Vertex-disjoint t-claws in graphs

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Abstract. Let $\delta(G)$ denote the minimum degree of a graph G. We prove that for $t \geq 4$ and $k \geq 2$, a graph G of order at least $(t+1)k + \frac{11}{6}t^2$ with $\delta(G) \geq k + t - 1$ contains k pairwise vertex-disjoint copies of $K_{1,t}$.

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§1. Introduction

We consider only undirected graphs without loops or multiple edges. For a graph G, we denote by $V(G), E(G)$ and $\delta(G)$ the vertex set, the edge set and the minimum degree of G, respectively. A graph F is called a t-claw if F is isomorphic to $K_{1,t}$.

Let H be a fixed connected graph, and let $k \geq 2$ be a fixed integer. In this paper, we are concerned with the existence of k pairwise vertex-disjoint copies of H in a graph G. The main theorem of this paper deals with the case where $|V(G)| > k|V(H)|$, but we start with results which deal with the case where $|V(G)| = k|V(H)|$. For $H = K_t$ with $t \geq 2$, Hajnal and Szemerédi [6] proved that if $|V(G)| = kt$ and $\delta(G) \geq \frac{t-1}{t}|V(G)|$, then G contains k pairwise vertex-disjoint copies of K_t (see also Corrádi and Hajnal [1]). For $H = P_t$ with $t(\geq 3)$ odd, it is easy to see that if $|V(G)| = kt$ and $\delta(G) \geq \frac{|V(G)|-2}{2}$, then G contains k pairwise vertex-disjoint copies of P_t (for results concerning the case where it is assumed that G is connected, the reader is referred to Johansson [7] and Enomoto, Kaneko and Tuza [4]).

Note that $P_3 \cong K_{1,2}$ is a 2-claw. Thus letting $t = 2$ in the above result, we obtain the following proposition.

Proposition 1. *Let $k \geq 2$ be an integer, and let G be a graph of order $3k$ such that $\delta(G) \geq (3k-2)/2$. Then G contains k pairwise vertex-disjoint 2-claws.*

In the case where H is a 3-claw, Egawa, Fujita and Ota [2] proved the following theorem.

Theorem 2 (Egawa, Fujita and Ota [2]). *Let $k \geq 2$ be an integer, and let G be a graph of order $4k$ such that $\delta(G) \geq 2k$. Then G contains k pairwise vertex-disjoint 3-claws, unless k is odd and G is isomorphic to $K_{2k,2k}$.*

In Proposition 1 and Theorem 2, the condition on the minimum degree is sharp. However, if we assume that the order of G is slightly greater than $3k$ or $4k$, then a much weaker condition on the minimum degree guarantees the existence of k pairwise vertex-disjoint 2-claws or 3-claws.

Theorem 3 (Ota [8]). *Let $k \geq 2$ be an integer, and let G be a graph of order at least $3k + 2$ such that $\delta(G) \geq k + 1$. Then G contains k pairwise vertex-disjoint 2-claws.*

Theorem 4 (Egawa and Ota [3]). *Let $k \geq 2$ be an integer, and let G be a graph of order at least $4k + 6$ such that $\delta(G) \geq k + 2$. Then G contains k pairwise vertex-disjoint 3-claws.*

Based on these results, Ota [8] made the following conjecture.

Conjecture 5 (Ota [8]). *Let $t \geq 2$, $k \geq 2$ be integers, and let G be a graph of order at least $(t+1)k + t^2 - t$ such that $\delta(G) \geq k + t - 1$. Then G contains k pairwise vertex-disjoint t-claws.*

As is shown in [8], in this conjecture, the condition on the minimum degree of G is sharp in the sense that for any fixed t and k, there exists a graph of arbitrarily large order which has minimum degree $k + t - 2$ but does not contain k vertex-disjoint t-claws and, if k is sufficiently large compared with t, then the condition on the order of G is also sharp in the sense that there exists a graph G with $|V(G)| = (t+1)k + t^2 - t - 1$ and $\delta(G) \geq k + t - 1$ such that G does not contain k vertex-disjoint t-claws. Theorems 3 and 4 above show that the conjecture is true for $t = 2, 3$. For $t \geq 4$, Ota [8; Theorem 1] proved the following theorem.

Theorem 6 (Ota [8]). *Let $t \geq 4$, $k \geq 2$ be intgers, and let G be a graph of order at least $(t+1)k + 2t^2 - 3t - 1$ such that $\delta(G) \geq k + t - 1$. Then G contains k pairwise vertex-disjoint t-claws.*

The coefficient -3 of t in the lower bound on $|V(G)|$ was improved to -4 by Fujita in [5].

Theorem 7 (Fujita [5]). *Let $t \geq 4$, $k \geq 2$ be intgers, and let G be a graph of order at least $(t+1)k + 2t^2 - 4t + 2$ such that $\delta(G) \geq k + t - 1$. Then G contains k pairwise vertex-disjoint t-claws.*

The purpose of this paper is to improve the coefficient of t^2 as follows.

Main Theorem *Let $t \geq 4$, $k \geq 2$ be intgers, and let G be a graph of order at least $(t+1)k + \frac{11}{6}t^2$ such that $\delta(G) \geq k + t - 1$. Then G contains k pairwise vertex-disjoint t-claws.*

We need the following notation and terminology. Let G be a graph. For a vertex $v \in V(G)$, we denote by $N(v) = N_G(v)$ and $d_G(v)$ the set of vertices adjacent to v and the degree of v, respectively; thus $d_G(v) = |N_G(v)|$. For $S \subseteq V(G)$, we let $\langle S \rangle = \langle S \rangle_G$ denote the subgraph of G induced by S. For disjoint subsets S and T of $V(G)$, we let $E(S,T) = E_G(S,T)$ denote the set of edges of G joining a vertex in S and vertex in T. When S or T contains of a single vertex, say $S = \{x\}$ or $T = \{y\}$, we write $E(x,T)$ or $E(S,y)$ for $E(S,T)$.

§2.   Preparation for the proof of the main theorem

By way of contradiction, suppose that there exists a graph G with $|V(G)| \geq (t+1)k + \frac{11}{6}t^2$ and $\delta(G) \geq k+t-1$ such that G does not contain k pairwise vertex-disjoint t-claws. By Theorem 7, we have $|V(G)| \leq (t+1)k + 2t^2 - 4t + 1$. Since $\lceil \frac{11}{6}t^2 \rceil \geq 2t^2 - 4t + 2$ for $4 \leq t \leq 23$, this implies $t \geq 24$. We may assume that G is an edge-maximal counterexample. Then G contains $k-1$ vertex-disjoint t-claws, say $C^{(1)}, C^{(2)}, \ldots, C^{(k-1)}$. Set $H = G - (\bigcup_{i=1}^{k-1} V(C^{(i)}))$. Let $P^{(1)}, P^{(2)}, \ldots, P^{(s)}$ be the K_t components of H, i.e.,the components of H isomorphic to K_t. Define $U = \bigcup_{\alpha=1}^{s} V(P^{(\alpha)})$ and $W = V(H) - U$. We may assume that $C^{(1)}, C^{(2)}, \ldots, C^{(k-1)}$ are chosen so that $|E(\langle W \rangle)| + \frac{2}{t}|E(\langle U \rangle)|$ is as large as possible.

By assumption, H contains no t-claw, or equivalently, every vertex of H has degree at most $t-1$. We define $n = |V(H)|$. Since $n = |V(G)| - (t+1)(k-1)$, we have $\frac{11}{6}t^2 + t + 1 \leq n \leq 2t^2 - 3t + 2$. For each i, let $a^{(i)}$ be the center of $C^{(i)}$ and $B^{(i)} = \{b_1^{(i)}, b_2^{(i)}, \ldots, b_t^{(i)}\}$ be the set of leaves of $C^{(i)}$. In the following argument, we sometimes fix i and set $C = C^{(i)}$. In such cases, we write $a, B, b_1, b_2, \ldots, b_t$ instead of $a^{(i)}, B^{(i)}, b_1^{(i)}, b_2^{(i)}, \ldots, b_t^{(i)}$, respectively.

We first state seven lemmas concerning the number of edges between $V(C^{(i)})$ and $V(H)$, which are proved in [5; Lemmas 2.1 through 2.7]. Fix i with $1 \leq i \leq k-1$. Thus as mentioned in the preceding paragraph, a denotes the center of $C = C^{(i)}$, and $B = \{b_1, b_2, \ldots, b_t\}$ denotes the set of leaves of C.

Lemma 2.1. *Let $v \in V(H)$, and suppose that $d_H(v) + |E(B,v)| \geq t$. Then $|E(a, V(H) - \{v\} - N_H(v))| \leq t - 1 - d_H(v)$.*

Lemma 2.2. *If $E(a, V(H)) \neq \emptyset$, then $|E(b_p, V(H))| \leq t$ for every $b_p \in B$.*

Lemma 2.3. *If $|((N(b_p) \cup N(b_q)) \cap V(H)| \geq 2t - 1$ for $b_p, b_q \in B$ with $p \neq q$, then $|E(b_p, V(H))| \leq t - 2$ or $|E(b_q, V(H))| \leq t - 2$.*

Lemma 2.4. *Let $v \in V(H)$, and suppose that $d_H(v) + |E(B, v)| \geq t + 1$. Then $|E(b_p, V(H) - \{v\} - N_H(v))| \leq t - 2$ for every $b_p \in B$.*

Lemma 2.5. *Let P be a K_t component of H, and suppose that there exists $v \in V(H) - V(P)$ such that $d_H(v) + |E(V(C), v)| \geq t + 1$. Then $E(V(C), V(P)) = \emptyset$.*

Lemma 2.6. *Let P be a K_t component of H, and suppose that there exists $v \in V(H) - V(P)$ such that $|E(V(C), v)| \geq 2$. Then $E(B, V(P)) = \emptyset$, and hence it follows that $|E(V(C), V(P))| \leq t$.*

Lemma 2.7. *Let P be a K_t component of H, and suppose that $E(V(C), V(H) - V(P)) \neq \emptyset$. Then $|E(V(C), V(P))| \leq t$.*

In the rest of this section, we consider the case where $s \geq t + 1$. For each α with $1 \leq \alpha \leq t + 1$, we take $u_\alpha \in V(P^{(\alpha)})$. Since

$$\sum_{i=1}^{k-1} \sum_{\alpha=1}^{t+1} |E(V(C^{(i)}), u_\alpha)| = \sum_{\alpha=1}^{t+1} (d_G(u_\alpha) - (t-1)) \geq (t+1)k,$$

there exists an index i with $1 \leq i \leq k - 1$ such that $\sum_{\alpha=1}^{t+1} |E(V(C^{(i)}), u_\alpha)| > t + 1$. Then there exist two edges $x u_\alpha$ and $y u_\beta$ joining $V(C^{(i)})$ and $\{u_1, u_2, \ldots, u_{t+1}\}$ with $x, y \in V(C^{(i)})$, $x \neq y$ and $\alpha \neq \beta$. Replacing $C^{(i)}$ by t-claws contained in $\langle \{x\} \cup V(P^{(\alpha)}) \rangle$ and $\langle \{y\} \cup V(P^{(\beta)}) \rangle$, we obtain k vertex-disjoint t-claws in G. This is a contradiction.

§3.   The case where $s = t$

We continue with the notation of the preceding section. In order to prove the main theorem, we shall choose some $C^{(i)}$'s and show that they together with some vertices in H contain more t-claws, which contradicts the assumption that G is a counterexample. In this section, we consider the case where $s = t$. For each α with $1 \leq \alpha \leq t$, we take a vertex $u_\alpha \in V(P^{(\alpha)})$, and let $v \in W$. Define

$$J = \{i \mid 1 \leq i \leq k - 1, \ |E(V(C^{(i)}), \{u_1, u_2, \ldots, u_t, v\})| \geq t + 2\}.$$

The following two lemmas are proved in [5; Lemmas 3.1 and 3.2].

Lemma 3.1. *Suppose that $C = C^{(i)}$ satisfies $|E(V(C), \{u_1, u_2, \ldots, u_t, v\})| \geq t + 2$. Then the following hold.*

(i) $2 \leq |E(V(C), v)| \leq t$.

(ii) $E(B, \{u_1, u_2, \ldots, u_t, v\}) = \emptyset$.

Lemma 3.2. $\sum_{i \in J} |E(V(C^{(i)}), v)| \geq |J| + t + 1$.

We may assume that $J = \{1, 2, \ldots, m\}$ where $m = |J|$, and $|E(V(C^{(1)}), v)| \geq |E(V(C^{(2)}), v)| \geq \cdots \geq |E(V(C^{(m)}), v)| \geq 2$. By Lemmas 3.1(i) and 3.2, there exists $l \in J$ with $2 \leq l \leq m$ such that

$$(3.1) \qquad \sum_{i=1}^{l} (|E(V(C^{(i)}), v)| - 1) \geq t$$

and such that $\sum_{i=1}^{l-1} (|E(V(C^{(i)}), v)| - 1) \leq t - 1$. By Lemma 3.1(i), we also have $\sum_{j=1}^{i} (|E(V(C^{(j)}), v)| - 1) \leq t - 1$ for each $1 \leq i \leq l - 1$, and $l - 1 \leq \sum_{i=1}^{l-1} (|E(V(C^{(i)}), v)| - 1) \leq t - 1$. Thus $2 \leq l \leq t$. The following lemma is proved in [5; Lemma 3.3].

Lemma 3.3. *We have $|E(a^{(i)}, \{u_1, u_2, \ldots, u_t\})| \geq i$ for each $1 \leq i \leq l$.*

Now by Lemma 3.3, we may assume that we can take l independent edges $a^{(i)} u_i$, $1 \leq i \leq l$. On the other hand, (3.1) implies that $\sum_{i=1}^{l} |E(B^{(i)}, v)| \geq t$. Hence we can take $X \subset N(v) \cap \left(\bigcup_{i=1}^{l} B^{(i)} \right)$ with $|X| = t$. Then each of $\langle X \cup \{v\} \rangle$ and $\langle \{a^{(i)}\} \cup V(P^{(i)}) \rangle$ for $1 \leq i \leq l$ contains a t-claw. These are $l + 1$ vertex-disjoint t-claws in $\langle (\bigcup_{i=1}^{l} V(C^{(i)})) \cup V(H) \rangle$, which contradicts the assumption that G is a counterexample.

§4. Counting argument

Throughout the rest of this paper, we assume that $s \leq t - 1$. In this section, we find a good vertex in H that can be used later to find an extra t-claw. The lemmas proved in this section are actually proved in [5], but we include their proofs for the convenience of the reader. Recall that U is the set of vertices contained in the K_t components of H, and $W = V(H) - U$. We define

$$I = \{i \mid 1 \leq i \leq k - 1, \ E(V(C^{(i)}), W) = \emptyset\},$$
$$J = \{i \mid 1 \leq i \leq k - 1, \ i \notin I, \ |E(V(C^{(i)}), V(H))| \geq n - s + 1\}.$$

Note that since $n \geq \frac{11}{6} t^2 + t + 1$ and $s \leq t - 1$, we have $|E(V(C^{(i)}), V(H))| \geq \frac{11}{6} t^2 + 3$ for each $i \in J$.

Lemma 4.1. *There exists $v \in W$ such that*

$$d_H(v) + \sum_{i \in J} |E(V(C^{(i)}), v)| \geq |J| + t.$$

Proof. Suppose that

$$(4.1) \qquad d_H(v) + \sum_{i \in J} |E(V(C^{(i)}), v)| \leq |J| + t - 1 \ \text{ for all } \ v \in W.$$

We first claim that $|E(V(C^{(i)}), U)| \leq t(t+1)$ for each $i \in I$. If $V(C^{(i)})$ is joined by edges to at most one component of $\langle U \rangle$, then the claim is obvious. If $V(C^{(i)})$ is joined to at least two components of $\langle U \rangle$, then by Lemma 2.7, $|E(V(C^{(i)}), U)| \leq ts < t(t+1)$. Thus the claim follows. Note that this claim implies that

$$(4.2) \qquad \sum_{i \in I} |E(V(C^{(i)}), U)| \leq t(t+1)|I|.$$

For $i \in J$, since $E(V(C^{(i)}), W) \neq \emptyset$, it follows from Lemma 2.7 that $|E(V(C^{(i)}), U)| \leq ts$. Hence

$$(4.3) \qquad \sum_{i \in J} |E(V(C^{(i)}), U)| \leq ts|J|.$$

By the definition of I,

$$(4.4) \qquad \sum_{i \in I} |E(V(C^{(i)}), W)| = 0.$$

By (4.1),

$$(4.5) \qquad \sum_{v \in W} \left(d_H(v) + \sum_{i \in J} |E(V(C^{(i)}), v)| \right) \leq (|J| + t - 1)(n - ts).$$

For $i \notin I \cup J$, we have $|E(V(C^{(i)}), V(H))| \leq n - s$ by the definition of J. Hence

$$(4.6) \qquad \sum_{i \notin I \cup J} |E(V(C^{(i)}), V(H))| \leq (n - s)(k - 1 - |I| - |J|).$$

Now we estimate the following weighted sum of the degrees of vertices in H in two ways: $\frac{t-1}{t} \sum_{u \in U} d_G(u) + \sum_{v \in W} d_G(v)$. First, since $\delta(G) \geq k + t - 1$,

$$(4.7) \qquad \frac{t-1}{t} \sum_{u \in U} d_G(u) + \sum_{v \in W} d_G(v) \geq (k + t - 1)\left(\frac{t-1}{t}|U| + |W| \right)$$

$$= (k + t - 1)(n - s).$$

On the other hand, by (4.2) through (4.6),

$$\frac{t-1}{t} \sum_{u \in U} d_G(u) + \sum_{v \in W} d_G(v)$$

$$= \frac{t-1}{t} \sum_{u \in U} \left(d_H(u) + \sum_{i=1}^{k-1} |E(V(C^{(i)}), u)| \right)$$

$$+ \sum_{v \in W} \left(d_H(v) + \sum_{i=1}^{k-1} |E(V(C^{(i)}), v)| \right)$$

$$= \frac{t-1}{t} \left(\sum_{u \in U} d_H(u) + \left(\sum_{i \in I} + \sum_{i \in J} + \sum_{i \notin I \cup J} \right) |E(V(C^{(i)}), U)| \right)$$

$$+ \sum_{v \in W} \left(d_H(v) + \sum_{i \in J} |E(V(C^{(i)}), v)| \right) + \sum_{i \notin I \cup J} |E(V(C^{(i)}), W)|$$

$$\leq \frac{t-1}{t} \left(\sum_{u \in U} d_H(u) + \sum_{i \in I} |E(V(C^{(i)}), U)| + \sum_{i \in J} |E(V(C^{(i)}), U)| \right)$$

$$+ \sum_{v \in W} \left(d_H(v) + \sum_{i \in J} |E(V(C^{(i)}), v)| \right) + \sum_{i \notin I \cup J} |E(V(C^{(i)}), V(H))|$$

$$\leq \frac{t-1}{t} (t(t-1)s + t(t+1)|I| + ts|J|) + (|J| + t - 1)(n - ts)$$

$$+ (n - s)(k - 1 - |I| - |J|)$$

$$= (k + t - 1)(n - s) + (t^2 - 1)|I| - (n - s)(|I| + 1)$$

$$\leq (k + t - 1)(n - s) + (t^2 - 1)|I| - \left(\frac{11}{6} t^2 + 2 \right)(|I| + 1)$$

$$= (k + t - 1)(n - s) - \left(\frac{5}{6} t^2 + 3 \right)|I| - \left(\frac{11}{6} t^2 + 2 \right).$$

This contradicts (4.7), which completes the proof of Lemma 4.1. $\qquad \square$

In the following argument, we consider the vertices in W satisfying the condition in Lemma 4.1. We define

$$W_0 = \{ v \in W \mid d_H(v) + \sum_{i \in J} |E(V(C^{(i)}), v)| \geq |J| + t \},$$

which is not empty by Lemma 4.1. We also define

$$W_1 = \{ v \in W \mid \text{there exists } i \in J \text{ such that } d_H(v) + |E(V(C^{(i)}), v)| \geq t + 1 \},$$
$$W_2 = \{ v \in W - W_1 \mid \text{there exists } J_0 \subset J \text{ with } 2 \leq |J_0| \leq t - d_H(v)$$
$$\text{such that } d_H(v) + \sum_{i \in J_0} |E(V(C^{(i)}), v)| \geq |J_0| + t \}.$$

Lemma 4.2. *The following statements hold:*

(i) $W_0 \subset W_1 \cup W_2$.

(ii) *If v is a vertex in W_0 with $d_H(v) = t - 1$, then $v \in W_1$.*

Proof. Suppose that $v \in W_0$. By the definition of W_0,

$$\sum_{i \in J}(|E(V(C^{(i)}), v)| - 1) \geq t - d_H(v).$$

Thus there exists $J_0 \subset J$ with $1 \leq |J_0| \leq t - d_H(v)$ such that $|E(V(C^{(i)}), v)| - 1 \geq 1$ for each $i \in J_0$ and

$$\sum_{i \in J_0}(|E(V(C^{(i)}), v)| - 1) \geq t - d_H(v).$$

This proves (i). Further if $d_H(v) = t - 1$, then $|J_0| = 1$. Thus (ii) holds. □

Lemma 4.3. *Suppose that $W_1 = \emptyset$. Fix $C = C^{(i)}$ with $i \in J$, and let $b_p \in B$. Suppose that $E(B, U) = \emptyset$ and $|E(b_p, V(H))| \geq t + 1$, and let $x_1, x_2, \ldots, x_{t-1}$ be $t - 1$ vertices in $N(b_p) \cap V(H)$. Then the following inequality holds:*
$$|E(a, V(H))| + |E(b_p, V(H))| + \sum_{i=1}^{t-1}(d_H(x_i) + |E(V(C), x_i)|)$$
$$\geq |E(V(C), V(H))| + t - 1 + |E(\langle a, x_1, x_2, \ldots, x_{t-1}\rangle)|.$$

Proof. First we claim that $|E(B - \{b_p\}, V(H) - \{x_1, x_2, \ldots, x_{t-1}\})| \leq \sum_{i=1}^{t-1} d_H(x_i) - e$, where $e = |E(\langle x_1, x_2, \ldots, x_{t-1}\rangle)|$. We replace C by the t-claw with center b_p contained in $\langle a, b_p, x_1, x_2, \ldots, x_{t-1}\rangle$. Let $H' = \langle (V(H) - \{x_1, x_2, \ldots, x_{t-1}\}) \cup (V(C) - \{a, b_p\})\rangle$, and let U' be the union of the vertex sets of the K_t components of H'. Also we set $S = (B - \{b_p\}) \cap U'$.

If $S = \emptyset$, then the claim immediately follows from the maximality of $|E(\langle W\rangle)| + \frac{2}{t}|E(\langle U\rangle)|$. Thus we may assume that $S \neq \emptyset$. Let $y \in N(b_p) \cap (V(H) - \{x_1, \ldots, x_{t-1}\})$. If there exists $b_q \in S$ such that $b_q y \notin E(G)$, then each of $\langle \{b_q, a\} \cup N_{H'}(b_q)\rangle$ and $\langle \{b_p, x_1, x_2, \ldots, x_{t-1}, y\}\rangle$ contains a t-claw, a contradiction. Thus $by \in E(G)$ for every $b \in S$. Hence there exists a K_t component P' of H' such that $\{y\} \cup S \subset V(P')$. Note that $|N(b_p) \cap (V(H) - \{x_1, \ldots, x_{t-1}\})| \geq 2$ by the assumption that $|E(b_p, V(H))| \geq t + 1$. Since the above observation holds for any choice of $y \in N(b_p) \cap (V(H) - \{x_1, \ldots, x_{t-1}\})$, it follows that $1 \leq |S| \leq t - 2$. This implies that $\langle V(C) - \{b_p\}\rangle \not\cong K_t$. On the other hand, since $W_1 = \emptyset$ and $d_H(y) + |E(V(C), y)| \geq d_H(y) + |E(B, y)| = |E(y, \{x_1, x_2, \ldots, x_{t-1}\})| + |E(y, V(H) - \{x_1, x_2, \ldots, x_{t-1}\})| + |S| + 1 = |E(y, \{x_1, x_2, \ldots, x_{t-1}\})| + d_{P'}(y) + 1 = |E(y, \{x_1, x_2, \ldots, x_{t-1}\})| + t$, we obtain $E(y, \{x_1, x_2, \ldots, x_{t-1}\}) = \emptyset$ and $d_H(y) = t - |E(B, y)|$.

Now replace C by the t-claw contained in $\langle b_p, x_1, x_2, \ldots, x_{t-1}, y\rangle$, and set $H'' = \langle (V(C) - \{b_p\}) \cup (V(H) - \{x_1, x_2, \ldots, x_{t-1}, y\})\rangle$. Then since $\langle V(C) -$

$\{b_p\}\rangle$ is connected and not isomorphic to K_t, the union of the vertex sets of the K_t components of H'' coincides with U. Therefore it follows from the maximality of $|E(\langle W \rangle)| + \frac{2}{t}|E(\langle U \rangle)|$ that

$$
\begin{aligned}
0 \leq &\left(\sum_{i=1}^{t-1} d_H(x_i) + d_H(y) - e \right) - \{ (|E(B - \{b_p\}, V(H) - \{x_1, x_2, \ldots, x_{t-1}\})| \\
&- |E(B - \{b_p\}, y)|) + |E(a, B - \{b_p\})| \} \\
= &\left(\sum_{i=1}^{t-1} d_H(x_i) + d_H(y) - e \right) - \{ (|E(B - \{b_p\}, V(H) - \{x_1, x_2, \ldots, x_{t-1}\})| \\
&- (|E(B, y)| - 1)) + (t - 1) \} \\
= &\left(\sum_{i=1}^{t-1} d_H(x_i) - e \right) - |E(B - \{b_p\}, V(H) - \{x_1, x_2, \ldots, x_{t-1}\})|,
\end{aligned}
$$

as claimed. Consequently

$$
\begin{aligned}
&|E(a, V(H))| + |E(b_p, V(H))| + \sum_{i=1}^{t-1} |E(V(C), x_i)| \\
= &|E(V(C), V(H)| + |E(\{a, b_p\}, \{x_1, x_2, \ldots, x_{t-1}\})| \\
&\qquad - |E(V(C) - \{a, b_p\}, V(H) - \{x_1, x_2, \ldots, x_{t-1}\})| \\
\geq &|E(V(C), V(H))| + (t - 1 + |E(a, \{x_1, x_2, \ldots, x_{t-1}\})|)) - \left(\sum_{i=1}^{t-1} d_H(x_i) - e \right) \\
= &|E(V(C), V(H)| + t - 1 + |E(\langle a, x_1, x_2, \ldots, x_{t-1} \rangle)| - \sum_{i=1}^{t-1} d_H(x_i).
\end{aligned}
$$

This completes the proof of Lemma 4.3. $\qquad \square$

§5. Property of J_0

We continue with the notation of the preceding sections. In this section and the next section, we consider the case where $W_1 = \emptyset$.

Case 1: $W_1 = \emptyset$.

We take a vertex $v \in W_0$, and fix it. By Lemma 4.2(i), $v \in W_2$. Also by Lemma 4.2(ii), $d_H(v) \leq t - 2$. By the definition of W_2, there exists $J_0 \subseteq J$ with $2 \leq |J_0| \leq t - d_H(v)$ such that $d_H(v) + \sum_{i \in J_0}(|E(V(C^{(i)}), v)| - 1) \geq t$. We choose such a subset J_0 of J so that $|J_0|$ is as small as possible. Then

$$(5.1) \qquad d_H(v) + \sum_{j \in J_0 - \{i\}} (|E(V(C^{(j)}), v)| - 1) \leq t - 1 \ \text{ for each } \ i \in J_0,$$

and hence

(5.2) $|E(V(C^{(i)}), v)| \geq 2$ for each $i \in J_0$.

By (5.1) and (5.2), we have

(5.3) $d_H(v) + \sum_{i \in J_0 - M} (|E(V(C^{(i)}), v)| - 1) \leq t - |M|$

for any nonempty subset M of J_0. By Lemma 2.6, (5.2) implies

(5.4) $E(B^{(i)}, U) = \emptyset$ for each $i \in J_0$.

Lemma 5.1. *For each $C = C^{(i)}$ with $i \in J_0$, one of the following statements hold:*

(i) $|E(a, V(H) - \{v\} - N_H(v))| \geq t - d_H(v)$; or

(ii) $E(a, V(H)) = \emptyset$ and $|E(b_p, W)| \geq \frac{5}{6}t^2 + 3t + 1$ for some $b_p \in B$.

Proof. Since $i \in J_0 \subset J$,

(5.5) $|E(V(C), V(H))| \geq n - s + 1 \geq \frac{11}{6}t^2 + 3$.

Suppose that (i) does not hold. Then $|E(a, V(H))| \leq (t - 1 - d_H(v)) + (1 + |N_H(v)|) = t$. If $E(a, V(H)) \neq \emptyset$, then by Lemma 2.2, $|E(b, V(H))| \leq t$ for every $b \in B$, and hence $|E(V(C), V(H))| = |E(a, V(H))| + |E(B, V(H))| \leq t + t^2$, which contradicts (5.5). Thus $E(a, V(H)) = \emptyset$. This together with (5.4) implies $E(V(C), V(H)) = E(B, W)$. Hence by (5.5), there exists $b_p \in B$ such that $|E(b_p, W)| \geq t + 1$. Take $x_1, x_2, \ldots, x_{t-1} \in N(b_p) \cap W$. Since $W_1 = \emptyset$, $d_H(x_i) + |E(x_i, V(C))| \leq t$ for each $1 \leq i \leq t - 1$. Consequently by Lemma 4.3, $|E(b_p, W)| = |E(b_p, V(H))| \geq |E(V(C), V(H))| + t - 1 - (\sum_{i=1}^{t-1}\{d_H(x_i) + |E(x_i, V(C))|\}) \geq \frac{11}{6}t^2 + 3 + t - 1 - t(t-1) = \frac{5}{6}t^2 + 2t + 2 > 2t - 1$. By Lemma 2.3, this implies that $|E(b, W)| \leq t - 2$ for each $b \in B - \{b_p\}$. Therefore it follows from (5.5) that $|E(b_p, W)| \geq \frac{11}{6}t^2 + 3 - (t-1)(t-2) \geq \frac{5}{6}t^2 + 3t + 1$. This completes the proof of Lemma 5.1. $\square$

We may assume that $J_0 = \{i \mid 1 \leq i \leq |J_0|\}$. We may also assume that there exists an integer h with $0 \leq h \leq |J_0|$ such that $C = C^{(i)}$ satisfies (i) in Lemma 5.1 for all $1 \leq i \leq h$, and $C = C^{(i)}$ satisfies (ii) in Lemma 5.1 for all $h + 1 \leq i \leq |J_0|$. Let $J_{0,1} = \{i \mid 1 \leq i \leq h\}$ and $J_{0,2} = \{i \mid h + 1 \leq i \leq |J_0|\}$. For $C = C^{(i)}$ with $i \in J_{0,2}$, we may assume that $b_1^{(i)}$ is the vertex $b_p^{(i)}$ satisfying the condition of Lemma 5.1(ii). Our first aim is to obtain an upper bound for

$\sum_{i \in J_{0,2}} |E(V(C^{(i)}), V(H))|$ (see Lemma 5.4). Recall that $d_H(v) \leq t-2$. Thus it follows from (5.4) and Lemma 5.1(ii) that for each $i \in J_{0,2}$,

$$(5.6) \quad |E(b_1^{(i)}, V(H) - \{v\} - N_H(v))| = |E(b_1^{(i)}, W)| - |E(b_1^{(i)}, \{v\} \cup N_H(v))|$$
$$\geq |E(b_1^{(i)}, W)| - 1 - d_H(v)$$
$$\geq |E(b_1^{(i)}, W)| - t + 1$$
$$\geq \frac{5}{6}t^2 + 2t + 2.$$

Lemma 5.2. $|J_{0,2}| \geq \frac{5}{6}t - 1$.

Proof. Suppose that $|J_{0,2}| \leq \lceil \frac{5}{6}t \rceil - 2$. Recall that

$$\sum_{i \in J_0} (|E(V(C^{(i)}), v)| - 1) \geq t - d_H(v).$$

This inequality implies that for each i $(1 \leq i \leq |J_0|)$, we can choose a subset $X^{(i)} \subset N(v) \cap V(C^{(i)})$ so that

$$|X^{(i)}| \leq |E(V(C^{(i)}), v)| - 1, \quad \sum_{i=1}^{|J_0|} |X^{(i)}| = t - d_H(v)$$

and

$$a^{(i)} \notin X^{(i)} \text{ for } 1 \leq i \leq h, \quad b_1^{(i)} \notin X^{(i)} \text{ for } h+1 \leq i \leq |J_0|.$$

Then we can find a t-claw with center v in $\langle \{v\} \cup N_H(v) \cup \bigcup_{i=1}^{|J_0|} X^{(i)} \rangle$. Note that by the condition in Lemma 5.1(ii), we have $a^{(i)} \notin X^{(i)}$ also for $h+1 \leq i \leq |J_0|$.

We define $Y^{(i)} = V(C^{(i)}) - X^{(i)}$ for $1 \leq i \leq h$ and $Y^{(i)} = \{a^{(i)}, b_1^{(i)}\}$ for $h+1 \leq i \leq |J_0|$. We take disjoint subsets $Z^{(i)}$ of $V(H) - \{v\} - N_H(v)$ for $1 \leq i \leq |J_0|$ so that

$$Z^{(i)} \subset N(a^{(i)}), \quad |Z^{(i)}| = |X^{(i)}| \text{ for } 1 \leq i \leq h,$$
$$Z^{(i)} \subset N(b_1^{(i)}), \quad |Z^{(i)}| = t - 1 \text{ for } h+1 \leq i \leq |J_0|.$$

This can be done by determining $Z^{(i)}$ from $i = 1$ up to $|J_0|$, because for $1 \leq i \leq h$,

$$\sum_{j=1}^{i} |X^{(j)}| \leq \sum_{j=1}^{|J_0|} |X^{(j)}| = t - d_H(v) \leq |E(a^{(i)}, V(H) - \{v\} - N_H(v))|$$

by Lemma 5.1(i) and, if $J_{0,2} \neq \emptyset$, then for $h + 1 \leq i \leq |J_0|$,

$$\sum_{j=1}^{h} |X^{(j)}| + (t-1)(i-h) \leq \sum_{j=1}^{h}(|E(V(C^{(j)}), v)| - 1) + (t-1)|J_{0,2}|$$
$$\leq (t - |J_{0,2}|) + (t-1)|J_{0,2}| = t + (t-2)|J_{0,2}|$$
$$\leq t + (t-2)\left(\left\lceil \frac{5}{6}t \right\rceil - 2\right) \leq t + (t-2)\left(\frac{5}{6}t - 1\right)$$
$$\leq \frac{5}{6}t^2 - \frac{10}{6}t + 2$$
$$\leq |E(b_1^{(i)}, V(H) - \{v\} - N_H(v))|$$

by (5.3) and (5.6). Then for each i with $1 \leq i \leq |J_0|$, $\langle Y^{(i)} \cup Z^{(i)} \rangle$ contains a t-claw with center $a^{(i)}$ or $b_1^{(i)}$ depending on whether $i \leq h$ or $i \geq h+1$. Obviously these t-claws and the t-claw in $\langle \{v\} \cup N_H(v) \cup \bigcup_{i=1}^{|J_0|} X^{(i)} \rangle$ are pairwise vertex-disjoint. This contradicts the assumption that G is a counterexample, and completes the proof of Lemma 5.2. $\square$

Lemma 5.3.

$$\sum_{i \in J_{0,2}} |E(b_1^{(i)}, W)| \leq \frac{7}{6}t^3.$$

Proof. Suppose that $\sum_{i \in J_{0,2}} |E(b_1^{(i)}, W)| > \lfloor \frac{7}{6}t^3 \rfloor$. Set

$$A = \left\{ i \in J_{0,2} \mid |E(b_1^{(i)}, W)| \geq t^2 - \frac{5}{6}t + 2 \right\}.$$

We show that

(5.7) $$|A| \geq \left\lfloor \frac{1}{6}t \right\rfloor + 1.$$

Suppose that $|A| \leq \frac{1}{6}t$. Recall that $n \leq 2t^2 - 3t + 2$ (see the second paragraph of Section 2). Since $|J_{0,2}| \leq |J_0| \leq t$ and $A \subset J_{0,2}$, we get

$$\sum_{i \in J_{0,2}} |E(b_1^{(i)}, W)| = \sum_{i \in A} |E(b_1^{(i)}, W)| + \sum_{i \in J_{0,2} - A} |E(b_1^{(i)}, W)|$$
$$\leq |A|(2t^2 - 3t + 2) + (t - |A|)\left(t^2 - \frac{5}{6}t + 2\right)$$
$$= |A|\left(t^2 - \frac{13}{6}t\right) + t\left(t^2 - \frac{5}{6}t + 2\right)$$
$$\leq \frac{1}{6}t\left(t^2 - \frac{13}{6}t\right) + t\left(t^2 - \frac{5}{6}t + 2\right)$$
$$= \frac{7}{6}t^3 + 2t - \frac{43}{36}t^2 \leq \frac{7}{6}t^3,$$

a contradiction. Thus (5.7) is proved.

By (5.7), we may assume that there exists an integer h' with $h \leq h' \leq |J_0| - \lfloor \frac{1}{6}t \rfloor - 1$ such that $|E(b_1^{(i)}, W)| \leq t^2 - \lfloor \frac{5}{6}t \rfloor + 1$ for all $h+1 \leq i \leq h'$, and $|E(b_1^{(i)}, W)| \geq t^2 - \frac{5}{6}t + 2$ for all $h'+1 \leq i \leq |J_0|$. Let $J'_{0,2} = \{i \mid h+1 \leq i \leq h'\}$ and $J''_{0,2} = \{i \mid h' + 1 \leq i \leq |J_0|\}$ (if $h' = h$, then $J'_{0,2} = \emptyset$). Recall that $d_H(v) + |J_0| \leq t$. Hence it follows from Lemma 5.2 and Lemma 5.1(ii) that for each $i \in J'_{0,2}$,

$$(5.8) \qquad \begin{aligned} |E(b_1^{(i)}, W - \{v\} - N_H(v))| &\geq |E(b_1^{(i)}, W)| - 1 - d_H(v) \\ &\geq |E(b_1^{(i)}, W)| - 1 + |J_0| - t \\ &\geq \left(\frac{5}{6}t^2 + 3t + 1\right) - 1 + \left(\frac{5}{6}t - 1\right) - t \\ &= \frac{5}{6}t^2 + \frac{17}{6}t - 1, \end{aligned}$$

and it follows from Lemma 5.2 and the choice of h' that for each $i \in J''_{0,2}$,

$$(5.9) \qquad \begin{aligned} |E(b_1^{(i)}, W - \{v\} - N_H(v))| &\geq |E(b_1^{(i)}, W)| - 1 - d_H(v) \\ &\geq |E(b_1^{(i)}, W)| - 1 + |J_0| - t \\ &\geq \left(t^2 - \frac{5}{6}t + 2\right) - 1 + (\frac{5}{6}t - 1) - t \\ &= t^2 - t. \end{aligned}$$

We now argue as in the proof of Lemma 5.2. For each i ($1 \leq i \leq |J_0|$), we can choose a subset $X^{(i)} \subset N(v) \cap V(C^{(i)})$ so that

$$|X^{(i)}| \leq |E(V(C^{(i)}), v)| - 1, \quad \sum_{i=1}^{|J_0|} |X^{(i)}| = t - d_H(v)$$

and

$$a^{(i)} \notin X^{(i)} \text{ for } 1 \leq i \leq h, \quad b_1^{(i)} \notin X^{(i)} \text{ for } h + 1 \leq i \leq |J_0|.$$

Then we can find a t-claw with center v in $\langle \{v\} \cup N_H(v) \cup \bigcup_{i=1}^{|J_0|} X^{(i)} \rangle$. Note that by the condition in Lemma 5.1(ii), we have $a^{(i)} \notin X^{(i)}$ also for $h+1 \leq i \leq |J_0|$.

We define $Y^{(i)} = V(C^{(i)}) - X^{(i)}$ for $1 \leq i \leq h$ and $Y^{(i)} = \{a^{(i)}, b_1^{(i)}\}$ for $h + 1 \leq i \leq |J_0|$. We take disjoint subsets $Z^{(i)}$ of $V(H) - \{v\} - N_H(v)$ for $1 \leq i \leq |J_0|$ so that

$$Z^{(i)} \subset N(a^{(i)}), \ |Z^{(i)}| = |X^{(i)}| \text{ for } 1 \leq i \leq h,$$
$$Z^{(i)} \subset N(b_1^{(i)}), \ |Z^{(i)}| = t - 1 \text{ for } h + 1 \leq i \leq |J_0|.$$

This can be done by determining $Z^{(i)}$ from $i = 1$ up to $|J_0|$, because for $1 \le i \le h$,

$$\sum_{j=1}^{i} |X^{(j)}| \le \sum_{j=1}^{|J_0|} |X^{(j)}| = t - d_H(v) \le |E(a^{(i)}, V(H) - \{v\} - N_H(v))|$$

by Lemma 5.1(i), and for $h+1 \le i \le h'$, since $|J'_{0,2}| = h' - h \le h' \le t - \lfloor \frac{1}{6}t \rfloor - 1$, we have

$$\sum_{j=1}^{h} |X^{(j)}| + (t-1)(i-h) \le \sum_{j=1}^{h}(|E(V(C^{(j)}), v)| - 1) + (t-1)|J'_{0,2}|$$

$$\le (t - |J_{0,2}|) + (t-1)\left(t - \left\lfloor \frac{1}{6}t \right\rfloor - 1\right)$$

$$\le \frac{1}{6}t + 1 + (t-1)\left(t - \left\lfloor \frac{1}{6}t \right\rfloor - 1\right)$$

$$\le \frac{1}{6}t + 1 + \frac{5}{6}t(t-1) = \frac{5}{6}t^2 - \frac{4}{6}t + 1$$

$$\le |E(b_1^{(i)}, V(H) - \{v\} - N_H(v))|$$

by (5.3), Lemma 5.2 and (5.8) and, for $h' + 1 \le i \le |J_0|$,

$$\sum_{j=1}^{h} |X^{(j)}| + (t-1)(i-h) \le (t-1)i$$

$$\le (t-1)|J_0|$$

$$\le (t-1)t$$

$$\le |E(b_1^{(i)}, V(H) - \{v\} - N_H(v))|$$

by (5.9). Then for each i with $1 \le i \le |J_0|$, $\langle Y^{(i)} \cup Z^{(i)} \rangle$ contains a t-claw with center $a^{(i)}$ or $b_1^{(i)}$ depending on whether $i \le h$ or $i \ge h + 1$. Obviously, these t-claws and the t-claw in $\langle \{v\} \cup N_H(v) \cup \bigcup_{i=1}^{|J_0|} X^{(i)} \rangle$ are pairwise vertex-disjoint. This contradicts the assumption that G is a counterexample, and completes the proof of Lemma 5.3. $\square$

Lemma 5.4.

$$\sum_{i \in J_{0,2}} |E(V(C^{(i)}), W)| \le \frac{13}{6}t^3 - 3t^2 + 2t.$$

Proof. For each $i \in J_{0,2}$, we have $|E(b, W)| \le t - 2$ for every $b \in B^{(i)} - \{b_1^{(i)}\}$ by Lemma 2.2. Also $E(V(C^{(i)}), W) = E(B^{(i)}, W)$ for each $i \in J_{0,2}$ by the first

assertion of Lemma 5.1(ii). Hence by Lemma 5.3,

$$
\begin{aligned}
\sum_{i \in J_{0,2}} |E(V(C^{(i)}), W)| &= \sum_{i \in J_{0,2}} |E(b_1^{(i)}, W)| + \sum_{i \in J_{0,2}} |E(B^{(i)} - \{b_1^{(i)}\}, W)| \\
&\leq \frac{7}{6} t^3 + |J_{0,2}|(t-1)(t-2) \\
&\leq \frac{7}{6} t^3 + t(t-1)(t-2) \\
&= \frac{13}{6} t^3 - 3t^2 + 2t,
\end{aligned}
$$

as desired. $\qquad\square$

We now consider edges among the $C^{(i)}$.

Lemma 5.5. *For each $i \in J_{0,2}$, there exists $l \in \{1, 2, \ldots, k-1\} - \{i\}$ such that $|E(a^{(i)}, V(C^{(l)}))| \geq 2$.*

Proof. By the definition of $J_{0,2}$, we have $E(a^{(i)}, V(H)) = \emptyset$. Hence from the assumption that $\delta(G) \geq k + t - 1$, it follows that

$$
\sum_{j \in \{1, \ldots, k-1\} - \{i\}} |E(a^{(i)}, V(C^{(j)}))| \geq (k + t - 1) - t = k - 1,
$$

which immediately implies the desired conclusion. $\qquad\square$

Having Lemma 5.5 in mind, we define

$$
L = \{l \mid 1 \leq l \leq k-1, \ |E(a^{(i)}, V(C^{(l)}))| \geq 2 \ \text{for some} \ i \in J_{0,2} - \{l\}\}.
$$

Lemma 5.6. *We have $|E(V(C^{(l)}), V(H))| \leq t(t-2) + 1$ for each $l \in L$; in particular $L \cap J = \emptyset$.*

Proof. Let $l \in L$. By the definition of L, there exists $i \in J_{0,2} - \{l\}$ such that $|E(a^{(i)}, V(C^{(l)}))| \geq 2$. Then $N(a^{(i)}) \cap B^{(l)} \neq \emptyset$. Take $b_p^{(l)} \in N(a^{(i)}) \cap B^{(l)}$.

Claim 5.6.1 $E(a^{(l)}, V(H)) = \emptyset$.

Proof. Soppose that $E(a^{(l)}, V(H)) \neq \emptyset$. Take $x \in N(a^{(l)}) \cap V(H)$. Then each of $\langle \{a^{(i)}, b_p^{(l)}\} \cup (B^{(i)} - \{b_1^{(i)}\}) \rangle$ and $\langle \{a^{(l)}, x\} \cup (B^{(l)} - \{b_p^{(l)}\}) \rangle$ contains a t-claw and, since $|E(b_1^{(i)}, W)| \geq \frac{5}{6} t^2 + 3t + 1$ by the definition of $J_{0,2}$, $\langle \{b_1^{(i)}\} \cup (N(b_1^{(i)}) \cap (V(H) - \{x\})) \rangle$ also contains a t-claw, a contradiction. $\qquad\square$

Claim 5.6.2 $|E(b_p^{(l)}, V(H))| \leq t - 1$.

Proof. Suppose that $|E(b_p^{(l)}, V(H))| \geq t$. Since $|E(V(C^{(l)}), a^{(i)})| \geq 2$, there exists $x \in N(a^{(i)}) \cap V(C^{(l)})$ with $x \neq b_p^{(l)}$. Also take a subset X of $N(b_p^{(l)}) \cap V(H)$ with $|X| = t$. Then each of $\langle \{a^{(i)}, x\} \cup (B^{(i)} - \{b_1^{(i)}\}) \rangle$, $\langle \{b_p^{(l)}\} \cup X \rangle$ and $\langle \{b_1^{(i)}\} \cup (N(b_1^{(i)}) \cap (V(H) - X)) \rangle$ contains a t-claw , a contradiction. $\square$

Claim 5.6.3 $|E(b, V(H))| \leq t - 2$ *for every* $b \in B^{(l)} - \{b_p^{(l)}\}$.

Proof. Suppose that there exists $b_q^{(l)} \in B^{(l)} - \{b_p^{(l)}\}$ such that $|E(b_q^{(l)}, V(H))| \geq t - 1$. Take a subset X of $N(b_q^{(l)}) \cap V(H)$ with $|X| = t - 1$. Then each of $\langle \{a^{(i)}, b_p^{(l)}\} \cup (B^{(i)} - \{b_1^{(i)}\}) \rangle$, $\langle \{a^{(l)}, b_q^l\} \cup X \rangle$ and $\langle \{b_1^{(i)}\} \cup (N(b_1^{(i)}) \cap (V(H) - X)) \rangle$ contains a t-claw, a contradiction. $\square$

Combining Claims 5.6.1 through 5.6.3, we obtain $|E(V(C^{(l)}), V(H))| \leq t - 1 + (t - 1)(t - 2) = t(t - 2) + 1$. Since $|E(V(C^{(j)}), V(H))| \geq \frac{11}{6}t^2 + 3$ for each $j \in J$ by the definition of J, we also get $l \notin J$. This completes the proof of Lemma 5.6. $\square$

Lemma 5.7. *Let* $l \in L$, *and let* $i \in J_{0,2}$ *be an index such that* $|E(a^{(i)}, V(C^{(l)}))| \geq 2$. *Then* $E(a^{(j)}, V(C^{(l)})) = \emptyset$ *for every* $j \in J_{0,2} - \{i\}$.

Proof. Suppose that there exists $j \in J_{0,2} - \{i\}$ such that $E(a^{(j)}, V(C^{(l)})) \neq \emptyset$, and take $y \in N(a^{(j)}) \cap V(C^{(l)})$. Since $|E(a^{(i)}, V(C^{(l)}))| \geq 2$, there exists $x \in N(a^{(i)}) \cap V(C^{(l)})$ with $x \neq y$. Then each of $\langle \{a^{(i)}, x\} \cup (B^{(i)} - \{b_1^{(i)}\}) \rangle$ and $\langle \{a^{(j)}, y\} \cup (B^{(j)} - \{b_1^{(j)}\}) \rangle$ contains a t-claw. Since $i, j \in J_{0,2}$, we can take disjoint subsets $X^{(i)}$ and $X^{(j)}$ of $V(H)$ such that

$$|X^{(i)}| = |X^{(j)}| = t, \quad X^{(i)} \subset N(b_1^{(i)}), \quad X^{(j)} \subset N(b_1^{(j)}).$$

Then each of $\langle \{b_1^{(i)}\} \cup X^{(i)} \rangle$ and $\langle \{b_1^{(j)}\} \cup X^{(j)} \rangle$ contains a t-claw, and thus we get a contradiction. $\square$

Note that it follows from Lemmas 5.2, 5.5 and 5.7 that

$$(5.10) \qquad\qquad |L| \geq |J_{0,2}| \geq \frac{5}{6}t - 1.$$

§6.   Another counting argument

In this section, we complete the proof for Case 1. Let $J_0, J_{0,1}, J_{0,2}, L$ be as in the preceding section. Set $I' = I - L$ and $J' = J - J_{0,2}$. Thus

$$\{1, \ldots, k - 1\} = I' \cup J' \cup J_{0,2} \cup L \cup (\{1, \ldots, k - 1\} - I - J - L)$$
(disjoint union).

Lemma 6.1. *There exists $v' \in W$ such that*

$$d_H(v') + \sum_{i \in J'} |E(V(C^{(i)}), v')| \geq |J'| + t.$$

Proof. We argue as in the proof of Lemma 4.1. Suppose that

(6.1) $d_H(v') + \displaystyle\sum_{i \in J'} |E(V(C^{(i)}), v')| \leq |J'| + t - 1$ for all $v' \in W.$

As in the proof of Lemma 4.1, we have $|E(V(C^{(i)}), U)| \leq t(t+1)$ for $i \in I'$. Hence

(6.2) $$\sum_{i \in I'} |E(V(C^{(i)}), U)| \leq t(t+1)|I'|.$$

For $i \in J'$, since $E(V(C^{(i)}), W) \neq \emptyset$, it follows from Lemma 2.7 that $|E(V(C^{(i)}), U)| \leq ts$. Hence

(6.3) $$\sum_{i \in J'} |E(V(C^{(i)}), U)| \leq ts|J'|.$$

By (5.4) and Lemma 5.1(ii),

(6.4) $$\sum_{i \in J_{0,2}} |E(V(C^{(i)}), U)| = 0.$$

By the definition of I,

(6.5) $$\sum_{i \in I'} |E(V(C^{(i)}), W)| = 0.$$

By (6.1),

(6.6) $$\sum_{v' \in W} \left(d_H(v') + \sum_{i \in J'} |E(V(C^{(i)}), v')| \right) \leq (|J'| + t - 1)(n - ts).$$

By Lemma 5.6,

(6.7) $$\sum_{i \in L} |E(V(C^{(i)}), V(H))| \leq (t^2 - 2t + 1)|L|.$$

For $i \notin I' \cup J' \cup J_{0,2} \cup L$, we have $|E(V(C^{(i)}), V(H))| \leq n - s$ by the definition of J. Hence

(6.8) $$\sum_{i \notin I' \cup J' \cup J_{0,2} \cup L} |E(V(C^{(i)}), V(H))|$$

$$\leq (n - s)(k - 1 - |I'| - |J'| - |J_{0,2}| - |L|).$$

Now since $\delta(G) \geq k + t - 1$,

$$(6.9) \qquad \frac{t-1}{t} \sum_{u \in U} d_G(u) + \sum_{v' \in W} d_G(v') \geq (k+t-1)\Big(\frac{t-1}{t}|U| + |W|\Big)$$
$$= (k+t-1)(n-s).$$

On the other hand, by (6.2) through (6.8), Lemma 5.4 and (5.10) ,

$$\frac{t-1}{t} \sum_{u \in U} d_G(u) + \sum_{v' \in W} d_G(v')$$
$$= \frac{t-1}{t} \sum_{u \in U} \Big(d_H(u) + \sum_{i=1}^{k-1} |E(V(C^{(i)}), u)|\Big)$$
$$\qquad + \sum_{v' \in W} \Big(d_H(v') + \sum_{i=1}^{k-1} |E(V(C^{(i)}), v')|\Big)$$
$$= \frac{t-1}{t} \Bigg(\sum_{u \in U} d_H(u) + \Big(\sum_{i \in I'} + \sum_{i \in J'} + \sum_{i \in L} + \sum_{i \notin I' \cup J' \cup J_{0,2} \cup L}\Big)|E(V(C^{(i)}), U)|\Bigg)$$
$$\qquad + \sum_{v' \in W} \Big(d_H(v') + \sum_{i \in J'} |E(V(C^{(i)}), v')|\Big)$$
$$\qquad + \sum_{i \in J_{0,2}} |E(V(C^{(i)}), W)| + \sum_{i \in L} |E(V(C^{(i)}), W)|$$
$$\qquad + \sum_{i \notin I' \cup J' \cup J_{0,2} \cup L} |E(V(C^{(i)}), W)|$$
$$\leq \frac{t-1}{t} \Bigg(\sum_{u \in U} d_H(u) + \sum_{i \in I'} |E(V(C^{(i)}), U)| + \sum_{i \in J'} |E(V(C^{(i)}), U)|\Bigg)$$
$$\qquad + \sum_{v' \in W} \Big(d_H(v') + \sum_{i \in J'} |E(V(C^{(i)}), v')|\Big) + \sum_{i \in J_{0,2}} |E(V(C^{(i)}), W)|$$
$$\qquad + \sum_{i \in L} |E(V(C^{(i)}), V(H))| + \sum_{i \notin I' \cup J' \cup J_{0,2} \cup L} |E(V(C^{(i)}), V(H))|$$
$$\leq \frac{t-1}{t}(t(t-1)s + t(t+1)|I'| + ts|J'|) + (|J'| + t - 1)(n - ts)$$
$$\qquad + \Big(\frac{13}{6}t^3 - 3t^2 + 2t\Big) + (t^2 - 2t + 1)|L|$$
$$\qquad + (n-s)(k - 1 - |I'| - |J'| - |J_{0,2}| - |L|)$$
$$= (k+t-1)(n-s) - (n - s - t^2 + 1)|I'| - (n - s)$$
$$\qquad - (n-s)(|J_{0,2}| + |L|) + \Big(\frac{13}{6}t^3 - 3t^2 + 2t\Big) + (t^2 - 2t + 1)|L|$$

$$\leq (k+t-1)(n-s) - \left(\frac{11}{6}t^2 + 2\right) - \left(\frac{11}{6}t^2 + 2\right)(|J_{0,2}| + |L|)$$
$$+ \left(\frac{13}{6}t^3 - 3t^2 + 2t\right) + (t^2 - 2t + 1)|L|$$
$$= (k+t-1)(n-s) + \frac{13}{6}t^3 - \frac{29}{6}t^2 + 2t - 2$$
$$- \left(\frac{11}{6}t^2 + 2\right)|J_{0,2}| - \left(\frac{11}{6}t^2 + 2 - t^2 + 2t - 1\right)|L|$$
$$\leq (k+t-1)(n-s) + \frac{13}{6}t^3 - \frac{29}{6}t^2 + 2t - 2$$
$$- \left(\frac{11}{6}t^2 + 2\right)\left(\frac{5}{6}t - 1\right) - \left(\frac{5}{6}t^2 + 2t + 1\right)\left(\frac{5}{6}t - 1\right)$$
$$= (k+t-1)(n-s) - \frac{2}{36}t^3 - \frac{23}{6}t^2 + \frac{9}{6}t + 1.$$

This contradicts (6.9), which completes the proof of Lemma 6.1. $\qquad\square$

Now by Lemma 6.1, there exists $v' \in W$ such that $d_H(v') + \sum_{i \in J'} |E(V(C^{(i)}), v')| \geq |J'| + t$, i.e., $\sum_{i \in J'}(|E(V(C^{(i)}), v')| - 1) \geq t - d_H(v')$. Then since $W_1 = \emptyset$, there exists $J_0' \subset J'$ with $2 \leq |J_0'| \leq t - d_H(v') \leq t$ such that $\sum_{i \in J_0'}(|E(V(C^{(i)}), v')| - 1) \geq t - d_H(v')$. We choose J_0' so that $|J_0'|$ is as small as possible. Arguing as in the proof of Lemmas 5.1 and 5.2, we see that there exist at least $\frac{5}{6}t - 1$ indices $i \in J_0'$ such that $E(a^{(i)}, V(H)) = \emptyset$ and $|E(b_p^{(i)}, W)| \geq \frac{5}{6}t^2 + 3t + 1$ for some $b_p^{(i)} \in B^{(i)}$. Set

$$J_{0,2}' = \left\{ i \in J_0' \ \Big| \ E(a^{(i)}, V(H)) = \emptyset, \text{ and there exists } b_p^{(i)} \in B^{(i)} \right.$$
$$\left. \text{such that } |E(b_p^{(i)}, W)| \geq \frac{5}{6}t^2 + 3t + 1 \right\}.$$

Thus $|J_{0,2}'| \geq \frac{5}{6}t - 1$. For $i \in J_{0,2}'$ we may assume that $b_1^{(i)}$ satisfies the condition $|E(b_1^{(i)}, W)| \geq \frac{5}{6}t^2 + 3t + 1$. By the definition of J', $J_{0,2} \cap J_{0,2}' = \emptyset$. Hence $|J_{0,2} \cup J_{0,2}'| \geq \frac{10}{6}t - 2 \geq t + 1$. Let K be a subset of $J_{0,2} \cup J_{0,2}'$ such that $|K| = t + 1$.

Lemma 6.2. *For each $i \in K$, $|E(a^{(i)}, V(C^{(j)}))| \leq 1$ for every $j \in K - \{i\}$.*

Proof. Suppose that $|E(a^{(i)}, V(C^{(j)}))| \geq 2$. Take $x \in N(a^{(i)}) \cap V(C^{(j)})$ with $x \neq b_1^{(j)}$. Then $\langle \{a^{(i)}, x\} \cup (B^{(i)} - \{b_1^{(i)}\}) \rangle$ contains a t-claw. Since $|E(b_1^{(i)}, W)| \geq \frac{5}{6}t^2 + 3t + 1$ and $|E(b_1^{(j)}, W)| \geq \frac{5}{6}t^2 + 3t + 1$, we can take disjoint subsets $X^{(i)}$ and $X^{(j)}$ of $V(H)$ such that $|X^{(i)}| = |X^{(j)}| = t$, $X^{(i)} \subset N(b_1^{(i)})$, $X^{(j)} \subset N(b_1^{(j)})$. Then each of $\langle \{b_1^{(i)}\} \cup X^{(i)} \rangle$ and $\langle \{b_1^{(j)}\} \cup X^{(j)} \rangle$ contains a t-claw, a contradiction. $\qquad\square$

We are now in a position to complete the proof for Case 1. Set $K' = \{1, 2, \ldots, k - 1\} - K$. Then $|K'| = k - 1 - (t + 1) = k - t - 2$. Since $E(a^{(i)}, V(H)) = \emptyset$ for each $i \in K$ and $\delta(G) \geq k + t - 1$, it follows from Lemma 6.2 that

$$
\begin{aligned}
\left| E(\{a^{(i)} \mid i \in K\}, \bigcup_{j \in K'} V(C^{(j)})) \right| &= \sum_{i \in K} \left| E(a^{(i)}, \bigcup_{j \in K'} V(C^{(j)})) \right| \\
&\geq |K|(k + t - 1 - t - t) \\
&= |K|(k - t - 1) \\
&= |K|(|K'| + 1) \\
&= (|K'| + 1)(t + 1).
\end{aligned}
$$

Hence there exists an index $j \in K'$ such that $|E(\{a^{(i)} \mid i \in K\}, V(C^{(j)}))| > t + 1$. This implies that there exist two edges $xa^{(l)}$ and $ya^{(m)}$ joining $V(C^{(j)})$ and $\{a^{(i)} \mid i \in K\}$ with $x, y \in V(C^{(j)})$, $x \neq y$ and $l \neq m$. Then each of $\langle \{a^{(l)}, x\} \cup (B^{(l)} - \{b_1^{(l)}\}) \rangle$ and $\langle \{a^{(m)}, y\} \cup (B^{(m)} - \{b_1^{(m)}\}) \rangle$ contains a t-claw. Since $l, m \in K \subset J_{0,2} \cup J'_{0,2}$, we can take disjoint subsets $X^{(l)}$ and $X^{(m)}$ of $V(H)$ such that $|X^{(l)}| = |X^{(m)}| = t$, $X^{(l)} \subset N(b_1^{(l)})$, $X^{(m)} \subset N(b_1^{(m)})$. Then each of $\langle \{b_1^{(l)}\} \cup X^{(l)} \rangle$ and $\langle \{b_1^{(m)}\} \cup X^{(m)} \rangle$ contains a t-claw. This contradicts the assumption that G is a counterexample. This concludes the discussion for Case 1.

§7.   Proof of the main theorem

In this section, we consider the case where $W_1 \neq \emptyset$.

Case 2: $W_1 \neq \emptyset$.

Let $v \in W_1$. By the definition of W_1, we can take a t-claw $C = C^{(i)}$ with $i \in J$ such that $d_H(v) + |E(V(C), v)| \geq t + 1$. By Lemma 2.5, we have $E(V(C), U) = \emptyset$, and hence

$$(7.1) \qquad |E(V(C), W)| = |E(V(C), V(H))| = n - s + 1 \geq \frac{11}{6}t^2 + 3.$$

Lemma 7.1. $E(a, W) = \emptyset$.

Proof. Suppose that $E(a, W) \neq \emptyset$. Then by Lemma 2.2, $|E(b, W)| \leq t$ for each $b \in B$. On the other hand, we see from Lemma 2.1 that $|E(a, W)| = |E(a, W - \{v\} - N_H(v))| + |E(a, \{v\} \cup N_H(v))| \leq (t - 1 - d_H(v)) + (1 + |N_H(v)|) = t$. Hence $|E(V(C), W)| = |E(a, W)| + |E(B, W)| \leq t + t^2$, which contradicts (7.1). $\qquad \square$

Note that it follows from Lemma 7.1 that

$$(7.2) \qquad\qquad d_H(v) + |E(B,v)| \geq t + 1.$$

Hence by Lemma 2.4,

$$(7.3) \qquad |E(b, W - \{v\} - N_H(v))| \leq t - 2 \text{ for every } b \in B,$$

which implies that

$$(7.4) \qquad\qquad |E(b, W)| \leq 2t - 2 \text{ for every } b \in B.$$

By (7.1) and Lemma 7.1, we also have

$$(7.5) \qquad\qquad |E(B, W)| \geq \frac{11}{6}t^2 + 3.$$

Hence $|E(B, W - \{v\} - N_H(v))| \geq \frac{11}{6}t^2 + 3 - |B|(1 + d_H(v)) \geq \frac{5}{6}t^2 + 3$, which together with (7.3) implies that

$$(7.6) \qquad E(B - \{b\}, W - \{v\} - N_H(v)) \neq \emptyset \text{ for every } b \in B.$$

Set $S = \{b \in B \mid |E(b, W)| \geq \frac{11}{6}t\}$. Note that $S \neq \emptyset$ by (7.5).

Case 2.1: $d_H(v) \leq \lfloor \frac{5}{6}t \rfloor$.

Take $b_p \in S$. By (7.2), $\langle \{v\} \cup N_H(v) \cup (N(v) \cap (B - \{b_p\})) \rangle$ contains a t-claw. Since $d_H(v) \leq \lfloor \frac{5}{6}t \rfloor$, it follows from the definition of S that

$$|E(b_p, W - \{v\} - N_H(v))| \geq \frac{11}{6}t - \left(1 + \left\lfloor \frac{5}{6}t \right\rfloor \right)$$
$$= t - 1.$$

Hence $\langle \{a, b_p\} \cup (N(b_p) \cap (W - \{v\} - N_H(v))) \rangle$ also contains a t-claw, a contradiction.

Case 2.2: $d_H(v) \geq \lfloor \frac{5}{6}t \rfloor + 1$.

Write $d_H(v) = \lfloor \frac{5}{6}t \rfloor + 1 + h$. Since $d_H(v) \leq t - 1$, we have $0 \leq h \leq \lceil \frac{1}{6}t \rceil - 2$. By (7.2),

$$(7.7) \qquad\qquad |E(B, v)| \geq t + 1 - \left(\left\lfloor \frac{5}{6}t \right\rfloor + 1 + h \right)$$
$$= \left\lceil \frac{1}{6}t \right\rceil - h.$$

Set $T = \{w \in N_H(v) \mid d_H(w) \geq \lfloor \frac{5}{6}t \rfloor + 1\}$.

Lemma 7.2. *We have $|E(B,w)| \leq \lceil \frac{1}{6}t \rceil$ for all $w \in T \cup \{v\}$.*

Proof. Suppose that there exists $w \in T \cup \{v\}$ such that $|E(B,w)| \geq \lceil \frac{1}{6}t \rceil + 1$. Since $d_H(w) \geq \lfloor \frac{5}{6}t \rfloor + 1$, we can take a subset X of $N_H(w)$ such that $|X| = \lfloor \frac{5}{6}t \rfloor$. Take $b_p \in S$. Then $|E(B - \{b_p\}, w)| \geq \lceil \frac{1}{6}t \rceil$. Hence $\langle \{w\} \cup X \cup (N(w) \cap (B - \{b_p\}))\rangle$ contains a t-claw. By the definition of S, $|E(b_p, W - \{w\} - X)| \geq |E(b_p, W)| - 1 - |X| \geq \frac{11}{6}t - 1 - \lfloor \frac{5}{6}t \rfloor \geq t - 1$. Hence $\langle \{a, b_p\} \cup (N(b_p) \cap (W - \{w\} - X))\rangle$ contains a t-claw, a contradiction. $\square$

Lemma 7.3. *We have $d_H(w) + |E(B,w)| \leq t$ for all $w \in N_H(v) - T$.*

Proof. Suppose that there exists $w \in N_H(v) - T$ such that $d_H(w) + |E(B,w)| \geq t + 1$. Take $b_p \in S$. Then $\langle \{w\} \cup N_H(w) \cup (N(w) \cap (B - \{b_p\}))\rangle$ contains a t-claw. Since $w \notin T$, $d_H(w) \leq \lfloor \frac{5}{6}t \rfloor$. Hence it follows from the definition of S that

$$|E(b_p, W - \{w\} - N_H(w))| \geq \frac{11}{6}t - \left(1 + \left\lfloor \frac{5}{6}t \right\rfloor\right)$$
$$= t - 1.$$

Consequently $\langle \{a, b_p\} \cup (N(b_p) \cap (W - \{w\} - N_H(w)))\rangle$ contains a t-claw, a contradiction. $\square$

By Lemmas 7.2 and 7.3,

$$|E(B, \{v\} \cup N_H(v))|$$
$$= |E(B,v)| + \sum_{w \in T} |E(B,w)| + \sum_{w \in N_H(v) - T} |E(B,w)|$$
$$\leq \left\lceil \frac{1}{6}t \right\rceil + |T| \left\lceil \frac{1}{6}t \right\rceil + \sum_{w \in N_H(v) - T} (t - d_H(w))$$
$$\leq \frac{1}{6}t + 1 + |T| \left(\frac{1}{6}t + 1\right) + |N_H(v) - T|t - \sum_{w \in N_H(v) - T} d_H(w)$$
$$= \frac{1}{6}t + 1 + \frac{1}{6}t|T| + |T| + |N_H(v)|t - t|T| - \sum_{w \in N_H(v) - T} d_H(w)$$
$$= \frac{1}{6}t + 1 - \frac{5}{6}t|T| + |T| + \left(\left\lfloor \frac{5}{6}t \right\rfloor + 1 + h\right)t - \sum_{w \in N_H(v) - T} d_H(w)$$
$$\leq \frac{5}{6}t^2 + ht + \frac{7}{6}t - \frac{5}{6}t|T| + |T| + 1 - \sum_{w \in N_H(v) - T} d_H(w).$$

Hence by (7.5),

(7.8)

$$
\begin{aligned}
&|E(B, W - \{v\} - N_H(v))| \\
&\quad = |E(B, W)| - |E(B, \{v\} \cup N_H(v))| \\
&\quad \geq \frac{11}{6}t^2 + 3 - \left(\frac{5}{6}t^2 + ht + \frac{7}{6}t - \frac{5}{6}t|T| + |T| + 1 - \sum_{w \in N_H(v) - T} d_H(w) \right) \\
&\quad = t^2 - \frac{7}{6}t - ht + \frac{5}{6}t|T| - |T| + 2 + \sum_{w \in N_H(v) - T} d_H(w).
\end{aligned}
$$

By (7.7), we can take a subset B' of $N(v) \cap B$ with $|B'| = \lceil \frac{1}{6}t \rceil - h - 1$. Then since $|T| \leq d_H(v) \leq t - 1$, $h \leq \lceil \frac{1}{6}t \rceil - 2$ and $t \geq 24$, it follows from (7.3) and (7.8) that

(7.9)

$$
\begin{aligned}
&|E(B - B', W - \{v\} - N_H(v))| \\
&\quad \geq t^2 - \frac{7}{6}t - ht + \frac{5}{6}t|T| - |T| + 2 + \sum_{w \in N_H(v) - T} d_H(w) \\
&\qquad\qquad - \left(\left\lceil \frac{1}{6}t \right\rceil - h - 1 \right)(t - 2) \\
&\quad \geq t^2 - \frac{7}{6}t - ht + \frac{5}{6}t|T| - |T| + 2 + \sum_{w \in N_H(v) - T} d_H(w) - \left(\frac{1}{6}t - h \right)(t - 2) \\
&\quad = \frac{5}{6}t^2 - \frac{5}{6}t - \frac{1}{6}t|T| + 2 - 2h + |T|(t - 1) + \sum_{w \in N_H(v) - T} d_H(w) \\
&\quad \geq \frac{5}{6}t^2 - \frac{5}{6}t - \frac{1}{6}t(t - 1) + 2 - 2h + |T|(t - 1) + \sum_{w \in N_H(v) - T} d_H(w) \\
&\quad = \frac{4}{6}t^2 - \frac{4}{6}t + 2 - 2h + |T|(t - 1) + \sum_{w \in N_H(v) - T} d_H(w) \\
&\quad \geq \frac{2}{3}t^2 - \frac{2}{3}t + 2 - 2\left(\left\lceil \frac{1}{6}t \right\rceil - 2 \right) + |T|(t - 1) + \sum_{w \in N_H(v) - T} d_H(w) \\
&\quad \geq \frac{2}{3}t^2 - t + 4 + |T|(t - 1) + \sum_{w \in N_H(v) - T} d_H(w) \\
&\quad > |T|(t - 1) + \sum_{w \in N_H(v) - T} d_H(w).
\end{aligned}
$$

On the other hand,

$$(7.10) \qquad |E(\langle\{v\} \cup N_H(v)\rangle)| + |E(N_H(v), W - \{v\} - N_H(v))|$$

$$\leq \sum_{w \in N_H(v)} d_H(w) = \sum_{w \in T} d_H(w) + \sum_{w \in N_H(v) - T} d_H(w)$$

$$\leq |T|(t-1) + \sum_{w \in N_H(v) - T} d_H(w).$$

Replace C by the t-claw with center v contained in $\langle\{v\} \cup N_H(v) \cup B'\rangle$. Let $H' = \langle(V(H) - \{v\} - N_H(v)) \cup (V(C) - B')\rangle$, and let U' be the union of the vertex sets of the K_t components of H'. Also set $W' = V(H') - U'$. Then by Lemma 7.1 (and by (7.6) if $|B'| = 1$), $\{a\} \cup (B - B')$ is not contained in a K_t components of H', which means $U' = U$. Therefore by (7.9) and (7.10), $|E(\langle W'\rangle)| + \frac{2}{t}|E(\langle U'\rangle)| > |E(\langle W\rangle)| + \frac{2}{t}|E(\langle U\rangle)|$. This contradicts the maximality of $|E(\langle W\rangle)| + \frac{2}{t}|E(\langle U\rangle)|$, which completes the proof for Case 2.2.

This completes the proof of the main theorem.

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