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An upper bound of the basis number of the semi-strong product of bipartite graphs

M.M.M. Jaradat

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Abstract. A basis of the cycle space, $\mathcal{C}(G)$, of a graph G is called a d-fold if each edge of G occurs in at most d cycles of the basis. The basis number, $b(G)$, of a graph G is defined to be the least integer d such that G has a d-fold basis for its cycle space. MacLane proved that a graph G is planar if and only if $b(G) \leq 2$. Schmeichel showed that for $n \geq 5$, $b(K_n \bullet P_2) \leq 1 + b(K_n)$. Ali proved that for $n, m \geq 5$, $b(K_n \bullet K_m) \leq 3 + b(K_m) + b(K_n)$. Jaradat proved that for any two bipartite graphs G and H, $b(G \wedge H) \leq 5 + b(G) + b(H)$. In this paper we give an upper bound of the basis number of the semi-strong product of bipartite graphs. Also, we give an example where the bound is achieved.

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§1. Introduction.

Bases of a cycle spaces of graphs have a variety of applications in science and engineering. For example, applications occur in structural flexibility analysis (see [13]), electrical networks (see [6]), and in chemical structures (see [7]). The basis number of a graph is one of the numbers which give rise to a better understanding and interpretations of geometric properties of a graph (see [14]).

In general, required bases are not well behaved under graph operations, that is, the basis numbers, $b(G)$, of graphs are not monotonic (see [15]). A global upper bound $b(G) \leq 2\gamma(G) + 2$ where $\gamma(G)$ is the genus of G is proven in [15].

In this paper, we construct a basis of the cycle spaces of the semi-strong product of bipartite graphs and we give an upper bound of the basis number of the same. Moreover, we give the basis number of the semi-strong product of a class of graphs.

§2. Definitions and preliminaries.

Throughout this paper, we assume that graphs are finite, undirected, simple and connected. We adopt the standard notation $\Delta(G)$ for the maximum degree of the vertices of G. Our terminologies and notations will be as in [4]. Given a graph G, let $e_1, e_2, \ldots, e_{|E(G)|}$ be an ordering of its edges. Then a subset S of $E(G)$ corresponds to a $(0,1)$-vector $(b_1, b_2, \ldots, b_{|E(G)|})$ in the usual way with $b_i = 1$ if $e_i \in S$, and $b_i = 0$ if $e_i \notin S$. These vectors form an $|E(G)|$-dimensional vector space, denoted by $(Z_2)^{|E(G)|}$, over the field of integers modulo 2. The vectors in $(Z_2)^{|E(G)|}$ which correspond to the cycles in G generate a subspace called the *cycle space* of G and denoted by $\mathcal{C}(G)$. We shall say that the cycles themselves, rather than the vectors corresponding to them, generate $\mathcal{C}(G)$. It is known that if r is the number of components of G, then $\dim \mathcal{C}(G) = |E(G)| - |V(G)| + r$ (see [5]).

A basis of $\mathcal{C}(G)$ is called *d-fold* if each edge of G occurs in at most d of the cycles in the basis. The basis number of G, $b(G)$, is the smallest non-negative integer d such that $\mathcal{C}(G)$ has a d-fold basis. The *required basis* of $\mathcal{C}(G)$ is a basis that is $b(G)$-fold. Let G and H be two graphs, $\varphi : G \longrightarrow H$ be an isomorphism and $\mathcal{B}$ be a (required) basis of $\mathcal{C}(G)$. Then $\{\varphi(c)|c \in \mathcal{B}\}$ is called the *corresponding (required) basis* of $\mathcal{B}$ in H. The *complement* of a spanning subgraph H of a graph G is the graph obtained from G by deleting the edges of H. The first use of the basis number of a graph was the theorem of MacLane when he proved that a graph G is planar if and only if $b(G) \leq 2$. Schmeichel proved that there are graphs with arbitrary large basis numbers. Moreover, Schmeichel proved that $b(K_n) \leq 3$.

Let G_1 and G_2 be two graphs. (1) The *direct product* $G = G_1 \wedge G_2$ is the graph with the vertex set $V(G) = V(G_1) \times V(G_2)$ and the edge set $E(G) = \{(u_1, u_2)(v_1, v_2)|u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)\}$. (2) The *semi-strong* product $G = G_1 \bullet G_2$ is the graph with the vertex set $V(G) = V(G_1) \times V(G_2)$ and the edge set $E(G) = \{(u_1, u_2)(v_1, v_2)|u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$ or $u_1 = v_1$ and $u_2v_2 \in E(G_2)\}$. (3) The *cartesian product* $G = G_1 \times G_2$ is the graph with the vertex set $V(G) = V(G_1) \times V(G_2)$ and the edge set $E(G) = \{(u_1, u_2)(v_1, v_2)|u_1v_1 \in E(G_1)$ and $u_2 = v_2$ or $u_2v_2 \in E(G_2)$ and $u_1 = v_1\}$. It is clear that the semi-strong product is non-commutative.

Studying the upper bound of the basis number of product graphs was the main interest of many authors. Schmeichel [15] proved the following results.

Theorem 2.1 (Schmeichel). *For each $n \geq 5$, $b(K_n \bullet P_2) \leq 1 + b(K_n)$.*

Ali [1] studied the basis number of the semi-strong product of complete graphs and he gave the following result:

Theorem 2.2 (Ali). *For each $n, m \geq 5$, $b(K_n \bullet K_m) \leq 3 + b(K_m) + b(K_n)$.*

The direct product was studied by Jaradat [9] who proved the following results.

Theorem 2.3 (Jaradat). *For each bipartite graphs G and H, $b(G \wedge H) \leq 5 + b(G) + b(H)$.*

For more papers on the basis number of graph product, we refer readers to [1], [2], [3], [10] and [11]. Based on the above results, one is naturally led to the following questions: Does there exist an upper bound of the basis number of the semi-strong product of two trees independent of their order? Does there exist an upper bound of the basis number of the semi-strong product of two bipartite graphs with respect to the basis number of the factors? These questions will be solved in the affirmative. Moreover, we will give an example to show the upper bounds is achieved. The method employed in this paper is based in part on ideas of Ali [1], Jaradat [9] and Schmeichel [15].

The author organized this paper as follows: In section 3, the author gives an upper bound of the basis number of the semi-strong product of two trees independent of their orders. In section 4, the author gives an upper bound of the basis number of the semi-strong product of bipartite graphs and gives an example where the bound is achieved.

Throughout this paper $f_B(e)$ stands for the number of cycles in B containing e, $E(B) = \cup_{b \in B} E(b)$ where $B \subseteq \mathcal{C}(G)$ and $\mathcal{B}_G$ stands for a required basis of G.

§3. The semi-strong product of two trees.

In this section, we give an upper bound of the basis number of the semi-strong product of two trees. Let e and $e^{'}$ be two edges such that $e = uv$, we define $\mathcal{A}_{ee^{'}}$ to be the cycle consists of the edge set $E(e \wedge e^{'}) \cup \{e^{'}_u, e^{'}_v\}$ where $e^{'}_u = u \times e^{'}$ and $e^{'}_v = v \times e^{'}$. Also, for any graph H we define

$$\mathcal{A}_e^H = \bigcup_{e^{'} \in E(H)} \mathcal{A}_{ee^{'}}.$$

Lemma 3.1. *Let T be a tree and $e = uv$ be an edge. Then $\mathcal{A}_e^T$ is a linearly independent subset of the cycle space $\mathcal{C}(e \bullet T)$. Moreover, any linear combination of cycles of $\mathcal{A}_e^T$ must contain at least two edges of two different copies of T, i.e. at least one edge of the copy $u \times T$ and at least one edge of the copy $v \times T$.*

Proof. The first part, $\mathcal{A}_e^T$ is linearly independent, follows from being that $E(\mathcal{A}_{ee^{'}}) \cap E(\mathcal{A}_{ee^{''}}) = \phi$ for each $e^{'} \neq e^{''}$. The second part follows from the fact that, any non-trivial linear combination of cycles of $\mathcal{A}_e^T$ must contains at least one element of $\mathcal{A}_e^T$, say $\mathcal{A}_{ee'}$, and noting that, $e_u^{'} \in E(\mathcal{A}_{ee'}) \cap E(u \times T)$, $e_v^{'} \in E(\mathcal{A}_{ee'}) \cap E(v \times T)$ and $e_u^{'}, e_v^{'}$ belong to no other element of $\mathcal{A}_e^T$. The proof is complete. □

Let G and H be two trees. We define

$$\mathcal{A}_G^H = \bigcup_{e \in E(G)} \mathcal{A}_e^H$$

where $\mathcal{A}_e^H$ is the linearly independent subset of $\mathcal{C}(e \bullet T_2)$ as defined above.

Lemma 3.2. *For any two trees T_1 and T_2, we have that $\mathcal{A}_{T_1}^{T_2}$ is a linearly independent subset of $\mathcal{C}(T_1 \bullet T_2)$.*

Proof. Let m be the size of T_1. We prove the lemma using induction on m. If $m = 1$, then T_1 is an edge, say e. Thus, $\mathcal{A}_{T_1}^{T_2} = \mathcal{A}_e^{T_2}$. Therefore, by Lemma 3.1, $\mathcal{A}_{T_1}^{T_2}$ is linearly independent. Assume that m is greater than or equal to 2 and it is true for less than m. Now, let v be an end vertex of T_1 and $e^* = uv \in E(T_1)$. Let $T_1^{'} = T_1 - v$. Then $T_1^{'}$ is a tree of size $m - 1$. Thus, by induction step and Lemma 3.1, both of $\mathcal{A}_{T_1^{'}}^{T_2} = \bigcup_{e \in E(T_1^{'})} \mathcal{A}_e^{T_2}$ and $\mathcal{A}_{e^*}^{T_2}$ are linearly independent. Note that, $e^* \bullet T_2 = (v \times T_2) \cup (u \times T_2) \cup (e^* \wedge T_2)$ and $(u \times T_2)$ is a tree. Thus any linear combination of cycles of $\mathcal{A}_{e^*}^{T_2}$ must contain at least one edge of $E(v \times T_2) \cup E(e^* \wedge T_2)$. On the other hand any linear combination of $\mathcal{A}_{T_1^{'}}^{T_2}$ contains no edge of $E(v \times T_2) \cup E(e^* \wedge T_2)$ because $E(v \times T_2) \cup E(e^* \wedge T_2) \subseteq E(e^* \bullet T_2)$ and

$$\left(\bigcup_{e \in E(T_1^{'})} E(e \bullet T_2) \right) \cap E(e^* \bullet T_2) = E(u \times T_2).$$

Therefore, $\mathcal{A}_{T_1}^{T_2} = (\bigcup_{e \in E(T_1^{'})} \mathcal{A}_e^{T_2}) \cup \mathcal{A}_{e^*}^{T_2}$ is linearly independent. The proof is complete. □

We remark that knowing the number of components in a graph is very important in finding the dimension (basis) of the cycle space of a graph. So we give the following result which is easy to prove.

Lemma 3.3. *If G and H are two connected graphs, then $G \bullet H$ is connected.*

The following proposition of Jaradat (see [9]) will be used frequently in our work.

Proposition 3.4. *For each tree T of order ≥ 3, there is a set of paths $S(T) = \left\{ P_3^{(1)}, P_3^{(2)}, \ldots, P_3^{(m)} \right\}$, which called a path sequence, such that*

(i) *each* $P_3^{(i)}$ *is a path of length* 2,

(ii) $\bigcup_{i=1}^{m} E(P_3^{(i)}) = E(T)$,

(iii) *every edge* $uv \in E(T)$ *appears in at most three paths of* $S(T)$,

(iv) *each* $P_3^{(j)}$ *contains exactly one edge which is not in* $\bigcup_{i=1}^{j-1} P_3^{(i)}$,

(v) *if* uv *appears in three paths of* $S(T)$, *then the paths have forms of either* uva, uvb *and* cuv *or* auv, buv *or* uvc,

(vi) *every edge with an end vertex occurs in at most two paths of* $S(T)$.

(vii) $m = |V(T)| - 2 = |E(T)| - 1$.

Proposition 3.5. *Let* T *be a tree and* $S(T)$ *be a path sequence satisfying the conditions* (i) *to* (vii) *of Proposition* 3.4. *Then,* $P_3^{(|V(T)|-2)}$ *contains an edge which appears in no other paths of* $S(T)$ *and incidents with an end vertex of* T.

Proof. By (iv) of Proposition 3.4, $P_3^{(|V(T)|-2)}$ contains an edge which appears in no other paths of $S(T)$, say ab, and the other edge appears in at least one more path of $S(T)$. Assume that each end vertex of $P_3^{(|V(T)|-2)}$ is not an end vertex of T. Then $T - ab$ is a graph consisting of two components. Moreover, by (ii), each components contains at least one path of $S(T)$. Thus, condition (iv) does not satisfy on $S(T) - \{P_3^{(|V(T)|-2)}\}$. On the other hand, since ab appears only in $P_3^{(|V(T)|-2)}$, the condition (iv) is still satisfying on $S(T) - \{P_3^{(|V(T)|-2)}\}$. This is a contradiction. The proof is complete. □

Let $e = uv$ be an edge and T be a tree with $S(T) = \{P_3^{(1)} = a_1b_1c_1, P_3^{(2)} = a_2b_2c_2, \ldots, P_3^{(|V(T)|-2)} = a_{|V(T)|-2}\, b_{|V(T)|-2}\, c_{|V(T)|-2}\}$ as in Proposition 3.4. For each $j = 1, 2, \ldots, |V(T)| - 2$, we define

$$\mathcal{B}_{(uv)P_3^{(j)}} = \{(u, a_j)(u, b_j)(u, c_j)(v, b_j)(u, a_j)\} \text{ and}$$
$$\mathcal{B}_{(uv)T} = \cup_{j=1}^{|V(T)|-2} \mathcal{B}_{(uv)P_3^{(j)}}.$$

Lemma 3.6. $\mathcal{B}_{(uv)T}$ *is linearly independent subset of* $\mathcal{C}(e \bullet T)$.

Proof. We use induction on $|S(T)|$ to show that $\mathcal{B}_{(uv)T}$ is linearly independent. If $|S(T)| = 1$, then $\mathcal{B}_{(uv)T}$ consists only of one cycle and so it is linearly independent. By induction step on $|S(T)|$ and noting that $\mathcal{B}_{(uv)P_3^{(|V(T)|-2)}}$ consists only of one cycle, we have that both of $\cup_{i=1}^{|V(T)|-3} \mathcal{B}_{(uv)P_3^{(i)}}$ and $\mathcal{B}_{(uv)P_3^{(|V(T)|-2)}}$ are linearly independent. By Proposition 3.5, $P_3^{(|V(T)|-2)}$ contains an edge, say $b_{|V(T)|-2}c_{|V(T)|-2}$, which does not appear in any other path of $S(T)$. Thus, $(u, b_{|V(T)|-2})(u, c_{|V(T)|-2})$ occurs only in $\mathcal{B}_{(uv)P_3^{(|V(T)|-2)}}$. Therefore, $\mathcal{B}_{(uv)P_3^{(|V(T)|-2)}}$ can not be written as a linear combination of cycles of $\cup_{i=1}^{|V(T)|-3}\mathcal{B}_{(uv)P_3^{(i)}}$. And so, $\mathcal{B}_{(uv)T}$ is linearly independent. The proof is complete. □

The following proposition (See [8], and [9]) will be used frequently in the sequel.

Proposition 3.7. *Let G be a bipartite graph and P_2 be a path of order 2. Then $G \wedge P_2$ consists of two components G_1 and G_2 each of which is isomorphic to G.*

The graph $T_1 \bullet T_2$ contains the graph $T_1 \wedge T_2$ as a subgraph. Moreover, $V(T_1 \bullet T_2) = V(T_1 \wedge T_2)$ and $E(T_1 \bullet T_2) = E(T_1 \wedge T_2) \cup M$ where $M = \cup_{u \in V(T_1)} E(u \times T_2)$.

Theorem 3.8. *For each two trees T_1 and T_2, we have*

$$b(T_1 \bullet T_2) \leq \max\left\{\left\{\begin{array}{ll} 3, & \text{if both of } T_1 \text{ and } T_2 \text{ are paths,} \\ 4, & \text{if } T_2 \text{ is a path,} \\ 5, & \text{if } T_1 \text{ is a path,} \\ 6, & \text{if both of } T_1 \text{ and } T_2 \text{ are not paths.} \end{array}\right\}, \Delta(T_1)\right\}.$$

Proof. Let $S(T_1) = \{ Q_3^{(1)} = u_1v_1w_1,\ Q_3^{(2)} = u_2v_2w_2, \ldots,\ Q_3^{(|V(T_1)|-2)} = u_{|V(T_1)|-2}\, v_{|V(T_1)|-2}\, w_{|V(T_1)|-2}\}$ and $S(T_2) = \{P_3^{(1)} = a_1b_1c_1, P_3^{(2)} = a_2b_2c_2, \ldots, P_3^{(|V(T_2)|-2)} = a_{|V(T_2)|-2}\, b_{|V(T_2)|-2}\, c_{|V(T_2)|-2}\}$ be path sequences of T_1 and T_2 as in Proposition 3.4, respectively. Let $\mathcal{B}(T_1 \wedge T_2) = \bigcup_{j=1}^{(|V(T_2)|-2)} \bigcup_{i=1}^{(|V(T_1)|-2)} \mathcal{B}_{i,j}$ where $\mathcal{B}_{i,j} = \{(u_i, b_j)(v_i, a_j)(w_i, b_j)(v_i, c_j)(u_i, b_j)\}$. Then, by Lemma 2.1 of Jaradat [8], $\mathcal{B}(T_1 \wedge T_2)$ is a basis for $\mathcal{C}(T_1 \wedge T_2)$, hence it is linearly independent subset of $\mathcal{C}(T_1 \bullet T_2)$. By Proposition 3.5, we may assume that $w_{|V(T_1)|-2}v_{|V(T_1)|-2}$ is an edge of T_1 which appears only on $Q_3^{(|V(T_1)|-2)}$ and $w_{|V(T_1)|-2}$ is an end vertex of T_1. Define $\mathcal{B}(T_1 \bullet T_2) = \mathcal{B}(T_1 \wedge T_2) \bigcup \mathcal{A}_{T_1}^{T_2} \bigcup \mathcal{B}_{(w_{|V(T_1)|-2}v_{|V(T_1)|-2})T_2}$ where $\mathcal{A}_{T_1}^{T_2}$ and $\mathcal{B}_{(w_{|V(T_1)|-2}v_{|V(T_1)|-2})T_2}$ are defined as in Lemma 3.2 and 3.6. Since $E(\mathcal{B}_{(w_{|V(T_1)|-2}v_{|V(T_1)|-2})T_2}) \subseteq E(w_{|V(T_1)|-2} \times T_2) \cup E(w_{|V(T_1)|-2}v_{|V(T_1)|-2} \wedge T_2)$ and since $E(w_{|V(T_1)|-2}v_{|V(T_1)|-2} \wedge T_2)$ is an edge set of a forest (Proposition 3.7), as a result any linear combination of cycles of $\mathcal{B}_{(w_{|V(T_1)|-2}v_{|V(T_1)|-2})T_2}$ must contain at least one edge of $w_{|V(T_1)|-2} \times T_2$ which is not in any cycle of $\mathcal{B}(T_1 \wedge T_2)$. Thus $\mathcal{B}(T_1 \wedge T_2) \bigcup \mathcal{B}_{(w_{|V(T_1)|-2}v_{|V(T_1)|-2})T_2}$ is linearly independent. We now show that the cycles of $\mathcal{A}_{T_1}^{T_2}$ are linearly independent of the cycles of $\mathcal{B}(T_1 \wedge T_2) \bigcup \mathcal{B}_{(w_{|V(T_1)|-2}v_{|V(T_1)|-2})T_2}$. Let

$$\mathcal{F} = \sum_{e \in A \subseteq E(T_1)} \mathcal{F}_e \pmod 2$$

where $\mathcal{F}_e$ is a linear combinations of cycles of $\mathcal{A}_e^{T_2}$. Since each $\mathcal{F}_e$ contain at least two edges of two different copies of T_2 (Lemma 3.1) and since $E(\mathcal{A}_e^{T_2}) \cap E(\mathcal{A}_{e'}^{T_2})$ is a subset of a one copy of T_2, as a result $\mathcal{F}$ must contain at least two edges of two different copies of T_2. On the other hand any linear combination of $\mathcal{B}(T_1 \wedge T_2) \cup \mathcal{B}_{(w_{|V(T_1)|-2}v_{|V(T_1)|-2})T_2}$ may contain edges of at most one copy of T_2, in fact of $w_{|E(T_1)|-1} \times T_2$. Thus, any linear combination of $\mathcal{A}_{T_1}^{T_2}$ can not be written as a linear combination of cycles of $\mathcal{B}(T_1 \wedge T_2) \cup \mathcal{B}_{(w_{|V(T_1)|-2}v_{|V(T_1)|-2})T_2}$. Therefore, $\mathcal{B}(T_1 \bullet T_2)$ is linearly independent. Since

$$\begin{aligned}
&|\mathcal{B}(T_1 \bullet T_2)| \\
&= |\mathcal{B}(T_1 \wedge T_2)| + |\mathcal{A}_{T_1}^{T_2}| + |\mathcal{B}_{(w_{|V(T_1)|-2}v_{|V(T_1)|-2})T_2}| \\
&= \dim \mathcal{C}(T_1 \wedge T_2) + \sum_{e \in E(T_1)} \sum_{e' \in E(T_2)} |\mathcal{A}_{ee'}| + \sum_{i=1}^{|V(T_2)|-2} |\mathcal{B}_{(w_{|V(T_1)|-2}v_{|V(T_1)|-2})P_3^{(i)}}| \\
&= 2|E(T_1)||E(T_2)| - |V(T_1)||V(T_2)| + 2 + |E(T_2)|(|V(T_1)| - 1) + |V(T_2)| - 2 \\
&= 2|E(T_1)||E(T_2)| + |E(T_2)||V(T_1)| - |V(T_1)||V(T_2)| + 1 \\
&= \dim \mathcal{C}(T_1 \bullet T_2),
\end{aligned}$$

$\mathcal{B}(T_1 \bullet T_2)$ is a basis for $\mathcal{C}(T_1 \bullet T_2)$. To complete the proof, we show that $\mathcal{B}(T_1 \bullet T_2)$ satisfies the fold which is stated in the theorem. Let $e \in E(T_1 \bullet T_2)$.

(1) If $e \in E(T_1 \wedge T_2) - E(w_{|E(T_2)|-1}v_{|E(T_2)|-1} \wedge T_2)$, then

$$f_{\mathcal{B}(T_1 \wedge T_2)}(e) \leq \begin{cases} 2, & \text{if both of } T_1 \text{ and } T_2 \text{ are paths,} \\ 3, & \text{if one of } T_1 \text{ and } T_2 \text{ is a path,} \\ 5, & \text{if both of } T_1 \text{ and } T_2 \text{ are not paths,} \end{cases}$$

$$f_{\mathcal{A}_{T_1}^{T_2}}(e) = 1$$

and

$$f_{\mathcal{B}_{(w_{|V(T_1)|-2}v_{|V(T_1)|-2})T_2}}(e) = 0.$$

(2) If $e \in E(w_{|E(T_2)|-1}v_{|E(T_2)|-1} \wedge T_2)$, then

$$f_{\mathcal{B}(T_1 \wedge T_2)}(e) \leq \begin{cases} 1, & \text{if } T_2 \text{ is a path,} \\ 2, & \text{if } T_2 \text{ is not a path,} \end{cases}$$

$$f_{\mathcal{A}_{T_1}^{T_2}}(e) = 1$$

and

$$f_{\mathcal{B}(w_{|V(T_1)|-2}v_{|V(T_1)|-2})T_2}(e) \leq \begin{cases} 1, & \text{if } T_2 \text{ is a path,} \\ 2, & \text{if } T_2 \text{ is not a path.} \end{cases}$$

(3) If $e \in E(u \times T_2)$ for any $u \in V(T_1)$ and $u \neq w_{|E(T_2)|-1}$, then

$$f_{\mathcal{B}(T_1 \wedge T_2)}(e) = 0,$$
$$f_{\mathcal{A}_{T_1}^{T_2}}(e) \leq \Delta(T_1)$$

and

$$f_{\mathcal{B}(w_{|V(T_1)|-2}v_{|V(T_1)|-2})T_2}(e) = 0.$$

(4) If $e \in E(w_{|E(T_2)|-1} \times T_2)$, then

$$f_{\mathcal{B}(T_1 \wedge T_2)}(e) = 0,$$
$$f_{\mathcal{A}_{T_1}^{T_2}}(e) = 1$$

and

$$f_{\mathcal{B}(w_{|V(T_1)|-2}v_{|V(T_1)|-2})T_2}(e) \leq \begin{cases} 2, & \text{if } T_2 \text{ is a path,} \\ 3, & \text{if } T_2 \text{ is not a path.} \end{cases}$$

The proof is complete. □

§4. The semi-strong product of two bipartite graphs.

In this section, we give an upper bound of the semi-strong product of two bipartite graphs with respect to the basis number of the factors. Let G be a graph. Then T_G stand for a spanning tree of G such that $\Delta(T_G) = \min\{\Delta(T) | T \text{ is a spanning tree of } G\}$ (See [2]).

Lemma 4.1. *If G is a bipartite graph and T is a tree, then*

$$b(G \bullet T) \leq \max \left\{ b(G) + \begin{cases} 3, & \text{if both of } T_G \text{ and } T \text{ are paths,} \\ 4, & \text{if } T \text{ is a path,} \\ 5, & \text{if } T_G \text{ is a path,} \\ 6, & \text{if both of } T_G \text{ and } T \text{ are not paths.} \end{cases}, \Delta(T_G) \right\}.$$

Proof. Let $\mathcal{B}(T_G \bullet T)$ be the basis of $\mathcal{C}(T_G \bullet T)$ as in Theorem 3.8. By Proposition 3.7, for each $e \in E(T)$, $G \wedge e$ consists of two components each of which is isomorphic to G. Thus, we set $\mathcal{B}_e = \mathcal{B}_e^{(1)} \cup \mathcal{B}_e^{(2)}$ where $\mathcal{B}_e^{(1)}$ and $\mathcal{B}_e^{(2)}$ are the corresponding required basis of $\mathcal{B}_G$ in the two components of G in $G \wedge e$. Let $\mathcal{T} = \bigcup_{e \in E(T)} \mathcal{B}_e$ and $\mathcal{B}(G \bullet T) = \mathcal{T} \bigcup \mathcal{B}(T_G \bullet T)$. Since $E(\mathcal{B}_e^{(1)}) \cap E(\mathcal{B}_e^{(2)}) = \phi$ for each $e \in E(T)$ and $E(\mathcal{B}_e) \cap E(\mathcal{B}_{e'}) = \phi$ for each $e \neq e'$, we get that $\mathcal{T}$ is linearly independent. We now show that $\mathcal{T}$ is linearly independent of $\mathcal{B}(T_G \bullet T)$. Let $O = \sum_{e \in A \subseteq E(T)} \sum_{i=1}^{\alpha_e} c_e^{(i)}$ (mod2) where $c_e^{(i)} \in \mathcal{B}_e$. By Proposition 3.7, $T_G \wedge e$ is a forest. Thus, the ring sum $c_e^{(1)} \oplus c_e^{(2)} \oplus \cdots \oplus c_e^{(\alpha_e)}$ contains at least one edge of $E((G - T_G) \wedge e)$ where $G - T_G$ is the complement of T_G of G. Since $E(G \wedge e) \cap E(G \wedge e') = \phi$ for each $e \neq e'$, the ring sum $O = \oplus_{e \in A \subseteq E(T)} \oplus_i^{\alpha_i} c_e^{(i)}$ contains at least one edge of $E((G - T_G) \wedge T)$. On the other hand, no cycle of $\mathcal{B}(T_G \bullet T)$ contains such kind of edges. Thus, $\mathcal{B}(G \bullet T)$ is linearly independent. Since,

$$
\begin{aligned}
&|\mathcal{B}(G \bullet T)| \\
&= |\mathcal{B}(T_G \bullet T)| + |\mathcal{T}| \\
&= 2|E(T_G)||E(T)| + |E(T)||V(T_G)| - |V(T_G)||V(T)| + 1 + \sum_{e \in E(T)} |\mathcal{B}_e| \\
&= 2|E(T_G)||E(T)| + |E(T)||V(T_G)| - |V(T_G)||V(T)| + 1 + 2\dim \mathcal{C}(G)|E(T)| \\
&= 2|E(T)|(|E(T_G)| + \dim \mathcal{C}(G)) + |E(T)||V(G)| - |V(G)||V(T)| + 1 \\
&= \dim \mathcal{C}(G \bullet T),
\end{aligned}
$$

$\mathcal{B}(G \bullet T)$ is a basis for $\mathcal{C}(G \bullet T)$. To this end, we show that $\mathcal{B}(G \bullet T)$ satisfies the required fold. Let $e \in E(G \bullet T)$.

(1) if $e \in G \wedge T$, then

$$
f_{\mathcal{B}(T_G \bullet T)}(e) \leq \begin{cases} 3, & \text{if both of } T_G \text{ and } T \text{ are paths,} \\ 4, & \text{if } T \text{ is a path,} \\ 5, & \text{if } T_G \text{ is a path,} \\ 6, & \text{if both of } T_G \text{ and } T \text{ are not paths,} \end{cases}
$$

$$
f_{\mathcal{T}}(e) \leq b(G).
$$

(2) if $e \in u \times T$ for some $u \in V(G)$, then

$$
f_{\mathcal{B}(T_G \bullet T)}(e) \leq \max \left\{ \left\{ \begin{array}{ll} 3, & \text{if } T \text{ is a path,} \\ 4, & \text{if } T \text{ is not a path.} \end{array} \right\}, \Delta(T_G) \right\},
$$

$$
f_{\mathcal{T}}(e) = 0.
$$

The proof is complete. □

Theorem 4.2. *For each two bipartite graphs G and H, we have*

$$b(G\bullet H) \le \max\left\{\begin{array}{r} b(G)+b(H)+\left\{\begin{array}{ll} 3, & \text{if both of } T_G \text{ and } T_H \text{ are paths,}\\ 4, & \text{if } T_H \text{ is a path,}\\ 5, & \text{if } T_G \text{ is a path,}\\ 6, & \text{if both of } T_G \text{ and } T_H \text{ are not paths.}\end{array}\right\},\\ \Delta(T_G)+b(H)\end{array}\right\}.$$

Proof. Let $\mathcal{B}(G\bullet T_H)$ be the basis of $\mathcal{C}(G\bullet T_H)$ as in Lemma 4.1. For each $e\in E(G)$, let $\mathcal{B}_e=\mathcal{B}_e^{(1)}\cup\mathcal{B}_e^{(2)}$ be the corresponding required basis of $\mathcal{B}_H$ in the two components of H in $e\wedge H$. By using the same arguments as in Lemma 4.1 we can prove that each of $\mathcal{Z}=\bigcup_{e\in E(G)}\mathcal{B}_e$ and $\mathcal{B}(G\bullet T_H)\bigcup\mathcal{Z}$ is linearly independent. Now, for each $u\in V(G)$, let $\mathcal{B}_u$ be the corresponding required basis of $\mathcal{B}_H$ in $u\times H$. Set $\mathcal{V}=\bigcup_{u\in V(G)}\mathcal{B}_u$, and $\mathcal{B}(G\bullet H)=\mathcal{B}(G\bullet T_H)\bigcup\mathcal{Z}\bigcup\mathcal{V}$. Since $E(\mathcal{B}_u)\cap E(\mathcal{B}_w)=\phi$ whenever $u\neq w$, we conclude that $\mathcal{V}$ is linearly independent. Note that each linear combination of cycles of $\mathcal{V}$ contains at least one edge of $E(u\times(H-T_H))$ for some $u\in V(G)$ where $H-T_H$ is the complement of T_H of H, on the other hand no cycle of $\mathcal{B}(G\bullet T_H)\cup\mathcal{Z}$ contains such an edge. Therefore, $\mathcal{B}(G\bullet H)$ is linearly independent. Since

$$\begin{aligned}
|\mathcal{B}(G\bullet H)| &= |\mathcal{B}(G\bullet T_H)|+|\mathcal{Z}|+|\mathcal{V}|\\
&= 2|E(G)||E(T_H)|+|V(G)||E(T_H)|-|V(G)||V(T_H)|+1\\
&\quad+\sum_{e\in E(G)}|\mathcal{B}_e|+\sum_{u\in V(G)}|\mathcal{B}_u|\\
&= 2|E(G)||E(T_H)|+|V(G)||E(T_H)|-|V(G)||V(T_H)|+1\\
&\quad+2|E(G)|\dim\mathcal{C}(H)+|V(G)|\dim\mathcal{C}(H)\\
&= 2|E(G)|(|E(T_H)|+\dim\mathcal{C}(H))\\
&\quad+|V(G)|(|E(T_H)|+\dim\mathcal{C}(H))-|V(G)||V(H)|+1\\
&= \dim\mathcal{C}(G\bullet H),
\end{aligned}$$

$\mathcal{B}(G\bullet H)$ is a basis for $\mathcal{C}(G\bullet H)$. To this end, it is an easy task to see that $\mathcal{B}(G\bullet H)$ satisfied the required fold which is stated in the theorem. The proof is complete. □

Now, we give an example where the bounds in Theorems 3.8 and 4.2 are achieved.

Corollary 4.3. *For each two paths P_n and P_m, we have $b(P_n \bullet P_m) = 3$ if $n \geq 3$ and $m \geq 4$.*

Proof. By Theorem 4.2 and MacLane's result, it is enough to show that $P_n \bullet P_m$ is non planar. Let $P_n = u_1u_2 \ldots u_n$ and $P_m = v_1v_2 \ldots v_m$. Consider the subgraph H whose vertex set $\{(u_1, v_2), (u_2, v_1), (u_3, v_2), (u_2, v_2), (u_1, v_3), (u_2, v_3), (u_3, v_3), (u_2, v_4)\}$ and whose edge set consists of the following nine paths: $P_1 = (u_1, v_2)(u_2, v_1)$, $P_2 = (u_2, v_1)(u_3, v_2)$, $P_3 = (u_2, v_1)(u_2, v_2)$, $P_4 = (u_2, v_2)(u_2, v_3)$, $P_5 = (u_2, v_3)(u_3, v_2)$, $P_6 = (u_1, v_2)(u_2, v_3)$, $P_7 = (u_2, v_2)(u_3, v_3)$, $P_8 = (u_3, v_2)\ (u_3, v_3)$ and $P_9 = (u_1, v_2)(u_1, v_3)(u_2, v_4)(u_3, v_3)$. Then H is homeomorphic to $K_{3,3}$. Therefore $P_n \bullet P_m$ is non planar. The proof is complete. □

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M.M.M. Jaradat
Department of Mathematics, Yarmouk University
Irbid-Jordan
E-mail: mmjst4@yu.edu.jo