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On certain bases for Ariki-Koike algebras arising from canonical bases for $U_v(\mathfrak{sl}_m)$

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Abstract. Frenkel, Khovanov and Kirillov showed that the parabolic Kazhdan-Lusztig basis of Iwahori-Hecke algebra associated to $\mathfrak{S}_n$ can be obtained as the canonical basis of a weight subspace of $V^{\otimes n}$, where V is the vector representation of the quantum group $U_v(\mathfrak{sl}_m)$. In this paper, a similar problem for the case of Ariki-Koike algebra $\mathcal{H}_{n,r}$ is discussed. We construct a certain basis of $\mathcal{H}_{n,r}$, which is fixed by the involution and is closely related to the canonical basis of $V^{\otimes n}$, by making use of the representation of $\mathcal{H}_{n,r}$ on $V^{\otimes n}$. In the case where $r = 2$, i.e., in the case of Iwahori-Hecke algebra of type B_n, this gives a basis different from the Kazhdan-Lusztig basis of $\mathcal{H}_{n,r}$.

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Contents

§0. Introduction

Let $U_v = U_v(\mathfrak{sl}_m)$ be the quantum group associated to the Lie algebra $\mathfrak{sl}_m$, and V the vector representation of U_v. Let $\mathcal{H}_n$ be the Iwahori-Hecke algebra associated to the symmetric group $\mathfrak{S}_n$. Then the n-fold tensor space $V^{\otimes n}$

turns out to be a $U_v \otimes \mathcal{H}_n$-module. Each weight subspace $V_\lambda^{\otimes n}$ of $V^{\otimes n}$ is $\mathcal{H}_n$-stable, and is naturally isomorphic to an induced module M_J from a linear representation of some parabolic subalgebra $\mathcal{H}_J$ of $\mathcal{H}_n$. A parabolic Kazhdan-Lusztig basis on M_J was defined by Deodhar [D], by generalizing the notion of Kazhdan-Lusztig basis of $\mathcal{H}_n$ introduced by Kazhdan and Lusztig [KL].

The notion of canonical basis for highest weight modules of U_v was introduced by Lusztig [L], which is a union of canonical bases for each weight subspace. In the case of highest weight module $V^{\otimes n}$, Frenkel, Khovanov and Kirillov [FKK] showed that the canonical basis of the weight subspace $V_\lambda^{\otimes n}$ coincides with the Kazhdan-Lusztig basis of M_J under the above isomorphism. Note that $\mathcal{H}_n$ has a standard basis $\{T_\sigma \mid \sigma \in \mathfrak{S}_n\}$, and the Kazhdan-Lusztig basis of $\mathcal{H}_n$ is characterized by the property that the transition matrix between this basis and the standard basis is of the unitriangular shape, and that it is fixed by a certain involution on $\mathcal{H}_n$, called the bar involution. In turn, $V^{\otimes n}$ has also a standard basis consisting of the tensor product of the given basis of V, and the canonical basis on $V^{\otimes n}$ is characterized by a certain involution ψ on it, together with some additional property related to the standard basis. The important step for proving the result in [FKK] is to show that these two involutions coincide with under the isomorphism $M_J \simeq V_\lambda^{\otimes n}$.

Let $W_{n,r}$ be the complex reflection group $\mathfrak{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$, and $\mathcal{H}_{n,r}$ the associated cyclotomic Hecke algebra, i.e., the Ariki-Koike algebra associated to $W_{n,r}$. In the case where $r = 1$, $\mathcal{H}_{n,r} \simeq \mathcal{H}_n$, and $\mathcal{H}_{n,r}$ is isomorphic to the Iwahori-Hecke algebra of type B_n if $r = 2$. $\mathcal{H}_{n,r}$ contains $\mathcal{H}_n$ as a subalgebra, and in [SS] the action of $\mathcal{H}_n$ on $V^{\otimes n}$ was extended to the action of $\mathcal{H}_{n,r}$. Each weight space $V^{\otimes n}$ is again $\mathcal{H}_{n,r}$-stable. The aim of this paper is to extend the result of [FKK] to the case of certain induced $\mathcal{H}_{n,r}$-modules. One of our main results is Theorem 2.4, which asserts that the bar involution of $\mathcal{H}_{n,r}$ is compatible with the involution ψ on $V^{\otimes n}$. By making use of this fact, one can show, in Theorem 4.3, that the weight subspace $V_\lambda^{\otimes n}$ is isomorphic to an $\mathcal{H}_{n,r}$-module M_J induced from a “non-parabolic” subalgebra $\mathcal{H}_J$ of $\mathcal{H}_{n,r}$, and that the canonical basis of $V_\lambda^{\otimes n}$ determines a basis of M_J fixed by the bar involution of $\mathcal{H}_{n,r}$. This may be regarded as a non-parabolic analogue of the result of [FKK].

However, if one focuses on the $\mathcal{H}_{n,r}$-module M_J induced from the parabolic subalgebra $\mathcal{H}_J$ of $\mathcal{H}_{n,r}$, for example $\mathcal{H}_{n,r}$ itself, the situation is much more complicated. There is no natural notion of standard basis nor Kazhdan-Lusztig basis of $\mathcal{H}_{n,r}$ for $r > 2$. Moreover, M_J turns out to be a direct sum of various weight subspaces $V_\lambda^{\otimes n}$. In order to treat these cases, we make use of the new generators of $\mathcal{H}_{n,r}$ introduced by [S]. By using the direct sum decomposition $M_J = \bigoplus_\lambda V_\lambda^{\otimes n}$, one can define two bases of M_J inherited from the standard basis and the canonical basis of $\bigoplus_\lambda V_\lambda^{\otimes n}$. As a special case, we can construct two bases of $\mathcal{H}_{n,r}$ in Theorem 4.7; the one has a property that the action of generators of $\mathcal{H}_{n,r}$ on this basis is explicitly described, and the other has a property that it is fixed by the bar involution on $\mathcal{H}_{n,r}$, and the transition matrix between these two bases is described by various parabolic Kazhdan-Lusztig polynomials of type A.

We remark that even in the case where $r = 2$ (i.e., the case of Iwahori-Hecke algebras of type B_n), our basis does not coincide with the Kazhdan-Lusztig basis of $\mathcal{H}_{n,r}$. In section 5, we discuss the relationship between these two bases, with the standard basis and the Kazhdan-Lusztig basis of $\mathcal{H}_{n,r}$. In particular we show in Proposition 5.2 that the parabolic Kazhdan-Lusztig polynomials of type B_n can be determined uniquely by various parabolic Kazhdan-Lusztig polynomials of type A, together with the information on the transition matrix between the standard basis of $\mathcal{H}_{n,r}$ and the standard basis of $\bigoplus_\lambda V_\lambda^{\otimes n}$.

§1. Review on Ariki-Koike algebras

1.1. Let $K = \mathbb{Q}(v, u_1, \dots, u_r)$ be a field of rational functions in variables $v, u_1, \dots, u_r$. Let $W = W_{n,r}$ be the complex reflection group $\mathfrak{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$, and $\mathcal{H}_{n,r}$ the Ariki-Koike algebra associated to W. $\mathcal{H}_{n,r}$ is the associative algebra over K with generators $a_1, \dots, a_n$, and relations

$$
\begin{aligned}
&(a_1 - u_1)(a_1 - u_2)\cdots(a_1 - u_r) = 0, && \\
&(a_i - v)(a_i + v^{-1}) = 0 && (2 \le i \le n), \\
&a_1a_2a_1a_2 = a_2a_1a_2a_1, && \\
&a_ia_{i+1}a_i = a_{i+1}a_ia_{i+1} && (2 \le i < n), \\
&a_ia_j = a_ja_i && (|i-j| \ge 2).
\end{aligned}
\tag{1.1.1}
$$

It is known that the subalgebra $\mathcal{H}_n$ of $\mathcal{H}_{n,r}$ generated by $a_2, \dots, a_n$ is isomorphic to the Hecke algebra associated to the symmetric group $\mathfrak{S}_n$ with standard generators.

1.2. Let $U_v = U_v(\mathfrak{sl}_m)$ be the quantum group associated to the Lie algebra $\mathfrak{sl}_m$ with generators E_i, F_i, K_i $(1 \le i \le m-1)$ and standard relations. Apriori, U_v is an associative algebra over $\mathbb{Q}(v)$, but for later discussion, we regard them as an algebra over K by an extension of scalars.

Let V be an m-dimensional vector space over K with basis $e_1, \dots, e_m$. The vector representation of U_v on V is defined by

$$
\begin{aligned}
&E_ie_{i+1} = e_i, \quad E_ie_j = 0 \quad j \ne i+1, \\
&F_ie_i = e_{i+1}, \quad F_ie_j = 0 \quad j \ne i,
\end{aligned}
$$

$$K_ie_j = \begin{cases} ve_i & j = i, \\ v^{-1}e_{i+1} & j = i+1, \\ e_j & j \neq i, i+1. \end{cases}$$

It is known that U_v has a Hopf algebra structure with comultiplication $\Delta : U_v \to U_v \otimes U_v$ given by

$$\begin{aligned} \Delta(K_i) &= K_i \otimes K_i, \\ \Delta(E_i) &= E_i \otimes 1 + K_i \otimes E_i, \\ \Delta(F_i) &= F_i \otimes K_i^{-1} + 1 \otimes F_i. \end{aligned}$$

For a positive integer n, we consider the tensor space $V^{\otimes n}$ on which $U_v^{\otimes n}$ acts naturally. We define inductively an algebra homomorphism $\Delta^{(k)} : U_v \to U_v^{\otimes k}$, by starting from $\Delta^{(2)} = \Delta$ and by putting $\Delta^{(k)} = (\Delta^{(k-1)} \otimes \mathrm{id}) \circ \Delta$ for each $k \geq 3$. By using $\Delta^{(n)}$, one can define an action of U_v on $V^{\otimes n}$.

1.3. In [Ji], Jimbo constructed an action of $\mathcal{H}_n$ on $V^{\otimes n}$, commuting with the action of $U_v(\mathfrak{sl}_m)$. Let us fix integers $m_1, \dots, m_r$ such that $\sum m_i = m$, and consider a Levi subalgebra $\mathfrak{g} = \mathfrak{sl}_{m_1} \oplus \cdots \oplus \mathfrak{sl}_{m_r}$ of $\mathfrak{sl}_m$. The action of $\mathcal{H}_n$ was extended by [SS] to the action of $\mathcal{H}_{n,r}$ on $V^{\otimes n}$ so that it commutes with the action of the subalgebra $U_v(\mathfrak{g})$ of $U_v(\mathfrak{sl}_m)$. We consider the decomposition $V = \bigoplus_i V_i$ with $\dim V_i = m_i$. We assume that a basis $\{e_j^{(k)}\}$ $(1 \leq j \leq m_k)$ of V_k is chosen for $k = 1, \dots, r$, and that

$$e_1^{(1)}, \dots, e_{m_1}^{(1)}, e_1^{(2)}, \dots, e_{m_2}^{(2)}, \dots, e_1^{(r)}, \dots, e_{m_r}^{(r)}$$

gives the basis $e_1, \dots, e_m$ of V in this order. The construction of the action of $\mathcal{H}_{n,r}$ on $V^{\otimes n}$ is given as follows. Let T be the element in $\mathrm{End}(V \otimes V)$ defined by

$$T(e_i \otimes e_j) = \begin{cases} ve_j \otimes e_i & \text{if } i = j, \\ e_j \otimes e_i & \text{if } i > j, \\ e_j \otimes e_i + (v - v^{-1})e_i \otimes e_j & \text{if } i < j. \end{cases} \tag{1.3.1}$$

Next we define a map $b : \{1, 2, \dots, m\} \to \mathbb{N}$ by $b(j) = k$ whenever $e_j \in V_k$. Let $wt : V \to V$ be a linear operator defined by $wt(e_j) = u_{b(j)}e_j$. Let us define linear operators, σ, S on $V^{\otimes 2}$ as follows.

$$\sigma(e_i \otimes e_j) = e_j \otimes e_i,$$

$$S(e_i \otimes e_j) = \begin{cases} T(e_i \otimes e_j) & \text{if } b(i) = b(j), \\ \sigma(e_i \otimes e_j) & \text{if } b(i) \neq b(j). \end{cases}$$

Using these operators, we define operators $T_i, \sigma_i, S_i, \omega_j \in \operatorname{End} V^{\otimes n}, (2 \le i \le n), (1 \le j \le n)$, by the condition,

$$
\begin{aligned}
T_i &= \mathrm{id}_V^{\otimes(i-2)} \otimes T \otimes \mathrm{id}_V^{\otimes(n-i)},\\
\sigma_i &= \mathrm{id}_V^{\otimes(i-2)} \otimes \sigma \otimes \mathrm{id}_V^{\otimes(n-i)},\\
S_i &= \mathrm{id}_V^{\otimes(i-2)} \otimes S \otimes \mathrm{id}_V^{\otimes(n-i)},\\
\omega_j &= \mathrm{id}_V^{\otimes(j-1)} \otimes wt \otimes \mathrm{id}_V^{\otimes(n-j)}.
\end{aligned}
\tag{1.3.2}
$$

We now define an operator T_1 on $V^{\otimes n}$ by

$$
T_1 = T_2^{-1} \cdots T_n^{-1} S_n \cdots S_2 \omega_1. \tag{1.3.3}
$$

Then it is shown in [SS, Th.3.2] that $\tau : a_i \mapsto T_i$ $(1 \le i \le n)$ gives rise to a representation of $\mathcal{H}_{n,r}$ on $V^{\otimes n}$.

Let $^- : K \to K$ be the unique $\mathbb{Q}$-algebra involution such that $\bar{v} = v^{-1}, \bar{u}_i = u_i^{-1}$ for $i = 1, \dots, r$. We say that a map ϕ on a K vector space X is antilinear if $\phi(\lambda x) = \bar{\lambda}\phi(x)$ for $\lambda \in K, x \in X$. One can check by (1.1.1) that there exists a unique antilinear $\mathbb{Q}$-algebra automorphism $a \mapsto \bar{a}$ on $\mathcal{H}_{n,r}$ such that $\bar{a}_i = a_i^{-1}$ $(1 \le i \le n)$. We call this map the bar involution on $\mathcal{H}_{n,r}$.

1.4. Recall that $\mathcal{H}_{n,r}$ has an alternative presentation given in [S, Th.3.7] as follows. (However, we remark that this presentation only admits a specialization of the type $\varphi : K \to K'$, where K' is a field such that $\varphi(\xi_i)$ are all distinct.) $\mathcal{H}_{n,r}$ is generated by $\{a_2, \dots, a_n, \xi_1, \dots, \xi_n\}$, subject to the following relations.

$$
\begin{aligned}
(a_i - v)(a_i + v^{-1}) &= 0 && (2 \le i \le n)\\
(\xi_i - u_1) \cdots (\xi_i - u_r) &= 0 && (1 \le i \le n)\\
a_i a_{i+1} a_i &= a_{i+1} a_i a_{i+1} && (2 \le i \le n)\\
a_i a_j &= a_j a_i && (|i - j| \ge 2)\\
\xi_i \xi_j &= \xi_j \xi_i && (1 \le i, j \le n),
\end{aligned}
\tag{1.4.1}
$$

$$
a_j \xi_j = \xi_{j-1} a_j + \varDelta^{-2} \sum_{c_1 < c_2} (u_{c_2} - u_{c_1})(v - v^{-1}) F_{c_1}(\xi_{j-1}) F_{c_2}(\xi_j), \tag{1.4.2}
$$

$$
a_j \xi_{j-1} = \xi_j a_j - \varDelta^{-2} \sum_{c_1 < c_2} (u_{c_2} - u_{c_1})(v - v^{-1}) F_{c_1}(\xi_{j-1}) F_{c_2}(\xi_j), \tag{1.4.3}
$$

$$
a_j \xi_k = \xi_k a_j \quad (k \ne j - 1, j), \tag{1.4.4}
$$

where $\varDelta = \prod_{i > j}(u_i - u_j)$ is the Vandermonde determinant with respect to the parameters $u_1, \dots, u_r$, and the sum in (1.4.2) or (1.4.3) is taken for all integers $1 \le c_1, c_2 \le r$. For each integer $1 \le c \le r$, $F_c(X)$ is a certain polynomial in a variable X with coefficients in $\mathbb{Z}[u_1, \dots, u_r]$, defined in [S, 3.3.2]. Note that the generators $a_2, \dots, a_n$ above may be identified with the generators appeared in 1.1.

Under the representation $\tau : \mathcal{H}_{n,r} \to \operatorname{End} V^{\otimes n}$, the generator ξ_i is mapped to ω_i for each i.

§2. Involutions associated to $\mathcal{H}_{n,r}$ and $U_v(\mathfrak{sl}_m)$

2.1 The operator T in (1.3.1) has its origin in the study of the universal R-matrix Θ attached to U_v. Let E be the orthogonal complement of $\sum \varepsilon_i$ in the Euclidean space $\mathbb{R}^m$ with the standard basis $\varepsilon_1, \dots, \varepsilon_m$. The root system Φ of $\mathfrak{sl}_m$ is given by the set $\{\varepsilon_i - \varepsilon_j \mid 1 \le i \ne j \le m\}$ with $\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid i < j\}$. Thus the root lattice $\mathbb{Z}\Phi$ is given by the $\mathbb{Z}$-submodule of E consisting of integral vectors. Let $(\ , \ )$ be the inner product of $\mathbb{R}^m$. The weight lattice Λ is given by the set of $\lambda \in E$ such that $(\lambda, \mu) \in \mathbb{Z}$ for any $\mu \in \Phi$. For $1 \le i \le m$, put $\bar{\varepsilon}_i = \varepsilon_i - \frac{1}{m}\sum_{j=1}^m \varepsilon_j \in \Lambda$. Then $\sum \bar{\varepsilon}_i = 0$, and $\bar{\varepsilon}_i$ is a weight with weight vector e_i. The weight lattice Λ is identified with the set $\mathbb{Z}^m/\mathbb{Z}(1, \dots, 1)$ by the correspondence $\lambda = \sum c_i\bar{\varepsilon}_i \leftrightarrow (c_1, \dots, c_m)$. For a U_v-module M, we denote by M_λ the weight subspace of M corresponding to $\lambda \in \Lambda$.

Let U_v^+ (resp. U_v^-) be the subalgebra of U_v generated by E_i, K_i (resp. F_i, K_i), respectively. For each $\mu \in \mathbb{Z}\Phi, \mu \ge 0$, we denote by $U_\mu^+, U_{-\mu}^-$ the weight subspace of $U_v^\pm$ with respect to μ or $-\mu$, respectively. Then there exists an element $\Theta_\mu \in U_{-\mu}^- \otimes U_\mu^+$ with $\Theta_0 = 1 \otimes 1$, for each μ, and $\Theta = \sum_{\mu \ge 0} \Theta_\mu$ (an element in a completion of $U_v \otimes U_v$, see [L, 4.1]) can be defined.

Let M and M' be finite dimensional U_v-modules. We fix an m-th root $v^{1/m}$ of v, and consider the extension field $K(v^{1/m})$ of K. (Accordingly, we regard U_v as the algebra over $K(v^{1/m})$ if needed). Following [Ja, 7.3, 7.9], we introduce a linear map $C' \in \operatorname{End} M \otimes M'$ ($\widetilde{f}$ in the notation of [Ja]). We define a map $f : \Lambda \times \Lambda \to K(v^{1/m})^*$ by

$$f(\lambda, \mu) = (v^{1/m})^{-m(\lambda,\mu)}$$

for all $\lambda, \mu \in \Lambda$. Note that $(\lambda, \mu) \in \frac{1}{m}\mathbb{Z}$. In particular, we have

$$f(\bar{\varepsilon}_i, \bar{\varepsilon}_j) = v^{1/m - \delta_{ij}}. \tag{2.1.1}$$

Now C' is defined, for $\lambda, \mu \in \Lambda$, by

$$C'(x \otimes y) = f(\lambda, \mu) x \otimes y$$

for all $x \in M_\lambda$ and $y \in M_\mu$.

The element Θ induces a well-defined map $\Theta_{M,M'} \in \operatorname{End} M \otimes M'$. It is known ([Ja, Th. 7.3]) that the map $\Theta_{M,M'}C'\sigma : M' \otimes M \to M \otimes M'$ gives rise to an isomorphism of U_v-modules, where $\sigma : M' \otimes M \to M \otimes M'$ is the permutation of factors.

2.2 The bar involution on K can be extended obviously to an involution on $K(v^{1/n})$. The bar involution $^-$ on U_v is an antilinear $\mathbb{Q}$-algebra automorphism on U_v defined on the generators by

$$\overline{E}_i = E_i, \quad \overline{F}_i = F_i, \quad \overline{K}_i = K_i^{-1}.$$

The bar involution is extended to $U_v \otimes U_v$ by $\overline{x \otimes y} = \bar{x} \otimes \bar{y}$. Let $\overline{\Theta} = {}^- \circ \Theta \circ {}^-$ be the bar conjugate of Θ. Then it is known by [L, 4.1] that $\Theta\overline{\Theta} = 1 \otimes 1$.

We consider the special case where $M = M' = V$, and write $\Theta_{V,V} \in \mathrm{End}(V \otimes V)$ simply as Θ. Then as is well-known (cf. [FKK, Prop. 2.1]), we have

$$(\Theta C' \sigma)^{-1} = v^{-1/m} T. \tag{2.2.1}$$

More precisely, the action of C' and $\Theta = \sum \Theta_\mu$ on $V \otimes V$ are described as follows. Since $e_i \in V$ is a weight vector with weight $\bar{\varepsilon}_i \in \Lambda$, by the property of Θ_μ (cf. [Ja, Chap. 7]), we have

$$\Theta_\mu(e_i \otimes e_j) = \begin{cases} (v^{-1} - v) e_j \otimes e_i & \text{if } \mu = \varepsilon_i - \varepsilon_j \text{ with } i < j, \\ e_i \otimes e_j & \text{if } \mu = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2.2}$$

It follows that

$$\Theta(e_i \otimes e_j) = \begin{cases} e_i \otimes e_j & \text{if } i \geq j, \\ e_i \otimes e_j - (v - v^{-1}) e_j \otimes e_i & \text{if } i < j. \end{cases} \tag{2.2.3}$$

Put $C = v^{-1/m} C'$. Then by (2.1.1), we have

$$C(e_i \otimes e_j) = \begin{cases} e_i \otimes e_j & i \neq j, \\ v^{-1} e_i \otimes e_j & i = j. \end{cases} \tag{2.2.4}$$

We define an antilinear involution ψ on $V^{\otimes n}$ inductively as follows: First define ψ on V by

$$\psi(\sum c_i e_i) = \sum \bar{c}_i e_i.$$

Next let W_1, W_2 be tensor powers of V, and assume that the involutions ψ on W_1, W_2 are already defined. We define ψ on $W_1 \otimes W_2$ by

$$\psi(w_1 \otimes w_2) = \Theta(\psi(w_1) \otimes \psi(w_2)).$$

Then it is shown in [L, 4.2.4, 27.3.6] that ψ on $V^{\otimes n}$ does not depend on the decomposition $V^{\otimes n} = W_1 \otimes W_2$, and it is compatible with the U_v-module structure of $V^{\otimes n}$ in the following sense: $\psi(ux) = \bar{u}\psi(x)$ for $u \in U_v, x \in V^{\otimes n}$.

In [FKK], Frenkel, Khovanov and Kirillov studied the relationship between Kazhdan-Lusztig basis of $\mathcal{H}_n$ and canonical basis of U_v by making use of the $\mathcal{H}_n \otimes U_v$-module $V^{\otimes n}$. In particular, they showed

Proposition 2.3 ([FKK, Prop. 2.4]). *The bar involution of $\mathcal{H}_n$ is compatible with the involution ψ on $V^{\otimes n}$, i.e., for any $a \in \mathcal{H}_n$, we have*

$$\psi \circ a = \overline{a} \circ \psi.$$

The main objective in this section is to extend this result to the case of $\mathcal{H}_{n,r}$. We shall show that

Theorem 2.4. *The bar involution on $\mathcal{H}_{n,r}$ is compatible with the involution ψ on $V^{\otimes n}$, i.e., for any $a \in \mathcal{H}_{n,r}$, we have*

$$\psi \circ a = \overline{a} \circ \psi. \tag{2.4.1}$$

2.5. The remainder of this section is devoted to the proof of Theorem 2.4. We denote by e_I with $I = (i_1, \dots, i_n)$ the vector $e_{i_1} \otimes \cdots \otimes e_{i_n}$ of $V^{\otimes n}$. Hence $\{e_I \mid I \in [1,m]^n\}$ gives a basis of $V^{\otimes n}$. (Here $[1,m]$ means the set $\{1, 2, \dots, m\}$). The symmetric group $\mathfrak{S}_n$ acts on $[1,m]^n$ by permuting the factors, compatible with the action on $V^{\otimes n}$, i.e. $\sigma(e_I) = e_{\sigma I}$ for $\sigma \in \mathfrak{S}_n$. If we denote by $m_I(i)$ the multiplicity of i occurring in $I = (i_1, \dots, i_n)$, then e_I is a weight vector of U_v-module $V^{\otimes n}$ with weight $\sum_i m_I(i)\bar{\varepsilon}_i$.

We define an antilinear involution $^-$ on $V^{\otimes n}$ by $\bar{x} = \sum_I \bar{c}_I e_I$ for $x = \sum_I c_i e_I$. Let Ψ_i be a linear map on $V^{\otimes n}$ defined by

$$\Psi_i = (\mathit{\Delta}^{(i-1)} \otimes 1)\mathit{\Theta}) \otimes 1^{\otimes(n-i)}.$$

Then it follows from the definition that the involution ψ can be expressed as

$$\psi = \Psi_n \Psi_{n-1} \cdots \Psi_2 \circ {}^-. \tag{2.5.1}$$

In order to describe the involution ψ, first we shall concentrate on the map $\Psi_n = (\mathit{\Delta}^{(n-1)} \otimes 1)\mathit{\Theta}$. We prepare some notation. By $z \mapsto z_{ij}$, we denote the embedding $U_v^{\otimes 2} \to U_v^{\otimes n}$ subject to the i-th and j-th factors, i.e., for $z = a \otimes b \in U_v \otimes U_v$, we put

$$z_{ij} = x_1 \otimes x_2 \otimes \cdots \otimes x_n$$

with $x_i = a, x_j = b$ and $x_k = 1$ for $k \neq i, j$.

For $\alpha,\beta\in\mathbb{Z}\Phi$ with $\alpha\ge\beta\ge 0$, we define $X^i_{\alpha,\beta}\in U_v^{\otimes n}$ by

$$X^i_{\alpha,\beta}=(1^{\otimes(n-i)}\otimes K_\alpha^{-1}\otimes 1^{\otimes(i-1)})(\Theta_{\alpha-\beta})_{n-i,n}\qquad(2\le i\le n-1),\tag{2.5.2}$$

where $K_\alpha=\prod K_i^{m_i}$ if $\alpha=\sum_i m_i(\varepsilon_i-\varepsilon_{i+1})$. The following lemma is a generalization of [J, Lemma 7.4]. The proof is reduced to the case $n=3$ by making use of the relation $\Delta^{(n)}\otimes 1=(\Delta\otimes 1^{\otimes(n-1)})(\Delta^{(n-1)}\otimes 1)$ (note that this relation is different from the defining relation for $\Delta^{(n)}$ in 1.2). The case $n=3$ follows from Lemma 7.4 in [J].

Lemma 2.6. *For all $\mu\in\mathbb{Z}\Phi$ with $\mu\ge 0$, we have*

$$(\Delta^{(n-1)}\otimes 1)\Theta_\mu=\sum(\Theta_{\mu-\nu_1})_{n-1,n}X^2_{\nu_1,\nu_2}\cdots X^{n-1}_{\nu_{n-2},\nu_{n-1}},$$

where the sum is taken over all the sequences $\mu\ge\nu_1\ge\cdots\ge\nu_{n-1}=0$ such that $\nu_i\in\mathbb{Z}\Phi$.

2.7. We shall describe $\Psi_n=(\Delta^{(n-1)}\otimes 1)\Theta$. For $I=(i_1,\dots,i_n)$, let $(\eta_1,\dots,\eta_n)$ be the sequence defined by $\eta_k=\varepsilon_{i_k}$. Let us define a linear map $\Theta^\sharp_{k,n}$ on $V^{\otimes n}$, for $k=1,\dots,n-1$, by

$$\Theta^\sharp_{k,n}(e_I)=\begin{cases}e_I & \text{if } i_k\ge i_n,\\ e_I+v^{-(\eta_k-\eta_n,\eta_{k+1}+\cdots+\eta_{n-1})}(v^{-1}-v)e_{I'} & \text{if } i_k<i_n,\end{cases}$$

where $I'=(k,n)I$. (In the case where $k=n-1$, we understand that the inner product in the second formula is equal to 0).

We have the following lemma.

Lemma 2.8. *As operators on $V^{\otimes n}$, we have*

$$(\Delta^{(n-1)}\otimes 1)\Theta=\Theta^\sharp_{n-1,n}\Theta^\sharp_{n-2,n}\cdots\Theta^\sharp_{1,n}.\tag{2.8.1}$$

Proof. First we compute $X^{n-k}_{\alpha,\beta}(e_I)$ for $I=(i_1,\dots,i_n)$. By (2.2.2) and (2.5.2), we see that

(2.8.2)
$$X^{n-k}_{\alpha,\beta}(e_I)=\begin{cases}v^{-(\alpha,\eta_{k+1})}(v^{-1}-v)e_{I'} & \text{if } \alpha-\beta=\eta_k-\eta_n>0 \text{ and } i_k<i_n,\\ v^{-(\alpha,\eta_{k+1})}e_I & \text{if } \alpha=\beta,\\ 0 & \text{otherwise,}\end{cases}$$

with $I'=(k,n)I$.

Next we compute $(\Delta^{(n-1)}\otimes 1)\Theta(e_I)$. For a fixed $I=(i_1,\dots,i_n)$, let $\mathcal{P}_I$ be the set of subsets $\mathbf{p}=\{p_1<\cdots<p_k\}$ of $\{1,\dots,n-1\}$ such that $i_{p_k}<\cdots<i_{p_2}<i_{p_1}<i_n$. We put $k=|\mathbf{p}|$. For $\mathbf{p}\in\mathcal{P}_I$, let

$$I(\mathbf{p})=(p_k,n)\cdots(p_2,n)(p_1,n)I=(n,p_1,p_2,\dots,p_k)I.$$

Then it follows from Lemma 2.6 and (2.8.2) that

$$(\varDelta^{(n-1)}\otimes 1)\varTheta(e_I)=\sum_{\mathbf{p}\in\mathcal{P}_I}v^{-c_{\mathbf{p}}}(v^{-1}-v)^{|\mathbf{p}|}e_{I(\mathbf{p})},$$

where

$$\begin{aligned}c_{\mathbf{p}}&=(\eta_{p_1}-\eta_n,\eta_{p_1+1}+\cdots+\eta_{p_2})\\&\quad+(\eta_{p_2}-\eta_n,\eta_{p_2+1}+\cdots+\eta_{p_3})\\&\quad+\quad\cdots\cdots\\&\quad+(\eta_{p_k}-\eta_n,\eta_{p_k+1}+\cdots+\eta_{n-1}).\end{aligned}$$

On the other hand, the right hand side of (2.8.1) is easily computed. We have

$$\varTheta^{\sharp}_{n-1,n}\varTheta^{\sharp}_{n-2,n}\cdots\varTheta^{\sharp}_{1,n}(e_I)=\sum_{\mathbf{p}\in\mathcal{P}_I}v^{-d_{\mathbf{p}}}(v^{-1}-v)^{|\mathbf{p}|}e_{I(\mathbf{p})}$$

with

$$\begin{aligned}d_{\mathbf{p}}&=(\eta_{p_1}-\eta_n,\eta_{p_1+1}+\cdots+\eta_{n-1})\\&\quad+(\eta_{p_2}-\eta_{p_1},\eta_{p_2+1}+\cdots+\eta_{n-1})\\&\quad+\quad\cdots\cdots\\&\quad+(\eta_{p_k}-\eta_{p_{k-1}},\eta_{p_k+1}+\cdots+\eta_{n-1}).\end{aligned}$$

But then we have

$$d_{\mathbf{p}}=\sum_{j=1}^{k}(\eta_{p_j},\eta_{p_j+1}+\cdots+\eta_{p_{j+1}})-(\eta_n,\eta_{p_1+1}+\cdots+\eta_{n-1}),$$

where we use the convention that $p_{k+1}=n-1$. This implies that $c_{\mathbf{p}}=d_{\mathbf{p}}$ for any $\mathbf{p}\in\mathcal{P}_I$, and the lemma follows. $\square$

2.9. For a fixed $1\le i,j\le n$ with $i\ne j$, we define an embedding $\operatorname{End}V^{\otimes 2}\to\operatorname{End}V^{\otimes n}, x\mapsto x_{ij}$, in a similar way as in 2.5; x_{ij} denotes the transformation on $V^{\otimes n}$ which acts on i-th and j-th factors of $V^{\otimes n}$ via the map x, and acts trivially on other factors. Then it is easy to see for any $\sigma\in\mathfrak{S}_n$ that

$$\sigma x_{ij}\sigma^{-1}=x_{\sigma(i)\sigma(j)}. \tag{2.9.1}$$

In later discussions, we consider the operators $\varTheta_{ij},C_{ij},T_{ij},S_{ij}$ for $\varTheta,C,T,S\in\operatorname{End}V^{\otimes 2}$, respectively. In particular, we note that $T_{i-1,i}=T_i$ (resp. $S_{i-1,i}=$

S_i) for $i = 2, \dots, n$ in the notation of (1.3.2). We also note that $\varTheta_{i-1,i} = \varTheta^\sharp_{i-1,i}$ by (2.2.3) and 2.7. However, $\varTheta^\sharp_{ij}$ does not mean the embedding in general.

Let $\overline{\varTheta}^\sharp_{ij}$ be the linear transformation on $V^{\otimes n}$ defined by $\overline{\varTheta}^\sharp_{ij} = {}^- \circ \varTheta^\sharp_{ij} \circ {}^-$. Then $\overline{\varTheta}^\sharp_{ij}$ coincides with the map defined in 2.7, but by replacing v by v^{-1}. The bar operation on $V^{\otimes 2}$ is compatible with the bar operation on $\varTheta$. It follows that $\overline{\varTheta}_{ij} = {}^- \circ \varTheta_{ij} \circ {}^-$. The following relations are easily verified.

$$\begin{aligned} {}^- \circ \omega_i \circ {}^- &= \omega_i^{-1}, \\ {}^- \circ C_{ij} \circ {}^- &= C_{ij}^{-1}, \\ {}^- \circ T_{ij} \circ {}^- &= T_{ji}^{-1}, \\ {}^- \circ S_{ij} \circ {}^- &= S_{ji}^{-1}. \end{aligned} \tag{2.9.2}$$

For a pair k, n such that $1 \le k \le n-1$, we put

$$D_{k,n} = \sigma_{k,n} C_{k,n} C_{k+1,n} \cdots C_{n-1,n},$$

where $\sigma_{k,n}$ denotes the cyclic permutation $(k, k+1, \dots, n)$. We have the following lemma.

Lemma 2.10. *For $1 \le k \le n-1$, we have*

$$\overline{\varTheta}^\sharp_{k,n} \overline{\varTheta}^\sharp_{k,n-1} \cdots \overline{\varTheta}^\sharp_{k,k+1} = D_{k,n} T_n T_{n-1} \cdots T_{k+1}. \tag{2.10.1}$$

Proof. First consider the case where $k = n-1$. It follows from (2.2.1) that we have

$$T_n = (n-1, n) C^{-1}_{n-1,n} \overline{\varTheta}^\sharp_{n-1,n} \tag{2.10.2}$$

since $\varTheta\overline{\varTheta} = 1$ and $\varTheta_{n-1,n} = \varTheta^\sharp_{n-1,n}$. Since $D_{n-1,n} = (n-1, n) C_{n-1,n} = C_{n-1,n}(n-1, n)$, we have $\overline{\varTheta}^\sharp_{n-1,n} = D_{n-1,n} T_n$ as asserted.

Next we show that

$$\overline{\varTheta}^\sharp_{k,n} D_{k,n-1} = D_{k,n} T_n \qquad \text{for } 1 \le k \le n-2. \tag{2.10.3}$$

By (2.9.1) we have

$$\begin{aligned} (n-1, n) & C_{k,n} C_{k+1,n} \cdots C_{n-1,n} T_n \\ &= C_{k,n-1} C_{k+1,n-1} \cdots C_{n-2,n-1} C_{n-1,n} (n-1, n) T_n \\ &= C_{k,n-1} C_{k+1,n-1} \cdots C_{n-2,n-1} \overline{\varTheta}^\sharp_{n-1,n}. \end{aligned}$$

The last formula follows from (2.10.2). In order to show (2.10.3), we have only to check that

$$\overline{\varTheta}^\sharp_{k,n} \sigma_{k,n-1} C_{k,n-1} \cdots C_{n-2,n-1} = \sigma_{k,n-1} C_{k,n-1} \cdots C_{n-2,n-1} \overline{\varTheta}^\sharp_{n-1,n}.$$

It is easy to evaluate the maps on both sides at e_I. For e_I with $I = (i_1, \dots, i_n)$, they have the common values

$$v^{-(\eta_k+\eta_{k+1}+\cdots+\eta_{n-2},\eta_{n-1})}e_{I'}$$

if $i_{n-1} \geq i_n$, and

$$v^{-(\eta_k+\eta_{k+1}+\cdots+\eta_{n-2},\eta_{n-1})}e_{I'} + v^{-(\eta_k+\eta_{k+1}+\cdots+\eta_{n-2},\eta_n)}(v - v^{-1})e_{I''}$$

if $i_{n-1} < i_n$, with $I' = \sigma_{k,n-1}I$ and $I'' = \sigma_{k,n}I'$. This proves (2.10.3).

Now the lemma is immediate by substituting $\overline{\varTheta}^\sharp_{k,i} = D_{k,i}T_iD^{-1}_{k,i-1}$ for $i \geq k+2$ by (2.10.3), and $\overline{\varTheta}^\sharp_{k,k+1} = D_{k,k+1}T_{k+1}$ into the left hand side of (2.10.1). □

By using Lemma 2.10, we can describe the involution ψ as follows.

Proposition 2.11. *Let $\sigma_0 = (1,n)(2,n-1)\cdots$ be the longest length element in $\mathfrak{S}_n$, and put*

$$\widehat{C} = \prod_{1\leq i<j\leq n} C_{ij}.$$

(Note that the operators C_{ij} commute with each other). Then we have

$$\psi = {}^- \circ \sigma_0\widehat{C}T_2(T_3T_2)\cdots(T_nT_{n-1}\cdots T_2).$$

Proof. By (2.5.1) and Lemma 2.8, we have (cf. 2.9)

$$\begin{aligned}{}^- \circ \psi &= {}^- \circ (\varTheta^\sharp_{n-1,n}\varTheta^\sharp_{n-2,n}\cdots\varTheta^\sharp_{1,n})(\varTheta^\sharp_{n-2,n-1}\varTheta^\sharp_{n-3,n-1}\cdots\varTheta^\sharp_{1,n-1})\cdots(\varTheta^\sharp_{12}) \circ {}^- \\ &= (\overline{\varTheta}^\sharp_{n-1,n}\overline{\varTheta}^\sharp_{n-2,n}\cdots\overline{\varTheta}^\sharp_{1,n})(\overline{\varTheta}^\sharp_{n-2,n-1}\overline{\varTheta}^\sharp_{n-3,n-1}\cdots\overline{\varTheta}^\sharp_{1,n-1})\cdots(\overline{\varTheta}^\sharp_{12}).\end{aligned}$$

It is clear that $\overline{\varTheta}^\sharp_{ij}$ and $\overline{\varTheta}^\sharp_{i'j'}$ commute with each other when $\{i,j\}\cap\{i',j'\} = \emptyset$. Hence we have

$$\begin{aligned}{}^- \circ \psi &= (\overline{\varTheta}^\sharp_{n-1,n})(\overline{\varTheta}^\sharp_{n-2,n}\overline{\varTheta}^\sharp_{n-2,n-1})\cdots(\overline{\varTheta}^\sharp_{1,n}\overline{\varTheta}^\sharp_{1,n-1}\cdots\overline{\varTheta}^\sharp_{12}) \\ &= (D_{n-1,n}T_n)(D_{n-2,n}T_nT_{n-1})\cdots(D_{1,n}T_nT_{n-1}\cdots T_2),\end{aligned}$$

where the second equality follows from Lemma 2.10. By definition of D_{ij}, and by using (2.9.1), the last formula is modified to

(2.11.1)
$${}^- \circ \psi = \sigma_0(C_{12}T_2)(C_{13}C_{23}T_3T_2)\cdots(C_{1,n}C_{2,n}\cdots C_{n-1,n}T_nT_{n-1}\cdots T_2).$$

Here we note that

(2.11.2) The product $C_{1,k}C_{2,k}\cdots C_{k-1,k}$ commutes with $T_2, T_3, \dots, T_{k-1}$.

In fact, (2.11.2) is reduced to showing that $C_{a-1,k}C_{a,k}$ commutes with T_a, and this follows from the fact that $C_{a-1,k}C_{a,k}$ acts on the subspace of $V^{\otimes n}$ generated by e_I and $e_{(a-1,a)I}$ by a scalar multiplication $v^{-(\eta_{a-1}+\eta_a,\eta_k)}$.

Now, by using (2.11.2), (2.11.1) is further modified to

$$^{-}\circ\psi = \sigma_0\cdot\prod_{i<j}C_{ij}\cdot T_2(T_3T_2)\cdots(T_n\cdots T_2).$$

This proves the proposition. □

2.12. We now proceed to the proof of Theorem 2.4. For the proof, it is enough to show (2.4.1) for the generators $a_1, \dots, a_n$. By Proposition 2.3, we know already that (2.4.1) holds for $a_2, \dots, a_n$. So, we have only to show it for a_1, i.e., to show that

$$\psi T_1 = T_1^{-1}\psi. \tag{2.12.1}$$

We shall show (2.12.1). Let $\widehat{C} = \prod C_{ij}$ be as in Proposition 2.11. First we note that

(2.12.2) $\widehat{C}$ commutes with T_{ij}, S_{ij}, σ for any $i \ne j$ and any $\sigma \in \mathfrak{S}_n$.

In fact, let $V_I^{\otimes n}$ be the subspace of $V^{\otimes n}$ generated by $\{e_{\sigma I} \mid \sigma \in \mathfrak{S}_n\}$ for a fixed $I = (i_1, \dots, i_n)$. Then $\widehat{C}$ acts on $V_I^{\otimes n}$ as a scalar multiplication by v^{-c} with $c = \sum_{i<j}(\eta_i, \eta_j)$. (2.12.2) follows from this.

By definition (1.3.3) and Proposition 2.11, we can write

(2.12.3)

$$\psi T_1 = Z\omega_1 \qquad \text{with } Z = {}^{-}\circ\sigma_0\widehat{C}T_2(T_3T_2)\cdots(T_{n-1}\cdots T_2)S_nS_{n-1}\cdots S_2.$$

We show that

$$Z = Z^{-1}. \tag{2.12.4}$$

In fact, by (2.9.2), (2.9.1) and (2.12.2), we have

$$\begin{aligned} Z^{-1} &= {}^{-}\circ(S_{21}S_{32}\cdots S_{n,n-1})(T_{21}T_{32}\cdots T_{n-1,n-2})\cdots(T_{21}T_{32})T_{21}\widehat{C}\sigma_0 \\ &= {}^{-}\circ\sigma_0\widehat{C}(S_nS_{n-1}\cdots S_2)(T_nT_{n-1}\cdots T_3)\cdots(T_nT_{n-1})T_n. \end{aligned}$$

It is known by [SS, Lemma 3.8] that

$$(S_nS_{n-1}\cdots S_2)T_j = T_{j-1}(S_nS_{n-1}\cdots S_2) \tag{2.12.5}$$

for $j = 3, \dots, n$. Therefore we have

$$Z^{-1} = {}^{-} \circ \sigma_0 \widehat{C}(T_{n-1} \cdots T_2) \cdots (T_{n-1}T_{n-2})T_{n-1}(S_n \cdots S_2).$$

Now by using the relations

$$(T_{n-1}T_{n-2} \cdots T_{n-a})T_i = T_{i-1}(T_{n-1}T_{n-2} \cdots T_{n-a})$$

for $i \geq n - a + 1$, which follows from the braid relations of $\mathcal{H}_n$, it is easy to see that

$$(T_{n-1} \cdots T_2) \cdots (T_{n-1}T_{n-2})T_{n-1} = T_2(T_3T_2) \cdots (T_{n-1} \cdots T_2).$$

Hence (2.12.4) holds.

Since ψ is an involution, by using (2.12.4), we have

$$T_1^{-1}\psi = (\psi T_1)^{-1} = \omega_1^{-1}Z^{-1} = \omega_1^{-1}Z.$$

Hence to prove (2.12.1), it is enough to show that $\omega_1^{-1}Z = Z\omega_1$. Note that $\sigma_0\omega_1\sigma_0^{-1} = \omega_n$ and that ω_n commutes with $\widehat{C}$ and $T_2, \dots, T_{n-1}$. Thus, by (2.9.2), we have

$$\omega_1^{-1}Z = {}^{-} \circ \sigma_0 \widehat{C} T_2(T_3T_2) \cdots (T_{n-1} \cdots T_2)\omega_n(S_n \cdots S_2). \tag{2.12.6}$$

Here we note the following formula.

$$\omega_i S_i = S_i \omega_{i-1} \qquad \text{for } i = 2, \dots, n. \tag{2.12.7}$$

In fact, it is enough to see the formula for the case where $n = i = 2$, and $S_i = S$. Now ω_2 and ω_1 act as a (common) scalar multiplication on $e_j \otimes e_k$ and $e_k \otimes e_j$ if $b(j) = b(k)$. S permutes $e_j \otimes e_k$ and $e_k \otimes e_j$ if $b(j) \neq b(k)$. (2.12.7) follows easily from these facts.

Now by applying (2.12.7), we have $\omega_n(S_n \cdots S_2) = (S_n \cdots S_2)\omega_1$. Hence (2.12.6) implies that $\omega_1^{-1}Z = Z\omega_1$, and (2.12.1) holds. The theorem is proved.

2.13. By making use of Theorem 2.4, combined with Proposition 2.11, one can describe the bar involution for generators $\{a_2, \dots, a_n, \xi_1, \dots, \xi_n\}$ of $\mathcal{H}_{n,r}$ given in 1.4.

Proposition 2.14. *Let $\{a_2, \dots, a_n, \xi_1, \dots, \xi_n\}$ be the generators of $\mathcal{H}_{n,r}$ given in* 1.4, *and put $x = a_2(a_3a_2) \cdots (a_na_{n-1} \cdots a_2)$. Then we have*

$$\begin{aligned} \bar{a}_i &= a_i^{-1} & &(2 \leq i \leq n), \\ \bar{\xi}_j &= x^{-1}\xi_{n-j+1}^{-1}x & &(1 \leq j \leq n). \end{aligned}$$

Proof. It is enough to show the formula for ξ_j. By Theorem 2.4, we have

$$\overline{\omega}_j = \psi^{-1} \circ \omega_j \circ \psi.$$

Note that $^- \circ \omega_j \circ {}^- = \omega_j^{-1}, \sigma_0\omega_j\sigma_0 = \omega_{n-j+1}$, and that ω_j commutes with $\widehat{C}$. Then by Proposition 2.11, we see that

$$\psi^{-1} \circ \omega_j \circ \psi = X^{-1}\omega_{n-j+1}^{-1}X$$

with $X = T_2(T_3T_2)\cdots(T_nT_{n-1}\cdots T_2)$. Since the representation τ is faithful if we choose $m_k \geq n$ for $k = 1, \ldots, r$, and since $\tau(\xi_j) = \omega_j$, $\tau(x) = X$, this gives the required formula for ξ_j. $\square$

§3. Kazhdan-Lusztig basis and Canonical basis

3.1. Let W be a Weyl group with a set of generators S. We denote by $\mathcal{H}$ the Hecke algebra associated to W. It is an associative algebra over $\mathbb{Q}(v)$ defined by generators a_s $(s \in S)$ and relations

$$(a_s - v)(a_s + v^{-1}) = 0 \tag{3.1.1}$$

together with usual braid relations. $\mathcal{H}$ has a basis $\{a_w \mid w \in W\}$, where $a_w = a_{s_1}\cdots a_{s_q}$ for a reduced expression $w = s_1\cdots s_q$.

For any subset J of S, we denote by W_J the parabolic subgroup in W generated by $s \in J$. Let W^J be the set of distinguished representatives in W/W_J. Hence W^J is the set of minimal elements w in wW_J with respect to the length $l(w)$ of W. Let $\mathcal{H}_J$ be the the subalgebra of $\mathcal{H}$ generated by a_s with $s \in J$. Then $\mathcal{H}_J$ is isomorphic to the Hecke algebra of W_J. Let φ be a homomorphism from $\mathcal{H}$ to $\mathbb{Q}(v)$ defined by $a_s \mapsto v$ for any $s \in S$. We denote by φ_J the restriction of φ on $\mathcal{H}_J$. Let M_J be the induced $\mathcal{H}$-module $\operatorname{Ind}_{\mathcal{H}_J}^{\mathcal{H}} \varphi_J$. Then by Deodhar [D], it is known that M_J has a basis $\{m_w \mid w \in W^J\}$ with the following properties,

$$a_s m_w = \begin{cases} m_{sw} + (v - v^{-1})m_w & \text{if } l(sw) < l(w), \\ m_{sw} & \text{if } l(sw) > l(w), sw \in W^J, \\ vm_w & \text{if } l(sw) > l(w), sw \notin W^J, \end{cases} \tag{3.1.2}$$

and $a_w m_1 = m_w$ for an identity element $1 \in W$ and $w \in W^J$. Note that $w \in W^J$ and $l(sw) < l(w)$ imply that $sw \in W^J$.

Let us define a bar involution on $\mathcal{H}$ by $\overline{v} = v^{-1}$ and $\overline{a}_s = a_s^{-1}$ as in 1.3. We also define a bar involution on M_J by the condition that $\overline{m}_e = m_e$ and that $\overline{hm} = \overline{h}\overline{m}$ for $h \in \mathcal{H}$, $m \in M_J$. Let $\le$ be the partial order on W^J induced from the Bruhat order on W. The Kazhdan-Lusztig basis $\{C^J_w \mid w \in W^J\}$ of M_J was introduced by Kazhdan-Lusztig [KL] for $M_\emptyset \simeq \mathcal{H}$, and then extended by Deodhar [D] to the case M_J. They are characterized by the following two properties.

$$C^J_w \in m_w + \sum_{\substack{x \in W^J \\ x < w}} v^{-1}\mathbb{Z}[v^{-1}]m_x \tag{3.1.3}$$

$$\overline{C}^J_w = C^J_w. \tag{3.1.4}$$

The parabolic Kazhdan-Lusztig polynomial $P^J_{x,w} \in \mathbb{Z}[q]$ is defined, following Deodhar, in terms of the coefficient $p_{x,w}$ of m_x in the expression of C^J_w as follows.

$$p_{x,w} = q^{l(w)-l(x))/2}\overline{P^J_{x,w}}(q) \qquad \text{with } v = q^{-1/2}.$$

Note that in [KL], [D], $\mathcal{H}$ is defined by the quadratic relation $(T_s - q)(T_s + 1)$ for an indeterminate q instead of (3.1.1). Then the relationship with our situation is given as follows; $v = q^{-1/2}$, and our a_s corresponds to $-vT_s$ in their setup. In particular, our C^J_σ corresponds to $(-1)^{l(\sigma)}C^J_\sigma$ under the notation of [KL], [D], and our m_w corresponds to $(-v)^{l(w)}m_w$ of [D].

3.2. For $x, w \in W^J$ such that $x < w$, we denote by $\mu(x, w)$ the coefficient of $q^{(l(w)-l(x)-1)/2}$ in $P^J_{x,w}(q)$. Note that $\deg P^J_{x,w} \le \frac{1}{2}(l(w)-l(x)-1)$. Let $s \in S$ be such that $l(sw) < l(w)$ for $w \in W^J$. Then C^J_w is determined inductively, with respect to the Bruhat order, by

$$C^J_w = (a_s + v^{-1})C^J_{sw} - \sum_{\substack{y \in W^J, y \le sw \\ sy < y \text{ or } sy \notin W^J}} (-1)^{l(w)-l(y)}\mu(y, sw)C^J_y \tag{3.2.1}$$

Now the action of a_s on C^J_w is given as follows; for $s \in S$ and $w \in W^J$,

$$a_s C^J_w = \begin{cases} -v^{-1}C^J_w + C^J_{sw} - \displaystyle\sum_{\substack{y < w \\ sy \le y \text{ or } sy \notin W^J}} (-1)^{l(w)-l(x)}\mu(y, w)C^J_y, \\ vC^J_w, \end{cases} \tag{3.2.2}$$

where the first equality occurs when $sw > w$ and $sw \in W^J$, and the second occurs when $sw < w$ or $sw \notin W^J$. In fact, (3.2.2) can be shown as in a similar way in [KL, 2.3] once we know that

$$a_s C^J_w = vC^J_w \quad \text{if } sw > w \text{ and } sw \notin W^J. \tag{3.2.3}$$

We show (3.2.3). By comparing the coefficients of m_w on both sides, we see that (3.2.3) is equivalent to the identities,

$$P^J_{sx,w} = P^J_{x,w} \quad \text{if } sx \in W^J. \tag{3.2.4}$$

(Note that $sx \in W^J$ if $sx < x$ and $x \in W^J$). Since we are in a setting $u = -1$ in the notation of [D], we have by [D, Prop. 3.4],

$$P^J_{x,w} = P_{xw_J, ww_J} \tag{3.2.5}$$

for $x, w \in W^J$, where w_J is the longest element in W^J, and $P_{x,y}$ is the original Kazhdan-Lusztig polynomial for W. By our assumption, $sw > w$ and $sw \notin W^J$. Then there exists $s' \in W_J$ such that $sw = ws'$ (e.g., [H, Lemma 7.2]), and we have $sww_J < ww_J$. It follows that $a_s C_{ww_J} = vC_{ww_J}$ by [KL, 2.3] (C_w is a Kazhdan-Lusztig basis for $\mathcal{H}$). This induces similar identities for P_{xw_J,ww_J} and P_{sxw_J,ww_J} as in (3.2.4). Now (3.2.4) follows from these identities in view of (3.2.5).

3.3. We assume that $W \simeq \mathfrak{S}_n$, and consider the $U_v \otimes \mathcal{H}_n$-module $V^{\otimes n}$. The weights of U_v on $V^{\otimes n}$ are given by $\lambda = (\lambda_1, \cdots, \lambda_m) \in \mathbb{Z}^m_{\geq 0}$ with $\sum \lambda_i = n$. The weight subspace $V^{\otimes n}_\lambda$ has a basis $\{e_I\}$, with $I = (i_1, \dots, i_n)$ such that $\sharp\{a \mid i_a = k\} = \lambda_k$. The involution ψ on $V^{\otimes n}$ stabilizes the subspace $V^{\otimes n}_\lambda$. Moreover $V^{\otimes n}_\lambda$ is an $\mathcal{H}_n$-submodule of $V^{\otimes n}$ generated by a single element e_{I_λ}, where

$$I_\lambda = (\underbrace{m, \dots, m}_{\lambda_m-\text{times}}, \dots, \underbrace{1, \dots, 1}_{\lambda_1-\text{times}}). \tag{3.3.1}$$

Let $\mathfrak{S}_\lambda \simeq \mathfrak{S}_{\lambda_m} \times \cdots \times \mathfrak{S}_{\lambda_1}$ be the stabilizer of I_λ in $\mathfrak{S}_n$. Then $\mathfrak{S}_\lambda$ is a parabolic subgroup W_J of $\mathfrak{S}_n$, and we denote by $\mathcal{H}_\lambda$ the parabolic subalgebra $\mathcal{H}_J$ corresponding to $\mathfrak{S}_\lambda$. It is easy to see that $\mathcal{H}_n$-module $V^{\otimes n}_\lambda$ is isomorphic to $M_J = \operatorname{Ind}^{\mathcal{H}_n}_{\mathcal{H}_\lambda} \varphi$.

Recall that $\{e_I \mid I \in [1,m]^n\}$ is the basis of $V^{\otimes n}$, which we call the standard basis of $V^{\otimes n}$. The canonical basis $\{b_I \mid I \in [1,m]^n\}$ of U_v-module $V^{\otimes n}$ is characterized by the following two properties ([L, Chap. 27]).

$$\begin{aligned} &b_I \in e_I + \sum_{I'} v^{-1}\mathbb{Z}[v^{-1}]e_{I'}, \\ &\psi(b_I) = b_I, \end{aligned} \tag{3.3.2}$$

where the sum in the first formula is taken over all I' having the same weight as I.

It is shown in [FKK] that the map $f : m_\sigma \mapsto e_{\sigma(I_\lambda)}$ gives an isomorphism $M_J \simeq V^{\otimes n}_\lambda$, which transfers the bar involution on M_J to the involution ψ on $V^{\otimes n}_\lambda$. We identify M_J with $V^{\otimes n}_\lambda$.

We define a partial order $I < I'$ on $[1,m]^n$ as the transitive closure of the relation

$$(\dots, a, \dots, b, \dots) < (\dots, b, \dots, a, \dots) \qquad \text{if } a > b.$$

Then we have the following.

Lemma 3.4. *Let $\sigma, \tau \in \mathfrak{S}_n^J$, and assume that $\sigma < \tau$. Then we have $\sigma(I_\lambda) < \tau(I_\lambda)$.*

Proof. The proof is reduced to the case where $\tau = \sigma s$ with a (not necessarily simple) reflection $s \in \mathfrak{S}_n$. So we assume that s is a transposition (p, q) with $1 \le p < q \le n$. Then it is easy to check that $\sigma^{-1}(p) < \sigma^{-1}(q)$ if $l(\sigma^{-1}) < l(s\sigma^{-1})$. If we write $I_\lambda = (i_1, \dots, i_n)$, we have

$$\sigma(I_\lambda) = (\dots, i_{\sigma^{-1}(p)}, \dots, i_{\sigma^{-1}(q)}, \dots), \quad \sigma s(I_\lambda) = (\dots, i_{\sigma^{-1}(q)}, \dots, i_{\sigma^{-1}(p)}, \dots).$$

Now by (3.3.1) and by our assumption, we have $i_{\sigma^{-1}(q)} \le i_{\sigma^{-1}(p)}$. The lemma follows from this. □

The following special case is worth mentioning.

Lemma 3.5 ([FKK, Lemma 2.1]). *Let $\sigma \in W^J$ and $s \in S$ be a transposition $(i, i+1)$. Let a and b be i-th and $(i+1)$-th entries of $\sigma(I_\lambda)$, respectively. Then we have*

$$\begin{aligned} &\text{if } a > b, && \text{then } s\sigma > \sigma \text{ and } s\sigma \in W^J, \\ &\text{if } a = b, && \text{then } s\sigma > \sigma \text{ and } s\sigma \notin W^J, \\ &\text{if } a < b, && \text{then } s\sigma < \sigma. \end{aligned}$$

The following result shows that the Kazhdan-Lusztig basis is obtained as a special case of the canonical basis of $V^{\otimes n}$.

Theorem 3.6 ([FKK, Th. 2.5]). *Assume that $W \simeq \mathfrak{S}_n$. Then, under the identification $M_J \simeq V_\lambda^{\otimes n}$, we have for each $\sigma \in W^J$,*

$$C_\sigma^J = b_{\sigma(I_\lambda)}.$$

Combining Theorem 3.6 with Lemma 3.4, we have the following refinement of (3.3.2).

Corollary 3.7. *Under the above notation, we have*

$$b_I \in e_I + \sum_{I' < I} v^{-1}\mathbb{Z}[v^{-1}] e_{I'}.$$

The following corollary is also immediate from (3.2.2), Lemma 3.5 and Theorem 3.6.

Corollary 3.8. *Let $s = (a, a+1)$ be a transposition. Then for $I = (i_1, \dots, i_n)$, we have*

$$T_s b_I = v b_I \quad \text{if } \ i_a \le i_{a+1}.$$

§4. $\mathcal{H}_{n,r}$-submodules of $V^{\otimes n}$

4.1. We now return to the setup in section 1, and assume that $W = W_{n,r}$. We consider the $\mathcal{H}_{n,r}$-module $V^{\otimes n}$ with the graded vector space $V = \bigoplus_{i=1}^r V_i$ as before. We prepare some notation in addition to 3.1. Let $S = \{s_1, \dots, s_n\}$ be the set of generators of $W_{n,r}$, where $t_1 = s_1$ has order r, and $s_2, \dots, s_n$ are generators of $\mathfrak{S}_n$ corresponding to transposition $(1,2), \dots, (n-1,n)$. We define $t_i \in W_{n,r}$ by $t_i = s_i \cdots s_2 t_1 s_2 \cdots s_i$ for $i = 2, \dots, n$. Then $t_1, \dots, t_n$ gives rise to a set of generators of the group $(\mathbb{Z}/r\mathbb{Z})^n$.

The basis vector of $V^{\otimes n}$ is given by e_I with $I = (i_1, \dots, i_n)$ as before. By 1.3, e_I can also be written as $e_{j_1}^{(\varepsilon_1)} \otimes \cdots \otimes e_{j_n}^{(\varepsilon_n)}$. In this case, we write I as $I = (j_1^{(\varepsilon_1)}, \dots, j_n^{(\varepsilon_n)})$. The weight λ of U_v on $V^{\otimes n}$ is expressed as $\lambda = (\lambda_1, \dots, \lambda_m)$ as in 3.3. In our situation, I determines a multi-composition $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$, with $\lambda^{(k)} = (\lambda_1^{(k)}, \dots, \lambda_{m_k}^{(k)}) \in \mathbb{Z}_{\ge 0}^{m_k}$, such that $\sum_{j,k} \lambda_j^{(k)} = n$; the correspondence is given by $\lambda_j^{(k)} = \sharp\{a \mid j_a = j, \varepsilon_a = k\}$. If one ignores the superscripts of $\lambda_j^{(k)}$, $\boldsymbol{\lambda}$ reduces to λ. We call $\boldsymbol{\lambda}$ the weight of e_I. We denote by $V_{\boldsymbol{\lambda}}^{\otimes n}$ the subspace of $V^{\otimes n}$ generated by e_I whose weight is $\boldsymbol{\lambda}$. It is easy to check that the action of $\mathcal{H}_{n,r}$ on $V^{\otimes n}$ stabilizes the subspace $V_{\boldsymbol{\lambda}}^{\otimes n}$. For the weight $\boldsymbol{\lambda}$, put

$$e_{\lambda^{(k)}} = \underbrace{e_{m_k}^{(k)} \otimes \cdots \otimes e_{m_k}^{(k)}}_{\lambda_{m_k}^{(k)}-\text{times}} \otimes \cdots \otimes \underbrace{e_1^{(k)} \otimes \cdots \otimes e_1^{(k)}}_{\lambda_1^{(k)}-\text{times}} \tag{4.1.1}$$

and define a vector $e_{\boldsymbol{\lambda}} \in V_{\boldsymbol{\lambda}}^{\otimes n}$ by

$$e_{\boldsymbol{\lambda}} = e_{\lambda^{(r)}} \otimes e_{\lambda^{(r-1)}} \otimes \cdots \otimes e_{\lambda^{(1)}}. \tag{4.1.2}$$

The stabilizer of $e_{\boldsymbol{\lambda}}$ in $\mathfrak{S}_n$ is isomorphic to

$$\mathfrak{S}_{\boldsymbol{\lambda}} = \mathfrak{S}_{\lambda^{(r)}} \times \mathfrak{S}_{\lambda^{(r-1)}} \times \cdots \times \mathfrak{S}_{\lambda^{(1)}}$$

with $\mathfrak{S}_{\lambda^{(k)}} = \mathfrak{S}_{\lambda^{(k)}_{m_k}} \times \cdots \times \mathfrak{S}_{\lambda^{(k)}_1}$. We define a subgroup $W_{\boldsymbol\lambda}$ of $W_{n,r}$ by $W_{\boldsymbol\lambda} = \mathfrak{S}_{\boldsymbol\lambda} \ltimes (\mathbb{Z}/r\mathbb{Z})^n$, i.e.,

$$W_{\boldsymbol\lambda} \simeq W_{\lambda^{(r)}} \times W_{\lambda^{(r-1)}} \times \cdots \times W_{\lambda^{(1)}}$$

with $W_{\lambda^{(k)}} = W_{\lambda^{(k)}_{m_k},r} \times \cdots \times W_{\lambda^{(k)}_1,r}$. Let $\mathcal{H}_{\boldsymbol\lambda}$ be the Ariki-Koike algebra associated to $W_{\boldsymbol\lambda}$, i.e.,

$$\mathcal{H}_{\boldsymbol\lambda} = \mathcal{H}_{\lambda^{(r)}} \otimes \mathcal{H}_{\lambda^{(r-1)}} \otimes \cdots \otimes \mathcal{H}_{\lambda^{(1)}} \tag{4.1.3}$$

with $\mathcal{H}_{\lambda^{(k)}} = \mathcal{H}_{\lambda^{(k)}_{m_k},r} \otimes \cdots \otimes \mathcal{H}_{\lambda^{(k)}_1,r}$. One can regard $\mathcal{H}_{\boldsymbol\lambda}$ as a subalgebra of $\mathcal{H}_{n,r}$ in a natural way, by making use of generators $\{\xi_1, \dots, \xi_n\}$ as discussed in [S, 4.2]. (For example, if $a + b = n$, then $\mathcal{H}_{a,r} \otimes \mathcal{H}_{b,r} \hookrightarrow \mathcal{H}_{n,r}$, where $\mathcal{H}_{a,r}$ (resp. $\mathcal{H}_{b,r}$) is the subalgebra of $\mathcal{H}_{n,r}$ generated by $s_2, \dots, s_a, \xi_1, \dots, \xi_a$, (resp. $s_{a+2}, \dots, s_n, \xi_{a+1}, \dots, \xi_n$), respectively.)

We can define a linear character $\varphi^{(k)}_n : \mathcal{H}_{n,r} \to K$ by

$$\begin{aligned} \varphi^{(k)}_n(a_i) &= v \qquad (2 \le i \le n), \\ \varphi^{(k)}_n(\xi_j) &= u_k \qquad (1 \le j \le n) \end{aligned}$$

(cf. [S, (3.3.3), 5.2]), and define $\varphi_{\lambda^{(k)}} : \mathcal{H}_{\lambda^{(k)}} \to K$ by $\varphi_{\lambda^{(k)}} = \varphi^{(k)}_{\lambda^{(k)}_{m_k}} \otimes \cdots \otimes \varphi^{(k)}_{\lambda^{(k)}_1}$. Then we define a linear character $\varphi_{\boldsymbol\lambda} : \mathcal{H}_{\boldsymbol\lambda} \to K$ by

$$\varphi_{\boldsymbol\lambda} = \varphi_{\lambda^{(r)}} \otimes \varphi_{\lambda^{(r-1)}} \otimes \cdots \otimes \varphi_{\lambda^{(1)}} \tag{4.1.4}$$

according to the embedding into $\mathcal{H}_{n,r}$ given in (4.1.3). Put $V^{\otimes n}_{\boldsymbol\lambda} = M_{\boldsymbol\lambda}$. Then we have the following result.

Proposition 4.2. *Let the notations be as above.*

(i) *$M_{\boldsymbol\lambda}$ is generated by $e_{\boldsymbol\lambda}$ as $\mathcal{H}_{n,r}$-module, and we have*

$$M_{\boldsymbol\lambda} = \mathcal{H}_{n,r} e_{\boldsymbol\lambda} \simeq \operatorname{Ind}^{\mathcal{H}_{n,r}}_{\mathcal{H}_{\boldsymbol\lambda}} \varphi_{\boldsymbol\lambda}$$

as $\mathcal{H}_{n,r}$-modules.

(ii) *$M_{\boldsymbol\lambda}$ has a basis $\{e_\sigma\}$ indexed by the set $\mathfrak{S}^J_n$ (here we regard $\mathfrak{S}_{\boldsymbol\lambda}$ as a parabolic subgroup $(\mathfrak{S}_n)_J$ of $\mathfrak{S}_n$). The action of $\mathcal{H}_{n,r}$ on this basis is given as follows:*

$$a_s e_\sigma = \begin{cases} e_{s\sigma} + (v - v^{-1}) e_\sigma & \text{if } l(s\sigma) < l(\sigma), \\ e_{s\sigma} & \text{if } l(s\sigma) > l(\sigma), s\sigma \in \mathfrak{S}^J_n, \\ v e_\sigma & \text{if } l(s\sigma) > l(\sigma), s\sigma \notin \mathfrak{S}^J_n, \end{cases}$$

$$\xi_j e_\sigma = u_{\varepsilon(j,\sigma)} e_\sigma,$$

where $\varepsilon(j, \sigma) \in \{1, \dots, r\}$ is given as follows; write $e_{\boldsymbol\lambda} = e_I$ as in 4.1, and put $\varepsilon(j, \sigma) = \varepsilon_j$ for $\sigma(I) = (j^{(\varepsilon_1)}_1, \dots, j^{(\varepsilon_n)}_n)$.

(iii) *There exists an involution* $^-$: $M_{\boldsymbol{\lambda}} \to M_{\boldsymbol{\lambda}}$ *satisfying the property that* $\overline{hm} = \bar{h}\overline{m}$ *for* $h \in \mathcal{H}_{n,r}, m \in M_{\boldsymbol{\lambda}}$, *and that* $\bar{e}_\sigma = e_\sigma$ *for* $\sigma = 1$.

Proof. Let λ be the weight of $e_{\boldsymbol{\lambda}}$ as U_v-module. Then $M_{\boldsymbol{\lambda}}$ coincides with $V_\lambda^{\otimes n}$ and $e_{\boldsymbol{\lambda}}$ is nothing but e_{I_λ} given in (3.3.1). Then by [FKK, Prop. 2.1], $V_\lambda^{\otimes n}$ is generated by e_{I_λ} as $\mathcal{H}_n$-module, and is isomorphic to M_J as in 3.3, for a parabolic subgroup $\mathfrak{S}_\lambda = \mathfrak{S}_n^J$. In particular, we see that $M_{\boldsymbol{\lambda}} = \mathcal{H}_{n,r}e_{\boldsymbol{\lambda}}$, and that

$$\dim V_{\boldsymbol{\lambda}}^{\otimes n} = |\mathfrak{S}_n^J| = \dim \operatorname{Ind}_{\mathcal{H}_{\boldsymbol{\lambda}}}^{\mathcal{H}_{n,r}} \varphi_{\boldsymbol{\lambda}}.$$

Since it is easy to see that $Ke_{\boldsymbol{\lambda}}$ is a one-dimensional $\mathcal{H}_{\boldsymbol{\lambda}}$-module affording $\varphi_{\boldsymbol{\lambda}}$, the first assertion follows.

Now we define a basis $\{e_\sigma \mid \sigma \in \mathfrak{S}_n^J\}$ in $M_{\boldsymbol{\lambda}}$ by using the basis $\{m_\sigma\}$ in M_J. Then we have $e_\sigma = m_\sigma = e_{\sigma(I_\lambda)}$ by [FKK]. The first three formula in (ii) now follows from (3.1.2). The last formula in (ii) follows by considering the action of ω_j on $e_{\sigma(I_\lambda)} \in V^{\otimes n}$.

The involution ψ on $V^{\otimes n}$ stabilizes the subspace $V_{\boldsymbol{\lambda}}^{\otimes n}$. We define the bar involution $^-$ on $M_{\boldsymbol{\lambda}} = V_{\boldsymbol{\lambda}}^{\otimes n}$ in terms of ψ. Then we have $\overline{hm} = \bar{h}\overline{m}$ by Theorem 2.4. Since $\psi(e_{\boldsymbol{\lambda}}) = e_{\boldsymbol{\lambda}}$, we have $\bar{e}_1 = e_1$. The proposition is proved. $\square$

The following result is an analogue to the case of $\mathcal{H}_{n,r}$ of the result of Frenkel, Khovanov and Kirillov (cf. Theorem 3.6) concerning the Kazhdan-Lusztig basis of $\mathcal{H}_n$ and canonical basis of U_q, and also of the parabolic Kazhdan-Lusztig basis of Deodhar (cf. 3.1). But note that $\mathcal{H}_{\boldsymbol{\lambda}}$ is no longer a parabolic subalgebra of $\mathcal{H}_{n,r}$.

Theorem 4.3. *Let* $M_{\boldsymbol{\lambda}} \simeq \operatorname{Ind}_{\mathcal{H}_{\boldsymbol{\lambda}}}^{\mathcal{H}_{n,r}} \varphi_{\boldsymbol{\lambda}}$ *be the induced* $\mathcal{H}_{n,r}$*-module. Then there exists a unique basis* $\{b_\sigma \mid \sigma \in \mathfrak{S}_n^J\}$ *in* $M_{\boldsymbol{\lambda}}$ *satisfying the following properties.*

$$b_\sigma \in e_\sigma + \sum_{\substack{\tau \in \mathfrak{S}_n^J \\ \tau < \sigma}} v^{-1}\mathbb{Z}[v^{-1}]e_\tau,$$

$$\bar{b}_\sigma = b_\sigma.$$

The coefficient $p_{\tau,\sigma}$ *of* e_τ *in the expression of* b_σ *is given by the parabolic Kazhdan-Lusztig polynomial for the case of* $\mathfrak{S}_n^J \subset \mathfrak{S}_n$ *just as in* 3.1.

Proof. By Theorem 3.6, canonical basis $\{b_{\sigma(I_\lambda)} \mid \sigma \in \mathfrak{S}_n^J\}$ gives rise to a basis of $V_\lambda^{\otimes n}$, which corresponds to the parabolic Kazhdan-Lusztig basis $\{C_\sigma^J\}$ in M_J. Hence, if we define the basis $\{b_\sigma\}$ in $M_{\boldsymbol{\lambda}} = V_\lambda^{\otimes n}$ in terms of $\{b_{\sigma(I_\lambda)}\}$, the assertions in the theorem follow from 3.1 and Proposition 4.2. $\square$

4.4. We now pass to a more general situation. Take an integer $t \geq 0$ such that $t \leq m_k$ for $k = 1, \dots, r$. Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ be an r-tuple of compositions as in 4.1, but here we assume that $\lambda^{(k)} = \emptyset$ for $k \neq r$, and that $\lambda^{(r)} = (\lambda^{(r)}_{t+1}, \dots, \lambda^{(r)}_{m_r}) \in \mathbb{Z}^{m_r - t}_{\geq 0}$. We consider a pair $(\boldsymbol{\lambda}; \mathbf{c})$, with $\mathbf{c} = (c_1, \dots, c_t) \in \mathbb{Z}^t_{>0}$ a composition such that $\sum_{j,k} \lambda^{(k)}_j + \sum_i c_i = n$. We put $c = \sum c_i$. We denote by $M_{\boldsymbol{\lambda},\mathbf{c}}$ the subspace of $V^{\otimes n}$ generated by e_I with $I = (j_1^{(\varepsilon_1)}, \dots, j_n^{(\varepsilon_n)})$ such that $\lambda^{(r)}_j = \sharp\{a \mid j_a = j, \varepsilon_a = r\}$ and that $c_i = \sharp\{a \mid j_a = i\}$. Then $M_{\boldsymbol{\lambda},\mathbf{c}}$ is a direct sum of various weight spaces $V^{\otimes n}_{\boldsymbol{\nu}}$, and so has a structure of $\mathcal{H}_{n,r}$-module. The decomposition of $M_{\boldsymbol{\lambda},\mathbf{c}}$ into $V^{\otimes n}_{\boldsymbol{\nu}}$ is described more precisely as follows. Let $\boldsymbol{\nu}$ be a pair $(\boldsymbol{\lambda}; \boldsymbol{\mu})$, where $\boldsymbol{\lambda}$ is as above, and $\boldsymbol{\mu} = (\mu^{(1)}, \dots, \mu^{(r)})$ is an r-tuples of compositions $\mu^{(k)} = (\mu^{(k)}_1, \dots, \mu^{(k)}_t) \in \mathbb{Z}^t_{\geq 0}$ such that $\sum_{k=1}^r \mu^{(k)}_i = c_i$ for $1 \leq i \leq t$. We denote by $\mathcal{P}_{\boldsymbol{\lambda},\mathbf{c}}$ the set of such $\boldsymbol{\nu} = (\boldsymbol{\lambda}; \boldsymbol{\mu})$. Note that $\boldsymbol{\nu}$ can be written, by rearranging the entries, as $(\nu^{(1)}, \dots, \nu^{(r)})$ with $\nu^{(r)} = (\mu^{(r)}_1, \dots, \mu^{(r)}_t, \lambda^{(r)}_{t+1}, \dots, \lambda^{(r)}_{m_r})$, and $\nu^{(k)} = (\mu^{(k)}_1, \dots, \mu^{(k)}_t, 0, \dots, 0)$ for $k \neq r$. Hence it determines an $\mathcal{H}_{n,r}$-subspace $V^{\otimes n}_{\boldsymbol{\nu}}$. It is easy to check that

$$M_{\boldsymbol{\lambda},\mathbf{c}} = \bigoplus_{\boldsymbol{\nu} \in \mathcal{P}_{\boldsymbol{\lambda},\mathbf{c}}} V^{\otimes n}_{\boldsymbol{\nu}}.$$

We shall investigate the $\mathcal{H}_{n,r}$-module structure of $M_{\boldsymbol{\lambda},\mathbf{c}}$. For each $\boldsymbol{\nu} \in \mathcal{P}_{\boldsymbol{\lambda},\mathbf{c}}$, we define $e^{\boldsymbol{\nu}} \in V^{\otimes n}_{\boldsymbol{\nu}}$ by $e^{\boldsymbol{\nu}} = e_{\boldsymbol{\lambda}} \otimes e^{\boldsymbol{\mu}}$, where $e_{\boldsymbol{\lambda}} = e_{\lambda^{(r)}}$ is defined just as in (4.1.1), by restricting the factors in between $e^{(r)}_{m_r}$ and $e^{(r)}_{t+1}$. $e^{\boldsymbol{\mu}} \in V^{\otimes c}$ is defined by $e^{\boldsymbol{\mu}} = E_1 \otimes E_2 \otimes \cdots \otimes E_t$, with

$$E_i = (e^{(1)}_{t-i+1})^{\mu^{(1)}_i} \otimes \cdots \otimes (e^{(r)}_{t-i+1})^{\mu^{(r)}_i} \in V^{\otimes c_i}. \tag{4.4.1}$$

Now $e^{\boldsymbol{\nu}}$ can be written as $e^{\boldsymbol{\nu}} = e_I$ for some I, and we denote by $b^{\boldsymbol{\nu}}$ the canonical basis $b_I \in V^{\otimes n}_{\boldsymbol{\nu}}$ corresponding to e_I. We define $m_{\boldsymbol{\lambda},\mathbf{c}} \in M_{\boldsymbol{\lambda},\mathbf{c}}$ by

$$m_{\boldsymbol{\lambda},\mathbf{c}} = \sum_{\boldsymbol{\nu} \in \mathcal{P}_{\boldsymbol{\lambda},\mathbf{c}}} b^{\boldsymbol{\nu}}. \tag{4.4.2}$$

We define a subalgebra $\mathcal{H}_{\boldsymbol{\lambda},\mathbf{c}}$ of $\mathcal{H}_{n,r}$ by $\mathcal{H}_{\boldsymbol{\lambda},\mathbf{c}} = \mathcal{H}_{\boldsymbol{\lambda}} \otimes \mathcal{H}_{\mathbf{c}}$, where $\mathcal{H}_{\boldsymbol{\lambda}} = \mathcal{H}_{\lambda^{(r)}}$ is defined as in (4.1.3), by modifying the definition of $\mathcal{H}_{\lambda^{(r)}}$ appropriately, and $\mathcal{H}_{\mathbf{c}}$ is defined by

$$\mathcal{H}_{\mathbf{c}} = \mathcal{H}_{c_1} \otimes \cdots \otimes \mathcal{H}_{c_t}. \tag{4.4.3}$$

(Remember that $\mathcal{H}_i$ is the Iwahori-Hecke algebra of type A_{i-1}). We define a linear character $\varphi_{\boldsymbol{\lambda},\mathbf{c}}$ of $\mathcal{H}_{\boldsymbol{\lambda},\mathbf{c}}$ by $\varphi_{\boldsymbol{\lambda},\mathbf{c}} = \varphi_{\boldsymbol{\lambda}} \otimes \varphi_{\mathbf{c}}$, where $\varphi_{\boldsymbol{\lambda}} = \varphi_{\lambda^{(r)}}$ is given as in (4.1.4). $\varphi_{\mathbf{c}}$ is given by $\varphi_{\mathbf{c}} = \varphi_{c_1} \otimes \cdots \otimes \varphi_{c_t}$, where φ_n is the linear character of $\mathcal{H}_n$ defined by $\varphi_n(a_j) = v$ for all generators a_j.

Under these notations, we have the following result.

Proposition 4.5. *$M_{\boldsymbol\lambda,\mathbf c}$ is generated by $m_{\boldsymbol\lambda,\mathbf c}$ as $\mathcal H_{n,r}$-module, and we have*

$$M_{\boldsymbol\lambda,\mathbf c} = \mathcal H_{n,r} m_{\boldsymbol\lambda,\mathbf c} \simeq \operatorname{Ind}_{\mathcal H_{\boldsymbol\lambda,\mathbf c}}^{\mathcal H_{n,r}} \varphi_{\boldsymbol\lambda,\mathbf c}, \qquad \psi(m_{\boldsymbol\lambda,\mathbf c}) = m_{\boldsymbol\lambda,\mathbf c}.$$

Proof. It is clear that $m_{\boldsymbol\lambda,\mathbf c}$ is fixed by ψ. We show the first two equalities. First we note that

$$hm_{\boldsymbol\lambda,\mathbf c} = \varphi_{\boldsymbol\lambda,\mathbf c}(h) m_{\boldsymbol\lambda,\mathbf c} \quad \text{for } h \in \mathcal H_{\boldsymbol\lambda,\mathbf c}. \tag{4.5.1}$$

In fact to show (4.5.1), it is enough to see, for each $\boldsymbol\nu \in \mathcal P_{\boldsymbol\lambda,\mathbf c}$, that

$$hb^{\boldsymbol\nu} = \begin{cases} \varphi_{\boldsymbol\lambda}(h) b^{\boldsymbol\nu} & \text{if } h \in \mathcal H_{\boldsymbol\lambda}, \\ \varphi_{\mathbf c}(h) b^{\boldsymbol\nu} & \text{if } h \in \mathcal H_{\mathbf c}. \end{cases} \tag{4.5.2}$$

We show (4.5.2). In view of (4.4.1) and Corollary 3.8, we see that $a_j b^{\boldsymbol\nu} = vb^{\boldsymbol\nu}$ for all generators $a_j \in \mathcal H_{c_i}$. This implies the second equality in (4.5.2). Next we consider the first equality. By (modified form of) (4.1.1), (4.1.2), together with (4.4.1), we see that $e^{\boldsymbol\nu} = e_I$, where I is of the form $I = (i_1, \dots, i_n)$ with $i_1 \ge i_2 \ge \cdots \ge i_{n-c}$, and with $i_{n-c} > i_k$ for all $c+1 \le k \le n$. Then by Corollary 3.7, $b^{\boldsymbol\nu}$ is written as a linear combination of $e_{I'}$, where $e_{I'}$ is of the form $e_{\boldsymbol\lambda} \otimes e^{\boldsymbol\mu'}$, for some $e^{\boldsymbol\mu'} \in V^{\otimes c}$. Then as in the case of Proposition 4.2, one can check that $he_{I'} = \varphi_{\boldsymbol\lambda}(h) e_{I'}$ for $h \in \mathcal H_{\boldsymbol\lambda}$. The first equality follows from this, and so (4.5.2) holds.

Next we show that

$$M_{\boldsymbol\lambda,\mathbf c} = \mathcal H_{n,r} m_{\boldsymbol\lambda,\mathbf c}. \tag{4.5.3}$$

Let ζ be a primitive r-th root of unity. By the specialization $v \mapsto 1, u_i \mapsto \zeta^i$, $\mathcal H_{n,r}$ turns out to be the group algebra $\mathbb C W_{n,r}$. (Note that in order to apply the specialization argument, one has to replace $\mathcal H_{n,r}$ by its "integral form" defined over a subring $R_1 = \mathbb Z[v, v^{-1}, u_1, \dots, u_r, \varDelta^{-1}]$ of K as in [S, 3.6]. Accordingly one needs to replace V by its R_1-lattice with basis e_i. All the ingredients up to now make sense for this setup, and we use them freely without referring R_1 in the discussion below.)

Let $\overline V = \bigoplus \overline V_i$ be the $\mathbb C$-vector space with $\dim \overline V_i = m_i$. We denote by $\{\overline e_j^{(i)}\}$ the basis of V_i. Then the $\mathcal H_{n,r}$-module $V^{\otimes n}$ is specialized to the $\mathbb C W_{n,r}$-module $\overline V^{\otimes n}$. Let t_i be as in 4.1. Then the action of t_i on $\overline V^{\otimes n}$ is given by $t_i e_I = \zeta^{\varepsilon_i} e_I$ for $I = (j_1^{(\varepsilon_1)}, \dots, j_n^{(\varepsilon_n)})$, which is the specialization of ξ_i on $V^{\otimes n}$. The previous construction for $M_{\boldsymbol\lambda,\mathbf c} = \bigoplus_{\boldsymbol\nu} V_{\boldsymbol\nu}^{\otimes n}$ makes sense, and by the specialization we have a $W_{n,r}$-module $\overline M_{\boldsymbol\lambda,\mathbf c} = \bigoplus_{\boldsymbol\nu} \overline V_{\boldsymbol\nu}^{\otimes n}$. Let $\overline e^{\boldsymbol\nu}, \overline b^{\boldsymbol\nu}, \overline m_{\boldsymbol\lambda,\mathbf c}$ be the elements in $\overline M_{\boldsymbol\lambda,\mathbf c}$ obtained from $e^{\boldsymbol\nu}, b^{\boldsymbol\nu}, m_{\boldsymbol\lambda,\mathbf c}$ by the specialization.

To show (4.5.3), it is enough to see that

$$\overline M_{\boldsymbol\lambda,\mathbf c} = \mathbb C W_{n,r} \overline m_{\boldsymbol\lambda,\mathbf c}. \tag{4.5.4}$$

We show (4.5.4). We prepare some notation. For $I = (j_1^{(\varepsilon_1)}, \dots, j_n^{(\varepsilon_n)})$, we call $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$ the signature of e_I, and $\mathbf{j} = (j_1, \dots, j_n)$ the foot of e_I. Put $\mathcal{U} = \mathbb{C}W_{n,r}\overline{m}_{\boldsymbol{\lambda},\mathbf{c}}$. Now $\overline{m}_{\boldsymbol{\lambda},\mathbf{c}}$ can be written as $\overline{m}_{\boldsymbol{\lambda},\mathbf{c}} = \sum_{\boldsymbol{\varepsilon}\in[1,r]^n} \overline{m}(\boldsymbol{\varepsilon})$, where $\overline{m}(\boldsymbol{\varepsilon})$ is a linear combination of vectors $\bar{e}_I$ whose signature is $\boldsymbol{\varepsilon}$. Note that $t_1, \dots, t_n$ are generators of the subgroup $(\mathbb{Z}/r\mathbb{Z})^n$ of $W_{n,r}$, and $e_{j_1}^{(\varepsilon_1)} \otimes \cdots \otimes e_{j_n}^{(\varepsilon_n)}$ generates a one dimensional representation $\varphi_{\boldsymbol{\varepsilon}}$ of $(\mathbb{Z}/r\mathbb{Z})^n$ given by $t_i \mapsto \zeta^{\varepsilon_i}$. It follows that each $\overline{m}(\boldsymbol{\varepsilon})$ belongs to $\mathcal{U}$.

Let us consider the partial order $<$ on $[1,m]^n$ defined in 3.3. Let $F(\boldsymbol{\varepsilon})$ be the set of vectors in $\overline{M}_{\boldsymbol{\lambda},\mathbf{c}}$ consisting of $\bar{e}_I$ with signature $\boldsymbol{\varepsilon}$, together with the vectors $e_{I'}$ obtained from those e_I by permuting the factors. Clearly $\bigcup_{\boldsymbol{\varepsilon}} F(\boldsymbol{\varepsilon})$ gives rise to a basis of $\overline{M}_{\boldsymbol{\lambda},\mathbf{c}}$. We show, by backward induction on the partial order of the set of signatures, that $F(\boldsymbol{\varepsilon}) \subset \mathcal{U}$. Take $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$ and assume that $F(\boldsymbol{\varepsilon}') \subset \mathcal{U}$ holds for any $\boldsymbol{\varepsilon}' > \boldsymbol{\varepsilon}$. Now $\overline{m}(\boldsymbol{\varepsilon})$ can be written as

$$\overline{m}(\boldsymbol{\varepsilon}) \in \overline{e}_I + \sum_{I'} \mathbb{C}\overline{e}_{I'}, \tag{4.5.5}$$

where the foot $\mathbf{j}$ of I is given by

$$\mathbf{j} = (\underbrace{m_r, \dots, m_r}_{\lambda^{(r)}_{m_r}-\text{times}}, \cdots, \underbrace{t+1, \dots, t+1}_{\lambda^{(r)}_{t+1}-\text{times}}, \underbrace{t, \dots, t}_{c_1-\text{times}}, \dots, \underbrace{1, \dots, 1}_{c_t-\text{times}}), \tag{4.5.6}$$

and $e_{I'}$ is a summand of some $b^{\boldsymbol{\nu}} = b_{I''}$, not equal to $e^{\boldsymbol{\nu}}$. Thus $e_{I'}$ is obtained from $e_{I''}$ by permuting the factors. Note that $e_{I''}$ has the same foot as (4.5.6), and we have $I' < I''$ by Corollary 3.7. Let $\boldsymbol{\varepsilon}''$ be the signature of I''. As in the proof of (4.5.2), one can write $e_{I''} = e_{\boldsymbol{\lambda}} \otimes e^{\boldsymbol{\mu}}$ and $e_{I'} = e_{\boldsymbol{\lambda}} \otimes e^{\boldsymbol{\mu}'}$. Since $(t, \dots, t, \dots, 1, \dots, 1)$ is ordered decreasingly, the condition that $I' < I''$ implies that $\boldsymbol{\varepsilon} < \boldsymbol{\varepsilon}''$. It follows, by induction, that $e_{I''} \in \mathcal{U}$. By operating $\mathfrak{S}_n$, we see that $e_{I'} \in \mathcal{U}$ also. This implies that e_I and all its permutations of factors belong to $\mathcal{U}$. Hence we have $F(\boldsymbol{\varepsilon}) \subset \mathcal{U}$. Thus (4.5.4), and so (4.5.3) holds.

It is easy to see that $\dim M_{\boldsymbol{\lambda},\mathbf{c}} = \dim \overline{M}_{\boldsymbol{\lambda},\mathbf{c}} = |W_{n,r}|/|W_{\boldsymbol{\lambda},\mathbf{c}}|$, where $W_{\boldsymbol{\lambda},\mathbf{c}}$ is the subgroup of $W_{n,r}$ corresponding to the subalgebra $\mathcal{H}_{\boldsymbol{\lambda},\mathbf{c}}$ of $\mathcal{H}_{n,r}$. Then (4.5.1) and (4.5.3) implies that $M_{\boldsymbol{\lambda},\mathbf{c}} \simeq \operatorname{Ind}_{\mathcal{H}_{\boldsymbol{\lambda},\mathbf{c}}}^{\mathcal{H}_{n,r}} \varphi_{\boldsymbol{\lambda},\mathbf{c}}$. The proposition is proved. $\square$

4.6. The space $M_{\boldsymbol{\lambda},\mathbf{c}}$ can be decomposed into a direct sum of weight spaces $V_{\boldsymbol{\nu}}^{\otimes n}$. Hence in view of Proposition 4.2 and Theorem 4.3, $M_{\boldsymbol{\lambda},\mathbf{c}}$ have bases inherited from the basis $\{e_I\}$ and $\{b_I\}$ of various $V_{\boldsymbol{\nu}}^{\otimes n}$. In particular, a bar involution on $M_{\boldsymbol{\lambda},\mathbf{c}}$ can be defined, and one obtains a basis invariant under the bar involution.

Here we consider the special case where $M_{\boldsymbol{\lambda},\mathbf{c}}$ is isomorphic to the regular representation of $\mathcal{H}_{n,r}$. Hence we assume that $\boldsymbol{\lambda} = \emptyset$, and $\mathbf{c} = (1^n)$. So

$\mathcal{H}_{\boldsymbol{\lambda},\mathbf{c}} \simeq K$ and $\varphi_{\boldsymbol{\lambda},\mathbf{c}} = 1_K$. $\mathcal{P}_{\boldsymbol{\lambda},\mathbf{c}}$ is in bijection with the set $[1,r]^n$, and the vector $e^{\boldsymbol{\nu}} \in V_{\boldsymbol{\nu}}^{\otimes n}$ corresponding to $\boldsymbol{\nu} \in \mathcal{P}_{\boldsymbol{\lambda},\mathbf{c}}$ in 4.4 is given by

$$e^{\boldsymbol{\nu}} = e_n^{(\varepsilon_1)} \otimes e_{n-1}^{(\varepsilon_2)} \otimes \cdots \otimes e_1^{(\varepsilon_n)} \tag{4.6.1}$$

under the correspondence $\boldsymbol{\nu} \leftrightarrow \varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in [1,r]^n$. The basis of $V_{\boldsymbol{\nu}}^{\otimes n}$ is obtained by permuting the factors of $e^{\boldsymbol{\nu}}$. If we write $e^{\boldsymbol{\nu}} = e_I$ and $e_{\sigma(I)} = e_{\sigma,\varepsilon}$, then $\{e_{\sigma,\varepsilon} \mid \sigma \in \mathfrak{S}_n\}$ forms a basis of $V_{\boldsymbol{\nu}}^{\otimes n}$, and

$$\{e_{\sigma,\varepsilon} \mid \sigma \in \mathfrak{S}_n, \varepsilon \in [1,r]^n\}$$

gives rise to a basis of $M_{\boldsymbol{\lambda},\mathbf{c}}$. We define a partial order on the set $\{(\sigma,\varepsilon) \mid \sigma \in \mathfrak{S}_n\}$ (for a fixed ε) as follows. Let $\tau_0 \in \mathfrak{S}_n$ be an element such that $e^{\boldsymbol{\nu}} = e_{\tau_0(I)}$, where $I = (i_1, \dots, i_n)$ with $i_1 \geq \cdots \geq i_n$. Then we put $(\tau,\varepsilon) < (\sigma,\varepsilon)$ if $\tau\tau_0 < \sigma\tau_0$.

We write $m_0 = m_{\boldsymbol{\lambda},\mathbf{c}} = \sum b^{\boldsymbol{\nu}}$ as in (4.4.2). Then the map $h \mapsto hm_0$ gives an isomorphism $\mathcal{H}_{n,r} \xrightarrow{\sim} M_{\boldsymbol{\lambda},\mathbf{c}}$. We denote by the same symbol the basis of $\mathcal{H}_{n,r}$ obtained from the basis $\{e_{\sigma,\varepsilon}\}$ of $M_{\boldsymbol{\lambda},\mathbf{c}}$. Since m_0 is ψ-invariant, it follows from Theorem 2.4 that the bar involution on $\mathcal{H}_{n,r}$ can be identified, under the above isomorphism, with the involution ψ on $M_{\boldsymbol{\lambda},\mathbf{c}}$. Let $\{b_{\sigma,\varepsilon}\}$ be the basis of $\mathcal{H}_{n,r}$ obtained by transferring the canonical basis of $M_{\boldsymbol{\lambda},\mathbf{c}}$ attached to $\{e_{\sigma,\varepsilon}\} \subset M_{\boldsymbol{\lambda},\mathbf{c}}$. Then the following result is immediate from Theorem 4.3.

Theorem 4.7. *There exists a unique basis $\{b_{\sigma,\varepsilon} \mid \sigma \in \mathfrak{S}_n, \varepsilon \in [1,r]^n\}$ of $\mathcal{H}_{n,r}$ satisfying the following properties.*

$$\begin{aligned} b_{\sigma,\varepsilon} &\in e_{\sigma,\varepsilon} + \sum_{\substack{\tau \in \mathfrak{S}_n \\ (\tau,\varepsilon) < (\sigma,\varepsilon)}} v^{-1}\mathbb{Z}[v^{-1}] e_{\tau,\varepsilon}, \\ \overline{b}_{\sigma,\varepsilon} &= b_{\sigma,\varepsilon}. \end{aligned}$$

The coefficient $p_{(\tau,\varepsilon),(\sigma,\varepsilon)}$ of $e_{\tau,\varepsilon}$ in the expression of $b_{\sigma,\varepsilon}$ is described by the parabolic Kazhdan-Lusztig polynomials of type A for the weight space $V_{\boldsymbol{\nu}}^{\otimes n}$ under the correspondence $\boldsymbol{\nu} \leftrightarrow \varepsilon$ in (4.6.1).

§5. The case of Iwahori-Hecke algebras of type B_n

5.1. We consider the case where $W = W_{n,2}$ is the Weyl group of type B_n. We specify the parameters of $\mathcal{H}_{n,2}$ by putting $u_1 = -v^{-1}, u_2 = v$, so that $\mathcal{H}_{n,2}$ is the Hecke algebra $\mathcal{H}$ of W as given in 3.1. We discuss the relationship between Kazhdan-Lusztig basis of $\mathcal{H}$ and the previous basis.

Let us consider the subalgebra $\mathcal{H}_{\boldsymbol{\lambda},\mathbf{c}}$ of $\mathcal{H}$ as in 4,4, and assume that $\mathcal{H}_{\boldsymbol{\lambda},\mathbf{c}}$ is the subalgebra $\mathcal{H}_J$ associated to a parabolic subgroup W_J of W. We also assume that the linear character $\varphi_{\boldsymbol{\lambda},\mathbf{c}} : \mathcal{H}_J \to K$ in 4.4 is of the form φ_J in 3.1 (hence $\boldsymbol{\lambda} = (\lambda^{(1)}; \lambda^{(2)}) = (-; k)$ for some $k \geq 0$). Then the $\mathcal{H}$-submodule $M_{\boldsymbol{\lambda},\mathbf{c}} = \bigoplus V_{\boldsymbol{\nu}}^{\otimes n}$ of $V^{\otimes n}$ can be identified with M_J in 3.1, where $m_{\boldsymbol{\lambda},\mathbf{c}} \in M_{\boldsymbol{\lambda},\mathbf{c}}$ corresponds to $m_e \in M_J$. By Theorem 2.4 and Proposition 4.5, the bar involution on M_J given in 3.1 coincides with the involution ψ on $M_{\boldsymbol{\lambda},\mathbf{c}}$. We shall compare various bases on M_J. Put $m_\sigma = T_\sigma m_e$ for $\sigma \in W^J$. Then $\mathcal{M} = \{m_\sigma \mid \sigma \in W^J\}$ gives a basis of M_J. Let $\mathcal{C} = \{C_\sigma^J \mid \sigma \in W^J\}$ be the Kazhdan-Lusztig basis of M_J given in 3.1. We also put $\mathcal{E} = \{e_I \mid I \in \mathcal{I}_J\}$ and $\mathcal{B} = \{b_I \mid I \in \mathcal{I}_J\}$ the bases of M_J arising from the standard basis and the canonical basis of $\bigoplus_{\boldsymbol{\nu}} V_{\boldsymbol{\nu}}^{\otimes n}$. For a two bases $X = (x_i)$ and $Y = (y_j)$ of M_J indexed by the set $\mathcal{I}_J \simeq W^J$, we denote by $M(X, Y) = (a_{ij})$ the transition matrix from X to Y given by

$$y_i = \sum_{j \in \mathcal{I}} a_{ji} x_j.$$

Put

$$P_B = M(\mathcal{M}, \mathcal{C}), \quad P_A = M(\mathcal{E}, \mathcal{B}), \quad X = M(\mathcal{B}, \mathcal{C}), \quad Y = M(\mathcal{E}, \mathcal{M}).$$

We define a total order on W^J which is compatible with the converse of the Bruhat order on W^J, and consider the matrix $P_B = (p_{\tau,\sigma})_{\tau,\sigma \in W^J}$ with respect to this order. Then P_B is a lower unitriangular matrix. Moreover, $p_{\tau,\sigma} \in v^{-1}\mathbb{Z}[v^{-1}]$, and $p_{\tau,\sigma}$ represents the parabolic Kazhdan-Lusztig polynomials of type B_n associated to W_J up to a power of v. If we fix a total order on the set $\mathcal{I}_J$ compatible with the weight decomposition $M_J = \bigoplus_{\boldsymbol{\nu}} V_{\boldsymbol{\nu}}^{\otimes n}$, the matrix P_A is a block-wise diagonal matrix, and diagonal blocks correspond to the weights on M_J. The diagonal block $P_A^{\boldsymbol{\nu}}$ corresponding to the weight $\boldsymbol{\nu}$ is the matrix of the parabolic Kazhdan-Lusztig polynomials of type A associated to the parabolic subgroup $\mathfrak{S}_{\boldsymbol{\nu}}$ (up to powers of v), where $\mathfrak{S}_{\boldsymbol{\nu}}$ is the stabilizer of $e^{\boldsymbol{\nu}}$ in $\mathfrak{S}_n$.

We have the following.

Proposition 5.2. *The matrices P_A, P_B, X, Y satisfy the following relation.*

$$P_B = Y^{-1} P_A X. \tag{5.2.1}$$

Moreover, the matrices P_B and X are determined uniquely by P_A and Y. In other words, the parabolic Kazhdan-Lusztig polynomials of type B_n can be determined by various parabolic Kazhdan-Lusztig polynomials of type A and by the matrix Y.

Proof. It is clear that P_A, P_B, X, Y satisfy (5.2.1). We show that P_A and Y determine P_B and X uniquely. Write the equation (5.2.1) as

$$P_B X^{-1} = Y^{-1} P_A \tag{5.2.2}$$

and consider (5.2.2) as the matrix equation with unknown matrices P_B and X. We fix a bijection $W^J \simeq \mathcal{I}_J$, and write the matrices as $P_B = (p_{ij}), X^{-1} = (x_{ij})$ with $i, j \in \mathcal{I}_J$ along the order inherited from the order on W^J. Here $p_{ij} \in v^{-1}\mathbb{Z}[v^{-1}]$ and $x_{ij} \in \mathbb{Q}(v)$ such that $\bar{x}_{ij} = x_{ij}$. We determine the matrices P_B and X^{-1} row wisely. Suppose that the first $(i-1)$-rows of P_B and X^{-1} are determined. Since P_B is lower unitriangular, one can write

$$\sum_{j=1}^{i-1} p_{ij}\mathbf{x}_j + \mathbf{x}_i = \boldsymbol{\alpha}_i, \tag{5.2.3}$$

where $\mathbf{x}_j$ (resp. $\boldsymbol{\alpha}_j$) denotes the j-th row of X^{-1} (resp. $Y^{-1}P_A$), respectively. By applying the bar involution on (5.2.3), and by subtracting each other, one has

$$\sum_{j=1}^{i-1} (p_{ij} - \bar{p}_{ij})\mathbf{x}_j = \boldsymbol{\alpha}_i - \overline{\boldsymbol{\alpha}}_i. \tag{5.2.4}$$

Here $\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, \boldsymbol{\alpha}_i - \overline{\boldsymbol{\alpha}}_i$ are known vectors. Since $\mathbf{x}_1, \dots, \mathbf{x}_{i-1}$ are linearly independent, (5.2.4) determines $d_{ij} = p_{ij} - \bar{p}_{ij}$ uniquely. But since $p_{ij} \in v^{-1}\mathbb{Z}[v^{-1}]$, d_{ij} determines p_{ij} uniquely. Thus the i-th row of P_B is determined. By substituting p_{ij} into (5.2.3), the i-th row $\mathbf{x}_i$ is also determined. Thus the matrices P_B and X^{-1} are determined. □

5.3. The bases $\{e_{\sigma,\varepsilon}\}$ and $\{b_{\sigma,\varepsilon}\}$ appeared in Theorem 4.7 are nothing but the bases $\mathcal{E}$ and $\mathcal{B}$, respectively. In order to relate these bases to the Kazhdan-Lusztig basis, it is essential to know about the matrix X since the matrix Y is more or less simpler than X. It would be an interesting problem to study the matrix X. One might expect that X has a relatively simple form compared to the matrix P_B. We give below a simple example of the matrix X, i.e., the relation between parabolic Kazhdan-Lusztig basis and the canonical basis.

Assume that W is the Weyl group of type B_n, and let W_J be the parabolic subgroup of type B_{n-1}. We put $J = \{t_1, s_2, \dots, s_{n-1}\}$. Then the distinguished representatives W^J are given as

$$W^J = \{s_i \cdots s_n \mid 2 \le i \le n+1\} \cup \{s_i \cdots s_2 t_1 s_2 \cdots s_n \mid 1 \le i \le n\}$$

under the convention that $s_1 = s_{n+1} = 1$. Assume that $V = V_1 \oplus V_2$ with $\dim V_1 = 1, \dim V_2 = 2$. We fix bases $e_1^{(1)}$ of V_1 and $e_1^{(2)}, e_2^{(2)}$ of V_2, respectively.

We also write $e_3 = e_2^{(2)}, e_2 = e_1^{(2)}, e_1 = e_1^{(1)}$. Let us consider $M_{\boldsymbol{\lambda},\mathbf{c}}$ as in 4.4, where $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}) = (-; n-1)$ and $\mathbf{c} = (1)$ (i.e., $t = 1$). Then $M_{\boldsymbol{\lambda},\mathbf{c}}$ is isomorphic to the induced representation $\operatorname{Ind}_{\mathcal{H}_J}^{\mathcal{H}} \varphi_J$, where $\mathcal{H}_J$ is the parabolic subalgebra of $\mathcal{H} = \mathcal{H}_{n,2}$ of type B_{n-1} and φ_J is as in 3.1. It can be decomposed into the direct sum of weight spaces $M_{\boldsymbol{\lambda},\mathbf{c}} = V_{\boldsymbol{\nu}}^{\otimes n} \bigoplus V_{\boldsymbol{\nu}'}^{\otimes n}$, where $\boldsymbol{\nu} = (0, 1, n-1), \boldsymbol{\nu}' = (1, 0, n-1)$ as weights for U_v. We define, for $1 \le i \le n$, I_i, I_i' by

$$I_i = (\underbrace{2^{(2)}, \dots, 2^{(2)}}_{i-1\text{-times}}, 2^{(1)}, 2^{(2)}, \dots, 2^{(2)}),$$
$$I_i' = (\underbrace{2^{(2)}, \dots, 2^{(2)}}_{i-1\text{-times}}, 1^{(1)}, 2^{(2)}, \dots, 2^{(2)}).$$

Then $b^{\boldsymbol{\nu}} = b_{I_n} = e_{I_n}$ and $b^{\boldsymbol{\nu}'} = b_{I_n'} = e_{I_n'}$, and we have $m_{\boldsymbol{\lambda},\mathbf{c}} = b_{I_n} + b_{I_n'}$. The Kazhdan-Lusztig basis C_σ^J for $\sigma \in W^J$ can be expressed in terms of canonical basis, as

$$C_\sigma^J = \begin{cases} b_{I_{i-1}} + b_{I_{i-1}'} & \text{if } \sigma = s_i \cdots s_n, \\ (v^i + v^{-i}) b_{I_1} - b_{I_{i+1}} + b_{I_{i+1}'} & \text{if } \sigma = s_i \cdots s_2 t s_2 \cdots s_n. \end{cases}$$

This determines the matrix X completely.

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