one solution

$$\frac{\log \frac{3^k}{2^n} + \log \frac{\log 3}{\log 2}}{\log \frac{3}{2}} < \frac{0.173}{0.405} < 1.$$

Therefore, if $1 \leq b$, then $g(x)$ is monotone decreasing at b. $\blacksquare$

Corollary 2.5. *If $3/4 \geq 3^k/2^n$ and $a \geq b \geq 1$, then $g(b) > g(a)$.*

How is the case $1 > 3^k/2^n > 3/4$? We have the following.

Lemma 2.6. *For any positive integer n, there exists at most one integer k which satisfies $1 > 3^k/2^n > 3/4$. The number k is given by $k = \lfloor n \log_3 2 \rfloor$, if it exists. $\lfloor x \rfloor$ means the greatest integer not exceeding x.*

Proof. By assumption $1 > 3^k/2^n > 3/4$, we have

$$0 > k - n \log_3 2 > -\log_3 \frac{4}{3}.$$

This implies that

$$n \log_3 2 > k > n \log_3 2 - \log_3 \frac{4}{3}. \tag{2.3}$$

Therefore, if there exists a positive integer k, then $k = \lfloor n \log_3 2 \rfloor$. ∎

Lemma 2.7. *Let $\alpha_1, \alpha_2 > 1$ be multiplicatively independent real algebraic numbers, and $D = [\mathbf{Q}(\alpha_1, \alpha_2) : \mathbf{Q}]$. Let A_1, A_2 denote real numbers > 1 such that*

$$\log A_j \geq max\{h(\alpha_j), \frac{\log \alpha_j}{D}, \frac{1}{D}\}, \qquad j = 1, 2,$$

where $h(\alpha)$ is absolute logarithmic height of α. Let b_1, b_2 are positive integers, and put

$$\Lambda = b_1 \log \alpha_1 - b_2 \log \alpha_2.$$

Then

$$\log |\Lambda| \geq -32.31 D^4 (max\{\log B + 0.18, \frac{10}{D}, \frac{1}{2}\})^2 (\log A_1)(\log A_2),$$

where

$$B = \frac{b_1}{D \log A_2} + \frac{b_2}{D \log A_1}.$$

Lemma 2.7 was proved in [8]. Now, using this lemma over rational integers we have.

Corollary 2.8. *Let $\alpha_1, \alpha_2 > 1$ be relatively prime rational integers. Let A_1, A_2 denote real numbers > 1 such that*

$$\log A_j \geq max\{\log \alpha_j, 1\}, \qquad j = 1, 2.$$

Let b_1, b_2 are positive integers, and put

$$\Lambda = b_1 \log \alpha_1 - b_2 \log \alpha_2.$$

Then

$$\log|\Lambda| \geq -32.31(max\{\log B + 0.18, 10\})^2(\log A_1)(\log A_2),$$

where

$$B = \frac{b_1}{\log A_2} + \frac{b_2}{\log A_1}.$$

§3. Proof of Theorem 1.4

In this section we shall prove Theorem 1.4. Let m_i be as in (1.1), and consider the equation (2.1). We compare $(2^n - 3^k)(2^l - 3)$ with $(2^{l-1} - 1)(2^{n-x} - 3^{k-x})$. First, we consider $2^l - 3$ and $2^{l-1} - 1$. We have

$$2^l - 3 - (2^{l-1} - 1) = 2(2^{l-2} - 1) \geq 0$$

for $l \geq 2$. Thus,

$$2^l - 3 \geq 2^{l-1} - 1. \tag{3.1}$$

Next, we consider $2^n - 3^k$ and $2^{n-x} - 3^{k-x}$. We have

$$2^n - 3^k = g(0) > g(x) = 2^{n-x} - 3^{k-x}$$

for $3/4 \geq 3^k/2^n$, by Corollary 2.5 and $x \geq 2$. Therefore,

$$(2^n - 3^k)(2^l - 3) > (2^{l-1} - 1)(2^{n-x} - 3^{k-x}).$$

It follows that

$$1 > \frac{(2^{l-1} - 1)(2^{n-x} - 3^{k-x})}{(2^n - 3^k)(2^l - 3)} > 0.$$

This means that $m_2 - m_1$ in (2.1) is not an integer, since the denominator $(2^n - 3^k)(2^l - 3)$ is an odd integer. But m_1 and m_2 are distinct positive integers for $k \geq 2$, and so $m_2 - m_1$ is a positive integer too. This is a contradiction.

§4. Proof of Theorem 1.5

In this section we shall prove Theorem 1.5. Let $1 > 3^k/2^n > 3/4$, $l \geq 2$. Then k can be expressed as

$$k = \lfloor n \log_3 2 \rfloor = n \log_3 2 + c_1$$

for $\log_3 \frac{3}{4} < c_1 < 0$ by Lemma 2.3 and (2.3), if k exists. We estimate the size of x,

$$x = n\frac{l \log_3 2 - 1}{l - 1} + c_2 \quad \left(c_2 = \frac{l}{l-1} c_1\right),$$

by (2.3) and $n = x + l(k - x)$. Hence we have

$$2^{n-x} - 3^{k-x} = 2^{n(1-\frac{l\log_3 2-1}{l-1})-c_2} - 3^{n(\log_3 2-\frac{l\log_3 2-1}{l-1})+c_1-c_2}.$$

Since the second term on the right hand is much smaller than the first term, we get,

$$|2^{n-x} - 3^{k-x}| < 2^{n(1-\frac{l\log_3 2-1}{l-1})-c_2} \le 2^{n\log_3 \frac{9}{4}-c_2},$$

for $l \ge 2$. Then, it is easy to see

$$|2^{n-x} - 3^{k-x}| < 2^{n\log_3 \frac{9}{4}+\log_3 \frac{16}{9}}. \tag{4.1}$$

On the other hand, we consider the following linear form in two logarithm:

$$\Lambda = b_1 \log \alpha_1 - b_2 \log \alpha_2 = n \log 2 - k \log 3,$$

by putting $\alpha_1 = 2, \alpha_2 = 3, b_1 = n, b_2 = k$. Using the inequality

$$\frac{|\log x|}{2} < 1 - x,$$

for $1 > x > 3/4$, we have

$$\frac{|\Lambda|}{2} = \frac{1}{2}|k\log 3 - n\log 2| = \frac{1}{2}\left|\log \frac{3^k}{2^n}\right| < 1 - \frac{3^k}{2^n}.$$

And, it follows from Corollary 2.8 that

$$\log|\Lambda| \ge -32.31H^2 \log 3$$

Hence we have

$$|2^n - 3^k| > 2^{-32.31H^2\log_2 3+n-1} \tag{4.2}$$

where $H = \max\{\log B + 0.18, 10\}$, and

$$B = \frac{n}{\log 3} + k = \frac{n}{\log 3} + n\log_3 2 + c_1 = n\frac{1+\log 2}{\log 3} + c_1.$$

First, we assume $H = 10$. Then $9.82 > \log B$. The inequality

$$9.82 > \log B = \log\left(n\frac{1+\log 2}{\log 3} + c_1\right) > \log\left(n\frac{1+\log 2}{\log 3} + \log_3 \frac{3}{4}\right)$$

says

$$n < 11938. \tag{4.3}$$

Next, we assume $H = \log B + 0.18$. We note that $|2^n - 3^k| < |2^{n-x} - 3^{k-x}|$ by (2.2), and (3.1). Hence we have

$$2^{-32.31(\log(n\frac{1+\log 2}{\log 3}+\log_3 \frac{3}{4})+0.18)^2(\log_2 3)+n-1} < 2^{n\log_3 \frac{9}{4}+\log_3 \frac{16}{9}},$$

by (4.1), (4.2). It means

$$n < 22033. \tag{4.4}$$

From (4.3) and (4.4), we have the necessary condition

$$n < 22033.$$

Since

$$22033 > n > k = \lfloor n \log_3 2 \rfloor > x = \lfloor n \frac{l \log_3 2 - 1}{l - 1} \rfloor,$$

the number of (n, k, x, l) is finite.

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References

[1] C. Bohm and G. Sontacchi, On the existence of cycles of given length in integer sequence like $x_{n+1} = x_n/2$ if x_n even, and $x_{n+1} = 3x_n + 1$ otherwise, *Atti Accad. Naz Lincei Rend. Sci. Fis. Mat. Natur.* (8) **64** (1978), 260-264.

[2] R. P. Steiner, A Thorem on the Syracuse Ploblem, Proc. 7th Manitoba Conf. Numerical Mathematics and Computing 1977 Winnipeg(1978), 553-559.

[3] R. E. Crandall, On the $3x + 1$ problem, *Math. Comput.*32(1978), 1281-1292.

[4] C. J. Everett, Iteration of the number-theoretic function $f(2n) = n, f(2n+1) = 3n + 2$, *Advances in Math.* **25**(1977), 42-45.

[5] L. A. Garner, On the Collatz $3n+1$ algorithm, *Proc. Amer. Math. Soc.* **82**(1981), 19-22.

[6] J. C. Lagarias, The $3n+1$ conjecture and its generalizations,*Am. Math. Monthly* **92**(1985), 3-23.

[7] G. J. Wirsching, The Dynamical System Generated by the $3n + 1$ Function, Lecture Notes in Mathematics **1681**, Springer-Verlag(1998).

[8] M. Laurent, M. Mignotte et Y. Nesterenko, Formes linéairies en deux logarithmes et déterminante d'interpolation, *J. Number Theory* **55**(1995), 258-321.

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