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Generalized Derivation

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Abstract. In this paper, we give an extension of the orthogonality results to dominant operators and p-hyponormal or log-hyponormal operators, also we will generalize some commutativity results.

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§1. Introduction

Let $B(H)$ denote the Banach space of all bounded linear operators on a separable, infinite dimensional Hilbert space. In this paper, a bounded operator T is called normal if $T^*T = TT^*$. According to [12], a bounded operator is called dominant if

$$(T - zI)H \subseteq (T - zI)^* H, \text{for all } z \in \sigma(T),$$

where $\sigma(T)$ denotes the spectrum of T. This condition is equivalent to the existence of a positive constant M_z for each $z \in \mathbb{C}$ such that

$$(T - zI)(T - zI)^* \leq {M_z}^2 (T - zI)^*(T - zI).$$

If there exist a constant M such that $M_z \leq M$ for all $z \in \mathbb{C}$, then T is called M-hyponormal, and if $M = 1$, T is hyponormal. Easily we see the following inclusion relations:

$$\{\text{Normal}\} \subseteq \{\text{Hyponormal}\} \subseteq \{\text{M-Hyponormal}\} \subseteq \{\text{Dominant}\}\cdot$$

Also T is called p-hyponormal [1, 6, 7, 15] if $(T^*T)^p \geq (TT^*)^p$, log-hyponormal [13] if T is an invertible operator which satisfies $\log(T^*T) \geq \log(TT^*)$.

Throughout this paper, we consider the case $p \in (0,1]$. By definition, the restriction of M-hyponormal (resp. dominant) to its invariant subspace is always M-hyponormal (resp. dominant). The parallel for p-hyponormal have been obtained by the author [15], i.e., it is true that the restriction of p-hyponormal to its invariant subspace is always p-hyponormal.

The organization of the paper is as follows, in Section 2, we recall some results which will be used in the sequel. In Section 3, we study the orthogonality of certain operators.

Let $A, B \in B(H)$, we define the generalized derivation $\delta_{A,B}$ induced by A and B by

$$\delta_{A,B}(X) = AX - XB, \quad \text{for all } X \in B(H).$$

If $A = B$, we note $\delta_{A,B} = \delta_A$. Given subspaces M and N of a Banach space V with norm $\|.\|$, M is said to be orthogonal to N if $\|m + n\| \geq \|n\|$ for all $m \in M$ and $n \in N$ (see [2]).

J.H. Anderson and Foias [3] proved that if A and B are normal, S is an operator such that $AS = SB$, then

$$\|\delta_{A,B}(X) - S\| \geq \|S\|, \quad \text{for all } X \in B(H).$$

Where $\|\cdot\|$ is the usual operator norm. Hence the range of $\delta_{A,B}$ is orthogonal to the null space of $\delta_{A,B}$. The orthogonality here is understood to be in the sense of definition [2].

§2. Preliminaries

Definition 2.1 ([16]). *We say that $A \in B(H)$ is finite if the distance* $\operatorname{dist}(I, R(\delta_A)) \geq 1$ *from the identity to the range of δ_A.*

Definition 2.2. *If $A \in B(H)$, we note by $\sigma_{ra}(A)$ the reduisant approximate point spectrum, the set of scalars λ for which there exists a normalized sequence $\{x_n\} \subset H$ verifying*

$$(A - \lambda)x_n \to 0 \text{ and } (A - \lambda)^* x_n \to 0.$$

Remark 2.3. The reduisant approximate point spectrum $\sigma_{ra}(A)$ coincides with the approximate point spectrum $\sigma_a(A)$, when A is dominant [4].

Proposition 2.4. *Let $A \in B(H)$, if $\sigma_{ra}(A)$ is not empty, then A is finite.*

Proof. Let $\lambda \in \sigma_{ra}(A)$ and $\{x_n\}$ a normalized sequence such that $(A-\lambda)x_n \to 0$ and $(A - \lambda)^* x_n \to 0$. If $X \in B(H)$, then we have

$$\begin{aligned} \|AX - XA - I\| &= \|(A - \lambda)X - X(A - \lambda) - I\| \\ &\geq |\langle (A - \lambda)Xx_n, x_n\rangle - \langle X(A - \lambda)x_n, x_n\rangle - 1|. \end{aligned}$$

Letting $n \to \infty$, we obtain $\|AX - XA - I\| \geq 1$. □

Corollary 2.5. *Every dominant operator is finite.*

The following Fuglede-Putnam's Theorem is famous.

Theorem 2.6 (Fuglede-Putnam's Theorem [14]). *Let $A \in B(K)$ be dominant and $B^* \in B(H)$ be p-hyponormal or* log-*hyponormal on Hilbert spaces K and H respectively. If $C \in B(H, K)$ and $AC = CB$, then $A^*C = CB^*$.*

§3. Main results

Our goal is to investigate the orthogonality of $R(\delta_{A,B})$ (the range of $\delta_{A,B}$) and $\ker(\delta_{A,B})$ (the kernel of $\delta_{A,B}$) for certain operators. We prove that $R(\delta_{A,B})$ is orthogonal to $\ker(\delta_{A,B})$ when A is dominant and B^* is p-hyponormal or log-hyponormal. Before proving this result, we need the following serial propositions.

Proposition 3.1. *If A is dominant (resp. M-hyponormal) and N is a normal operator such that $AN = NA$, then for every $\lambda \in \sigma_p(N)$,*

$$|\lambda| \leq \operatorname{dist}(N, R(\delta_A)).$$

Proof. Let $\lambda \in \sigma_p(N)$ and M_λ the eigenspace associate to λ, since $NA = AN$, then $N^*A = AN^*$ by Fuglede's [8]. Hence M_λ reduces orthogonality A and N. Let $T \in L(H)$, according to the decomposition of $H = M_\lambda \oplus M_\lambda^\perp$, we write A, N and T as follows:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}, \quad \text{and} \quad T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}.$$

We have

$$\begin{aligned} \|N + AT - TA\| &= \left\| \begin{bmatrix} \lambda + A_1T_1 - T_1A_1 & * \\ * & * \end{bmatrix} \right\| \\ &\geq \|\lambda + A_1T_1 - T_1A_1\| \\ &\geq |\lambda| \left\| I + A_1(\frac{T_1}{\lambda}) - (\frac{T_1}{\lambda})A_1 \right\| \\ &\geq |\lambda|. \end{aligned}$$

□

In the sequel, we need the Berberian technique, it allows us to construct a Hilbert space which contains a given Hilbert space H on which we could speak about "approached eigenvectors" and those as regarded as eigenvectors.

Proposition 3.2 (Berberian technique [5]). *Let H be a complex Hilbert space, then there exists a Hilbert space $\hat{H} \supset H$ and $\varphi : B(H) \to B(\hat{H})$ $(A \mapsto \hat{A})$ satisfying: φ is an *-isometric isomorphism preserving the order such that:*

(1) $\varphi(A^*) = \varphi(A)^*$;

(2) $\varphi(I) = \hat{I}$;

(3) $\varphi(\alpha A + \beta B) = \alpha\varphi(A) + \beta\varphi(B)$;

(4) $\varphi(AB) = \varphi(A).\varphi(B)$;

(5) $\|\varphi(A)\| = \|A\|$;

(6) $\varphi(A) \leq \varphi(B)$ *if* $A \leq B$, $\forall A, B \in B(H), \alpha, \beta \in \mathbb{C}$;

(7) $\sigma(A) = \sigma(\hat{A})$, $\sigma_a(A) = \sigma_a(\hat{A}) = \sigma_p(\hat{A})$.

Proposition 3.3. *If A is dominant (resp. M-hyponormal), then for every normal operator N such that $AN = NA$, we have $\|N\| \leq \operatorname{dist}(N, R(\delta_A))$.*

Proof. Let $\lambda \in \sigma(N) = \sigma_a(N)$ [9], then from Proposition 3.3, $\hat{N}$ is normal and $\hat{A}$ is dominant, $\widehat{NA} = \hat{N}\hat{A} = \hat{A}\hat{N}$, also $\lambda \in \sigma_p(\hat{N})$. By Proposition 3.1, we obtain for every $T \in L(H)$

$$|\lambda| \leq \|\hat{N} + \hat{A}\hat{T} - \hat{T}\hat{A}\| = \|N + AT - TA\|.$$

Therefore

$$\sup_{\lambda \in \sigma(\hat{N})} |\lambda| = \|\hat{N}\| = \|N\| = r(N) \leq \|N + AT - TA\|.$$

□

Theorem 3.4. *If A is dominant and B^* is p-hyponormal or* log*-hyponormal, then for every $T \in \ker(\delta_{A,B})$, we have $\|T\| \leq \operatorname{dist}(T, R(\delta_{A,B}))$.*

Proof. Let $T \in \ker(\delta_{A,B})$, then by Theorem 2.5, $T \in \ker(\delta_{A^*,B^*})$. Thus,

$$ATT^* = TBT^* = TT^*A.$$

Applying Proposition 3.3, we obtain for all $X \in B(H)$

$$\begin{aligned} \|TT^*\| &= \|T\|^2 \\ &\leq \|TT^* + AXT^* - XT^*A\| \\ &\leq \|TT^* + AXT^* - XBT^*\| \\ &\leq \|T^*\|\|T + AX - XB\|. \end{aligned}$$

Hence

$$\|T\| \leq \|T + AX - XB\|.$$

□

Next, we prove some commutativity results. A. H. Moadjil [11] proved that if N is normal operator such that $N^2X = XN^2$ and $N^3X = XN^3$, for some $X \in B(H)$, then $NX = XN$. In [11] , A.H. Moadjil give a counterexample for proving that this result is not true for quasinormal operators [11] i.e., $A(A^*A) = (A^*A)A$. F. Kittaneh [10] generalize this results for subnormal operators [9] by taking A and B^* subnormal operators, i.e., if $A^2X = XB^2$ and $A^3X = XB^3$, for some $X \in B(H)$, then $AX = XB$. This results can be generalized to some several classes of operators as follows.

Theorem 3.5. *Let A be a dominant operator and B^* be a p-hyponormal or* log*-hyponormal. If $A^2X = XB^2$ and $A^3X = XB^3$, for some $X \in B(H)$, then $AX = XB$.*

Proof. Let $T = AX - XB$, then

$$\begin{aligned} A^2T &= A^3X - A^2XB = XB^3 - XB^3 = 0, \\ TB^2 &= AXB^2 - XB^3 = A^3X - A^3X = 0, \end{aligned}$$

and

$$ATB = A^2XB - AXB^2 = XB^3 - A^3X = 0.$$

Hence $A(AT - TB) = A^2T - ATB = 0$ and $(AT - TB)B = ATB - TB^2 = 0$. This yields that $AT - TB \in \ker(\delta_{A,B}) \cap R(\delta_{A,B}) = \{0\}$, therefore $AT - TB = 0$. Hence $T \in \ker(\delta_{A,B}) \cap R(\delta_{A,B}) = \{0\}$ is obtained by Theorem 3.4, this implies that $T = 0$. i.e., $AX = XB$. □

Remark 3.6. This result can be generalized to the pair (A, B) of operators such that $\ker(\delta_{A,B})$ is orthogonal to $R(\delta_{A,B})$, i.e., If $R(\delta_{A,B}) \cap Ker(\delta_{A,B}) = \{0\}$, then

$$\ker(\delta_{A^3,B^3}) \cap \ker(\delta_{A^2,B^2}) \subset \ker(\delta_{A,B}).$$

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