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Differential subordinations and superordinations for a comprehensive class of analytic functions

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Abstract. In the present investigation, we obtain some subordination and superordination results involving Hadamard product for certain normalized analytic functions in the open unit disk. Our results extend corresponding previously known results.

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§1. Introduction

Let $\mathcal{H}$ be the class of analytic functions in $U := \{z : |z| < 1\}$ and $\mathcal{H}(a, n)$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots .$$

Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Let $p, h \in \mathcal{H}$ and let $\phi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$. If p and $\phi(p(z), zp'(z), z^2p''(z); z)$ are univalent and if p satisfies the second order superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z), \tag{1.1}$$

then p is a solution of the differential superordination (1.1). (If f is subordinate to F, then F is superordinate to f.) An analytic function q is called a *subordinant* if $q \prec p$ for all p satisfying (1.1). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1.1) is said to be the best subordinant. Recently Miller and Mocanu[12] obtained conditions on h, q and ϕ for which the following implication holds:

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z).$$

For two functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z).$$

For $\alpha_j \in \mathbb{C} \quad (j = 1, 2, \ldots, l)$ and $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ $(j = 1, 2, \ldots m)$, the *generalized hypergeometric function* ${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$ is defined by the infinite series

$${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!}$$

$$(l \leq m + 1; l, m \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}),$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & (n = 0); \\ a(a+1)(a+2)\ldots(a+n-1), & (n \in \mathbb{N} := \{1, 2, 3 \ldots\}). \end{cases}$$

Corresponding to the function

$$h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := z\ {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z),$$

the Dziok-Srivastava operator [6] (see also [7, 20]) $H_m^l(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)$ is defined by the Hadamard product

$$\begin{aligned} H_m^l(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)f(z) &:= h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1} \ldots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \ldots (\beta_m)_{n-1}} \frac{a_n z^n}{(n-1)!}. \end{aligned} \tag{1.2}$$

For brevity, we write

$$H_m^l[\alpha_1]f(z) := H_m^l(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)f(z).$$

It is easy to verify from (1.2) that

$$z(H_m^l[\alpha_1]f(z))' = \alpha_1 H_m^l[\alpha_1 + 1]f(z) - (\alpha_1 - 1)H_m^l[\alpha_1]f(z). \tag{1.3}$$

Special cases of the Dziok-Srivastava linear operator includes the Hohlov linear operator [8], the Carlson-Shaffer linear operator $L(a, c)$ [5], the Ruscheweyh derivative operator D^n [18], the generalized Bernardi-Libera-Livingston linear integral operator (*cf.* [2], [9], [10]) and the Srivastava-Owa fractional derivative operators (*cf.* [16], [17]).

Using the results of Miller and Mocanu[12], Bulboacă [4] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators (see [3]). Recently many authors [1, 13, 14, 19] have used the results of Bulboacă [4] and shown some sufficient conditions applying first order differential subordinations and superordinations.

The main object of the present paper is to find sufficient condition for certain normalized analytic functions $f(z)$ in U such that $(f * \Psi)(z) \neq 0$ and f to satisfy

$$q_1(z) \prec \frac{(f * \Phi)(z)}{(f * \Psi)(z)} \prec q_2(z),$$

where q_1, q_2 are given univalent functions in U and $\Phi(z) = z + \sum_{n=2}^{\infty} \lambda_n z^n$, $\Psi(z) = z + \sum_{n=2}^{\infty} \mu_n z^n$ are analytic functions in U with $\lambda_n \geq 0$, $\mu_n \geq 0$ and $\lambda_n \geq \mu_n$. Further the results are extended to Dziok-Srivastava linear operator. Also we obtain number of known results as special cases.

§2. Subordination and Superordination Results

For our present investigation, we shall need the following:

Definition 2.1. [12] Denote by Q, the set of all functions f that are analytic and injective on $\overline{U} - E(f)$, where

$$E(f) = \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U - E(f)$.

Lemma 2.2. [11] *Let q be univalent in the unit disk U and θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set*

$$\psi(z) := zq'(z)\phi(q(z)) \quad \text{and} \quad h(z) := \theta(q(z)) + \psi(z).$$

Suppose that

1. *$\psi(z)$ is starlike univalent in U and*
2. *$Re\left\{\frac{zh'(z)}{\psi(z)}\right\} > 0$ for $z \in U$.*

If p is analytic with $p(0) = q(0)$, $p(U) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{2.1}$$

then

$$p(z) \prec q(z)$$

and q is the best dominant.

Lemma 2.3. [4] *Let q be convex univalent in the unit disk U and ϑ and φ be analytic in a domain D containing $q(U)$. Suppose that*

1. *$Re\{\vartheta'(q(z))/\varphi(q(z))\} > 0$ for $z \in U$ and*
2. *$\psi(z) = zq'(z)\varphi(q(z))$ is starlike univalent in U.*

If $p(z) \in \mathcal{H}[q(0), 1] \cap Q$, with $p(U) \subseteq D$, and $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in U and

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)), \tag{2.2}$$

then $q(z) \prec p(z)$ and q is the best subordinant.

Using Lemma 2.2, we first prove the following theorem.

Theorem 2.4. *Let $\Phi, \Psi \in \mathcal{A}$, $\gamma \neq 0$ and α, β be the complex numbers and $q(z)$ be convex univalent in U with $q(0) = 1$. Further assume that*

$$Re\left\{\frac{\beta q(z)}{\gamma} - \frac{zq'(z)}{q(z)} + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0 \quad (z \in U). \tag{2.3}$$

If $f \in \mathcal{A}$ satisfies

$$\Upsilon_1(f,\, \Phi, \Psi,\, \alpha,\, \beta,\, \gamma) \prec \alpha + \beta q(z) + \gamma\frac{zq'(z)}{q(z)}, \tag{2.4}$$

where

$$\begin{aligned}\Upsilon_1(f,\, \Phi, \Psi,\, \alpha,\, \beta,\, \gamma) := \alpha + \beta\frac{(f * \Phi)(z)}{(f * \Psi)(z)} \\ + \gamma\left[\frac{z(f * \Phi)'(z)}{(f * \Phi)(z)} - \frac{z(f * \Psi)'(z)}{(f * \Psi)(z)}\right],\end{aligned} \tag{2.5}$$

$(f * \Phi)(z) \neq 0$ and $(f * \Psi)(z) \neq 0$, then

$$\frac{(f * \Phi)(z)}{(f * \Psi)(z)} \prec q(z)$$

and q is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) := \frac{(f*\Phi)(z)}{(f*\Psi)(z)} \quad (z \in U). \tag{2.6}$$

Then the function $p(z)$ is analytic in U and $p(0) = 1$. Therefore, by making use of (2.6), we obtain

$$\alpha + \beta\frac{(f*\Phi)(z)}{(f*\Psi)(z)} + \gamma\left[\frac{z(f*\Phi)'(z)}{(f*\Phi)(z)} - \frac{z(f*\Psi)'(z)}{(f*\Psi)(z)}\right] = \alpha + \beta p(z) + \gamma\frac{zp'(z)}{p(z)}.$$

By using (2.7) in (2.4), we have

$$\alpha + \beta p(z) + \gamma\frac{zp'(z)}{p(z)} \prec \alpha + \beta q(z) + \gamma\frac{zq'(z)}{q(z)}. \tag{2.7}$$

By setting

$$\theta(w) := \alpha + \beta\omega \quad \text{and} \quad \phi(\omega) := \frac{\gamma}{\omega},$$

it can be easily observed that $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C} - \{0\}$ and that $\phi(w) \neq 0$. Hence the result now follows by an application of Lemma 2.2. $\square$

Taking $p(z) = \frac{H_m^l[\alpha_1+1](f*\Phi)(z)}{H_m^l[\alpha_1](f*\Psi)(z)}$ and $p(z) = \frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)}$ respectively we obtain the following two theorems.

Theorem 2.5. *Let $\Phi, \Psi \in \mathcal{A}$, $\gamma \neq 0$ and α, β be the complex numbers and $q(z)$ be convex univalent in Δ with $q(0) = 1$. Further assume that* (2.3) *holds true. If $f \in \mathcal{A}$ satisfies*

$$\Upsilon_2(f,\, \Phi, \Psi,\, \alpha,\, \beta, \gamma) \prec \alpha + \beta q(z) + \gamma\frac{zq'(z)}{q(z)}, \tag{2.8}$$

where

$$\Upsilon_2(f,\, \Phi, \Psi, \alpha,\, \beta, \gamma) :=$$

$$\begin{cases} \alpha + \beta\frac{H_m^l[\alpha_1+1](f*\Phi)(z)}{H_m^l[\alpha_1](f*\Psi)(z)} \\ +\gamma\left[(\alpha_1+1)\frac{H_m^l[\alpha_1+2](f*\Phi)(z)}{H_m^l[\alpha_1](f*\Phi)(z)} - \alpha_1\frac{H_m^l[\alpha_1+1](f*\Psi)(z)}{H_m^l[\alpha_1](f*\Psi)(z)} - 1\right], \end{cases} \tag{2.9}$$

then

$$\frac{H_m^l[\alpha_1+1](f*\Phi)(z)}{H_m^l[\alpha_1](f*\Psi)(z)} \prec q(z)$$

and q is the best dominant.

Theorem 2.6. *Let $\Phi, \Psi \in \mathcal{A}$, $\gamma \neq 0$ and α, β be the complex numbers and $q(z)$ be convex univalent in Δ with $q(0) = 1$. Further assume that* (2.3) *holds true. If $f \in \mathcal{A}$ satisfies*

$$\Upsilon_3(f,\, \Phi, \Psi,\, \alpha,\, \beta, \gamma) \prec \alpha + \beta q(z) + \gamma \frac{zq'(z)}{q(z)}, \tag{2.10}$$

where

$$\Upsilon_3(f,\, \Phi, \Psi, \alpha,\, \beta, \gamma) :=$$
$$\begin{cases} \alpha + \beta \frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)} \\ +\gamma \left[\alpha_1 \frac{H_m^l[\alpha_1+1](f*\Phi)(z)}{H_m^l[\alpha_1](f*\Phi)(z)} - (\alpha_1+1)\frac{H_m^l[\alpha_1+2](f*\Psi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)} + 1\right], \end{cases} \tag{2.11}$$

then

$$\frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)} \prec q(z)$$

and q is the best dominant.

When $l = 2, m = 1, \alpha_1 = a, \alpha_2 = 1$ and $\beta_1 = c$ in Theorem 2.5 and Theorem 2.6, we state the following corollaries for Carlson-Shaffer linear operator $L(a, c)$ [5].

Corollary 2.7. *Let $\Phi, \Psi \in \mathcal{A}$, $\gamma \neq 0$ and α, β be the complex numbers and $q(z)$ be convex univalent in Δ with $q(0) = 1$. Further assume that* (2.3) *holds true. If $f \in \mathcal{A}$ satisfies*

$$\Upsilon_4(f,\, \Phi, \Psi,\, \alpha,\, \beta, \gamma) \prec \alpha + \beta q(z) + \gamma \frac{zq'(z)}{q(z)}, \tag{2.12}$$

where

$$\Upsilon_4(f,\, \Phi, \Psi,\, \alpha,\, \beta, \gamma) := \begin{cases} \alpha + \beta \frac{L(a+1,c)(f*\Phi)(z)}{L(a,c)(f*\Psi)(z)} \\ +\gamma \left[(a+1)\frac{L(a+2,c)(f*\Phi)(z)}{L(a,c)(f*\Phi)(z)} - a\frac{L(a+1,c)(f*\Psi)(z)}{L(a,c)(f*\Psi)(z)} - 1\right], \end{cases} \tag{2.13}$$

then

$$\frac{L(a+1,c)(f*\Phi)(z)}{L(a,c)(f*\Psi)(z)} \prec q(z)$$

and q is the best dominant.

Corollary 2.8. *Let $\Phi, \Psi \in \mathcal{A}$, $\gamma \neq 0$ and α, β be the complex numbers and $q(z)$ be convex univalent in Δ with $q(0) = 1$. Further assume that* (2.3) *holds true. If $f \in \mathcal{A}$ satisfies*

$$\Upsilon_5(f,\, \Phi, \Psi,\, \alpha,\, \beta, \gamma) \prec \alpha + \beta q(z) + \gamma \frac{zq'(z)}{q(z)}, \tag{2.14}$$

where

$$\Upsilon_5(f,\, \Phi, \Psi,\, \alpha,\, \beta, \gamma) := \begin{cases} \alpha + \beta \frac{L(a,c)(f*\Phi)(z)}{L(a+1,c)(f*\Psi)(z)} \\ +\gamma \left[a\frac{L(a+1,c)(f*\Phi)(z)}{L(a,c)(f*\Phi)(z)} - (a+1)\frac{L(a+2,c)(f*\Psi)(z)}{L(a+1,c)(f*\Psi)(z)} + 1 \right], \end{cases} \tag{2.15}$$

then

$$\frac{L(a,c)(f*\Phi)(z)}{L(a+1,c)(f*\Psi)(z)} \prec q(z)$$

and q is the best dominant.

By fixing $\Phi(z) = \frac{z}{(1-z)^2}$ and $\Psi(z) = \frac{z}{1-z}$ in Theorem 2.4, we obtain the following corollary.

Corollary 2.9. *Let $\gamma \neq 0$, α, β be the complex numbers and q be convex univalent in U with $q(0) = 1$ and* (2.3) *holds true. If $f \in \mathcal{A}$ satisfies*

$$\alpha + (\beta - \gamma)\frac{zf'(z)}{f(z)} + \gamma \left(1 + \frac{zf''(z)}{f'(z)} \right) \prec \alpha + \beta q(z) + \gamma \frac{zq'(z)}{q(z)},$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z)$$

and q is the best dominant.

Specializing the values of $\alpha = 1$, $\beta = 0$, $q(z) = \frac{1}{(1-z)^{2b}}$ $(b \in C - \{0\})$, $\gamma = \frac{1}{b}$, $\Phi(z) = \frac{z}{1-z}$ and $\Psi(z) = z$ in Theorem 2.4, we have the following corollary as stated in [21].

Corollary 2.10. *Let b be a non zero complex number. If $f \in \mathcal{A}$ and*

$$1 + \frac{1}{b}\left[\frac{zf'(z)}{f(z)} - 1 \right] \prec \frac{1+z}{1-z},$$

then

$$\frac{f(z)}{z} \prec \frac{1}{(1-z)^{2b}}$$

and $\frac{1}{(1-z)^{2b}}$ is the best dominant.

Similarly for $\alpha = 1$, $\beta = 0$, $\gamma = \frac{1}{b}$, $q(z) = \frac{1}{(1-z)^{2b}}$ $\quad (b \in C - \{0\})$, $\Phi(z) = \frac{z}{(1-z)^2}$ and $\Psi(z) = z$ in Theorem 2.4, we have the following corollary as stated in [21].

Corollary 2.11. *Let b be a non zero complex number. If $f \in \mathcal{A}$ and*

$$1 + \frac{1}{b}\left[\frac{zf''(z)}{f'(z)}\right] \prec \frac{1+z}{1-z},$$

then

$$f'(z) \prec \frac{1}{(1-z)^{2b}}$$

and $\frac{1}{(1-z)^{2b}}$ is the best dominant.

Remark 2.12. For the choices $\Phi(z) = \frac{z}{(1-z)^2}$, $\Psi(z) = \frac{z}{(1-z)}$, $\alpha = 0$, $\beta > -1$, $\gamma = 1$ and $q(z) = \frac{k}{k+z}$ $(k > 1)$ in Theorem 2.4, we get the result obtained by Obradovic et.al., [15].

By taking $l = 2$, $m = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$ and $\beta_1 = 1$ in Theorem 2.5 and Theorem 2.6, we state the following corollaries.

Corollary 2.13. *Let $\Phi, \Psi \in \mathcal{A}$, $\gamma \neq 0$, α, β be the complex numbers and q be convex univalent in U with $q(0) = 1$ and (2.3) holds true. If $f \in \mathcal{A}$ satisfies*

$$(\alpha+\gamma) + \beta\frac{z(f*\Phi)'(z)}{(f*\Psi)(z)} + \gamma\left[\frac{z(f*\Phi)''(z)}{(f*\Phi)'(z)} - \frac{z(f*\Psi)'(z)}{(f*\Psi)(z)}\right] \prec \alpha + \beta q(z) + \gamma\frac{zq'(z)}{q(z)}$$

with $(f*\Psi)(z) \neq 0$ and $(f*\Phi)'(z) \neq 0$, then

$$\frac{z(f*\Phi)'(z)}{(f*\Psi)(z)} \prec q(z)$$

and q is the best dominant.

Corollary 2.14. *Let $\Phi, \Psi \in \mathcal{A}$ and $\gamma \neq 0$, α, β be the complex numbers. Let q be convex univalent in U with $q(0) = 1$ and (2.3) holds true. If $f \in \mathcal{A}$ satisfies*

$$(\alpha-\gamma) + \beta\frac{(f*\Phi)(z)}{z(f*\Psi)'(z)} + \gamma\left[\frac{z(f*\Phi)'(z)}{(f*\Phi)(z)} - \frac{z(f*\Psi)''(z)}{(f*\Psi)'(z)}\right] \prec \alpha + \beta q(z) + \gamma\frac{zq'(z)}{q(z)}$$

with $(f*\Phi)(z) \neq 0$ and $(f*\Psi)'(z) \neq 0$, then

$$\frac{(f*\Phi)(z)}{z(f*\Psi)'(z)} \prec q(z)$$

and q is the best dominant.

By fixing $\Phi(z) = \frac{z}{1-z}$ and $\Psi(z) = \frac{z}{1-z}$ in Theorem 2.5 and Theorem 2.6 we obtain the following corollaries.

Corollary 2.15. *Let $\gamma \neq 0$, α, β be the complex numbers and q be convex univalent in U with $q(0) = 1$ and (2.3) holds true. If $f \in \mathcal{A}$ satisfies*

$$\alpha + \beta \frac{H_m^l[\alpha_1 + 1]f(z)}{H_m^l[\alpha_1]f(z)} + \gamma \left[(\alpha_1 + 1)\frac{H_m^l[\alpha_1 + 2]f(z)}{H_m^l[\alpha_1]f(z)} - \alpha_1 \frac{H_m^l[\alpha_1 + 1]f(z)}{H_m^l[\alpha_1]f(z)} - 1\right] \prec \alpha + \beta q(z) + \gamma \frac{zq'(z)}{q(z)} ,$$

then

$$\frac{H_m^l[\alpha_1 + 1]f(z)}{H_m^l[\alpha_1]f(z)} \prec q(z)$$

and q is the best dominant.

Corollary 2.16. *Let $\gamma \neq 0$, α, β be the complex numbers and q be convex univalent in U with $q(0) = 1$ and (2.3) holds true. If $f \in \mathcal{A}$ satisfies*

$$\alpha + \beta \frac{H_m^l[\alpha_1]f(z)}{H_m^l[\alpha_1 + 1]f(z)} + \gamma \left[\alpha_1 \frac{H_m^l[\alpha_1 + 1]f(z)}{H_m^l[\alpha_1]f(z)} - (\alpha_1 + 1)\frac{H_m^l[\alpha_1 + 2]f(z)}{H_m^l[\alpha_1 + 1]f(z)} + 1\right] \prec \alpha + \beta q(z) + \gamma \frac{zq'(z)}{q(z)} ,$$

then

$$\frac{H_m^l[\alpha_1]f(z)}{H_m^l[\alpha_1 + 1]f(z)} \prec q(z)$$

and q is the best dominant.

By fixing $\Phi(z) = \frac{z}{1-z}$ and $\Psi(z) = \frac{z}{1-z}$ in Corollary 2.13, Corollary 2.14 and also $l = 2$, $m = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$ and $\beta_1 = 1$ in Corollary 2.15, Corollary 2.16 we obtain the following corollaries.

Corollary 2.17. *Let $\gamma \neq 0$, α, β be the complex numbers and q be convex univalent in U with $q(0) = 1$ and (2.3) holds true. If $f \in \mathcal{A}$ satisfies*

$$(\alpha + \gamma) + \beta \frac{zf'(z)}{f(z)} + \gamma \left[\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right] \prec \alpha + \beta q(z) + \gamma \frac{zq'(z)}{q(z)} ,$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z)$$

and q is the best dominant.

Corollary 2.18. *Let $\gamma \neq 0$, α, β be the complex numbers and q be convex univalent in U with $q(0) = 1$ and (2.3) holds true. If $f \in \mathcal{A}$ satisfies*

$$\alpha + \beta\frac{f(z)}{zf'(z)} - \gamma\left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right] \prec \alpha + \beta q(z) + \gamma\frac{zq'(z)}{q(z)} \ ,$$

then

$$\frac{f(z)}{zf'(z)} \prec q(z)$$

and q is the best dominant.

Theorem 2.19. *Let $\Phi, \Psi \in \mathcal{A}$ and $\gamma \neq 0$, α, β be the complex numbers. Let q be convex univalent in U with $q(0) = 1$. Assume that*

$$Re\ \{\overline{\gamma}\beta q(z)\} > 0. \tag{2.16}$$

Let $f \in \mathcal{A}$, $\frac{(f*\Phi)(z)}{(f*\Psi)(z)} \in H[q(0), 1] \cap Q$. Let $\Upsilon_1(f, \Phi, \Psi, \alpha, \beta, \gamma)$ be univalent in U and

$$\alpha + \beta q(z) + \gamma\frac{zq'(z)}{q(z)} \prec \Upsilon_1(f, \Phi, \Psi, \alpha, \beta, \gamma), \tag{2.17}$$

where $\Upsilon_1(f, \Phi, \Psi, \alpha, \beta, \gamma)$ is given by (2.5) with $(f*\Phi)(z) \neq 0$ and $(f*\Psi)(z) \neq 0$, then

$$q(z) \prec \frac{(f*\Phi)(z)}{(f*\Psi)(z)}$$

and q is the best subordinant.

Proof. Define the function $p(z)$ by

$$p(z) := \frac{(f*\Phi)(z)}{(f*\Psi)(z)}. \tag{2.18}$$

Simple computation from (2.18), we get,

$$\Upsilon_1(f, \Phi, \Psi, \alpha, \beta, \gamma) = \alpha + \beta p(z) + \gamma\frac{zp'(z)}{p(z)},$$

then

$$\alpha + \beta q(z) + \gamma\frac{zq'(z)}{q(z)} \prec \alpha + \beta p(z) + \gamma\frac{zp'(z)}{p(z)}.$$

By setting $\vartheta(\omega) = \alpha + \beta\omega$ and $\phi(\omega) = \frac{\gamma}{\omega}$, it is easily observed that $\vartheta(\omega)$ is analytic in $\mathbb{C}$. Also, $\phi(\omega)$ is analytic in $\mathbb{C} - \{0\}$ and that $\phi(\omega) \neq 0$.

Since $q(z)$ is convex univalent function, it follows that

$$\operatorname{Re}\left\{\frac{\vartheta'(q(z))}{\phi(q(z))}\right\} = \Re\left\{\overline{\gamma}\beta q(z)\right\} > 0, \quad z \in U.$$

Now Theorem 2.19 follows by applying Lemma 2.3. □

Theorem 2.20. *Let $\Phi, \Psi \in \mathcal{A}$. Let $\gamma \neq 0$, α and β be the complex numbers. Let q be convex univalent in U with $q(0) = 1$. Assume that (2.16) holds true. Let $f \in \mathcal{A}$, $\frac{H_m^l[\alpha_1+1](f*\Phi)(z)}{H_m^l[\alpha_1](f*\Psi)(z)} \in H[q(0),1] \cap Q$. Let $\Upsilon_2(f,\Phi,\Psi,\alpha,\beta,\gamma)$ be univalent in U and*

$$\alpha + \beta q(z) + \gamma\frac{zq'(z)}{q(z)} \prec \Upsilon_2(f,\Phi,\Psi,\alpha,\beta,\gamma), \tag{2.19}$$

where $\Upsilon_2(f,\Phi,\Psi,\alpha,\beta,\gamma)$ is given by (2.9), then

$$q(z) \prec \frac{H_m^l[\alpha_1+1](f*\Phi)(z)}{H_m^l[\alpha_1](f*\Psi)(z)}$$

and q is the best subordinant.

Theorem 2.21. *Let $\Phi, \Psi \in \mathcal{A}$. Let $\gamma \neq 0$, α and β be the complex numbers. Let q be convex univalent in U with $q(0) = 1$. Assume that (2.16) holds true. Let $f \in \mathcal{A}$, $\frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)} \in H[q(0),1] \cap Q$. Let $\Upsilon_3(f,\Phi,\Psi,\alpha,\beta,\gamma)$ be univalent in U and*

$$\alpha + \beta q(z) + \gamma\frac{zq'(z)}{q(z)} \prec \Upsilon_2(f,\Phi,\Psi,\alpha,\beta,\gamma), \tag{2.20}$$

where $\Upsilon_3(f,\Phi,\Psi,\alpha,\beta,\gamma)$ is given by (2.11), then

$$q(z) \prec \frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)}$$

and q is the best subordinant.

For the Choices of $p(z) = \frac{H_m^l[\alpha_1+1](f*\Phi)(z)}{H_m^l[\alpha_1](f*\Psi)(z)}$ and $p(z) = \frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)}$, the proofs of Theorem 2.20 and Theorem 2.21 are lines similar to the proof of Theorem 2.19, so we omitted the proofs of Theorems 2.20 and 2.21.

When $l = 2$, $m = 1$, $\alpha_1 = a$, $\alpha_2 = 1$ and $\beta_1 = c$ in Theorem 2.20 and Theorem 2.21, we state the following corollary.

Corollary 2.22. *Let $\Phi, \Psi \in \mathcal{A}$. Let $\gamma \neq 0$, α and β be the complex numbers. Let q be convex univalent in U with $q(0) = 1$ and (2.16) holds true. If $f \in \mathcal{A}$ and*

$$\alpha + \beta q(z) + \gamma \frac{zq'(z)}{q(z)} \prec \Upsilon_4(f, \Phi, \Psi, \alpha, \beta, \gamma),$$

where $\Upsilon_4(f, \Phi, \Psi, \alpha, \beta, \gamma)$ is given by (2.13), then

$$q(z) \prec \frac{L(a+1, c)(f * \Phi)(z)}{L(a, c)(f * \Psi)(z)}$$

and q is the best subordinant.

Corollary 2.23. *Let $\Phi, \Psi \in \mathcal{A}$. Let $\gamma \neq 0$, α and β be the complex numbers. Let q be convex univalent in U with $q(0) = 1$ and (2.16) holds true. If $f \in \mathcal{A}$ and*

$$\alpha + \beta q(z) + \gamma \frac{zq'(z)}{q(z)} \prec \Upsilon_5(f, \Phi, \Psi, \alpha, \beta, \gamma),$$

where $\Upsilon_5(f, \Phi, \Psi, \alpha, \beta, \gamma)$ is given by (2.15), then

$$q(z) \prec \frac{L(a, c)(f * \Phi)(z)}{L(a+1, c)(f * \Psi)(z)}$$

and q is the best subordinant.

When $l = 2$, $m = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$ and $\beta_1 = 1$ in Theorem 2.20 and Theorem 2.21, we derive the following corollaries.

Corollary 2.24. *Let $\Phi, \Psi \in \mathcal{A}$. Let $\gamma \neq 0$, α and β be the complex numbers. Let q be convex univalent in U with $q(0) = 1$ and (2.16) holds true. If $f \in \mathcal{A}$ and*

$$\alpha + \beta q(z) + \gamma \frac{zq'(z)}{q(z)} \prec (\alpha + \gamma) + \beta \frac{z(f * \Phi)'(z)}{(f * \Psi)(z)} + \gamma \left[\frac{z(f * \Phi)''(z)}{(f * \Phi)'(z)} - \frac{z(f * \Psi)'(z)}{(f * \Psi)(z)} \right]$$

with $(f * \Psi)(z) \neq 0$ and $(f * \Phi)'(z) \neq 0$, then

$$q(z) \prec \frac{z(f * \Phi)'(z)}{(f * \Psi)(z)}$$

and q is the best subordinant.

Corollary 2.25. *Let $\Phi, \Psi \in \mathcal{A}$. Let $\gamma \neq 0$, α and β be the complex numbers. Let q be convex univalent in U with $q(0) = 1$ and (2.16) holds true. If $f \in \mathcal{A}$ and*

$$\alpha + \beta q(z) + \gamma \frac{zq'(z)}{q(z)} \prec (\alpha - \gamma) + \beta \frac{(f * \Phi)(z)}{z(f * \Psi)'(z)} + \gamma \left[\frac{z(f * \Phi)'(z)}{(f * \Phi)(z)} - \frac{z(f * \Psi)''(z)}{(f * \Psi)'(z)} \right]$$

with $(f*\Phi)(z)\neq 0$ *and* $(f*\Psi)'(z)\neq 0$, *then*

$$q(z)\prec \frac{(f*\Phi)(z)}{z(f*\Psi)'(z)}$$

and q *is the best subordinant.*

By Taking $l=2$, $m=1$, $\alpha_1=1$, $\alpha_2=1$ and $\beta_1=1$ in Theorem 2.20 and Theorem 2.21 and by fixing $\Phi(z)=\Psi(z)=\frac{z}{1-z}$ in Corollary 2.24 and 2.25, we obtain the following corollaries.

Corollary 2.26. *Let* $\gamma\neq 0$, α *and* β *be the complex numbers. Let* q *be convex univalent in* U *with* $q(0)=1$ *and* (2.16) *holds true. If* $f\in\mathcal{A}$ *and*

$$\alpha+\beta q(z)+\gamma\frac{zq'(z)}{q(z)}\prec(\alpha+\gamma)+\beta\frac{zf'(z)}{f(z)}+\gamma\left[\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right],$$

then

$$q(z)\prec\frac{zf'(z)}{f(z)}$$

and q *is the best subordinant.*

Corollary 2.27. *Let* $\gamma\neq 0$, α *and* β *be the complex numbers. Let* q *be convex univalent in* U *with* $q(0)=1$ *and* (2.16) *holds true. If* $f\in\mathcal{A}$ *and*

$$\alpha+\beta q(z)+\gamma\frac{zq'(z)}{q(z)}\prec\alpha+\beta\frac{f(z)}{zf'(z)}-\gamma\left[1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right],$$

then

$$q(z)\prec\frac{f(z)}{zf'(z)}$$

and q *is the best subordinant.*

We Conclude this paper by stating the following sandwich results.

§3. Sandwich Results

Theorem 3.1. *Let* q_1 *and* q_2 *be convex univalent in* U, $\gamma\neq 0$ *and* α, β *be the complex numbers. Let* $\Phi,\Psi\in\mathcal{A}$. *Suppose* q_2 *satisfies* (2.3) *and* q_1 *satisfies* (2.16). *Moreover suppose* $\frac{(f*\Phi)(z)}{(f*\Psi)(z)}\in\mathcal{H}[1,1]\cap Q$ *and* $\Upsilon_1(f,\Phi,\Psi,\alpha,\beta,\gamma)$ *is univalent in* U. *If* $f\in\mathcal{A}$ *satisfies*

$$\alpha+\beta q_1(z)+\gamma\frac{zq_1'(z)}{q_1(z)}\prec\Upsilon_1(f,\Phi,\Psi,\alpha,\beta,\gamma)\prec\alpha+\beta q_2(z)+\gamma\frac{zq_2'(z)}{q_2(z)},$$

where $\Upsilon_1(f,\Phi,\Psi,\alpha,\beta,\gamma)$ *is given by* (2.5) *with* $(f*\Phi)(z)\neq 0$ *and* $(f*\Psi)(z)\neq 0$, *then*

$$q_1(z)\prec \frac{(f*\Phi)(z)}{(f*\Psi)(z)}\prec q_2(z)$$

and q_1, q_2 *are respectively the best subordinant and best dominant.*

By taking $q_1(z)=\frac{1+A_1z}{1+B_1z}$ $(-1\leq B_1<A_1\leq 1)$ and $q_2(z)=\frac{1+A_2z}{1+B_2z}$ $(-1\leq B_2<A_2\leq 1)$ in Theorem 3.1 we obtain the following result.

Corollary 3.2. *Let* $\Phi,\Psi\in\mathcal{A}$. *If* $f\in\mathcal{A}$, $\frac{(f*\Phi)(z)}{(f*\Psi)(z)}\in\mathcal{H}[1,1]\cap Q$ *and* $\Upsilon_1(f,\Phi,\Psi,\alpha,\beta,\gamma)$ *is univalent in U. Further*

$$\alpha+\beta\left(\frac{1+A_1z}{1+B_1z}\right)+\frac{\gamma(A_1-B_1)z}{(1+A_1z)(1+B_1z)}$$
$$\prec \Upsilon_1(f,\Phi,\Psi,\alpha,\beta,\gamma)$$
$$\prec \alpha+\beta\left(\frac{1+A_2z}{1+B_2z}\right)+\frac{\gamma(A_2-B_2)z}{(1+A_2z)(1+B_2z)}$$

where $\Upsilon_1(f,\Phi,\Psi,\alpha,\beta,\gamma)$ *is given by* (2.5) *with* $(f*\Phi)(z)\neq 0$ *and* $(f*\Psi)(z)\neq 0$, *then*

$$\frac{1+A_1z}{1+B_1z}\prec \frac{(f*\Phi)(z)}{(f*\Psi)(z)}\prec \frac{1+A_2z}{1+B_2z}$$

and $\frac{1+A_1z}{1+B_1z}$, $\frac{1+A_2z}{1+B_2z}$ *are respectively the best subordinant and best dominant.*

Theorem 3.3. *Let* q_1 *and* q_2 *be convex univalent in U,* $\gamma\neq 0$ *and* α, β *be the complex numbers. Let* $\Phi,\Psi\in\mathcal{A}$. *Suppose* q_2 *satisfies* (2.3) *and* q_1 *satisfies* (2.16). *Moreover suppose* $\frac{H_m^l[\alpha_1+1](f*\Phi)(z)}{H_m^l[\alpha_1](f*\Psi)(z)}\in\mathcal{H}[1,1]\cap Q$ *and* $\Upsilon_2(f,\Phi,\Psi,\alpha,\beta,\gamma)$ *is univalent in U. If* $f\in\mathcal{A}$ *satisfies*

$$\alpha+\beta q_1(z)+\gamma\frac{zq_1'(z)}{q_1(z)}\prec \Upsilon_2(f,\Phi,\Psi,\alpha,\beta,\gamma)\prec \alpha+\beta q_2(z)+\gamma\frac{zq_2'(z)}{q_2(z)},$$

where $\Upsilon_2(f,\Phi,\Psi,\alpha,\beta,\gamma)$ *is given by* (2.9), *then*

$$q_1(z)\prec \frac{H_m^l[\alpha_1+1](f*\Phi)(z)}{H_m^l[\alpha_1](f*\Psi)(z)}\prec q_2(z)$$

and q_1, q_2 *are respectively the best subordinant and best dominant.*

Theorem 3.4. *Let* q_1 *and* q_2 *be convex univalent in U,* $\gamma\neq 0$ *and* α, β *be the complex numbers. Let* $\Phi,\Psi\in\mathcal{A}$. *Suppose* q_2 *satisfies* (2.3) *and* q_1 *satisfies* (2.16). *Moreover suppose* $\frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)}\in\mathcal{H}[1,1]\cap Q$ *and* $\Upsilon_3(f,\Phi,\Psi,\alpha,\beta,\gamma)$ *is univalent in U. If* $f\in\mathcal{A}$ *satisfies*

$$\alpha+\beta q_1(z)+\gamma\frac{zq_1'(z)}{q_1(z)}\prec \Upsilon_3(f,\Phi,\Psi,\alpha,\beta,\gamma)\prec \alpha+\beta q_2(z)+\gamma\frac{zq_2'(z)}{q_2(z)},$$

where $\Upsilon_3(f,\Phi,\Psi,\alpha,\beta,\gamma)$ *is given by* (2.11)*, then*

$$q_1(z) \prec \frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)} \prec q_2(z)$$

and q_1, q_2 *are respectively the best subordinant and best dominant.*

By making use of Corollaries 2.7 and 2.22, we state the following corollary.

Corollary 3.5. *Let* q_1 *and* q_2 *be convex univalent in* U, $\gamma \neq 0$ *and* α, β *be the complex numbers. Let* $\Phi,\Psi \in \mathcal{A}$. *Suppose* q_2 *satisfies* (2.3) *and* q_1 *satisfies* (2.16). *Moreover suppose* $\frac{L(a+1,c)(f*\Phi)(z)}{L(a,c)(f*\Psi)(z)} \in \mathcal{H}[1,1]\cap Q$ *and* $\Upsilon_4(f,\Phi,\Psi,\alpha,\beta,\gamma)$ *is univalent in* U. *If* $f \in \mathcal{A}$ *satisfies*

$$\alpha+\beta q_1(z)+\gamma\frac{zq_1'(z)}{q_1(z)} \prec \Upsilon_4(f,\Phi,\Psi,\alpha,\beta,\gamma) \prec \alpha+\beta q_2(z)+\gamma\frac{zq_2'(z)}{q_2(z)},$$

where $\Upsilon_4(f,\Phi,\Psi,\alpha,\beta,\gamma)$ *is given by* (2.13)*, then*

$$q_1(z) \prec \frac{L(a+1,c)(f*\Phi)(z)}{L(a,c)(f*\Psi)(z)} \prec q_2(z)$$

and q_1, q_2 *are respectively the best subordinant and best dominant.*

By making use of Corollaries 2.8 and 2.23, we state the following corollary.

Corollary 3.6. *Let* q_1 *and* q_2 *be convex univalent in* U, $\gamma \neq 0$ *and* α, β *be the complex numbers. Let* $\Phi,\Psi \in \mathcal{A}$. *Suppose* q_2 *satisfies* (2.3) *and* q_1 *satisfies* (2.16). *Moreover suppose* $\frac{L(a,c)(f*\Phi)(z)}{L(a+1,c)(f*\Psi)(z)} \in \mathcal{H}[1,1]\cap Q$ *and* $\Upsilon_5(f,\Phi,\Psi,\alpha,\beta,\gamma)$ *is univalent in* U. *If* $f \in \mathcal{A}$ *satisfies*

$$\alpha+\beta q_1(z)+\gamma\frac{zq_1'(z)}{q_1(z)} \prec \Upsilon_5(f,\Phi,\Psi,\alpha,\beta,\gamma) \prec \alpha+\beta q_2(z)+\gamma\frac{zq_2'(z)}{q_2(z)},$$

where $\Upsilon_5(f,\Phi,\Psi,\alpha,\beta,\gamma)$ *is given by* (2.15)*, then*

$$q_1(z) \prec \frac{L(a,c)(f*\Phi)(z)}{L(a+1,c)(f*\Psi)(z)} \prec q_2(z)$$

and q_1, q_2 *are respectively the best subordinant and best dominant.*

By making use of Corollaries 2.13 and 2.24, we state the following corollary.

Corollary 3.7. *Let* q_1 *and* q_2 *be convex univalent in* U, $\gamma \neq 0$ *and* α, β *be the complex numbers. Let* $\Phi,\Psi \in \mathcal{A}$. *Suppose* q_2 *satisfies* (2.3) *and* q_1 *satisfies*

(2.16). *Moreover suppose* $\frac{z(f*\Phi)'(z)}{(f*\Psi)(z)} \in \mathcal{H}[1,1] \cap Q$ *and* $(\alpha+\gamma)+\beta\frac{z(f*\Phi)'(z)}{(f*\Psi)(z)}+\gamma\left[\frac{z(f*\Phi)''(z)}{(f*\Phi)'(z)}-\frac{z(f*\Psi)'(z)}{(f*\Psi)(z)}\right]$ *is univalent in* U. *If* $f \in \mathcal{A}$ *satisfies*

$$\begin{aligned}\alpha+\beta q_1(z)+\gamma\frac{zq_1'(z)}{q_1(z)} \\ \prec (\alpha+\gamma)+\beta\frac{z(f*\Phi)'(z)}{(f*\Psi)(z)}+\gamma\left[\frac{z(f*\Phi)''(z)}{(f*\Phi)'(z)}-\frac{z(f*\Psi)'(z)}{(f*\Psi)(z)}\right] \\ \prec \alpha+\beta q_2(z)+\gamma\frac{zq_2'(z)}{q_2(z)},\end{aligned}$$

with $(f*\Psi)(z) \neq 0$ *and* $(f*\Phi)'(z) \neq 0$, *then*

$$q_1(z) \prec \frac{z(f*\Phi)'(z)}{(f*\Psi)(z)} \prec q_2(z)$$

and q_1, q_2 *are respectively the best subordinant and best dominant.*

By making use of Corollaries 2.14 and 2.25, we state the following corollary.

Corollary 3.8. *Let* q_1 *and* q_2 *be convex univalent in* U, $\gamma \neq 0$ *and* α, β *be the complex numbers. Let* $\Phi, \Psi \in \mathcal{A}$. *Suppose* q_2 *satisfies* (2.3) *and* q_1 *satisfies* (2.16). *Moreover suppose* $\frac{(f*\Phi)(z)}{z(f*\Psi)'(z)} \in \mathcal{H}[1,1] \cap Q$ *and* $(\alpha-\gamma)+\beta\frac{(f*\Phi)(z)}{z(f*\Psi)'(z)}+\gamma\left[\frac{z(f*\Phi)'(z)}{(f*\Phi)(z)}-\frac{z(f*\Psi)''(z)}{(f*\Psi)'(z)}\right]$ *is univalent in* U. *If* $f \in \mathcal{A}$ *satisfies*

$$\begin{aligned}\alpha+\beta q_1(z)+\gamma\frac{zq_1'(z)}{q_1(z)} \\ \prec (\alpha-\gamma)+\beta\frac{(f*\Phi)(z)}{z(f*\Psi)'(z)}+\gamma\left[\frac{z(f*\Phi)'(z)}{(f*\Phi)(z)}-\frac{z(f*\Psi)''(z)}{(f*\Psi)'(z)}\right] \\ \prec \alpha+\beta q_2(z)+\gamma\frac{zq_2'(z)}{q_2(z)},\end{aligned}$$

with $(f*\Phi)(z) \neq 0$ *and* $(f*\Psi)'(z) \neq 0$, *then*

$$q_1(z) \prec \frac{(f*\Phi)(z)}{z(f*\Psi)'(z)} \prec q_2(z)$$

and q_1, q_2 *are respectively the best subordinant and best dominant.*

By making use of Corollaries 2.17 and 2.26, we state the following corollary.

Corollary 3.9. *Let* q_1 *and* q_2 *be convex univalent in* U, $\gamma \neq 0$ *and* α, β *be the complex numbers. Suppose* q_2 *satisfies* (2.3) *and* q_1 *satisfies* (2.16). *Moreover* *suppose* $\frac{zf'(z)}{f(z)} \in \mathcal{H}[1,1] \cap Q$ *and* $(\alpha+\gamma)+\beta\frac{zf'(z)}{f(z)}+\gamma\left[\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right]$ *is univalent in* U*. If* $f \in \mathcal{A}$ *satisfies*

$$\alpha + \beta q_1(z) + \gamma \frac{zq_1'(z)}{q_1(z)} \prec (\alpha+\gamma) + \beta\frac{zf'(z)}{f(z)} + \gamma\left[\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right] \prec \alpha + \beta q_2(z) + \gamma\frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z)$$

and q_1, q_2 *are respectively the best subordinant and best dominant.*

By making use of Corollaries 2.18 and 2.27, we state the following corollary.

Corollary 3.10. *Let* q_1 *and* q_2 *be convex univalent in* U, $\gamma \neq 0$ *and* α, β *be the complex numbers. Suppose* q_2 *satisfies* (2.3) *and* q_1 *satisfies* (2.16)*. Moreover suppose* $\frac{f(z)}{zf'(z)} \in \mathcal{H}[1,1] \cap Q$ *and* $\alpha+\beta\frac{f(z)}{zf'(z)}-\gamma\left[1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right]$ *is univalent in* U*. If* $f \in \mathcal{A}$ *satisfies*

$$\alpha + \beta q_1(z) + \gamma \frac{zq_1'(z)}{q_1(z)} \prec \alpha + \beta\frac{f(z)}{zf'(z)} - \gamma\left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right] \prec \alpha + \beta q_2(z) + \gamma\frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and q_1, q_2 *are respectively the best subordinant and best dominant.*

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