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Certain basic inequalities for submanifolds of locally conformal Kaehler space forms

Sungpyo Hong, Koji Matsumoto and Mukut Mani Tripathi

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Abstract. Certain basic inequalities, involving the squared mean curvature and one of the scalar curvature, the sectional curvature and the Ricci curvature for a submanifold of any Riemannian manifold, are obtained. Applying these results we obtain the corresponding inequalities for different kinds of submanifolds of a locally conformal Kaehler space form. Equality cases are also discussed. Finally, we also find a sufficient condition for a Lagrangian submanifold of a locally conformal Kaehler space form to be minimal.

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§1. Introduction

In [2], B.-Y. Chen recalled that one of the basic interests of submanifold theory is to establish simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold. Many famous results in differential geometry can be regarded as results in this respect. The main extrinsic invariant is the squared mean curvature and the main intrinsic invariants include the classical curvature invariants namely the scalar curvature, the sectional curvature and the Ricci curvature. There are also other important modern intrinsic invariants of (sub)manifolds introduced by B.-Y. Chen [7].

In the literature, we find several work done in establishing basic inequalities involving the squared mean curvature and one of the classical curvature invariants namely the scalar curvature, the sectional curvature and the Ricci curvature for different kind of submanifolds of real space forms and complex space forms. The first results in these directions were proved by B.-Y. Chen in

[2], [4] and [5]. To prove these kind of results, one needs an extra condition on the Riemannian curvature tensor of the ambient manifold, like its constancy in the case of real space forms and the constancy of holomorphic sectional curvature in the case of complex space forms.

On the other hand, in [5], B.-Y. Chen extends the notion of Ricci curvature to k-Ricci curvature ($2 \leq k \leq n$) in an n-dimensional Riemannian manifold. Since the notion of k-Ricci curvatures involves curvature functions that "interpolate" between the sectional curvature ($k = 2$) and the Ricci curvature ($k = n - 1$), it is natural to ask to study the role of k-Ricci curvatures in finding such inequalities for submanifolds.

Motivated by a result of B.-Y. Chen [5], a basic inequality, involving the Ricci curvature and the squared mean curvature of the submanifold of any Riemannian manifolds, was proved recently [10]. The goal was achieved by use of the concept of k-Ricci curvature.

In this paper, we find basic inequalities for a submanifold of any Riemannian manifold involving the squared mean curvature and one of the intrinsic invariants namely the scalar curvature and the sectional curvature of the submanifold. Then, we apply these results to find corresponding inequalities for different kinds of submanifolds of a locally conformal Kaehler space form. The paper is organized as follows. In section 2, we recall the definitions of Ricci curvature, k-Ricci curvature, scalar curvature, normalized scalar curvature. Then we give basic equations and definitions for a submanifolds. Section 3 contains a brief account of locally conformal Kaehler manifolds. In section 4, we find a basic inequality involving the scalar curvature and the squared mean curvature for submanifolds of a Riemannian manifold. Then, we apply this inequality to find a similar inequality for submanifolds of a locally conformal Kaehler space form. In section 5, first we establish a basic inequality involving sectional curvatures and the squared mean curvature for submanifolds of a Riemannian manifold, then by applying this inequality we find a similar inequality for submanifolds of a locally conformal Kaehler space form. In section 6, first we recall a basic inequality for submanifolds of a Riemannian manifold, which involves the Ricci curvature and the squared mean curvature of the submanifold. As an application, we find the corresponding inequality for submanifolds of a locally conformal Kaehler space form. In section 7, we find a sufficient condition for minimality of a Lagrangian submanifold of a locally conformal Kaehler space form such that the Lee form is tangential to the submanifold.

§2. Preliminaries

Let M be an n-dimensional Riemannian manifold equipped with a Riemannian metric g. The inner product of the metric g is denoted by $\langle , \rangle$. We denote the set of unit vectors in T_pM by T_p^1M; thus

$$T_p^1M = \{X \in T_pM \mid \langle X, X\rangle = 1\}.$$

Let $\{e_1, \dots, e_n\}$ be any orthonormal basis for T_p^1M. For a fixed $i \in \{1, \dots, n\}$, the *Ricci curvature* of e_i, denoted $\mathrm{Ric}\,(e_i)$, is defined by

$$\mathrm{Ric}\,(e_i) = \sum_{j\neq i}^{n} K_{ij}. \tag{2.1}$$

Let Π_k be a k-plane section of T_pM and X a unit vector in Π_k. We choose an orthonormal basis $\{e_1, \dots, e_k\}$ of Π_k such that $e_1 = X$. The Ricci curvature Ric_{Π_k} of Π_k at X is defined by [5]

$$\mathrm{Ric}_{\Pi_k}(X) = K_{12} + K_{13} + \cdots + K_{1k}. \tag{2.2}$$

$\mathrm{Ric}_{\Pi_k}(X)$ is called a *k-Ricci curvature.* The scalar curvature $\tau\,(\Pi_k)$ of the k-plane section Π_k is given by

$$\tau\,(\Pi_k) = \sum_{1\leq i<j\leq k} K_{ij}, \tag{2.3}$$

where $\{e_1, \dots, e_k\}$ is any orthonormal basis of the k-plane section Π_k. The scalar curvature $\tau(p)$ of M at p is identical with the scalar curvature of the tangent space T_pM of M at p, that is, $\tau\,(p) = \tau\,(T_pM)$. If Π_2 is a plane section, $\tau(\Pi_2)$ is simply the sectional curvature $K\,(\Pi_2)$ of Π_2. Geometrically, $\tau(\Pi_k)$ is the scalar curvature of the image $\exp_p(\Pi_k)$ of Π_k at p under the exponential map at p. We define the normalized scalar curvature $\tau_N\,(\Pi_k)$ of Π_k by

$$\tau_N\,(\Pi_k) = \frac{2\tau\,(\Pi_k)}{k\,(k-1)}. \tag{2.4}$$

The normalized scalar curvature at p is defined as [4]

$$\tau_N\,(p) = \frac{2\tau\,(p)}{n\,(n-1)}. \tag{2.5}$$

Then, we see that

$$\tau_N\,(p) = \tau_N\,(T_pM)\,.$$

Let M be an n-dimensional submanifold of an m-dimensional Riemannian manifold $\widetilde{M}$ equipped with a Riemannian metric $\widetilde{g}$. We use the inner product notation $\langle , \rangle$ for both the metrics $\widetilde{g}$ of $\widetilde{M}$ and the induced metric g on the submanifold M.

The Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y) \qquad \text{and} \qquad \widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N$$

for all $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, where $\widetilde{\nabla}$, ∇ and $\nabla^{\perp}$ are respectively the Riemannian, induced Riemannian and induced normal connections in $\widetilde{M}$, M and the normal bundle $T^{\perp}M$ of M respectively, and σ is the second fundamental form related to the shape operator A by $\langle \sigma(X,Y), N\rangle = \langle A_N X, Y\rangle$. The equation of Gauss is given by

$$(2.6) \qquad \begin{aligned} R(X,Y,Z,W) \;=\; & \widetilde{R}(X,Y,Z,W) + \langle \sigma(X,W), \sigma(Y,Z)\rangle \\ & - \langle \sigma(X,Z), \sigma(Y,W)\rangle \end{aligned}$$

for all $X, Y, Z, W \in \Gamma(TM)$, where $\widetilde{R}$ and R are the curvature tensors of $\widetilde{M}$ and M respectively.

The mean curvature vector H is given by $H = \frac{1}{n}\mathrm{trace}(\sigma)$. The submanifold M is *totally geodesic* in $\widetilde{M}$ if $\sigma = 0$, and *minimal* if $H = 0$. If $\sigma(X,Y) = g(X,Y)H$ for all $X, Y \in \Gamma(TM)$, then M is *totally umbilical.*

The *relative null space* of M at p is defined by [5]

$$\mathcal{N}_p = \{X \in T_pM \mid \sigma(X,Y) = 0 \text{ for all } Y \in T_pM\},$$

which is also known as the *kernel of the second fundamental form* at p [6].

§3. Locally conformal Kaehler space forms

A Hermitian manifold $\widetilde{M}$, equipped with a complex structure J and a Hermitian metric $\widetilde{g}$, is called a *locally conformal Kaehler manifold*, if $\widetilde{g}$ is conformal to some local Kaehler metric in the neighborhood of each point of $\widetilde{M}$, that is, if there is an open cover $\{\mathcal{U}_i\}_{i\in I}$ of $\widetilde{M}$ and a family $\{f_i\}_{i\in I}$ of C^{∞}-functions $f_i : \mathcal{U}_i \to \mathbf{R}$ so that each local metric $g_i = \exp(-2f_i)\widetilde{g}|_{\mathcal{U}_i}$ is a Kaehler metric on $\mathcal{U}_i$ [19]. Although, complex geometry deals primarily with Kaehler manifolds, there are some complex manifolds, such as for instance complex Hopf manifolds, which do not admit any global Kaehler metrics at all. For more details we refer to [13] and [9].

A necessary and sufficient condition for a Hermitian manifold to be a locally conformal Kaehler manifold is as follows.

Proposition 3.1 ([11])**.** *A Hermitian manifold $\widetilde{M}$ is a locally conformal Kaehler manifold if and only if there exists a global closed* 1*-form ω, called* *the Lee form, satisfying*

$$\begin{aligned}\left\langle\left(\widetilde{\nabla}_Z J\right) X, Y\right\rangle &= \{\omega(JX)\langle Y, Z\rangle - \omega(X)\langle JY, Z\rangle\} \\ &- \{\omega(JY)\langle X, Z\rangle - \omega(Y)\langle JX, Z\rangle\}\end{aligned}$$

for all $X, Y, Z \in \Gamma(T\widetilde{M})$, *where* $\langle,\rangle$ *denotes the inner product of the metric* $\widetilde{g}$.

On a locally conformal Kaehler manifold, a symmetric $(0,2)$-tensor $\widetilde{P}$ is defined by

$$\widetilde{P}(X, Y) = -\left(\widetilde{\nabla}_X \omega\right) Y - \omega(X)\,\omega(Y) + \frac{1}{2}\|\omega\|^2 \langle X, Y\rangle,$$

where $\|\omega\|$ denotes the length of the Lee form ω with respect to $\widetilde{g}$. The tensor field $\widetilde{P}$ is said to be *hybrid* if

$$\widetilde{P}(JX, Y) + \widetilde{P}(X, JY) = 0, \qquad X, Y \in \Gamma(T\widetilde{M}).$$

Proposition 3.2 ([13])**.** *In a locally conformal Kaehler manifold* $\widetilde{M}$ *of real dimension* $2m$, *the Ricci tensor* $\widetilde{S}$ *satisfies*

$$\widetilde{S}(JX, Y) + \widetilde{S}(X, JY) = 2(m-1)\left(\widetilde{P}(JX, Y) + \widetilde{P}(X, JY)\right)$$

for all $X, Y \in \Gamma(T\tilde{M})$. *Thus, the tensor filed* $\widetilde{P}$ *is hybrid if and only if the Ricci tensor* $\widetilde{S}$ *is hybrid.*

If the holomorphic sectional curvature of a locally conformal Kaehler manifold $\widetilde{M}$ is a real constant c, then $\widetilde{M}$ is said to be a *locally conformal Kaehler space form*, and is denoted by $\widetilde{M}(c)$. Under the assumption that $\widetilde{P}$ is hybrid, the Riemann curvature tensor $\widetilde{R}$ of $\widetilde{M}(c)$ is given by [11], [13]

$$\begin{aligned}(3.1)\quad \widetilde{R}(X, Y, Z, W) &= \frac{c}{4}\{\langle Y, Z\rangle\langle X, W\rangle - \langle X, Z\rangle\langle Y, W\rangle\} \\ &+ \frac{c}{4}\{\langle JY, Z\rangle\langle JX, W\rangle - \langle JX, Z\rangle\langle JY, W\rangle \\ &\qquad - 2\langle JX, Y\rangle\langle JZ, W\rangle\} \\ &+ \frac{3}{4}\{\langle Y, Z\rangle\widetilde{P}(X, W) - \langle X, Z\rangle\widetilde{P}(Y, W) \\ &\qquad + \widetilde{P}(Y, Z)\langle X, W\rangle - \widetilde{P}(X, Z)\langle Y, W\rangle\} \\ &- \frac{1}{4}\{\langle JY, Z\rangle\widetilde{P}(JX, W) - \langle JX, Z\rangle\widetilde{P}(JY, W) \\ &\qquad + \widetilde{P}(JY, Z)\langle JX, W\rangle - \widetilde{P}(JX, Z)\langle JY, W\rangle \\ &\qquad - 2\widetilde{P}(JX, Y)\langle JZ, W\rangle - 2\langle JX, Y\rangle\widetilde{P}(JZ, W)\}\end{aligned}$$

for all $X, Y, Z, W \in \Gamma(T\widetilde{M})$. Throughout this paper we assume that $\widetilde{P}$ is hybrid in a locally conformal Kaehler space form.

§4. Scalar curvature of submanifolds

Let M be an n-dimensional submanifold of an m-dimensional Riemannian manifold $\widetilde{M}$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space T_pM and e_r $(r = n+1, \ldots, m)$ belongs to an orthonormal basis $\{e_{n+1}, \ldots, e_m\}$ of the normal space $T_p^{\perp}M$. We put

$$\sigma_{ij}^r = \langle \sigma(e_i, e_j), e_r \rangle \quad \text{and} \quad \|\sigma\|^2 = \sum_{i,j=1}^{n} \langle \sigma(e_i, e_j), \sigma(e_i, e_j) \rangle .$$

Let K_{ij} and $\widetilde{K}_{ij}$ denote the sectional curvature of the plane section spanned by e_i and e_j at p in the submanifold M and in the ambient manifold $\widetilde{M}$ respectively. Thus, K_{ij} and $\widetilde{K}_{ij}$ are the intrinsic and extrinsic sectional curvature of the Span$\{e_i, e_j\}$ at p. In view of the equation (2.6) of Gauss, we have

$$K_{ij} = \widetilde{K}_{ij} + \sum_{r=n+1}^{m} \left(\sigma_{ii}^r \sigma_{jj}^r - (\sigma_{ij}^r)^2 \right) . \tag{4.1}$$

From (4.1) it follows that

$$2\tau(p) = 2\widetilde{\tau}(T_pM) + n^2 \|H\|^2 - \|\sigma\|^2 , \tag{4.2}$$

where

$$\widetilde{\tau}(T_pM) = \sum_{1 \le i < j \le n} \widetilde{K}_{ij}$$

denote the scalar curvature of the n-plane section T_pM in the ambient manifold $\widetilde{M}$. Thus, $\tau(p)$ and $\widetilde{\tau}(T_pM)$ are the intrinsic and extrinsic scalar curvature of the submanifold at p respectively.

In view of (4.2) it follows that for an n-dimensional submanifold M of a Riemannian manifold

$$\tau(p) \le \frac{1}{2} n^2 \|H\|^2 + \widetilde{\tau}(T_pM) \tag{4.3}$$

with equality if and only if M is totally geodesic.

Now, we recall the following algebraic Lemma.

Lemma 4.1 (Lemma 3.2, [18]). *If $a_1, \ldots, a_n$ are n $(n > 1)$ real numbers then*

$$\frac{1}{n} \left(\sum_{i=1}^{n} a_i \right)^2 \le \sum_{i=1}^{n} a_i^2 , \tag{4.4}$$

with equality holding if and only if $a_1 = a_2 = \cdots = a_n$.

Using Lemma 4.1 we shall improve the inequality (4.3). In fact, we have

Theorem 4.2. *For an n-dimensional submanifold M in a Riemannian manifold, at each point $p \in M$, we have*

$$\tau(p) \leq \frac{n(n-1)}{2} \|H\|^2 + \widetilde{\tau}(T_pM) \tag{4.5}$$

with equality if and only if p is a totally umbilical point.

Proof. We choose an orthonormal basis $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_m\}$ at p such that $e_1, \ldots, e_n$ are tangential to M at p and e_{n+1} is parallel to the mean curvature vector $H(p)$ and $e_1, \ldots, e_n$ diagonalize the shape operator $A_{e_{n+1}}$. Then the shape operators take the forms

$$A_{e_{n+1}} = \operatorname{diag}\left(\sigma_{11}^{n+1}, \sigma_{22}^{n+1}, \ldots, \sigma_{nn}^{n+1}\right), \tag{4.6}$$

$$A_{e_r} = \left(\sigma_{ij}^r\right), \quad \operatorname{trace} A_{e_r} = \sum_{i=1}^{n} \sigma_{ii}^r = 0 \tag{4.7}$$

for all $i, j = 1, \ldots, n$ and $r = n+2, \ldots, m$; and from (4.2), we get

$$2\tau(p) = 2\widetilde{\tau}(T_pM) + n^2 \|H\|^2 - \sum_{i=1}^{n} \left(\sigma_{ii}^{n+1}\right)^2 - \sum_{r=n+2}^{m} \sum_{i,j=1}^{n} (\sigma_{ij}^r)^2. \tag{4.8}$$

Using Lemma 4.1, we get

$$n \|H\|^2 \leq \sum_{i=1}^{n} \left(\sigma_{ii}^{n+1}\right)^2. \tag{4.9}$$

In view of (4.8) and (4.9), we have

$$\tau(p) \leq \frac{n(n-1)}{2} \|H\|^2 + \widetilde{\tau}(T_pM) - \frac{1}{2} \sum_{r=n+2}^{m} \sum_{i,j=1}^{n} (\sigma_{ij}^r)^2, \tag{4.10}$$

which implies (4.5). If the equality in (4.5) holds, then from Lemma 4.1 and (4.10) it follows that

$$\sigma_{11}^{n+1} = \sigma_{22}^{n+1} = \cdots = \sigma_{nn}^{n+1} \quad \text{and} \quad A_{e_r} = 0, \quad r = n+2, \ldots, m.$$

Therefore, p is a totally umbilical point. The converse is straightforward. □

Remark 4.3. Using an inequality for roots of a polynomial, B. Suceava proved Theorem 4.2 for a hypersurface (see Proposition 1, [16]). Then in general codimension case, he proved Theorem 4.2 with out any information about equality case (see Proposition 2, [16]). But our proof of Theorem 4.2 is very short and also includes equality case.

In view of Theorem 4.2, we have

Theorem 4.4. *For an n-dimensional submanifold M of a Riemannian manifold, at each point $p \in M$, we have*

$$\tau_N(p) \leq \|H\|^2 + \widetilde{\tau}_N(T_pM), \tag{4.11}$$

where τ_N is the normalized scalar curvature of M at p, and $\widetilde{\tau}_N(T_pM)$ denotes the normalized scalar curvature of T_pM in the ambient manifold $\widetilde{M}$. The equality in (4.11) *holds if and only if p is a totally umbilical point.*

Theorems 4.2 and 4.4 provide the following obstructions for a minimal immersion into a Riemannian manifold.

Theorem 4.5. *Let M be an n-dimensional submanifold of an m-dimensional Riemannian manifold $\widetilde{M}$. If the intrinsic scalar curvature* (*resp. intrinsic normalized scalar curvature*) *of M is greater than the extrinsic scalar curvature* (*resp. extrinsic normalized scalar curvature*)*, then M admits no minimal immersion into $\widetilde{M}$.*

If M is an n-dimensional submanifold of a real space form $R^m(c)$, then we have

$$2\widetilde{\tau}(T_pM) = n(n-1)c \qquad \text{and} \qquad \widetilde{\tau}_N(T_pM) = c.$$

Consequently, in view of Theorem 4.4 we have the following

Theorem 4.6 (Lemma 1, [4])**.** *Let M be an n-dimensional submanifold of a real space form $R^m(c)$. Then at each point $p \in M$, the normalized scalar curvature τ_N of M satisfies*

$$\tau_N(p) \leq \|H\|^2 + c,$$

with equality holding if and only if p is a totally umbilical point. Consequently, if the normalized scalar curvature of M is greater than c, then M admits no minimal immersion into the real space form $R^m(c)$.

Let M be a submanifold of an almost Hermitian manifold $(\widetilde{M}, J, \widetilde{g})$. For any $X \in T_pM$ we decompose JX into tangential and normal parts given by

$$JX = PX + FX, \qquad PX \in T_pM,\ FX \in T_p^{\perp}M; \tag{4.12}$$

thus PX is the tangential part of JX while FX is the normal part of JX. There are two well-known classes of submanifolds, namely, holomorphic (invariant) submanifolds and totally real (anti-invariant) submanifolds [20]. In the first case the tangent space of the submanifold remains invariant under the action of the almost complex structure J where as in the second case it is mapped into the normal space. Thus, M is invariant if $F = 0$, and it is anti-invariant if $P = 0$. The squared norm of P at $p \in M$ is defined to be $\|P\|^2 = \sum_{i,j=1}^{n} \langle Pe_i, e_j \rangle^2$, where $\{e_1, \ldots, e_n\}$ is any orthonormal basis of the tangent space T_pM.

Now, we study scalar curvature of submanifolds of locally conformal Kaehler space forms. We need the following Lemma.

Lemma 4.7. *Let M be an n-dimensional submanifold of an m-dimensional locally conformal Kaehler space form $\widetilde{M}(c)$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space T_pM and e_r belongs to an orthonormal basis $\{e_{n+1}, \ldots, e_m\}$ of the normal space $T_p^{\perp}M$. Then*

$$\widetilde{K}_{ij} = \frac{c}{4} + \frac{3c}{4} \langle Pe_i, e_j \rangle^2 + \frac{3}{4} \left\{ \widetilde{P}(e_i, e_i) + \widetilde{P}(e_j, e_j) \right\} + \frac{3}{2} \langle Pe_i, e_j \rangle \widetilde{P}(e_i, Je_j), \tag{4.13}$$

$$\begin{aligned} \widetilde{\mathrm{Ric}}_{(T_pM)}(e_i) &= (n-1)\frac{c}{4} + \frac{3c}{4} \|Pe_i\|^2 \\ &+ \frac{3}{4} \{(n-2)\widetilde{P}(e_i, e_i) + \mathrm{trace}(\widetilde{P}|_M)\} \\ &+ \frac{3}{2} \sum_{j=1}^{n} \langle Pe_i, e_j \rangle \widetilde{P}(e_i, Je_j), \end{aligned} \tag{4.14}$$

$$\begin{aligned} \widetilde{\tau}(T_pM) &= n(n-1)\frac{c}{8} + \frac{3c}{8} \|P\|^2 + \frac{3}{4}(n-1)\,\mathrm{trace}(\widetilde{P}|_M) \\ &+ \frac{3}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} \langle Pe_i, e_j \rangle \widetilde{P}(e_i, Je_j). \end{aligned} \tag{4.15}$$

Proof. Equation (4.13) follows from (3.1). Using $\widetilde{\mathrm{Ric}}_{(T_pM)}(e_i) = \sum_{j \neq i}^{n} \widetilde{K}_{ij}$ from (4.13), we get (4.14). Next, using $2\widetilde{\tau}(T_pM) = \sum_{i=1}^{n} \widetilde{\mathrm{Ric}}_{(T_pM)}(e_i)$ from (4.14), we get (4.15). □

In view of (4.15), the equation (4.2) becomes

$$\begin{aligned} 2\tau(p) &= n^2 \|H\|^2 - \|\sigma\|^2 + n(n-1)\frac{c}{4} \\ &+ \frac{3c}{4} \|P\|^2 + \frac{3}{2}(n-1)\,\mathrm{trace}(\widetilde{P}|_M) \\ &+ \frac{3}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \langle Pe_i, e_j \rangle \widetilde{P}(e_i, Je_j). \end{aligned} \tag{4.16}$$

In particular, if M is a totally real submanifold, then the equation (4.16) reduces to

$$2\tau(p) = n^2 \|H\|^2 - \|\sigma\|^2 + n(n-1)\frac{c}{4} + \frac{3}{2}(n-1)\operatorname{trace}(\widetilde{P}|_M), \tag{4.17}$$

which is the corrected version of the equation (2.2) namely

$$2\tau(p) = n^2 \|H\|^2 - \|\sigma\|^2 + \frac{1}{4}n(n-1)\left(c + 6\operatorname{trace}(\widetilde{P}|_M)\right) \tag{4.18}$$

of [14].

Next, we have

Theorem 4.8. *For an n-dimensional submanifold M of a locally conformal Kaehler space form $\widetilde{M}(c)$, at each point $p \in M$, we have*

$$\begin{aligned} \tau(p) \quad \le \quad & \frac{n(n-1)}{2}\|H\|^2 + n(n-1)\frac{c}{8} \\ & + \frac{3c}{8}\|P\|^2 + \frac{3}{4}(n-1)\operatorname{trace}(\widetilde{P}|_M) \\ & + \frac{3}{4}\sum_{i=1}^{n}\sum_{j=1}^{n}\langle Pe_i, e_j\rangle \widetilde{P}(e_i, Je_j), \end{aligned} \tag{4.19}$$

with equality if and only if p is a totally umbilical point.

Proof. Using (4.15) in (4.5) gives (4.19). □

Putting $P = 0$ in (4.19), we immediately get the following

Corollary 4.9. *For an n-dimensional totally real submanifold M of a locally conformal Kaehler space form $\widetilde{M}(c)$*

$$\tau(p) \le \frac{n(n-1)}{2}\|H\|^2 + n(n-1)\frac{c}{8} + \frac{3}{4}(n-1)\operatorname{trace}(\widetilde{P}|_M) \tag{4.20}$$

with equality if and only if p is a totally umbilical point.

We also have

Corollary 4.10. *If M is an n-dimensional invariant submanifold of a locally conformal Kaehler space form $\widetilde{M}(c)$, then at each point $p \in M$ it follows that*

$$\begin{aligned} \tau(p) \quad \le \quad & \frac{n(n-1)}{2}\|H\|^2 + n(n+2)\frac{c}{8} + \frac{3}{4}(n-1)\operatorname{trace}(\widetilde{P}|_M) \\ & + \frac{3}{4}\sum_{i=1}^{n}\sum_{j=1}^{n}\langle Pe_i, e_j\rangle \widetilde{P}(e_i, Je_j) \end{aligned} \tag{4.21}$$

with equality if and only if p is a totally umbilical point.

§5. Sectional curvature of submanifolds

First, we recall the following Lemma.

Lemma 5.1 (Lemma 3.1, [2])**.** *If $n \geq 2$ and $a_1, \ldots, a_n, a$ are real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + a\right), \tag{5.1}$$

then $2a_1a_2 \geq a$, with equality holding if and only if $a_1 + a_2 = a_3 = \cdots = a_n$.

Now, we establish an inequality for submanifolds M of a Riemannian manifold involving intrinsic invariants, namely the sectional curvature and the scalar curvature of M; and the main extrinsic invariant, namely the squared mean curvature as follows:

Theorem 5.2. *Let M be an n-dimensional $(n \geq 3)$ submanifold of an m-dimensional Riemannian manifold $\widetilde{M}$. Then, for each point $p \in M$ and each plane section $\Pi_2 \subset T_pM$, we have*

$$\tau - K(\Pi_2) \leq \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + \widetilde{\tau}(T_pM) - \widetilde{K}(\Pi_2). \tag{5.2}$$

The equality in (5.2) *holds at $p \in M$ if and only if there exist an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1}, \ldots, e_m\}$ of $T_p^{\perp}M$ such that* **(a)** *$\Pi_2 = \mathrm{Span}\{e_1, e_2\}$ and* **(b)** *the forms of shape operators $A_r \equiv A_{e_r}$, $r = n+1, \ldots, m$, become*

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & (a+b)I_{n-2} \end{pmatrix}, \tag{5.3}$$

$$A_r = \begin{pmatrix} c_r & d_r & 0 \\ d_r & -c_r & 0 \\ 0 & 0 & 0_{n-2} \end{pmatrix}, \qquad r \in \{n+2, \ldots, m\}. \tag{5.4}$$

Proof. Let $\Pi_2 \subset T_pM$ be a plane section. We choose an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ for T_pM and $\{e_{n+1}, \ldots, e_m\}$ for the normal space $T_p^{\perp}M$ at p such that $\Pi_2 = \mathrm{Span}\{e_1, e_2\}$ and the mean curvature vector H is in the direction of the normal vector to e_{n+1}. We rewrite (4.2) as

$$\begin{aligned} &\left(\sum_{i=1}^{n} \sigma_{ii}^{n+1}\right)^2 \\ = \; &(n-1)\left(\sum_{i=1}^{n} \left(\sigma_{ii}^{n+1}\right)^2 + \sum_{i \neq j} \left(\sigma_{ij}^{n+1}\right)^2 + \sum_{r=n+2}^{m} \sum_{i,j=1}^{n} \left(\sigma_{ij}^{r}\right)^2 + \Upsilon\right), \end{aligned} \tag{5.5}$$

where

$$\Upsilon = 2\tau - 2\widetilde{\tau}\,(T_pM) - \frac{n^2(n-2)}{n-1}\,\|H\|^2\,. \tag{5.6}$$

Applying Lemma 5.1 to (5.5), we get

$$2\sigma_{11}^{n+1}\sigma_{22}^{n+1} \geq \Upsilon + \sum_{i\neq j}\left(\sigma_{ij}^{n+1}\right)^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}\left(\sigma_{ij}^{r}\right)^2. \tag{5.7}$$

From equation (4.1) it also follows that

$$K\,(\Pi_2) = \widetilde{K}\,(\Pi_2) + \sigma_{11}^{n+1}\sigma_{22}^{n+1} - \left(\sigma_{12}^{n+1}\right)^2 + \sum_{r=n+2}^{m}\left(\sigma_{11}^{r}\sigma_{22}^{r} - (\sigma_{12}^{r})^2\right). \tag{5.8}$$

From (5.7) and (5.8) we have

$$\begin{aligned} K(\Pi_2) \;\geq\; & \widetilde{K}\,(\Pi_2) + \frac{1}{2}\Upsilon + \sum_{r=n+1}^{m}\sum_{j>2}\{(\sigma_{1j}^{r})^2 + (\sigma_{2j}^{r})^2\} \\ & + \frac{1}{2}\sum_{i\neq j>2}(\sigma_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j>2}(\sigma_{ij}^{r})^2 + \frac{1}{2}\sum_{r=n+2}^{m}(\sigma_{11}^{r}+\sigma_{22}^{r})^2, \end{aligned} \tag{5.9}$$

or

$$K(\Pi_2) \geq \widetilde{K}\,(\Pi_2) + \frac{1}{2}\Upsilon. \tag{5.10}$$

In view of (5.6) and (5.10), we get (5.2).

If the equality in (5.2) holds, then the inequalities given by (5.7) and (5.9) become equalities. In this case, we have

$$\left\{\begin{array}{l} \sigma_{1j}^{n+1} = 0,\ \sigma_{2j}^{n+1} = 0,\ \sigma_{ij}^{n+1} = 0, \quad i \neq j > 2; \\ \sigma_{1j}^{r} = \sigma_{2j}^{r} = \sigma_{ij}^{r} = 0,\ r = n+2,\dots,m; \quad i,j = 3,\dots,n; \\ \sigma_{11}^{n+2} + \sigma_{22}^{n+2} = \cdots = \sigma_{11}^{m} + \sigma_{22}^{m} = 0. \end{array}\right. \tag{5.11}$$

Now, we choose e_1 and e_2 so that $\sigma_{12}^{n+1} = 0$. Applying Lemma 5.1 we also have

$$\sigma_{11}^{n+1} + \sigma_{22}^{n+1} = \sigma_{33}^{n+1} = \cdots = \sigma_{nn}^{n+1}. \tag{5.12}$$

Thus, after choosing a suitable orthonormal basis $\{e_1,\dots,e_m\}$, the shape operator of M becomes of the form given by (5.3) and (5.4). The converse is easy to follow. □

If M is an n-dimensional submanifold in a real space form $R^m(c)$, then we have

$$\widetilde{\tau}(T_pM) - \widetilde{K}(\Pi_2) = \frac{1}{2}n(n-1)c - c = \frac{1}{2}(n+1)(n-2)c.$$

Then recalling the Chen invariant [2] $\delta_M(p) = \tau(p) - (\inf K)(p)$, in view of Theorem 5.2 we have a sharp inequality for submanifolds M in a real space form involving intrinsic invariant, namely Chen invariant of M; and the main extrinsic invariant, namely the squared mean curvature as follows:

Theorem 5.3 (Lemma 3.2, [2])**.** *Let M be an n-dimensional $(n \geq 3)$ submanifold of a real space form $R^m(c)$. Then*

$$\delta_M \equiv \tau - \inf K \leq \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{1}{2}(n+1)(n-2)c.$$

Equality holds if and only if, with respect to suitable orthonormal frame fields $e_1, \ldots, e_n$, $e_{n+1}, \ldots, e_m$, the forms of the shape operators $A_r = A_{e_r}$, $r = n+1, \ldots, m$ become (5.3) *and* (5.4)*.*

Theorem 5.3 is an improvement of a result of [8]. B.-Y. Chen also established similar inequality in Theorem 2 of [3] for a submanifold of a complex space form. Now, we apply Theorem 5.2, to get a similar results for submanifolds of locally conformal Kaehler space forms.

Theorem 5.4. *Let M be an n-dimensional $(n \geq 3)$ submanifold of a locally conformal Kaehler space form $\widetilde{M}(c)$. Then, for each point $p \in M$ and each plane section $\Pi_2 = \mathrm{Span}\{e_1, e_2\} \subset T_pM$, we have*

$$\begin{aligned}(5.13) \qquad \tau - K(\Pi_2) \leq\ & \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{1}{8}(n+1)(n-2)c \\ &+ \frac{3c}{8}\|P\|^2 - \frac{3c}{4}\langle Pe_1, e_2\rangle^2 \\ &+ \frac{3}{4}(n-1)\,\mathrm{trace}(\widetilde{P}|_M) - \frac{3}{4}\mathrm{trace}(\widetilde{P}|_{\Pi_2}) \\ &+ \frac{3}{4}\sum_{i=1}^{n}\sum_{j=1}^{n}\langle Pe_i, e_j\rangle\,\widetilde{P}(e_i, Je_j) \\ &- \frac{3}{2}\langle Pe_1, e_2\rangle\,\widetilde{P}(e_1, Je_2).\end{aligned}$$

The equality in (5.13) *holds at $p \in M$ if and only if there exist an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1}, \ldots, e_m\}$ of $T_p^{\perp}M$ such that the shape operators $A_r \equiv A_{e_r}$, $r = n+1, \ldots, m$, become of forms* (5.3) *and* (5.4)*.*

Proof. If M is an n-dimensional submanifold of a locally conformal Kaehler space form $\widetilde{M}(c)$, then from (4.13) and (4.15) we get

$$\begin{aligned}\widetilde{\tau}(T_pM) - \widetilde{K}(\Pi_2) &= \frac{1}{8}(n+1)(n-2)c \\ &+ \frac{3c}{8}\|P\|^2 - \frac{3c}{4}\langle Pe_1, e_2\rangle^2 \\ &+ \frac{3}{4}(n-1)\operatorname{trace}(\widetilde{P}|_M) - \frac{3}{4}\operatorname{trace}(\widetilde{P}|_{\Pi_2}) \\ &+ \frac{3}{4}\sum_{i=1}^{n}\sum_{j=1}^{n}\langle Pe_i, e_j\rangle \widetilde{P}(e_i, Je_j) \\ &- \frac{3}{2}\langle Pe_1, e_2\rangle \widetilde{P}(e_1, Je_2).\end{aligned}$$

Using the above equation in (5.2), we get (5.13). □

Theorem 5.5. *Let M be an n-dimensional $(n \geq 3)$ totally real submanifold of a locally conformal Kaehler space form $\widetilde{M}(c)$. Then, for each point $p \in M$ and each plane section $\Pi_2 \subset T_pM$, we have*

$$\begin{aligned}\tau - K(\Pi_2) \leq\ & \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{1}{8}(n+1)(n-2)c \\ &+ \frac{3}{4}(n-1)\operatorname{trace}(\widetilde{P}|_M) - \frac{3}{4}\operatorname{trace}(\widetilde{P}|_{\Pi_2}).\end{aligned} \tag{5.14}$$

The equality in (5.14) *holds at $p \in M$ if and only if there exist an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1}, \ldots, e_m\}$ of $T_p^{\perp}M$ such that* **(a)** $\Pi_2 = \operatorname{Span}\{e_1, e_2\}$ *and* **(b)** *the shape operators $A_r \equiv A_{e_r}$, $r = n+1, \ldots, m$, become of forms* (5.3) *and* (5.4).

Proof. Put $P = 0$ in (5.13). □

Remark 5.6. The inequality (5.14) is different from the inequality (2.3) in [14]. Instead of (4.17) the equation (4.18) is used in [14].

§6. Ricci curvature of submanifolds

First, we recall the Ricci inequality (6.1) in the following.

Theorem 6.1 ([10])**.** *Let M be an n-dimensional submanifold of a Riemannian manifold. Then, the following statements are true.*

(a) *For $X \in T_p^1 M$, it follows that*

$$\mathrm{Ric}\,(X) \leq \frac{1}{4}\, n^2 \|H\|^2 + \widetilde{\mathrm{Ric}}_{(T_pM)}\,(X)\,, \tag{6.1}$$

where $\widetilde{\mathrm{Ric}}_{(T_pM)}\,(X)$ is the n-Ricci curvature of T_pM at $X \in T_p^1 M$ with respect to the ambient manifold $\widetilde{M}$.

(b) *The equality case of* (6.1) *is satisfied by $X \in T_p^1 M$ if and only if*

$$\begin{cases} \sigma\,(X, Y) = 0, & \text{for all } Y \in T_pM \text{ orthogonal to } X, \\ 2\sigma\,(X, X) = nH\,(p)\,. \end{cases} \tag{6.2}$$

(c) *The equality case of* (6.1) *holds for all $X \in T_p^1 M$ if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

Proof. We put

$$\sigma''(X, Y) = \sigma(X, Y) - \frac{n}{2} g(X, Y) H$$

for any $X, Y \in T_pM$. Then for $X \in T_p^1 M$, we obtain

$$\begin{aligned} 0 \quad &\leq \quad \sum_{i=1}^{n} < \sigma''(X, e_i), \sigma''(X, e_i) > \\ &= \quad \sum_{i=1}^{n} < \sigma(X, e_i), \sigma(X, e_i) > -n < H, \sigma(X, X) > + \frac{n^2}{4} \parallel H \parallel^2 . \end{aligned}$$

According to the Gauss equation (2.6) and the above inequality, we can easily get our theorem. □

We immediately have the following

Corollary 6.2. *Let M be an n-dimensional submanifold of a Riemannian manifold. Then for $X \in T_p^1 M$ any two of the following three statements imply the remaining one.*

(a) *X satisfies the equality case of* (6.1)*.*

(b) *$H(p) = 0$.*

(c) *$X \in \mathcal{N}_p$.*

Now, we establish a basic relationship between the Ricci curvature and the squared mean curvature for a submanifold of a locally conformal Kaehler space form.

Theorem 6.3. *Let M be an n-dimensional submanifold of a locally conformal Kaehler space form $\widetilde{M}(c)$. Then the following statement are true.*

(a) *If $X \in T_p^1 M$, then*

$$\begin{aligned} 4\mathrm{Ric}(X) \leq\ & n^2\|H\|^2 + (3\|PX\|^2 + n - 1)c \\ & + 3(n-2)\widetilde{P}(X,X) + 3\,\mathrm{trace}(\widetilde{P}|_M) \\ & + 6\sum_{j=1}^{n} \langle PX, e_j\rangle\, \widetilde{P}(X, Je_j). \end{aligned} \tag{6.3}$$

(b) *If M is an invariant submanifold, then for any $X \in T_p^1 M$ it follows that*

$$\begin{aligned} 4\mathrm{Ric}(X) \leq\ & n^2\|H\|^2 + (n+2)c + 3(n-2)\widetilde{P}(X,X) \\ & + 3\,\mathrm{trace}(\widetilde{P}|_M) + 6\sum_{j=1}^{n} \langle PX, e_j\rangle\, \widetilde{P}(X, Je_j). \end{aligned} \tag{6.4}$$

(c) *If M is a totally real submanifold, then for any $X \in T_p^1 M$ it follows that*

$$4\mathrm{Ric}(X) \leq n^2\|H\|^2 + (n-1)c + 3(n-2)\widetilde{P}(X,X) + 3\,\mathrm{trace}(\widetilde{P}|_M). \tag{6.5}$$

(d) *If $H(p) = 0$, then $X \in T_p^1 M$ satisfies the equality cases of the inequalities* (6.3)*,* (6.4) *and* (6.5) *if and only if $X \in \mathcal{N}_p$.*

(e) *The equality cases of the inequalities* (6.3)*,* (6.4) *and* (6.5) *are satisfied for all $X \in T_p^1 M$ if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

Proof. Using (4.14) in the Ricci inequality (6.1), we find the inequality (6.3). If M is an invariant submanifold of an almost Hermitian manifold, then for a unit vector $X \in T_p M$, $\|PX\| = 1$. Using this in (6.3) gives (6.4). Putting $P = 0$ in (6.3), we get (6.5). Rest of the proof is straightforward. □

Remark 6.4. The inequality (6.5) in Theorem 6.3 is different from the inequality (3.1), namely

$$4\mathrm{Ric}(X) \leq n^2\|H\|^2 + (n-1)\,c + 3\widetilde{P}(X,X) + 3\left(n^2 - n - 1\right)\mathrm{trace}(\widetilde{P}|_M)$$

in the Theorem 1 of [15] and the inequality (6.1), namely

$$4\mathrm{Ric}(X) \leq n^2\|H\|^2 + (n-1)\,c + 6\,(n-1)\,\mathrm{trace}(\widetilde{P}|_M)$$

in the Theorem 6.1 of [1]. Instead of (4.17) the equation (4.18) is used in [1] and [15].

§7. Minimality of Lagrangian submanifolds

It is well known that an invariant submanifold of a Kaehler manifold is always minimal. In [12], K. Matsumoto proved that an invariant submanifold of a locally conformal Kaehler manifold is minimal if and only if the Lee form is tangent to the submanifold. In [6], B.-Y. Chen proved an inequality for maximum Ricci curvature for Lagrangian submanifolds of complex space forms, and proved that in the equality case the Lagrangian submanifolds must be minimal. In this section, we prove the following result for a Lagrangian submanifold M of a locally conformal Kaehler space form.

Theorem 7.1. *Let M be a Lagrangian submanifold of a $2n$-dimensional locally conformal Kaehler space form such that the Lee form is tangential to the submanifold. If the equality case of* (6.5) *is satisfied by a unit vector at every point of M, then M is a minimal submanifold.*

Proof. Note that if M is a Lagrangian submanifold of a locally conformal Kaehler manifold such that the Lee form is tangential to the submanifold, then

$$A_{FX}Y = A_{FY}X, \qquad X, Y \in T_pM. \tag{7.1}$$

Choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM such that e_1 satisfies the equality case of (6.5) at $p \in M$. Then, $\{e_{n+1}, \ldots, e_{2n}\}$ is an orthonormal basis of $T_p^\perp M$ such that $e_{n+j} = Fe_j$, $j \in \{1, \ldots, n\}$. From the second equation of (6.2), we get

$$2\sigma(e_1, e_1) = \sigma(e_1, e_1) + \sigma(e_2, e_2) + \cdots + \sigma(e_n, e_n),$$

which shows that

$$\sigma(e_1, e_1) = \sum_{j=2}^{n} \sigma(e_j, e_j). \tag{7.2}$$

From first equation of (6.2), we get

$$\sigma(e_1, e_j) = 0, \qquad j = 2, \ldots, n. \tag{7.3}$$

Let $Y = \sum_{j=1}^{n} a_j e_{n+j} = \sum_{j=1}^{n} a_j F e_j$ be an arbitrary vector in $T_p^\perp M$. Then using (7.2), (7.1) and (7.3) we obtain

$$\begin{aligned}
\langle \sigma(e_1, e_1), Y \rangle &= a_1 \langle \sigma(e_1, e_1), Fe_1 \rangle + \sum_{j=2}^{n} a_j \langle \sigma(e_1, e_1), Fe_j \rangle \\
&= a_1 \left\langle \sum_{j=2}^{n} \sigma(e_j, e_j), Fe_1 \right\rangle + \sum_{j=2}^{n} a_j \langle \sigma(e_1, e_1), Fe_j \rangle \\
&= a_1 \sum_{j=2}^{n} \langle A_{Fe_1} e_j, e_j \rangle + \sum_{j=2}^{n} a_j \langle A_{Fe_j} e_1, e_1 \rangle \\
&= a_1 \sum_{j=2}^{n} \langle A_{Fe_j} e_1, e_j \rangle + \sum_{j=2}^{n} a_j \langle A_{Fe_1} e_j, e_1 \rangle \\
&= a_1 \sum_{j=2}^{n} \langle \sigma(e_1, e_j), Fe_j \rangle + \sum_{j=2}^{n} a_j \langle \sigma(e_1, e_j), Fe_1 \rangle \\
&= a_1 \sum_{j=2}^{n} \langle 0, Fe_j \rangle + \sum_{j=2}^{n} a_j \langle 0, Fe_1 \rangle = 0.
\end{aligned}$$

Thus we get $\sigma(e_1, e_1) = 0$, which in view of the second equation of (6.2) shows that $H(p) = 0$. □

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Sungpyo Hong
Department of Mathematics, Pohang University of Science and Technology (POSTECH)
Pohang 790-784, Republic of Korea
E-mail: sungpyo@postech.ac.kr

Koji Matsumoto
Department of Mathematics, Faculty of Education, Yamagata University
Yamagata 990, Japan
E-mail: ej192@kdw.kj.yamagata-u.ac.jp

Mukut Mani Tripathi
Department of Mathematics, Pohang University of Science and Technology (POSTECH)
Pohang 790-784, Republic of Korea
Permanent Address:
Department of Mathematics and Astronomy, Lucknow University
Lucknow 226 007, India
E-mail: mmtripathi66@yahoo.com