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L^p estimates for rough parametric Marcinkiewicz integrals

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Abstract. We prove the L^p boundedness of a class of parametric Marcinkiewicz integral operators $\mathcal{M}^{\rho}_{\Omega,h}$ when h satisfies a certain integrability condition and Ω belongs to the block space $B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ for some $q>1, n\geq 2$. Also, we obtain the L^p boundedness for a class of rough parametric Marcinkiewicz integral operators $\mathcal{M}^{*,\rho}_{\Omega,h,\lambda}$ and $\mathcal{M}^{\rho}_{\Omega,h,S}$ related to the Littlewood-Paley g^*_λ-function and the area integral S, respectively. Our results are essential improvement and extension of some previously known results.

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§1. Introduction

Let $\mathbf{R}^n$ $(n\geq 2)$ be the $n-$dimensional Euclidean space and $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbf{R}^n$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(\cdot)$. For $x\in\mathbf{R}^n\backslash\{0\}$, let $x' = x/|x|$.

For a suitable C^1 function Ψ on $\mathbf{R}_+$ and a measurable function $h:\mathbf{R}_+\longrightarrow\mathbf{C}$ define the parametric Marcinkiewicz integral operator $\mathcal{M}^{\rho}_{\Omega,\Psi,h}$ by

(1.1)
$$\mathcal{M}^{\rho}_{\Omega,\Psi,h}f(x)=\left(\int_0^\infty\left|\frac{1}{t^\rho}\int_{|y|\leq t}f(x-\Psi(|y|)y')\frac{\Omega(y/|y|)}{|y|^{n-\rho}}h(|y|)dy\right|^2\frac{dt}{t}\right)^{1/2},$$

where $\rho=\alpha+i\beta$ ($\alpha,\beta\in\mathbf{R}$ with $\alpha>0$) and $f\in\mathcal{S}(\mathbf{R}^n)$, the space of Schwartz functions and Ω is defined on $\mathbf{S}^{n-1}$, $\Omega\in L^1(\mathbf{S}^{n-1})$ and satisfies the vanishing condition

$$\int_{\mathbf{S}^{n-1}} \Omega\left(x'\right) d\sigma\left(x'\right) = 0. \tag{1.2}$$

Throughout this article, we denote $\mathcal{M}^{\rho}_{\Omega,\Psi,h}$ by $\mathcal{M}^{\rho}_{\Omega,h}$ if $\Psi(t) \equiv t$, p' will denote the dual exponent to p, that is $1/p + 1/p' = 1$ and $\Delta_{\gamma}\left(\mathbf{R}^{+}\right)$ $(\gamma > 1)$ will denote the set of all measurable functions h on $\mathbf{R}^{+}$ such that

$$\sup_{R>0}\left(\frac{1}{R}\int_{0}^{R}\left|h\left(t\right)\right|^{\gamma} dt\right)^{1/\gamma} < \infty.$$

It is well-known that $\mathcal{M}^{1}_{\Omega,1}$ is the classical Marcinkiewicz integral operator of higher dimension, corresponding to the Littlewood-Paley g-function, introduced by E. Stein in [St1]. In 1958, Stein showed that if Ω is continuous and $\Omega \in \mathrm{Lip}_{\alpha}(\mathbf{S}^{n-1})$ $(0 < \alpha \leq 1)$, then $\mathcal{M}^{1}_{\Omega,1}$ is of type (p,p) $(1 < p \leq 2)$ and of weak type $(1,1)$. In [BCP], Benedek, Calderón, and Panzone proved that $\mathcal{M}^{1}_{\Omega,1}$ is of type (p,p) for $p \in (1,\infty)$ if $\Omega \in C^{1}\left(\mathbf{S}^{n-1}\right)$. Very recently, Al-Qassem and Al-Salman in [AA] showed that $\mathcal{M}^{1}_{\Omega,1}$ is of type (p,p) for $p \in (1,\infty)$ if $\Omega \in B^{(0,-1/2)}_{q}(\mathbf{S}^{n-1})$ and the condition $\Omega \in B^{(0,-1/2)}_{q}(\mathbf{S}^{n-1})$ is optimal in the sense that there exists an Ω which lies in $B^{(0,\upsilon)}_{q}(\mathbf{S}^{n-1})$ for all $-1 < \upsilon < -1/2$ such that $\mathcal{M}^{1}_{\Omega,1}$ is not bounded on $L^{2}(\mathbf{R}^{n})$. On the other hand, in 1960, Hörmander [Ho] proved that the parametric Marcinkiewicz operator $\mathcal{M}^{\rho}_{\Omega,1}$ is of type (p,p) for $p \in (1,\infty)$ if $\rho > 0$ and $\Omega \in \mathrm{Lip}_{\alpha}(\mathbf{S}^{n-1})$ $(0 < \alpha \leq 1)$. 1996, Sakamoto and Yabuta [SY] studied the L^{p} boundedness of the parametric Marcinkiewicz integral operator $\mathcal{M}^{\rho}_{\Omega,1}$ if ρ is complex and proved that $\mathcal{M}^{\rho}_{\Omega,1}$ is of type (p,p) for $p \in (1,\infty)$ if $\mathrm{Re}(\rho) = \alpha > 0$ and $\Omega \in \mathrm{Lip}_{\tau}(\mathbf{S}^{n-1})$ $(0 < \tau \leq 1)$.

In light of the above results, the question regarding the L^{p} boundedness of $\mathcal{M}^{\rho}_{\Omega,1}$ under a non smooth condition on Ω has remained unanswered. The main purpose of this article is to show that the L^{p} boundedness of the parametric Marcinkiewicz operator $\mathcal{M}^{\rho}_{\Omega,h}$ holds when Ω lacks regularity and even when an extra rough function h appears in the kernel. In fact, we are able to prove the following more general result.

Theorem 1.1. *Let $h \in \Delta_{\gamma}\left(\mathbf{R}_{+}\right)$ with $\gamma > 1$. Let Ψ be in $C^{2}([0,\infty))$, convex, and increasing function with $\Psi(0) = 0$. If $\Omega \in B^{(0,-1/2)}_{q}(\mathbf{S}^{n-1})$ and $\mathrm{Re}(\rho) = \alpha > 0$, then*

$$\left\|\mathcal{M}^{\rho}_{\Omega,\Psi,h}(f)\right\|_{L^{p}(\mathbf{R}^{n})} \leq C_{p}\left\|\Omega\right\|_{B^{(0,-1/2)}_{q}(\mathbf{S}^{n-1})}\left\|f\right\|_{L^{p}(\mathbf{R}^{n})} \tag{1.3}$$

is bounded on $L^p(\mathbf{R}^n)$ *for* $|1/p - 1/2| < \min\{1/\gamma', 1/2\}$.

Remarks. (a) We remark that on $\mathbf{S}^{n-1}$, for any $q > 1, 0 < \tau \leq 1$ and $-1 < \upsilon$, the following inclusions hold and are proper:

$$C^1(\mathbf{S}^{n-1}) \subset \mathrm{Lip}_\tau(\mathbf{S}^{n-1}) \subset L^q(\mathbf{S}^{n-1}) \subset L(\log^+ L)(\mathbf{S}^{n-1}) \subset H^1(\mathbf{S}^{n-1}), \tag{1.4}$$

$$\bigcup_{r>1} L^r(\mathbf{S}^{n-1}) \subset B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}). \tag{1.5}$$

With regard to the relationship between $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$ *and* $H^1(\mathbf{S}^{n-1})$ (for $\upsilon > -1$) remains open.

(b) We point out that the range of p given in Theorem 1.1 is the full range $(1, \infty)$ whenever $\gamma \geq 2$. Also, the result in Theorem 1.1 extends the result of Al-Qassem-Al-Salman [AA] who obtained Theorem 1.1 in the special case $h \equiv 1, \rho = 1$ and $\Psi(t) = t$ and also improves substantially the result of Sakamoto and Yabuta [SY].

The paper is organized as follows. Section 2 contains the definition of the block spaces $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$ as well as some of their important properties. The main estimates needed in the proofs of our results are established in Section 3. The proofs of Theorem 1.1 and additional results will be given in Sections 4–5. Throughout the rest of the paper the letter C will stand for a positive constant not necessarily the same one at each occurrence.

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§2. Some Definitions

The block spaces originated in the work of M. H. Taibleson and G. Weiss on the convergence of the Fourier series in connection with the developments of the real Hardy spaces. Below we shall recall the definition of block spaces on $\mathbf{S}^{n-1}$. For further background information about the theory of spaces generated by blocks and its applications to harmonic analysis, see the book [LTW].

Definition 2.1. *A* q*-block on* $\mathbf{S}^{n-1}$ *is an* L^q $(1 < q \leq \infty)$ *function* $b(x)$ *that satisfies*

$$\text{(i) } \mathrm{supp}(b) \subset I; \qquad \text{(ii) } \|b\|_{L^q} \leq |I|^{-1/q'},$$

where $|I| = \sigma(I)$, *and* $I = B(x_0', \theta_0) = \{x' \in \mathbf{S}^{n-1} : |x' - x_0'| < \theta_0\}$ *is a cap on*

$\mathbf{S}^{n-1}$ *for some* $x_0' \in \mathbf{S}^{n-1}$ *and* $\theta_0 \in (0,1]$.

Jiang and Lu introduced (see [LTW]) the class of block spaces $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$ (for $\upsilon > -1$) with respect to the study of homogeneous singular integral operators.

Definition 2.2. *The block space* $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$ *is defined by*

$$B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}) = \left\{ \Omega \in L^1(\mathbf{S}^{n-1}) : \Omega = \sum_{\mu=1}^{\infty} \lambda_\mu b_\mu ,\ M_q^{(0,\upsilon)}\left(\{\lambda_\mu\}\right) < \infty \right\},$$

where each λ_μ *is a complex number; each* b_μ *is a* q*-block supported on a cap* I_μ *on* $\mathbf{S}^{n-1}$, $\upsilon > -1$ *and*

$$M_q^{(0,\upsilon)}\left(\{\lambda_\mu\}\right) = \sum_{\mu=1}^{\infty} \left|\lambda_\mu\right| \left\{ 1 + \log^{(\upsilon+1)}(\left|I_\mu\right|^{-1}) \right\}. \tag{2.1}$$

Let $\|\Omega\|_{B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})} = \inf\{M_q^{(0,\upsilon)}\left(\{\lambda_\mu\}\right)$: $\Omega = \sum_{\mu=1}^{\infty} \lambda_\mu b_\mu$ and each b_μ is a q-block function supported on a cap I_μ on $\mathbf{S}^{n-1}\}$. Then $\|\cdot\|_{B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})}$ is a norm on the space $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$ and $(B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}), \|\cdot\|_{B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})})$ is a Banach space.

In their investigations of block spaces, Keitoku and Sato in [KS] showed that these spaces enjoy the following properties:for any $\upsilon > -1$ and $q > 1$,

$$\begin{aligned} B_q^{(0,\upsilon_2)}(\mathbf{S}^{n-1}) &\subset B_q^{(0,\upsilon_1)}(\mathbf{S}^{n-1}) \text{ if } \upsilon_2 > \upsilon_1 > -1; \\ B_{q_2}^{(0,\upsilon)}(\mathbf{S}^{n-1}) &\subset B_{q_1}^{(0,\upsilon)}(\mathbf{S}^{n-1}) \text{ if } 1 < q_1 < q_2; \\ \bigcup_{q>1} B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}) &\nsubseteq \bigcup_{q>1} L^q(\mathbf{S}^{n-1}). \end{aligned}$$

Definition 2.3. *For a suitable* C^1 *function* Ψ *on* $\mathbf{R}_+$, *a measurable function* $h : \mathbf{R}_+ \longrightarrow \mathbf{C}$ *and a suitable function* $\tilde{b}_\mu$ *on* $\mathbf{S}^{n-1}$ *we define the family of measures* $\{\sigma_{\tilde{b}_\mu,t} : t \in \mathbf{R}_+\}$ *and the maximal operator* $\sigma^*_{\tilde{b}_\mu}$ *on* $\mathbf{R}^n$ *by*

$$\int_{\mathbf{R}^n} f d\sigma_{\tilde{b}_\mu,t} = \frac{1}{t^\rho} \int_{\frac{1}{2}t<|y|\le t} f(\Psi(|y|)y')h(|y|)\frac{\tilde{b}_\mu(y')}{|y|^{n-\rho}}dy,$$

and

$$\sigma^*_{\tilde{b}_\mu} f(x) = \sup_{t\in\mathbf{R}_+} \left| \left|\sigma_{\tilde{b}_\mu,t}\right| * f(x) \right|,$$

where $\left|\sigma_{\tilde{b}_\mu,t}\right|$ *is defined in the same way as* $\sigma_{\tilde{b}_\mu,t}$, *but with* $\tilde{b}_\mu$ *replaced by* $\left|\tilde{b}_\mu\right|$ *and* h *replaced by* $|h|$.

§3. Main Estimates

Lemma 3.1. *Let $\mu \in \mathbf{N}$ and $h \in \Delta_\gamma(\mathbf{R}_+)$ for some γ with $1 < \gamma \le 2$. Let $\tilde{b}_\mu$ be a function on $\mathbf{S}^{n-1}$ satisfying* (i) $\int_{\mathbf{S}^{n-1}} \tilde{b}_\mu(y)d\sigma(y) = 0$; (ii) $\left\|\tilde{b}_\mu\right\|_q \le \left|I_\mu\right|^{-1/q'}$ *for some $q > 1$ and for some cap I_μ on $\mathbf{S}^{n-1}$ with $\left|I_\mu\right| < e^{-1}$; and* (iii) $\left\|\tilde{b}_\mu\right\|_1 \le 1$. *Assume that Ψ is in $C^2([0,\infty))$, convex, and an increasing function with $\Psi(0) = 0$. Then there exist constants C and $0 < \upsilon < 1/q'$ such that for all $k \in \mathbf{Z}$ and $\xi \in \mathbf{R}^n$ we have*

$$\left\|\sigma_{\tilde{b}_\mu,t}\right\| \le C; \tag{3.1}$$

$$\int_{\omega_\mu^k}^{\omega_\mu^{k+1}} \left|\hat{\sigma}_{\tilde{b}_\mu,t}(\xi)\right|^2 \frac{dt}{t} \le C\log(\left|I_\mu\right|^{-1}) \left|\Psi(\omega_\mu^{k-1})\xi\right|^{-\frac{2\upsilon}{\gamma' \log(\left|I_\mu\right|^{-1})}}; \tag{3.2}$$

$$\int_{\omega_\mu^k}^{\omega_\mu^{k+1}} \left|\hat{\sigma}_{\tilde{b}_\mu,t}(\xi)\right|^2 \frac{dt}{t} \le C\log(\left|I_\mu\right|^{-1}) \left|\Psi(\omega_\mu^{k+1})\xi\right|^{\frac{2\upsilon}{\gamma' \log(\left|I_\mu\right|^{-1})}}, \tag{3.3}$$

where $\omega_\mu = 2^{\log(\left|I_\mu\right|^{-1})}$ and $\left\|\sigma_{\tilde{b}_\mu,t}\right\|$ stands for the total variation of $\sigma_{\tilde{b}_\mu,t}$. The constant C is independent of k, μ, ξ and $\Psi(\cdot)$.

Proof. By (iii) and the definition of $\sigma_{\tilde{b}_\mu,t}$, one can easily see that (3.1) holds with a constant C independent of t and μ. Next we prove (3.2). By definition,

$$\hat{\sigma}_{\tilde{b}_\mu,t}(\xi) = \frac{1}{t^\rho} \int_{\frac{1}{2}t}^{t} \int_{\mathbf{S}^{n-1}} e^{-i\Psi(s)\xi\cdot x} \tilde{b}_\mu(x) \frac{h(s)}{s^{1-\rho}} d\sigma(x)\, ds.$$

By Hölder's inequality, a change of variable, the assumption $1 < \gamma \le 2$ and since

$$\left|\int_{\mathbf{S}^{n-1}} e^{-i\Psi(s)\xi\cdot x}\tilde{b}_\mu(x)d\sigma(x)\right| \le 1,$$

we obtain

$$\begin{aligned} \left|\hat{\sigma}_{\tilde{b}_\mu,t}(\xi)\right| &\le \left(\int_{\frac{1}{2}t}^{t} |h(s)|^\gamma \frac{ds}{s}\right)^{1/\gamma} \left(\int_{\frac{1}{2}t}^{t} \left|\int_{\mathbf{S}^{n-1}} e^{-i\Psi(s)\xi\cdot x}\tilde{b}_\mu(x)d\sigma(x)\right|^{\gamma'} \frac{ds}{s}\right)^{1/\gamma'} \\ &\le C\left(\int_{\frac{1}{2}t}^{t} \left|\int_{\mathbf{S}^{n-1}} e^{-i\Psi(s)\xi\cdot x}\tilde{b}_\mu(x)d\sigma(x)\right|^2 \frac{ds}{s}\right)^{1/\gamma'} \\ &= C\left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} \tilde{b}_\mu(x)\overline{\tilde{b}_\mu(y)} I_{\mu,t}(\xi,x,y)d\sigma(x)\, d\sigma(y)\right)^{1/\gamma'}, \end{aligned} \tag{3.4}$$

where

$$I_{\mu,t}(\xi,x,y)=\int_{1/2}^{1}e^{-i\Psi(ts)\xi\cdot(x-y)}\frac{ds}{s}.$$

Write $I_{\mu,t}(\xi,x,y)$ as

$$I_{\mu,t}(\xi,x,y)=\int_{1/2}^{1}G_t'(s)\frac{ds}{s},$$

where

$$G_t(s)=\int_{1/2}^{s}e^{-i\Psi(tw)\xi\cdot(x-y)}dw,\ 1/2\le s\le 1.$$

By the assumptions on Ψ and using the mean value theorem we have

$$\frac{d}{dw}\left(\Psi(tw)\right)=t\Psi'(tw)\ge\frac{\Psi(tw)}{w}\ge\frac{\Psi(t/2)}{s}\text{ for }1/2\le w\le s\le 1.$$

Thus by van der Corput's lemma, $|G_t(s)|\le\left|\frac{\Psi(t/2)\xi}{s}\right|^{-1}|\xi'\cdot(x-y)|^{-1}$. By integration by parts, we get

$$|I_{\mu,t}(\xi,x,y)|\le C\,|\Psi(t/2)\xi|^{-1}\left|\xi'\cdot(x-y)\right|^{-1},$$

which when combined with the trivial estimate $|I_{\mu,t}(\xi,x,y)|\le\log 2$ and choosing τ such that $0<\tau<1/q'$ yields to

$$|I_{\mu,t}(\xi,x,y)|\le|\Psi(t/2)\xi|^{-\tau}\left|\xi'\cdot(x-y)\right|^{-\tau}. \tag{3.5}$$

By Hölder's inequality and (ii) we get

$$\begin{aligned}\left|\hat{\sigma}_{\tilde{b}_\mu,t}(\xi)\right| &\le C\,|\Psi(t/2)\xi|^{-\tau/\gamma'}\left\|\tilde{b}_\mu\right\|_q^{2/\gamma'}\\ &\quad\times\left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}}\left|\xi'\cdot(x-y)\right|^{-\tau q'}d\sigma(x)\,d\sigma(y)\right)^{1/(q'\gamma')}\\ &\le C\,|\Psi(t/2)\xi|^{-\tau/\gamma'}\left|I_\mu\right|^{-2/(q'\gamma')}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_{\omega_\mu^k}^{\omega_\mu^{k+1}}\left|\hat{\sigma}_{\tilde{b}_\mu,t}(\xi)\right|^2\frac{dt}{t}\\ &\le C\min\{\log(\left|I_\mu\right|^{-1}),\log(\left|I_\mu\right|^{-1})\left|\Psi(\tfrac{1}{2}\omega_\mu^k)\xi\right|^{-2\tau/\gamma'}\left|I_\mu\right|^{-4/(q'\gamma')}\}\\ &\le C\log(\left|I_\mu\right|^{-1})\left|\Psi(\omega_\mu^{k-1})\xi\right|^{-\frac{2\tau}{\gamma'\log(\left|I_\mu\right|^{-1})}},\end{aligned}$$

which proves (3.2). To prove (3.3), we use the cancellation condition of $\tilde{b}_\mu$ to get

$$\left|\hat{\sigma}_{\tilde{b}_\mu,t}(\xi)\right| \leq \int_{\mathbf{S}^{n-1}} \int_{\frac{1}{2}t}^{t} \left|e^{-i\Psi(s)\xi\cdot x} - 1\right| |h(s)| \left|\tilde{b}_\mu(x)\right| \frac{ds}{s} d\sigma(x).$$

Hence, by (iii) and since Ψ is increasing we get

$$\left|\hat{\sigma}_{\tilde{b}_\mu,t}(\xi)\right| \leq C \left|\Psi(t)\xi\right|.$$

By using the same argument as above we get (3.3). The lemma is proved.

By the same argument as in [St3, p. 57] we get

Lemma 3.2. *Let φ be a nonnegative, decreasing function on $[0,\infty)$ with $\int_{[0,\infty)} \varphi(t)dt = 1$. Then*

$$\left|\int_{[0,\infty)} f(x - ty')\varphi(t)dt\right| \leq M_{y'} f(x),$$

where

$$M_{y'} f(x) = \sup_{R\in\mathbf{R}} \frac{1}{R} \int_0^R \left|f(x - sy')\right| ds$$

is the Hardy-Littlewood maximal function of f in the direction of y'.

Lemma 3.3. *Let $\mu \in \mathbf{N}$, $h \in \Delta_\gamma(\mathbf{R}_+)$ for some $\gamma > 1$. Assume that $\tilde{b}_\mu \in L^1(\mathbf{S}^{n-1})$ and Ψ is in $C^2([0,\infty))$, convex, and increasing function with $\Psi(0) = 0$. Then, for $\gamma' < p < \infty$, there exists a positive constant C_p such that*

$$\left\|\sigma^*_{\tilde{b}_\mu}(f)\right\|_{L^p(\mathbf{R}^n)} \leq C_p \left\|\tilde{b}_\mu\right\|_{L^1(\mathbf{S}^{n-1})} \|f\|_{L^p(\mathbf{R}^n)}. \tag{3.6}$$

Proof. By Hölder's inequality, we have

$$\begin{aligned}
&\left|\left|\sigma_{\tilde{b}_\mu,t}\right| * f(x)\right| \\
\leq\ & \left(\int_{\frac{1}{2}t}^{t} |h(s)|^\gamma \frac{ds}{s}\right)^{1/\gamma} \left(\int_{\frac{1}{2}t}^{t} \left|\int_{\mathbf{S}^{n-1}} \tilde{b}_\mu(y') f(x - \Psi(s)y') d\sigma(y')\right|^{\gamma'} \frac{ds}{s}\right)^{1/\gamma'} \\
\leq\ & C \left(\int_{\frac{1}{2}t}^{t} \int_{\mathbf{S}^{n-1}} \left|\tilde{b}_\mu(y')\right| \left|f(x - \Psi(s)y')\right|^{\gamma'} d\sigma(y') \frac{ds}{s}\right)^{1/\gamma'}.
\end{aligned}$$

Thus

$$\sigma^*_{\tilde{b}_\mu} f(x) \le C \left(\int_{\mathbf{S}^{n-1}} \left| \tilde{b}_\mu(y') \right| M_{\Psi,y'}(|f|^{\gamma'})(x) d\sigma(y') \right)^{1/\gamma'}, \tag{3.7}$$

where

$$M_{\Psi,y'} f(x) = \sup_{t\in\mathbf{R}_+} \frac{1}{t} \left| \int_0^t f(x - \Psi(s)y')ds \right|.$$

Without loss of generality, we may assume that $\Psi(t) > 0$ for all $t > 0$. By a change of variable we have

$$M_{\Psi,y'} f(x) \le \sup_{t\in\mathbf{R}_+} \left(\frac{1}{t} \int_0^{\Psi(t)} \left| f(x - sy') \right| \frac{ds}{\Psi'(\Psi^{-1}(s))} \right).$$

Since the function $\frac{1}{t\Psi'(\Psi^{-1}(s))}$ is non-negative, decreasing and its integral over $[0,\ \Psi(t)]$ is equal to 1, by Lemma 3.2 we obtain

$$M_{\Psi,y'} f(x) \le M_{y'} f(x). \tag{3.8}$$

By (3.7)-(3.8) and Minkowski's inequality for integrals we get

$$\left\| \sigma^*_{\tilde{b}_\mu}(f) \right\|_{L^p(\mathbf{R}^n)} \le C \left(\int_{\mathbf{S}^{n-1}} \left| \tilde{b}_\mu(y') \right| \left\| M_{y'}(|f|^{\gamma'}) \right\|_{L^{p/\gamma'}(\mathbf{R}^n)} d\sigma(y') \right)^{1/\gamma'}. \tag{3.9}$$

Since $M_{y'}$ is bounded $L^p(\mathbf{R}^n)$ with bound independent of y', we immediately get (3.6). This completes the proof of the lemma.

Lemma 3.4. *Let $\mu \in \mathbf{N}$, $h \in \Delta_\gamma(\mathbf{R}_+)$ for some $\gamma \in (1, 2]$. Assume that $\tilde{b}_\mu \in L^1(\mathbf{S}^{n-1})$ and Ψ is in $C^2([0,\infty))$, convex, and increasing function with $\Psi(0) = 0$. Then, for any p satisfying $|1/p - 1/2| < 1/\gamma'$, there exists a positive constant C_p such that*

$$\begin{aligned} &\left\| \left(\sum_{k\in\mathbf{Z}} \int_{\omega_\mu^k}^{\omega_\mu^{k+1}} \left| \sigma_{\tilde{b}_\mu,t} * g_k \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\mathbf{R}^n)} \\ \le\ & C_p (\log |I_\mu|^{-1})^{1/2} \left\| \tilde{b}_\mu \right\|_{L^1(\mathbf{S}^{n-1})} \left\| (\sum_{k\in\mathbf{Z}} |g_k|^2)^{1/2} \right\|_{L^p(\mathbf{R}^n)} \end{aligned} \tag{3.10}$$

holds for arbitrary functions $\{g_k(\cdot)\}_{k\in\mathbf{Z}}$ on $\mathbf{R}^n$. The constant C_p is independent of μ.

Proof. Assume first that $2 \le p < \frac{2\gamma}{2-\gamma}$. We use a similar argument as in the proof of Theorem 7.5 in [FP]. By duality there exists a nonnegative function f in $L^{(p/2)'}(\mathbf{R}^n)$ with $\|f\|_{(p/2)'} \le 1$ such that

$$\left\| \left(\sum_{k\in\mathbf{Z}} \int_{\omega_\mu^k}^{\omega_\mu^{k+1}} \left|\sigma_{\tilde{b}_\mu,t} * g_k\right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\mathbf{R}^n)}^2 = \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n} \int_{\omega_\mu^k}^{\omega_\mu^{k+1}} \left|\sigma_{\tilde{b}_\mu,t} * g_k(x)\right|^2 \frac{dt}{t} f(x)dx.$$

By Schwarz's inequality we get

$$\left|\sigma_{\tilde{b}_\mu,t} * g_k(x)\right|^2 \le \left(\int_{\frac{1}{2}t}^{t} \int_{\mathbf{S}^{n-1}} |g_k(x-\Psi(s)y)|\,|h(s)| \left|\tilde{b}_\mu(y)\right| d\sigma(y)\frac{ds}{s} \right)^2$$
$$\le C\left\|\tilde{b}_\mu\right\|_{L^1(\mathbf{S}^{n-1})} \left(\int_{\frac{1}{2}t}^{t} \int_{\mathbf{S}^{n-1}} |g_k(x-\Psi(s)y)|^2 \left|\tilde{b}_\mu(y)\right| |h(s)|^{2-\gamma} d\sigma(y)\frac{ds}{s} \right).$$

Therefore, by a change of variable we have

$$\left\| \left(\sum_{k\in\mathbf{Z}} \int_{\omega_\mu^k}^{\omega_\mu^{k+1}} \left|\sigma_{\tilde{b}_\mu,t} * g_k\right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\mathbf{R}^n)}$$

$$\le C\left(\log\left|I_\mu\right|^{-1}\right) \left\|\tilde{b}_\mu\right\|_{L^1(\mathbf{S}^{n-1})} \int_{\mathbf{R}^n} \left(\sum_{k\in\mathbf{Z}} |g_k(x)|^2 \right) \tilde{M}_{|h|^{2-\gamma},\tilde{b}_\mu} f(x)dx, \tag{3.11}$$

where

$$\tilde{M}_{|h|^{2-\gamma},\tilde{b}_\mu} f(x) = \sup_{t\in\mathbf{R}_+} \left(\int_{\frac{1}{2}t<|y|\le t} \left|f(x+\Psi(|y|)y')\right| |h(|y|)|^{2-\gamma} \frac{\left|\tilde{b}_\mu(y)\right|}{|y|^n} dy \right).$$

By Lemma 3.3 and noticing that $|h(\cdot)|^{2-\gamma} \in \Delta_{\gamma/(2-\gamma)}(\mathbf{R}_+)$ and $(p/2)' > \left(\frac{\gamma}{2-\gamma}\right)'$ we obtain

$$\left\|\tilde{M}_{|h|^{2-\gamma},\tilde{b}_\mu} f\right\|_{L^{(p/2)'}(\mathbf{R}^n)} \le C_p \left\|\tilde{b}_\mu\right\|_{L^1(\mathbf{S}^{n-1})} \|f\|_{L^{(p/2)'}(\mathbf{R}^n)} \le C_p \left\|\tilde{b}_\mu\right\|_{L^1(\mathbf{S}^{n-1})}. \tag{3.12}$$

Thus, by (3.11)–(3.12) and Hölder's inequality we get (3.10) for $2 \le p < \frac{2\gamma}{2-\gamma}$.

Now we need to prove (3.11) for the case $\frac{2\gamma}{3\gamma-2} < p < 2$. Let $E_{\mu,k} = [\omega_\mu^k, \omega_\mu^{k+1})$. By a duality argument, there exist functions $f = f_k(x,t)$ defined on $\mathbf{R}^n \times \mathbf{R}^+$ with $\left\| \left\| \|f_k\|_{L^2(E_{\mu,k},dt/t)} \right\|_{l^2} \right\|_{L^{p'}} \leq 1$ such that

$$\begin{aligned}
&\left\| \left(\sum_{k\in\mathbf{Z}} \int_{\omega_\mu^k}^{\omega_\mu^{k+1}} \left| \sigma_{\tilde{b}_\mu,t} * g_k \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_p \\
&= \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \int_{E_{\mu,k}} \left(\sigma_{\tilde{b}_\mu,t} * g_k(x) \right) f_k(x,t) \frac{dt}{t} dx \\
&\leq C_p (\log |I_\mu|^{-1})^{1/2} \left\| (\sum_{k\in\mathbf{Z}} |g_k|^2)^{1/2} \right\|_p \left\| (S(f))^{1/2} \right\|_{p'}, \qquad (3.13)
\end{aligned}$$

where

$$S(f)(x) = \sum_{k\in\mathbf{Z}} \int_{E_{\mu,k}} \left| \sigma_{\tilde{b}_\mu,t} * f_k(x,t) \right|^2 \frac{dt}{t}.$$

Now, since $p' > 2$, there exists a function $q \in L^{(p'/2)'}(\mathbf{R}^n)$ such that

$$\|S(f)\|_{p'/2} = \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n} \int_{E_{\mu,k}} \left| f_k(x,t) * \sigma_{\tilde{b}_\mu,t} \right|^2 \frac{dt}{t} q(x) dx.$$

By the same argument as above, we have

$$\begin{aligned}
&\|S(f)\|_{p'/2} \\
&\leq C \left\| \tilde{b}_\mu \right\|_{L^1(\mathbf{S}^{n-1})} \int_{\mathbf{R}^n} \tilde{M}_{|h|^{2-\gamma},\tilde{b}_\mu} q(x) \left(\sum_{k\in\mathbf{Z}} \int_{E_{\mu,k}} |f_k(x,t)|^2 \frac{dt}{t} \right) dx \\
&\leq C \left\| \tilde{b}_\mu \right\|_{L^1(\mathbf{S}^{n-1})} \left\| \left(\sum_{k\in\mathbf{Z}} \int_{E_{\mu,k}} |f_k(\cdot,t)|^2 \frac{dt}{t} \right) \right\|_{p'/2} \left\| \tilde{M}_{|h|^{2-\gamma},\tilde{b}_\mu} q \right\|_{(p'/2)'}.
\end{aligned}$$

By invoking Lemma 3.3 we obtain

$$\left\| \tilde{M}_{|h|^{2-\gamma},\tilde{b}_\mu}(q) \right\|_{(p'/2)'} \leq C_p \left\| \tilde{b}_\mu \right\|_{L^1(\mathbf{S}^{n-1})} \|q\|_{(p'/2)'} \leq C_p \left\| \tilde{b}_\mu \right\|^2_{L^1(\mathbf{S}^{n-1})}.$$

Thus by our choice of $f_k(x,t)$ we have

$$\|S(f)\|_{p'/2} \leq C_p \left\| \tilde{b}_\mu \right\|^2_{L^1(\mathbf{S}^{n-1})} \left\| \left(\sum_{k\in\mathbf{Z}} \int_{E_{\mu,k}} |f_k(\cdot,t)|^2 \frac{dt}{t} \right) \right\|_{p'/2} \leq C_p \left\| \tilde{b}_\mu \right\|^2_{L^1(\mathbf{S}^{n-1})}$$

which in turn along with (3.13) gives (3.10) for $\frac{2\gamma}{3\gamma-2} < p < 2$. The proof is complete.

§4. Conclusion

Assume that $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ for some $q > 1$ and satisfies (1.2). Thus Ω can be written as $\Omega = \sum_{\mu=1}^{\infty} \lambda_\mu b_\mu$, where $\lambda_\mu \in \mathbf{C}$, b_μ is a q-block supported on a cap I_μ on $\mathbf{S}^{n-1}$ and $M_q^{(0,-1/2)}\left(\{\lambda_\mu\}\right) < \infty$. To each block function $b_\mu(\cdot)$, let $\tilde{b}_\mu(\cdot)$ be a function defined by

$$\tilde{b}_\mu(x) = b_\mu(x) - \int_{\mathbf{S}^{n-1}} b_\mu(u) d\sigma(u). \tag{4.1}$$

Let $\mathbf{J} = \left\{\mu \in \mathbf{N} : \left|I_\mu\right| < e^{-1}\right\}$. Let $\tilde{b}_0 = \Omega - \sum_{\mu \in \mathbf{J}}^{\infty} \lambda_\mu \tilde{b}_\mu$. Then for some positive constant C, the following holds for all $\mu \in \mathbf{J} \cup \{0\}$:

$$\begin{aligned}
&(4.2) \quad \int_{\mathbf{S}^{n-1}} \tilde{b}_\mu(u)\, d\sigma(u) &&= 0,\\
&(4.3) \quad \left\|\tilde{b}_\mu\right\|_q &&\leq C \left|I_\mu\right|^{-1/q'},\\
&(4.4) \quad \left\|\tilde{b}_\mu\right\|_1 &&\leq C,\\
&(4.5) \quad \Omega &&= \sum_{\mu \in \mathbf{J} \cup \{0\}} \lambda_\mu \tilde{b}_\mu,
\end{aligned}$$

where $|I_0| = e^{-1}$.

By (4.5) we have

$$\mathcal{M}^\rho_{\Omega,\Psi,h}(f) \leq \sum_{\mu \in \mathbf{J} \cup \{0\}} \left|\lambda_\mu\right| \mathcal{M}^\rho_{\tilde{b}_\mu,\Psi,h}(f). \tag{4.6}$$

Therefore, Theorem 1.1 is proved if we can show that

$$\left\|\mathcal{M}^\rho_{\tilde{b}_\mu,\Psi,h}(f)\right\|_{L^p(\mathbf{R}^n)} \leq C_p (\log \left|I_\mu\right|^{-1})^{1/2} \|f\|_{L^p(\mathbf{R}^n)} \tag{4.7}$$

for $\mu \in \mathbf{J} \cup \{0\}$ and for p satisfying $|1/p - 1/2| < \min\{1/\gamma', 1/2\}$.

Since $\Delta_\gamma(\mathbf{R}_+) \subseteq \Delta_2(\mathbf{R}_+)$ for $\gamma \geq 2$, we may assume that $1 < \gamma \leq 2$. Therefore, it suffices to prove (4.7) for p satisfying $|1/p - 1/2| < 1/\gamma'$. For $k \in \mathbf{Z}$ and $\mu \in \mathbf{N}$, let $a_{\mu,k} = \Psi(\omega_\mu^k)$. We notice that $\{a_{\mu,k} : k \in \mathbf{Z}\}$ is a lacunary sequence with $a_{\mu,k+1}/a_{\mu,k} \geq \omega_\mu$. As in [AP], let $\{\Lambda_{k,\mu}\}_{-\infty}^{\infty}$ be a smooth partition of unity in $(0, \infty)$ adapted to the interval $\mathcal{I}_{k,\mu} = [a^{-1}_{\mu,k+1},$

$a^{-1}_{\mu,k-1}]$. To be precise, we require the following:

$$\begin{aligned}\Lambda_{k,\mu} &\in C^{\infty},\ 0\le\Lambda_{k,\mu}\le 1,\ \sum_k \Lambda_{k,\mu}(t)=1;\\ \operatorname{supp}\Lambda_{k,\mu} &\subseteq \mathcal{I}_{k,\mu},\ \left|\frac{d^s\Lambda_{k,\mu}(t)}{dt^s}\right|\le\frac{C_s}{t^s},\end{aligned}$$

where C_s is independent of the lacunary sequence $\{a_{\mu,k}:k\in\mathbf{Z}\}$. Let $\widehat{\Psi_{k,\mu}}(\xi)=\Lambda_{k,\mu}(|\xi|)$.

By Minkowski's inequality we have

$$\begin{aligned}\mathcal{M}^{\rho}_{\tilde{b}_\mu,\Psi,h}f(x) &= \left(\int_0^\infty\left|\sum_{k=0}^{\infty}2^{-k\rho}\sigma_{\tilde{b}_\mu,2^{-k}t}*f(x)\right|^2\frac{dt}{t}\right)^{1/2}\\ &\le\sum_{k=0}^{\infty}2^{-k\alpha}\left(\int_0^\infty\left|\sigma_{\tilde{b}_\mu,2^{-k}t}*f(x)\right|^2\frac{dt}{t}\right)^{1/2}\\ &=\left(\frac{1}{1-2^{-\alpha}}\right)\left(\int_0^\infty\left|\sigma_{\tilde{b}_\mu,t}*f(x)\right|^2\frac{dt}{t}\right)^{1/2}.\end{aligned}$$

Decompose

$$f*\sigma_{\tilde{b}_\mu,t}(x)=\sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{Z}}(\Psi_{k+j,\mu}*\sigma_{\tilde{b}_\mu,t}*f)(x)\chi_{[\omega^k_\mu,\omega^{k+1}_\mu)}(t):=\sum_{j\in\mathbf{Z}}Y_{j,\mu}(x,t)$$

and define

$$S_{j,\mu}f(x)=\left(\int_0^\infty|Y_{j,\mu}(x,t)|^2\frac{dt}{t}\right)^{1/2}.$$

Then

$$\mathcal{M}^{\rho}_{\tilde{b}_\mu,\Psi,h}(f)\le\left(\frac{1}{1-2^{-\alpha}}\right)\sum_{j\in\mathbf{Z}}S_{j,\mu}(f)$$

holds for $f\in\mathcal{S}(\mathbf{R}^n)$.

Thus, to prove (4.7), it is enough to show that

$$\|S_{j,\mu}(f)\|_{L^p(\mathbf{R}^n)}\le C(\log\left|I_\mu\right|^{-1})^{1/2}2^{-\alpha_p|j|}\,\|f\|_{L^p(\mathbf{R}^n)}\tag{4.8}$$

for some $\alpha_p>0$ and for p satisfying $|1/p-1/2|<1/\gamma'$.

To prove (4.8), let us first compute the L^2-norm of $S_{j,\mu}(f)$. By using Plancherel's theorem, we have

$$\begin{aligned}\|S_{j,\mu}(f)\|^2_{L^2(\mathbf{R}^n)} &= \sum_{k\in\mathbf{Z}}\int_{\mathbf{R}^n}\int_{\omega^k_\mu}^{\omega^{k+1}_\mu}\left|\Psi_{k+j,\mu}*\sigma_{\tilde{b}_\mu,t}*f(x)\right|^2\frac{dt}{t}dx\\ &\le\sum_{k\in\mathbf{Z}}\int_{\Gamma_{k+j,\mu}}\left(\int_{\omega^k_\mu}^{\omega^{k+1}_\mu}\left|\hat{\sigma}_{\tilde{b}_\mu,t}(\xi)\right|^2\frac{dt}{t}\right)\left|\hat{f}(\xi)\right|^2d\xi,\end{aligned}$$

where

$$\Gamma_{k,\mu} = \{\xi \in \mathbf{R}^n : |\xi| \in \mathcal{I}_{k,\mu}\}.$$

Thus, by Lemma 3.1 we have

$$\|S_{j,\mu}(f)\|_{L^2(\mathbf{R}^n)} \le C(\log \left|I_\mu\right|^{-1})^{1/2}\, 2^{-\frac{\alpha}{2}|j|}\, \|f\|_{L^2(\mathbf{R}^n)}\,. \tag{4.9}$$

Next, let us compute the L^p boundedness of the operator $S_{j,\mu}$. For $|1/p - 1/2| < 1/\gamma'$, we have

$$\begin{aligned} \|S_{j,\mu}(f)\|_{L^p(\mathbf{R}^n)} &\le C_p(\log \left|I_\mu\right|^{-1})^{1/2} \left\| \left(\sum_{k\in\mathbf{Z}} |\Psi_{k+j,\mu} * f|^2\right)^{1/2} \right\|_{L^p(\mathbf{R}^n)} \\ &\le C_p(\log \left|I_\mu\right|^{-1})^{1/2}\, \|f\|_{L^p(\mathbf{R}^n)}\,. \qquad (4.10) \end{aligned}$$

The last two inequalities are obtained by applying Lemma 3.4 and applying the Littlewood-Paley theory and Theorem 3 along with the remark that follows its statement in ([St2], p. 96).

Now by interpolation between (4.9) and (4.10) we get (4.8). This completes the proof of Theorem 1.1.

§5. Further results

As an application of Theorem 1.1, we get the L^p boundedness for a class of parametric Marcinkiewicz operators $\mathcal{M}^{*,\rho}_{\Omega,\Psi,h,\lambda}$ and $\mathcal{M}^{\rho}_{\Omega,\Psi,h,S}$ related to the Littlewood-Paley g^*_λ-function and the area integral S, respectively. The definition and the precise statement of the results regarding of these operators are given as follows:

Theorem 5.1. *Let $h \in \Delta_\gamma(\mathbf{R}_+)$ for some $\gamma > 1$. Let Ψ be in $C^2([0,\infty))$, convex, and increasing function with $\Psi(0) = 0$. If $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$, there exists $C_p > 0$ such that*

$$\left\|\mathcal{M}^{\rho}_{\Omega,\Psi,h,S}(f)\right\|_{L^p(\mathbf{R}^n)} + \left\|\mathcal{M}^{\rho,*}_{\Omega,\Psi,h,\lambda}(f)\right\|_{L^p(\mathbf{R}^n)} \le \frac{C_p}{(1-2^{-\alpha})}\, \|f\|_{L^p(\mathbf{R}^n)} \tag{5.1}$$

for $2 \le p < \infty$. Here $\alpha = \mathrm{Re}\rho > 0$, $\mathcal{M}^{\rho}_{\Omega,\Psi,h,S}$ and $\mathcal{M}^{\rho,*}_{\Omega,\Psi,h,\lambda}$ are defined by

$$\begin{aligned} \mathcal{M}^{\rho}_{\Omega,\Psi,h,S} f(x) &= \left(\int_{\Gamma(x)} \left|F^{\rho}_{\Omega,\Psi,h} f(t,y)\right|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}, \\ \mathcal{M}^{\rho,*}_{\Omega,\Psi,h,\lambda} f(x) &= \left(\int_{\mathbf{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} \left|F^{\rho}_{\Omega,\Psi,h} f(t,y)\right|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}, \end{aligned}$$

where $\lambda > 1$, $\Gamma(x) = \left\{(y,t) \in \mathbf{R}_{+}^{n+1} : |x-y| < t\right\}$ *and*

$$F_{\Omega,\Psi,h}^{\rho} f(t,x) = \frac{1}{t^{\rho}} \int_{|u|\leq t} f(x - \Psi(|u|)u') \frac{\Omega(u')}{|u|^{n-\rho}} h(|u|) du.$$

The proof of theorem 5.1 is based on the following lemma.

Lemma 5.2. Let $\lambda > 1$. *Then, for any nonnegative locally integrable function* g*, we have*

$$\int_{\mathbf{R}^n} \left(\mathcal{M}_{\Omega,\Psi,h,\lambda}^{\rho,*} f(x)\right)^2 g(x) dx \leq \frac{C}{(1-2^{-\alpha})} \int_{\mathbf{R}^n} |f(x)|^2 Mg(x) dx, \tag{5.2}$$

where M *denotes the usual Hardy-Littlewood maximal operators on* $\mathbf{R}^n$.

A proof of this lemma can be obtained by Theorem 1.1 and following a similar argument as in the proof of Theorem 5 in Torchinsky and Wang [TW].

Proof of Theorem 5.1. Since $\mathcal{M}_{\Omega,\Psi,h,S}^{\rho} f(x) \leq 2^{n\lambda} \mathcal{M}_{\Omega,\Psi,h,\lambda}^{\rho,*} f(x)$, we only consider the operator $\mathcal{M}_{\Omega,\Psi,h,\lambda}^{\rho,*}$. Let $g \equiv 1$ in (5.2). The by the L^{∞} boundedness of M we have

$$\int_{\mathbf{R}^n} \left(\mathcal{M}_{\Omega,\Psi,h,\lambda}^{\rho,*} f(x)\right)^2 dx \leq \frac{C}{(1-2^{-\alpha})} \int_{\mathbf{R}^n} |f(x)|^2 dx, \tag{5.3}$$

and hence we get $\mathcal{M}_{\Omega,\Psi,h,\lambda}^{\rho,*}$ is bounded on L^2. When $2 < p < \infty$, choose $g \in L^{(p/2)'}$ with $\|g\|_{(p/2)'} \leq 1$ such that

$$\left\|\mathcal{M}_{\Omega,\Psi,h,\lambda}^{\rho,*} f\right\|_p^2 = \left|\int_{\mathbf{R}^n} \left(\mathcal{M}_{\Omega,\Psi,h,\lambda}^{\rho,*} f(x)\right)^2 g(x) dx\right|.$$

Thus, by Lemma 5.2 and Hölder's inequality we get

$$\begin{aligned}
\left\|\mathcal{M}_{\Omega,\Psi,h,\lambda}^{\rho,*} f\right\|_p^2 &\leq \frac{C}{(1-2^{-\alpha})} \int_{\mathbf{R}^n} |f(x)|^2 Mg(x) dx \\
&\leq \frac{C}{(1-2^{-\alpha})} \|f\|_p^2 \|Mg\|_{(p/2)'} \\
&\leq \frac{C}{(1-2^{-\alpha})} \|\Omega\|_{B_q^{(0,-1/2)}(\mathbf{S}^{n-1})} \|f\|_p^2
\end{aligned}$$

which ends the proof of Theorem 5.1.

Remark. We point out that Theorem 5.1 extends and improves the corresponding results in [KY] where the authors of [KY] obtained that the operators $\mathcal{M}_{\Omega,\Psi,h,\lambda}^{\rho,*}$ *and* $\mathcal{M}_{\Omega,\Psi,h,S}^{\rho}$ are bounded on $L^p(\mathbf{R}^n)$ $(2 \leq p < \infty)$ if $\Psi(y) \equiv y$, $h \equiv 1$ *and* $\Omega \in \mathrm{Lip}_{\tau}(\mathbf{S}^{n-1})$ $(0 < \tau \leq 1)$.

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