SUT Journal of Mathematics
Vol. 43, No. 2 (2007), 267–285

(t,k)-Shredders in k-Connected Graphs

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(Received May 17, 2007)

Abstract. Let t, k be integers with $t \geq 3$ and $k \geq 1$. For a graph G, a subset S of $V(G)$ with cardinality k is called a (t,k)-shredder if $G-S$ consists of t or more components. In this paper, we show that if $t \geq 3$, $2(t-1) \leq k \leq 3t-5$ and G is a k-connected graph of order at least k^8, then the number of (t,k)-shredders of G is less than or equal to $((2t-1)(|V(G)|-f(|V(G)|)))/(2(t-1)^2)$, where $f(n)$ denotes the unique real number x with $x \geq k-1$ such that $n = 2(t-1)^2\binom{x}{k}+x$.

AMS 2000 *Mathematics Subject Classification.* 05C40.

Key words and phrases. (t,k)-Shredder, Upper bound.

§1. Introduction

In this paper, we consider only finite, undirected, simple graphs with no loops and no multiple edges.

Let $G = (V(G), E(G))$ be a graph. Let t, k be integers with $t \geq 3$ and $k \geq 1$. A subset S of $V(G)$ with cardinality k is called a *(t,k)-shredder* if $G-S$ consists of t or more components. In this paper, we are concerned with the number of (t,k)-shredders in k-connected graphs.

Before stating our result, we make the following definitions. For a real number x, we let

$$\binom{x}{k} = \left(\prod_{0\leq i\leq k-1}(x-i)\right) \Big/ k!.$$

For a real number n with $n \geq k-1$, we let $f_{t,k}(n)$ denote the unique real number x with $x \geq k-1$ such that

$$n = 2(t-1)^2\binom{x}{k} + x.$$

We start with known results concerning $(3,k)$-shredders. For $1 \leq k \leq 3$, the following result was proved by T. Jordán in [4].

Theorem 1. *Let k be an integer with $1 \leq k \leq 3$, and let G be a k-connected graph. Then unless $k = 3$ and $G \cong K_{3,3}$, the number of $(3, k)$-shredders of G is less than or equal to $(|V(G)| - k - 1)/2$.*

Subsequently the following two results were proved in [2].

Theorem 2. *Let G be a 4-connected graph of order $n \geq 2200$. Then the number of $(3, 4)$-shredders of G is less than or equal to $5(n - f_{3,4}(n))/8$.*

Theorem 3. *Let k be an integer with $k \geq 5$, and let G be a k-connected graph. Then the number of $(3, k)$-shredders of G is less than $2|V(G)|/3$.*

In Theorems 1 and 2, the upper bound on the number of $(3, k)$-shredders is best possible; as for Theorem 3, the bound itself is not best possible, but the coefficient $2/3$ of $|V(G)|$ in the bound is best possible (see [2], [4], [5]).

In [6], Theorem 1 was generalized to (t, k)-shredders as follows.

Theorem 4. *Let t, k be integers with $t \geq 3$ and $1 \leq k \leq 2t - 3$, and let G be a k-connected graph of order $n \geq 2k + 1$. Then the number of (t, k)-shredders of G is less than or equal to $(n - k - 1)/(t - 1)$.*

Similarly the following generalization of Theorem 3 was proved by G. Liberman and Z. Nutov in [5].

Theorem 5. *Let t, k be integers with $t \geq 3$ and $k \geq 3t - 4$, and let G be a k-connected graph. Then the number of (t, k)-shredders of G is less than $2|V(G)|/(2t - 3)$.*

The bound $(n - k - 1)/(t - 1)$ in Theorem 4 is best possible. Also modifications of examples constructed in [2] show that in Theorem 5, the coefficient $2/(2t - 3)$ of $|V(G)|$ in the bound is best possible. The purpose of this paper is to generalize Theorem 2 to (t, k)-shredders as follows.

Main Theorem. *Let t, k be integers with $t \geq 3$ and $2(t - 1) \leq k \leq 3t - 5$, and let G be a k-connected graph of order $n \geq k^8$. Then the number of (t, k)-shredders of G is less than or equal to*

$$\big((2t - 1)(n - f_{t,k}(n))\big) \ / \ (2(t - 1)^2).$$

We here include a discussion concerning the condition $2(t-1) \leq k \leq 3t-5$ on k. In view of Theorem 4, it is natural to assume $k \geq 2(t - 1)$. On the other hand, the fact that the coefficient $2/(2t-3)$ in Theorem 5 is sharp shows that the conclusion of the Main Theorem does not hold if $k \geq 3t - 4$. Thus the upper bound $3t - 5$ on k in the assumption of the Main Theorem is best possible.

The organization of the paper is as follows. In Section 2, we discuss the sharpness of the bound $\big((2t-1)(n-f_{t,k}(n))\big)/(2(t-1)^2)$. Section 3 and Section 4 contain preliminary results. We prove the Main Theorem in Section 5.

§2. Examples

In the Main Theorem, the bound $\big((2t-1)(n-f_{t,k}(n))\big)/(2(t-1)^2)$ is best possible in the sense that there are infinitely many graphs which attain the bound. To see this, let $m \geq k+1$ be an integer, and let W be a set of cardinality m. Let $\mathscr{R}$ denote the set of all subsets of cardinality k of W, and write $\mathscr{R} = \{R_1, \ldots, R_{\binom{m}{k}}\}$. For each p with $1 \leq p \leq \binom{m}{k}$, write $R_p = U_p \cup V_p$ with $|U_p| = |V_p| = k-t+1$. Define a graphs G of order

$$|W| + 2(t-1)^2|\mathscr{R}| = m + 2(t-1)^2\binom{m}{k}$$

by

$$\begin{aligned} V(G) = W \cup \Bigg(& \bigcup_{1 \leq p \leq \binom{m}{k}} \{a_{p,i,j} \mid 1 \leq i,\, j \leq t-1\}\Bigg) \\ & \cup \Bigg(\bigcup_{1 \leq p \leq \binom{m}{k}} \{b_{p,i,j} \mid 1 \leq i,\, j \leq t-1\}\Bigg), \\ E(G) = & \bigcup_{1 \leq p \leq \binom{m}{k}} \{a_{p,h,i}b_{p,h,j}, a_{p,h,i}u, b_{p,h,j}v \mid 1 \leq h,\, i,\, j \leq t-1, \\ & \qquad u \in U_p,\, v \in V_p\} \cup \{xy \mid x,\, y \in W,\, x \neq y\}. \end{aligned}$$

Then G is k-connected and, in addition to the members of $\mathscr{R}$, G has $2(t-1)|\mathscr{R}|$ (t,k)-shredders

$$\begin{aligned} &\{a_{p,i,j} \mid 1 \leq j \leq t-1\} \cup V_p \quad (1 \leq i \leq t-1,\, 1 \leq p \leq \tbinom{m}{k}), \\ &\{b_{p,i,j} \mid 1 \leq j \leq t-1\} \cup U_p \quad (1 \leq i \leq t-1,\, 1 \leq p \leq \tbinom{m}{k}). \end{aligned}$$

Hence the total number of (t,k)-shredders of G is

$$(2(t-1)+1)\binom{m}{k} = \big((2t-1)(|V(G)| - f_{t,k}(|V(G)|))\big) \,/\, (2(t-1)^2).$$

§3. Preliminary results

Throughout this section, let t, k be integers with $t \geq 3$ and $k \geq 2(t-1)$, let G be a k-connected graph, and let $\mathscr{S}$ denote the set of (t,k)-shredders of G.

For each $S \in \mathscr{S}$, we define $\mathscr{K}(S)$, $\mathscr{L}(S)$ and $L(S)$ as follows. Let $S \in \mathscr{S}$. We let $\mathscr{K}(S)$ denote the set of components of $G-S$. Write $\mathscr{K}(S) =$

$\{H_1, \dots, H_s\}$ ($s = |\mathscr{K}(S)|$). We may assume $|V(H_1)| \geq |V(H_2)| \geq \cdots \geq |V(H_s)|$ (any such labeling will do). Under this notation, we let $\mathscr{L}(S) = \mathscr{K}(S) - \{H_1\}$ and $L(S) = \bigcup_{2 \leq i \leq s} V(H_i)$; thus $L(S) = \bigcup_{C \in \mathscr{L}(S)} V(C)$. Now let $\mathscr{L} = \bigcup_{S \in \mathscr{S}} \mathscr{L}(S)$. A member F of $\mathscr{L}$ is said to be *saturated* if there exists a subset $\mathscr{C}$ of $\mathscr{L} - \{F\}$ such that $V(F) = \bigcup_{C \in \mathscr{C}} V(C)$.

Let S, $T \in \mathscr{S}$ with $S \neq T$. We say that S *meshes* with T if S intersects with at least two members of $\mathscr{K}(T)$. It is easy to see that if S meshes with T, then T intersects with all members of $\mathscr{K}(S)$, and hence T meshes with S and S intersects with all members of $\mathscr{K}(T)$ (see [1; Lemma 4.3 (1)]). We define an auxiliary graph $\mathscr{G}$ by

$$V(\mathscr{G}) = \mathscr{S},$$
$$E(\mathscr{G}) = \{ST \mid S, T \in \mathscr{S},\ S \neq T,\ S \text{ and } T \text{ mesh with each other}\}.$$

We start with easy observations.

Lemma 3.1. *Let $S \in \mathscr{S}$. Then for each $x \in S$ and each $C \in \mathscr{K}(S)$, there is an edge of G joining x and a vertex of C.*

Proof. If $xy \notin E(G)$ for any $y \in C$, then $G - (S - \{x\})$ is disconnected, which contradicts the assumption that G is k-connected. □

Lemma 3.2. *Let S, $T \in \mathscr{S}$ with $S \neq T$, and suppose that $ST \in E(\mathscr{G})$. Then the following hold.*

(i) *For each $C \in \mathscr{K}(S)$ and each $D \in \mathscr{K}(T)$, there is an edge of G joining a vertex of C and a vertex of D.*

(ii) *The subgraph of G induced by $L(S) \cup L(T)$ is connected.*

Proof. Since $ST \in E(\mathscr{G})$, we have $S \cap V(D) \neq \emptyset$. Hence (i) follows from Lemma 3.1, and (ii) follows from (i). □

Lemma 3.3. *Let S, $T \in \mathscr{S}$ with $S \neq T$, and suppose that $ST \in E(\mathscr{G})$. Then $|S \cap L(T)| \geq t - 1$ and $|L(S) \cap T| \geq t - 1$.*

Proof. Since $ST \in E(\mathscr{G})$, $S \cap V(D) \neq \emptyset$ for all $D \in \mathscr{K}(T)$. Since $|\mathscr{L}(T)| \geq t-1$, this implies $|S \cap L(T)| \geq |\mathscr{L}(T)| \geq t-1$. Similarly $|L(S) \cap T| \geq t-1$. □

Note that a (t, k)-shredder is a $(3, k)$-shredder. Thus the following five lemmas follow from [4; Lemmas 2.1 and 3.1] (see also [2; Lemmas 3.2 through 3.6]).

Lemma 3.4. *Let S, $T \in \mathscr{S}$ with $S \neq T$, and suppose that $ST \in E(\mathscr{G})$. Then the following hold.*

(i) $S \supseteq L(T)$ *or* $T \supseteq L(S)$.

(ii) $L(S) \cap L(T) = \emptyset$.

Lemma 3.5. *Let* S, $T \in \mathscr{S}$ *with* $S \neq T$, *and suppose that* $ST \notin E(\mathscr{G})$. *Then one of the following holds:*

(i) $L(S) \cap L(T) = \emptyset$, $(L(S) \cup L(T)) \cap (S \cup T) = \emptyset$, *and no edge of* G *joins a vertex in* $L(S)$ *and a vertex in* $L(T)$;

(ii) *there exists* $C \in \mathscr{L}(S)$ *such that* $V(C) \supseteq L(T)$ *(so* $L(S) \supseteq L(T)$*); or*

(iii) *there exists* $D \in \mathscr{L}(T)$ *such that* $V(D) \supseteq L(S)$ *(so* $L(T) \supseteq L(S)$*).*

Lemma 3.6. *Let* S, $T \in \mathscr{S}$ *with* $S \neq T$, *and suppose that* $ST \notin E(\mathscr{G})$ *and* $L(S) \not\subseteq L(T)$. *Then* $S \cap L(T) = \emptyset$.

Lemma 3.7. *Let* C, $D \in \mathscr{L}$. *Then one of the following holds:*

(i) $V(C) \cap V(D) = \emptyset$;

(ii) $V(C) \supseteq V(D)$; *or*

(iii) $V(D) \supseteq V(C)$.

Lemma 3.8. *Let* $F \in \mathscr{L}$. *Suppose that* F *is saturated, and let* $\mathscr{C}$ *be a subset of* $\mathscr{L} - \{F\}$ *with minimum cardinality such that* $V(F) = \bigcup_{C \in \mathscr{C}} V(C)$. *Then the following hold.*

(i) $\mathscr{C} = \bigcup_{S \in \mathscr{T}} \mathscr{L}(S)$ *for some subset* $\mathscr{T}$ *of* $\mathscr{S}$ *(so* $V(F) = \bigcup_{S \in \mathscr{T}} L(S)$*).*

(ii) $|\mathscr{T}| \geq 2$, *and the subgraph induced by* $\mathscr{T}$ *in* $\mathscr{G}$ *is connected.*

We can prove the following lemma by arguing as in the proof of [3; Lemma 2.12].

Lemma 3.9. *Let* S, $T \in \mathscr{S}$, *and suppose that* $ST \in E(\mathscr{G})$ *and* $L(T) \not\subseteq S$. *Then* $|S \cap L(T)| \geq 2t - 3$.

Proof. Since $L(T) \not\subseteq S$, it follows form Lemma 3.4 (i) that $L(S) \subseteq T$ which, in particular, implies $L(S) \cap L(T) = \emptyset$. Hence $(V(G) - S - L(S)) \cap L(T) \neq \emptyset$. Write $\mathscr{L}(T) = \{F_1, \ldots, F_a\}$ ($a = |\mathscr{L}(T)| \geq t - 1$). We may assume $(V(G) - S - L(S)) \cap V(F_1) \neq \emptyset$. Then $(S \cap V(F_1)) \cup (T - L(S))$ separates $(V(G) - S - L(S)) \cap V(F_1)$ from the rest. Hence $|(S \cap V(F_1)) \cup (T - L(S))| \geq k$, which implies $|S \cap V(F_1)| \geq k - |T - L(S)| = |T| - |T - L(S)| = |L(S) \cap T|$. Therefore

$$|S \cap V(F_1)| \geq t - 1 \tag{3.1}$$

by Lemma 3.3. Since $S \cap V(F_i) \neq \emptyset$ for each i by the definition of meshing, we now obtain $|S \cap L(T)| = \sum_{1 \leq i \leq a} |S \cap V(F_i)| = |S \cap V(F_1)| + \sum_{2 \leq i \leq a} |S \cap V(F_i)| \geq t - 1 + a - 1 \geq 2t - 3$. □

Lemma 3.10. *Suppose that $2(t-1) \leq k \leq 3t-5$ and $|V(G)| > (k^2+6k+1)/4$. Let S, $T \in \mathscr{S}$, and suppose that $ST \in E(\mathscr{G})$. Then the following hold.*

(i) *If we write $\mathscr{K}(S) - \mathscr{L}(S) = \{C\}$ and $\mathscr{K}(T) - \mathscr{L}(T) = \{D\}$, then $V(C) \cap V(D) \neq \emptyset$.*

(ii) *$L(S) \subseteq T, L(T) \subseteq S$.*

(iii) *$t - 1 \leq |L(S)| \leq k - t + 1$, $t - 1 \leq |L(T)| \leq k - t + 1$.*

Proof. In view of Lemma 3.4, we may assume $L(S) \subseteq T$. Then $L(S) \cap V(D) = \emptyset$. To prove (i), suppose that $V(C) \cap V(D) = \emptyset$. Then $V(D) \subseteq S$, and hence $|V(D)| = |S \cap V(D)| \leq |S| - |S \cap L(T)|$. By the definition of meshing, $|\mathscr{L}(T)| \leq |S \cap L(T)|$. Since D is the largest component in $\mathscr{K}(T)$, we obtain $|L(T)| \leq |\mathscr{L}(T)||V(D)| \leq |S \cap L(T)|(k - |S \cap L(T)|)$, and hence $|V(G)| = |V(D)| + |T| + |L(T)| \leq -|S \cap L(T)|^2 + (k-1)|S \cap L(T)| + 2k = -\big(|S \cap L(T)| - (k-1)/2\big)^2 + (k^2+6k+1)/4 \leq (k^2+6k+1)/4$. This contradicts the assumption that $|V(G)| > (k^2+6k+1)/4$. Thus (i) is proved. To prove (ii), suppose that $L(T) \not\subseteq S$. By Lemma 3.9, $|S \cap L(T)| \geq 2t - 3$. Since $V(C) \cap V(D) \neq \emptyset$ by (i), we get

$$|S \cap V(D)| \geq t - 1 \tag{3.2}$$

by arguing as in the proof of (3.1). Consequently $k \geq |S \cap L(T)| + |S \cap V(D)| \geq 3t - 4$, which contradicts the assumption that $k \leq 3t - 5$. Thus (ii) is proved. Now by (ii) and (3.2), $t-1 \leq |\mathscr{L}(T)| \leq |L(T)| \leq |S| - |S \cap V(D)| \leq k - (t-1)$. Similarly $t - 1 \leq |L(S)| \leq |T| - |V(C) \cap T| \leq k - (t-1)$, which proves (iii). □

Lemma 3.11. *Suppose that $2(t-1) \leq k \leq 3t-5$ and $|V(G)| > (k^2+6k+1)/4$. Let $T \in \mathscr{S}$, and suppose that $\deg_{\mathscr{G}}(T) \geq 1$, i.e., there exists $T' \in \mathscr{S} - \{T\}$ such that $TT' \in E(\mathscr{G})$. Then there is no $S \in \mathscr{S} - \{T\}$ such that $L(S) \subseteq L(T)$.*

Proof. Suppose that there exists $S \in \mathscr{S} - \{T\}$ such that $L(S) \subseteq L(T)$. Then $ST \notin E(\mathscr{G})$ by Lemma 3.4, and hence it follows Lemma 3.5 that there exists $M \in \mathscr{L}(T)$ such that $L(S) \subseteq V(M)$. This implies

$$\begin{aligned}|L(T)| &= \sum_{F \in \mathscr{L}(T) - \{M\}} |V(F)| + |V(M)| \\ &\geq (|\mathscr{L}(T)| - 1) + |L(S)| \\ &\geq (t - 1 - 1) + (t - 1) = 2t - 3.\end{aligned}$$

On the other hand, since $\deg_{\mathscr{G}}(T) \geq 1$, $|L(T)| \leq k - t + 1$ by Lemma 3.10 (iii). Consequently $2t - 3 \leq |L(T)| \leq k - t + 1$, which contradicts the assumption $k \leq 3t - 5$. □

§4. Numerical results

In this section, we state preliminary lemmas, most of which are Numerical results. Throughout this section, we let t, k be as in the Main Theorem. Also for simplicity, we write $f(n)$ for $f_{t,k}(n)$. The following lemma is easily verified, and we omit its proof (see the proof of Lemma 4.2):

Lemma 4.1. *Let a, x, x' be real numbers such that $a \leq k + 2$ and $k + 1 \leq x < x'$. Then*

$$\binom{x}{k} - ax < \binom{x'}{k} - ax'.$$

Let α denote the real number with $k+2 < \alpha \leq k+3$ such that $\binom{\alpha}{k} = (k+1)\alpha$. The existence of α follows from the fact that we have

$$\binom{k+2}{k} < (k+1)(k+2) \text{ and } \binom{k+3}{k} \geq (k+1)(k+3).$$

Lemma 4.2. *Let x, x' be real numbers with $\alpha \leq x < x'$. Then*

$$\begin{aligned}&(t-1)\binom{x}{k} - \big((k+1)(t-1)(2t-1)+1\big)x \\ &< (t-1)\binom{x'}{k} - \big((k+1)(t-1)(2t-1)+1\big)x'.\end{aligned}$$

Proof. We define $h(x)$ by $h(x) = (t-1)\binom{x}{k} - \big((k+1)(t-1)(2t-1)+1\big)x$. Then $h'(\alpha) = (t-1)(k+1)\alpha \sum_{0 \leq i \leq k-1} \big(1/(\alpha - i)\big) - \big((k+1)(t-1)(2t-1)+1\big)$. We show that $h'(\alpha) > 0$. Since $\alpha/(\alpha - i) \geq (k+3)/(k+3-i)$ for each $0 \leq i \leq k-1$ and since $2(t-1) \leq k$, $h'(\alpha) \geq (t-1)(k+1)(k+3) \sum_{0 \leq i \leq k-1} \big(1/(k+3-i)\big) - \big((k+$

$1)^2(t-1)+1\big) > (t-1)\big((k+1)(k+3)\sum_{0\le i\le k-1}\big(1/(k+3-i)\big) - \big((k+1)^2+1\big)\big)$. Thus it suffices to show

$$\sum_{0\le i\le k-1} 1/(k+3-i) > (k+1)/(k+3) + 1/\big((k+1)(k+3)\big). \tag{4.1}$$

It is easy to verify (4.1) for $4 \le k \le 6$. On the other hand, if $k \ge 7$, $\sum_{0\le i\le k-1}\big(1/(k+3-i)\big) \ge \sum_{4\le i\le 10}(1/i) > 1 > (k+1)/(k+3) + 1/\big((k+1)(k+3)\big)$. Hence (4.1) holds, and we therefore obtain $h'(\alpha) > 0$. Since we clearly have $h''(x) > 0$ for all $x \ge \alpha$, we now see that $h'(x) > 0$ for $x \ge \alpha$, and hence the desired inequality holds. □

For convenience, we restate Lemma 4.1 in the following form:

Lemma 4.3. *Let a, m, b, b' be real numbers such that $a \le k+2$, $b' < b$ and $(t-1)b \le m-(k+1)$. Then*

$$\binom{m-(t-1)b}{k} + (t-1)ab < \binom{m-(t-1)b'}{k} + (t-1)ab'.$$

Lemma 4.4. *Let $n \ge k^8$ be a real number. Then the following hold.*

(i) (a) $f(n) > k+6$.
 (b) *If* $k=4$, $f(n) > 11$.

(ii) $f(n) < n/\big(\big(2(t-1)^2(k+1)+1\big)(2t-1)\big)$.

Proof. Statement (i) (a) follows from the inequality $2(t-1)^2\binom{k+6}{k} + k + 6 \le (k^2\binom{k+6}{k})/2+k+6 < k^8$. Similarly (i) (b) follows from the fact that $8\binom{11}{4}+11 < 4^8$. Note that $n/((2(t-1)^2(k+1)+1)(2t-1)) - f(n) = ((2(t-1))/\big((2(t-1)^2(k+1)+1)(2t-1)\big))\big((t-1)\binom{f(n)}{k} - \big((k+1)(t-1)(2t-1)+1\big)f(n)\big)$. Thus (ii) is equivalent to the inequality

$$(t-1)\binom{f(n)}{k} - \big((k+1)(t-1)(2t-1)+1\big)f(n) > 0. \tag{4.2}$$

Assume for the moment that $k \ge 5$. By (i) (a) and Lemma 4.2, (4.2) follows if we prove $(t-1)\binom{k+6}{k} - \big((k+1)(t-1)(2t-1)+1\big)(k+6) > 0$. In view of the assumption that $2(t-1) \le k$, it suffices to show $\binom{k+6}{k} - ((k+1)^2+1)(k+6) > 0$, which holds because $\binom{k+6}{k} = (k+1)(k+2)(k+6)\big((k+5)(k+4)(k+3)/720\big) \ge (k+1)(k+2)(k+6)$. Similarly if $k=4$, then by (i) (b) and Lemma 4.2, (4.2) follows from the fact that $\binom{11}{4} - \big((4+1)^2+1\big)\cdot 11 > 0$. □

Lemma 4.5. *Let n, m, b_j $(0 \le j \le t-1)$ be nonnegative real numbers with $n \ge k^8$ such that*

$$0 \le \sum_{0\le j\le t-2} (t-1-j)b_j \le m-(k+1),$$

$$\sum_{1\le j\le t-1} b_j \le \binom{m-\sum_{0\le j\le t-2}(t-1-j)b_j}{k} + (k+1)\sum_{0\le j\le t-2}(t-1-j)b_j,$$

$$2(t-1)\sum_{1\le j\le t-1} jb_j \le n-m.$$

Then

$$(n-m)/(t-1) + \sum_{0\le j\le t-1} b_j \le \big((2t-1)(n-f(n))\big)/(2(t-1)^2).$$

Proof. If we let $c_0 = \sum_{0\le i\le t-2}\big((t-1-i)/(t-1)\big)b_i$, $c_j = 0$ $(1 \le j \le t-2)$, $c_{t-1} = \sum_{1\le i\le t-1}(ib_i)/(t-1)$, then the c_j $(0 \le j \le t-1)$ satisfy the assumptions of the lemma, and $\sum_{0\le j\le t-1} b_j = \sum_{0\le j\le t-1} c_j$. Thus we may assume $b_j = 0$ for every $1 \le j \le t-2$. Then we have

$$0 \le (t-1)b_0 \le m-(k+1) \tag{4.3}$$

$$b_{t-1} \le \binom{m-(t-1)b_0}{k} + (k+1)(t-1)b_0 \tag{4.4}$$

$$2(t-1)^2 b_{t-1} \le n-m \tag{4.5}$$

Case 1. $m \le f(n)$.

By (4.4),

$$b_0 + b_{t-1} \le \binom{m-(t-1)b_0}{k} + (t-1)\big(k+1+1/(t-1)\big)b_0.$$

Since $k+1+1/(t-1) < k+2$ and since $0 \le (t-1)b_0 \le m-(k+1)$ by (4.3), we get

$$\binom{m-(t-1)b_0}{k} + (t-1)\big(k+1+1/(t-1)\big)b_0 \le \binom{m}{k}$$

by applying Lemma 4.3 with $a = k+1+1/(t-1)$, $b = b_0$ and $b' = 0$. Hence $b_0 + b_{t-1} \le \binom{m}{k}$. Therefore we obtain

$$\begin{aligned}(n-m)/(t-1) + b_0 + b_{t-1} &\le n/(t-1) + \binom{m}{k} - m/(t-1)\\ &\le n/(t-1) + \binom{f(n)}{k} - f(n)/(t-1)\\ &= \big((2t-1)(n-f(n))\big)/(2(t-1)^2)\end{aligned}$$

by Lemma 4.1.

Case 2. $m > f(n)$.

Subcase 2.1. $b_{t-1} \leq ((k+1)n)/\big(2(t-1)^2(k+1)+1\big)$.

By (4.3),

$$\begin{aligned}(n-m)/(t-1)+b_0+b_{t-1} & \\ &\leq (n-m)/(t-1)+\big(m-(k+1)\big)/(t-1) \\ &\qquad\qquad + ((k+1)n)/\big(2(t-1)^2(k+1)+1\big) \\ &< n/(t-1)+((k+1)n)/(2(t-1)^2(k+1)+1).\end{aligned}$$

Since $\big((k+1)n\big)/(2(t-1)^2(k+1)+1) < \big((n-(2t-1)f(n)\big)/(2(t-1)^2)$ by Lemma 4.4 (ii), this implies $(n-m)/(t-1)+b_0+b_{t-1} < \big((2t-1)(n-f(n))\big)/(2(t-1)^2)$.

Subcase 2.2. $b_{t-1} > \big((k+1)n\big)/(2(t-1)^2(k+1)+1)$.

Let α be as in the paragraph preceding Lemma 4.2. By (4.5) and the assumption of this subcase, $m < n/(2(t-1)^2(k+1)+1)$, and hence $b_{t-1} > (k+1)m$, which implies

$$\begin{aligned}\binom{m-(m-\alpha)}{k}+(k+1)(m-\alpha) &= (k+1)m \\ &< b_{t-1} \\ &\leq \binom{m-(t-1)b_0}{k}+(k+1)(t-1)b_0.\end{aligned}$$

We here consider the function $g(x) = \binom{m-(t-1)x}{k}+(t-1)(k+1)x$. Then the above inequality is written in the form

$$g((m-\alpha)/(t-1)) < b_{t-1} \leq g(b_0); \tag{4.6}$$

in particular,

$$g((m-\alpha)/(t-1)) < g(b_0). \tag{4.7}$$

Since $\alpha > k+2$ by the definiton of α, we have

$$m-\alpha < m-(k+1). \tag{4.8}$$

Since the function $g(x)$ is monotonely decreasing in the interval $x \leq (m-(k+1))/(t-1)$ by Lemma 4.3, it follows from (4.7), (4.8) and (4.3) that $b_0 < (m-\alpha)/(t-1)$. Hence it follows from (4.6) that there exists b_0' with $b_0 \leq b_0' < (m-\alpha)/(t-1)$ such that $g(b_0') = b_{t-1}$, i.e.,

$$b_{t-1} = \binom{m-(t-1)b_0'}{k}+(k+1)(t-1)b_0'.$$

Thus by replacing the number b_0 in the statement of the lemma by b'_0, we may assume that equality holds in (4.4); that is to say, we have

$$b_{t-1} = \binom{m-(t-1)b_0}{k} + (k+1)(t-1)b_0 \tag{4.9}$$

and

$$m-(t-1)b_0 > \alpha. \tag{4.10}$$

Since $m > f(n)$, $b_{t-1} < (n-f(n))/(2(t-1)^2) = \binom{f(n)}{k}$ by (4.5), and hence

$$\binom{m-(t-1)b_0}{k} < \binom{f(n)}{k}$$

by (4.9), which implies

$$m-(t-1)b_0 < f(n). \tag{4.11}$$

Now by (4.9) and (4.5),

$$\begin{aligned} b_{t-1} &+ 2(t-1)^2(k+1)b_{t-1} \\ &\leq \binom{m-(t-1)b_0}{k} + (k+1)(t-1)b_0 + (k+1)(n-m) \\ &= \binom{m-(t-1)b_0}{k} - (k+1)(m-(t-1)b_0) + (k+1)n, \end{aligned}$$

and hence

$$\begin{aligned} b_{t-1} \leq \Bigg(\binom{m-(t-1)b_0}{k} &- (k+1)(m-(t-1)b_0) \\ &+ (k+1)n\Bigg) \Big/ \big(2(t-1)^2(k+1)+1\big), \end{aligned}$$

which implies

$$\begin{aligned} &(n-m)/(t-1) + b_0 + b_{t-1} \\ &\leq (n-m)/(t-1) + b_0\Bigg(\binom{m-(t-1)b_0}{k} \\ &\qquad - (k+1)(m-(t-1)b_0) + (k+1)n\Bigg) \Big/ \big(2(t-1)^2(k+1)+1\big) \\ &= \Bigg(((k+1)(t-1)(2t-1)+1)n + (t-1)\binom{m-(t-1)b_0}{k} \\ &- \big((k+1)(t-1)(2t-1)+1\big)(m-(t-1)b_0)\Bigg) \Big/ \big((2(t-1)^2(k+1)+1)(t-1)\big). \end{aligned}$$

Consequently it follows from Lemma 4.2 and (4.10) and (4.11) that

$$\begin{aligned}
&(n-m)/(t-1)+b_0+b_{t-1}\\
&< \Bigg(\big((k+1)(t-1)(2t-1)+1\big)n+(t-1)\binom{f(n)}{k}\\
&- \big((k+1)(t-1)(2t-1)+1\big)f(n)\Bigg)/\big((2(t-1)^2(k+1)+1)(t-1)\big)\\
&= \big((2t-1)(n-f(n))\big)/(2(t-1)^2). \qquad \square
\end{aligned}$$

Lemma 4.6. *Let x, y, x', y' be real numbers such that $k \le x' < x \le y < y'$ and $x+y=x'+y'$. Then*

$$\binom{x}{k}+\binom{y}{k}<\binom{x'}{k}+\binom{y'}{k}.$$

Proof. The function $\varphi(x)=\binom{x}{k}$ is strictly convex in the interval $x \ge k$. Hence $(\binom{x}{k}-\binom{x'}{k})/(x-x') < (\binom{y'}{k}-\binom{y}{k})/(y'-y)$. Since $x-x'=y'-y$, this implies $\binom{x}{k}+\binom{y}{k}<\binom{x'}{k}+\binom{y'}{k}$. $\square$

Repeated applications of Lemma 4.6 yield:

Lemma 4.7. *Let $x_1,\ldots,x_{b+1}$ be real numbers such that $x_i \ge k+1$ for all $1 \le i \le b+1$, and let $x=\sum_{1\le i\le b+1} x_i$. Then*

$$\sum_{1\le i\le b+1}\binom{x_i}{k} \le b\binom{k+1}{k}+\binom{x-(k+1)b}{k}=\binom{x-(k+1)b}{k}+(k+1)b.$$

Proof. We proceed by induction on b. If $b=0$, the lemma clearly holds. We may assume $b \ge 1$. Then by the induction hypothesis,

$$\begin{aligned}
\sum_{1\le i\le b}\binom{x_i}{k}+\binom{x_{b+1}}{k} &\le (b-1)\binom{k+1}{k}\\
&\quad+\binom{\sum_{1\le i\le b} x_i-(k+1)(b-1)}{k}+\binom{x_{b+1}}{k}.
\end{aligned}$$

Note that $k+1 \le \sum_{1\le i\le b} x_i-(k+1)(b-1) \le x-(k+1)b$ and $k+1 \le x_{b+1} \le x-(k+1)b$. Hence, whether $\sum_{1\le i\le b} x_i-(k+1)(b-1) \le x_{b+1}$ or $x_{b+1} \le \sum_{1\le i\le b} x_i-(k+1)(b-1)$, we obtain

$$\binom{\sum_{1\le i\le b} x_i-(k+1)(b-1)}{k}+\binom{x_{b+1}}{k} \le \binom{k+1}{k}+\binom{x-(k+1)b}{k}$$

by Lemma 4.6. Therefore

$$\begin{aligned}\sum_{1\le i\le b}\binom{x_i}{k}+\binom{x_{b+1}}{k} &\le (b-1)\binom{k+1}{k}+\binom{k+1}{k}+\binom{x-(k+1)b}{k}\\ &= b\binom{k+1}{k}+\binom{x-(k+1)b}{k}.\end{aligned}$$ □

Lemma 4.8. *Let $b \ge 0$ be an integer (we allow the possibility that $b = 0$). Let W be a finite set. Let $Z_1, \dots, Z_b$; $Q_1, \dots, Q_b$ be subsets of W such that $Z_i \cap Z_j = \emptyset$ for all i, j with $1 \le i < j \le b$ and such that $|Q_i| \le k$ for all $1 \le i \le b$. Let $\mathscr{R}$ be a family of subsets of cardinality k of W such that for each $R \in \mathscr{R}$ and for each $1 \le i \le b$, we have either $R \cap Z_i = \emptyset$ or $R \cap (W - (\bigcup_{1\le j\le i} Z_j) - Q_i) = \emptyset$. Then the following hold.*

(i) $|\mathscr{R}| \le \left(\sum_{1\le i\le b}\binom{|Z_i|+k}{k}\right)+\binom{|W|-\left|\bigcup_{1\le i\le b} Z_i\right|}{k}.$

(ii) *If $Z_i \ne \emptyset$ for all $1 \le i \le b$ and $|W| - |\bigcup_{1\le i\le b} Z_i| \ge k+1$, then* $|\mathscr{R}| \le \binom{|W|-b}{k}+(k+1)b.$

Proof. We first prove (i). If $b = 0$, (i) clearly holds. Thus we may assume $b \ge 1$. We proceed by induction on b. Set

$$\begin{aligned}\mathscr{R}' &= \{R \in \mathscr{R} \mid R \cap Z_1 = \emptyset\},\\ \mathscr{T} &= \{R \in \mathscr{R} \mid R \cap (W - Z_1 - Q_1) = \emptyset\}.\end{aligned}$$

By assumption, $\mathscr{R} = \mathscr{R}' \cup \mathscr{T}$. Hence

$$|\mathscr{R}| \le |\mathscr{T}| + |\mathscr{R}'| \le \binom{|Z_1|+k}{k}+\binom{|W|-|Z_1|}{k},$$

which shows that (i) holds for $b = 1$. Thus we may assume $b \ge 2$. Set $W' = W - Z_1$, and set $Z'_i = Z_{i+1}$ and $Q'_i = Q_{i+1} - Z_1$ for each $1 \le i \le b-1$. Then $\mathscr{R}'$, W', the Z'_i and the Q'_i satisfy the assumptions of the lemma with b replaced by $b-1$. Hence by the induction hypothesis,

$$\begin{aligned}|\mathscr{R}'| &\le \left(\sum_{1\le i\le b-1}\binom{|Z'_i|+k}{k}\right)+\binom{|W'|-\left|\bigcup_{1\le i\le b-1} Z'_i\right|}{k}\\ &= \left(\sum_{2\le i\le b}\binom{|Z_i|+k}{k}\right)+\binom{|W-Z_1|-\left|\bigcup_{2\le i\le b} Z_i\right|}{k}.\end{aligned}$$

Therefore

$$\begin{aligned} |\mathscr{R}| &\le |\mathscr{T}| + |\mathscr{R}'| \\ &\le \binom{|Z_1|+k}{k} + \left(\sum_{2\le i\le b}\binom{|Z_i|+k}{k}\right) + \binom{|W-Z_1| - \left|\bigcup_{2\le i\le b} Z_i\right|}{k}. \end{aligned}$$

This proves (i). Since $(\sum_{1\le i\le b}(|Z_i|+k)) + (|W| - |\bigcup_{1\le i\le b} Z_i|) = |W| + kb$, (ii) follows from (i) and Lemma 4.7. □

§5. Proof of the main theorem

In this section, we let t, k, G, n be as in the Main Theorem, and follow the notation introduced in Section 3. Also as in Section 4, we write $f(n)$ for $f_{t,k}(n)$. Since $\big((2t-1)(n-f(n))\big)/(2(t-1)^2) > n/(t-1)$ by Lemma 4.4 (ii), we may assume $|\mathscr{S}| > n/(t-1)$.

Let $\mathscr{H}_1, \dots, \mathscr{H}_a$ be the nontrivial components of $\mathscr{G}$. For each $1 \le p \le a$, write $V(\mathscr{H}_p) = \{T_{p,1}, \dots, T_{p,|V(\mathscr{H}_p)|}\}$ (here $V(\mathscr{H}_p)$ denotes the vertex set of $\mathscr{H}_p$, so $V(\mathscr{H}_p) \subseteq \mathscr{S}$ by the definition of $\mathscr{G}$), and let F_p denote the subgraph of G induced by $\bigcup_{1\le i\le |V(\mathscr{H}_p)|} L(T_{p,i})$. Let $W = V(G) - \bigcup_{1\le p\le a} V(F_p)$.

The following claim follows immediately from Lemma 3.2.

Claim 5.1. *F_p is connected for all p with $1 \le p \le a$.*

Claim 5.2. *$V(F_p) \cap V(F_q) = \emptyset$ and $E(V(F_p), V(F_q)) = \emptyset$ for all p, q with $1 \le p < q \le a$.*

Proof. Take $T_{p,i} \in \mathscr{H}_p$ and $T_{q,j} \in \mathscr{H}_q$. Then $T_{p,i}T_{q,j} \notin E(\mathscr{G})$, and hence $L(T_{p,i}) \cap L(T_{q,j}) = \emptyset$ and $E(L(T_{p,i}), L(T_{q,j})) = \emptyset$ by Lemmas 3.5 and 3.11. Since $T_{p,i}$ and $T_{q,j}$ are arbitrary, this means

$$V(F_p) \cap V(F_q) = \emptyset \text{ and } E(V(F_p), V(F_q)) = \emptyset.$$ □

For each $1 \le p \le a$, $|V(F_p)| = \sum_{1\le i\le |V(\mathscr{H}_p)|} |L(T_{p,i})|$ by Lemmas 3.4, 3.5 and 3.11, and hence $(t-1)|V(\mathscr{H}_p)| \le |V(F_p)|$ by Lemma 3.10 (iii). Consequently

$$(t-1)\sum_{1\le p\le a} |V(\mathscr{H}_p)| \le \sum_{1\le p\le a} |V(F_p)|. \tag{5.1}$$

By Claim 5.2,

$$|W| = n - \sum_{1 \le p \le a} |V(F_p)|. \tag{5.2}$$

By (5.1) and (5.2),

$$\sum_{1 \le p \le a} |V(\mathscr{H}_p)| \le (n - |W|)/(t-1). \tag{5.3}$$

Since $|V(\mathscr{H}_p)| \ge 2$ for each p, it follows from (5.3) that

$$a \le (n - |W|)/(2(t-1)). \tag{5.4}$$

Set $\mathscr{R} = \mathscr{S} - \bigcup_{1 \le p \le a} V(\mathscr{H}_p)$.

Claim 5.3. *Let $S \in \mathscr{S} - V(\mathscr{H}_p)$. Then $S \cap V(F_p) = \emptyset$.*

Proof. Let $T \in V(\mathscr{H}_p)$. Then $ST \notin E(\mathscr{G})$. Hence $S \cap L(T) = \emptyset$ by Lemmas 3.6 and 3.11. Thus $S \cap V(F_p) = S \cap (\bigcup_{T \in V(\mathscr{H}_p)} L(T)) = \emptyset$. □

Claim 5.4. *Let $S \in \mathscr{R}$. Then $S \subseteq W$.*

Proof. This is because $S \cap V(F_p) = \emptyset$ for each $1 \le p \le a$ by Claim 5.3. □

Claim 5.5. *Let $S \in \mathscr{R}$, and let $C \in \mathscr{K}(S) - \{F_1, \ldots, F_a\}$. Then the following holds.*

(i) *If $C \in \mathscr{L}(S)$, then C is not saturated.*

(ii) *If we let $A = \{p \mid V(F_p) \cap V(C) \ne \emptyset\}$, then $V(C) - W = \bigcup_{p \in A} V(F_p)$.*

Proof. Let A be as in (ii). Then by Claims 5.1 and 5.3, $V(F_p) \subseteq V(C)$ for each $p \in A$, and hence $\bigcup_{p \in A} V(F_p) \subseteq V(C) - W$. Thus (ii) is proved. Now let $C \in \mathscr{L}(S)$, and suppose that C is saturated. By Lemma 3.8, there exists $\mathscr{T} \subseteq \mathscr{S}$ with $|\mathscr{T}| \ge 2$ such that $V(C) = \bigcup_{M \in \mathscr{T}} L(M)$ and such that the subgraph induced by $\mathscr{T}$ in $\mathscr{G}$ is connected. Then there exists p such that $\mathscr{T} \subseteq V(\mathscr{H}_p)$, and hence $V(C) \subseteq V(F_p)$. By (ii), this implies $V(C) = V(F_p)$, which contradicts the assumption that $C \notin \{F_1, \ldots, F_a\}$. □

Set

$$\mathscr{Q}_i = \{S \in \mathscr{R} \mid |\mathscr{K}(S) \cap \{F_1, \ldots, F_a\}| = i\} \quad (0 \le i \le t-2),$$
$$\mathscr{Q}_{t-1} = \{S \in \mathscr{R} \mid |\mathscr{K}(S) \cap \{F_1, \ldots, F_a\}| \ge t-1\}$$

and let $b_i = |\mathscr{Q}_i|$ for each i. Since $\mathscr{K}(S) \cap \mathscr{K}(T) = \emptyset$ for any $S, T \in \mathscr{S}$ with $S \neq T$, we have

$$\sum_{1 \le i \le t-1} i b_i \le a. \tag{5.5}$$

By (5.4) and (5.5)

$$2(t-1) \sum_{1 \le i \le t-1} i b_i \le n - |W|. \tag{5.6}$$

If $|W| \le k$, then $\binom{|W|}{k} \le |W|/k \le |W|/(t-1)$, and hence it follows from (5.3) and Claim 5.4 that $|\mathscr{S}| \le (n - |W|)/(t-1) + \binom{|W|}{k} \le n/(t-1)$, which contradicts the assumption that $|\mathscr{S}| > n/(t-1)$. Thus

$$|W| \ge k + 1. \tag{5.7}$$

Now label the members of $\bigcup_{0 \le i \le t-2} \mathscr{Q}_i$ as $Q_1, \dots, Q_h$ ($h = \sum_{0 \le i \le t-2} b_i$) so that

$$L(Q_j) \not\subseteq L(Q_i) \text{ for any } i,\ j \text{ with } 1 \le i < j \le h \tag{5.8}$$

(it is possible that $h = 0$). In the case where $h \ge 2$, if possible, we choose our labeling so that $L(Q_{h-1}) \not\subseteq L(Q_h)$. For each $1 \le i \le h$, let j_i ($0 \le j_i \le t-2$) be the index such that $Q_i \in \mathscr{Q}_{j_i}$, and take $C_{i,1}, \dots, C_{i,t-1-j_i} \in \mathscr{L}(Q_i) - \{F_1, \dots, F_a\}$ (the existence of such components follows from the definition of $\mathscr{Q}_{j_i}$). Let $W_0 = \emptyset$. For i with $1 \le i \le h$, we define $X_{i,l}$ ($1 \le l \le t-1-j_i$) and W_i inductively as follows: $X_{i,l} = (V(C_{i,l}) \cap W) - W_{i-1}$, $W_i = W_{i-1} \cup (\bigcup_{1 \le l \le t-1-j_i} X_{i,l})$. Then

$$W \supseteq W_h = \bigcup_{1 \le i \le h} \Bigg(\bigcup_{1 \le l \le t-1-j_i} X_{i,l} \Bigg) \quad \text{(disjoint union)}. \tag{5.9}$$

Arguing as in [2; Claims 6.3 and 6.4 and 6.5], we obtain the following three claims. We include sketches of their proofs for the convenience of the reader.

Claim 5.6. *$X_{i,l} \neq \emptyset$ for every i, l with $1 \le i \le h$ and $1 \le l \le t-1-j_i$.*

Proof. Set $A = \{p \mid V(F_p) \cap V(C_{i,l}) \neq \emptyset\}$. By Claim 5.5 (ii), $V(C_{i,l}) - W = \bigcup_{p \in A} V(F_p)$. Set $J = \{j \mid 1 \le j \le i-1, L(Q_j) \subseteq V(C_{i,l})\}$. Suppose that $X_{i,l} = \emptyset$. Then $(V(C_{i,l}) \cap W) - W_{i-1} = \emptyset$, and hence $V(C_{i,l}) \cap W \subseteq W_{i-1} \subseteq \bigcup_{1 \le j \le i-1} L(Q_j)$. On the other hand, for each $1 \le j \le i-1$ with $j \notin J$, $L(Q_j) \cap V(C_{i,l}) = \emptyset$ by (5.8) and Lemma 3.5 (note that $\{Q_\alpha \mid 1 \le \alpha \le h\} \subseteq \mathscr{R}$, and thus $Q_i Q_j \notin E(\mathscr{G})$ by the definition of $\mathscr{R}$). Consequently $V(C_{i,l}) \cap W \subseteq \bigcup_{j \in J} L(Q_j) \subseteq V(C_{i,l})$, and hence $V(C_{i,l}) = (\bigcup_{p \in A} V(F_p)) \cup (\bigcup_{j \in J} L(Q_j))$. Since $V(F_p) = \bigcup_{T \in V(\mathscr{H}_p)} L(T)$ for each $p \in A$, this means that $V(C_{i,l})$ is saturated, which contradicts Claim 5.5 (i). $\square$

Claim 5.7. *Suppose that either $h \geq 2$ and $L(Q_{h-1}) \subseteq L(Q_h)$, or $h = 1$, and let $C \in \mathscr{K}(Q_h) - \{C_{h,1}, \dots, C_{h,t-1-j_h}, F_1, \dots, F_a\}$. Then $(V(C) \cap W) - W_h \neq \emptyset$.*

Proof. Since C and the $C_{h,l}$ $(1 \leq l \leq t - 1 - j_h)$ are distinct members of $\mathscr{K}(Q_h)$, $(V(C) \cap W) \cap (W_h - W_{h-1}) = \emptyset$. Thus it suffices to show that $(V(C) \cap W) - W_{h-1} \neq \emptyset$. Suppose that

$$(V(C) \cap W) - W_{h-1} = \emptyset. \tag{5.10}$$

If $C \in \mathscr{L}(Q_h)$, we can get a contradiction by arguing as in the proof of Claim 5.6. Thus we may assume $C \notin \mathscr{L}(Q_h)$. Then

$$V(C) \cap L(Q_h) = \emptyset. \tag{5.11}$$

Assume for the moment that $h \geq 2$ and $L(Q_{h-1}) \subseteq L(Q_h)$. Then by the choice of our labeling mentioned immediately after (5.8), we have $L(Q'_{h-1}) \subseteq L(Q'_h)$ for any labeling $Q'_1, \dots, Q'_h$ of $\bigcup_{0 \leq i \leq t-2} \mathscr{Q}_i$ which satisfies (5.8). This implies $L(Q_i) \subseteq L(Q_h)$ for all $1 \leq i \leq h - 1$. Hence by (5.11), $V(C) \cap L(Q_i) = \emptyset$ for all $1 \leq i \leq h - 1$ which, in view of (5.10), implies that

$$V(C) \cap W = (V(C) \cap W) - W_{h-1} = \emptyset. \tag{5.12}$$

Note that if $h = 1$, then (5.10) immediately implies (5.12). Thus (5.12) holds. But in view of Claim 5.5 (ii) and Claim 5.2, (5.12) implies that $C = F_p$ for some p with $1 \leq p \leq a$, which contradicts the assumption that $C \notin \{F_1, \dots, F_a\}$. □

Claim 5.8. $|W_h| \leq |W| - (k + 1)$.

Proof. If $h = 0$, the claim immediately follows from (5.7). Thus we may assume $h \geq 1$. By (5.8) and Lemma 3.6, $Q_h \cap L(Q_i) = \emptyset$ for all i, and hence

$$Q_h \cap W_h = \emptyset. \tag{5.13}$$

Assume first that $h \geq 2$ and $L(Q_{h-1}) \not\subseteq L(Q_h)$. Then by (5.8) and Lemma 3.6, we obtain $Q_{h-1} \cap W_h = \emptyset$. Since $Q_{h-1}, Q_h \subseteq W$ by Claim 5.4, this together with (5.13) implies that $|W_h| \leq |W| - |Q_h \cup Q_{h-1}| \leq |W| - (k + 1)$. Assume now that $h \geq 2$ and $L(Q_{h-1}) \subseteq L(Q_h)$ or $h = 1$. Let C be as in Claim 5.7. Then since $Q_h \subseteq W$ by Claim 5.4, Claim 5.7 and (5.13) imply that $|W_h| \leq |W| - |Q_h| - |(V(C) \cap W) - W_h| \leq |W| - (k + 1)$. □

Claim 5.9. $\sum_{0 \leq j \leq t-2} (t - 1 - j) b_j \leq |W| - (k + 1)$.

Proof. Recall that for each $1 \le i \le h$, j_i denotes the index such that $Q_i \in \mathscr{Q}_{j_i}$, and thus $b_j = |\{i \mid 1 \le i \le h, j_i = j\}|$ for each $0 \le j \le t-2$. Therefore by (5.9) and Claims 5.6 and 5.8,

$$\begin{aligned}\sum_{0 \le j \le t-2} (t-1-j)b_j &= \sum_{1 \le i \le h} (t-1-j_i) \\ &\le \sum_{1 \le i \le h} \left(\sum_{1 \le l \le t-1-j_i} |X_{i,l}| \right) \\ &= \left| \bigcup_{1 \le i \le h} \left(\bigcup_{1 \le l \le t-1-j_i} X_{i,l} \right) \right| \\ &= |W_h| \le |W| - (k+1). \qquad \square\end{aligned}$$

Claim 5.10. *For any i, l with $1 \le i \le h$ and $1 \le l \le t-1-j_i$, no member of $\bigcup_{0 \le j \le t-1} \mathscr{Q}_j$ intersects with both $X_{i,l}$ and $W - W_{i-1} - X_{i,l} - Q_i$.*

Proof. Recall that $\{Q_\alpha \mid 1 \le \alpha \le h\} = \bigcup_{0 \le j \le t-2} \mathscr{Q}_j \subseteq \bigcup_{0 \le j \le t-1} \mathscr{Q}_j = \mathscr{R}$. Also note that a vertex in $X_{i,l}$ and a vertex in $W - W_{i-1} - X_{i,l} - Q_i$ belong to distinct components of $G - Q_i$. Since no two members of $\mathscr{R}$ mesh with each other by the definition of $\mathscr{R}$, this means that no member of $\bigcup_{0 \le j \le t-1} \mathscr{Q}_j$ intersects with both $X_{i,l}$ and $W - W_{i-1} - X_{i,l} - Q_i$. $\square$

In view of Lemma 4.8 (ii), Claim 5.10 together with Claims 5.6 and 5.8 implies

$$(5.14) \quad \sum_{0 \le j \le t-1} b_j \le \binom{|W| - \sum_{0 \le j \le t-2} (t-1-j)b_j}{k} + (k+1) \sum_{0 \le j \le t-2} (t-1-j)b_j.$$

We now obtain

$$\begin{aligned}|\mathscr{S}| &= \sum_{1 \le p \le a} |V(\mathscr{H}_p)| + \sum_{0 \le i \le t-1} b_i \\ &\le (n - |W|)/(t-1) + \sum_{0 \le i \le t-1} b_i \quad \text{(by (5.3))} \\ &\le \big((2t-1)(n - f(n))\big) \;/\; (2(t-1)^2) \\ &\qquad \text{(by (5.6), (5.14), Claim 5.9 and Lemma 4.5).}\end{aligned}$$

This completes the proof of the Main Theorem.

References

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