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A modified second order Bonferroni approximation in elliptical populations with unequal sample sizes

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Abstract. The approximate upper percentile of Hotelling's T^2-type statistic is derived in order to construct simultaneous confidence intervals for pairwise multiple comparisons and comparisons with a control under elliptical populations. The accuracy and conservativeness of the first and the modified second order Bonferroni approximations are evaluated via a Monte Carlo simulation study.

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§1. Introduction

Simultaneous confidence intervals for pairwise multiple comparisons and comparisons with a control among mean vectors are considered under k independent elliptical populations with unequal sample sizes. In order to construct them, it is necessary to obtain the upper percentile of $T^2_{\max}$ which is Hotelling's T^2-type statistic. $T^2_{\max}$ is reduced to the multivariate Studentized range statistic under the normal distribution and equal sample sizes, see Roy and Bose [6]. However, it is difficult to obtain upper percentiles exactly even when populations have the multivariate normal distribution. In order to obtain conservative approximate simultaneous confidence intervals, Bonferroni's inequality is applied to T^2-type statistic. Under elliptical populations with equal sample sizes, the first and the modified second order Bonferroni approximations are discussed by Seo [7]. For unequal sample sizes, the first order Bonferroni approximation is discussed by Okamoto and Seo [5]. The first order Bonferroni approximation becomes conservative too much when the number of populations or the kurtosis parameter is large. In this paper, the modified second order Bonferroni approximation is discussed for unequal sample sizes. In Section 2, simultaneous confidence intervals for pairwise multiple comparisons and the first order Bonferroni approximation are discussed. The modified second order Bonferroni approximation is derived in Section 3. In Section 4, simultaneous confidence intervals for comparisons with a control and the first and the modified second order Bonferroni approximations are also obtained as well as preceding section. Finally, the accuracy and conservativeness of the first and the modified second order Bonferroni approximations are evaluated via a Monte Carlo simulation study in Section 5.

For the j-th population, a $p \times 1$ random vector $\boldsymbol{x}^{(j)}$ is said to have an elliptical distribution with parameters $\boldsymbol{\mu}^{(j)}$ $(p\times 1)$ and $\Lambda^{(j)}$ $(p\times p)$ if its density function is

$$f(\boldsymbol{x}^{(j)}) = c_p^{(j)}|\Lambda^{(j)}|^{-\frac{1}{2}}g_j\left\{(\boldsymbol{x}^{(j)}-\boldsymbol{\mu}^{(j)})'{\Lambda^{(j)}}^{-1}(\boldsymbol{x}^{(j)}-\boldsymbol{\mu}^{(j)})\right\}$$

for some non-negative function g_j, where $c_p^{(j)}$ is a normalizing constant and $\Lambda^{(j)}$ is a positive definite. The characteristic function of the vector $\boldsymbol{x}^{(j)}$ is $\phi_j(\boldsymbol{t}) = \exp(i\boldsymbol{t}'\boldsymbol{\mu}^{(j)})\psi_j(\boldsymbol{t}'\Lambda^{(j)}\boldsymbol{t})$ for some function ψ_j, and $\mathrm{E}[\boldsymbol{x}^{(j)}] = \boldsymbol{\mu}^{(j)}$ and $\Sigma^{(j)} = \mathrm{Cov}[\boldsymbol{x}^{(j)}] = -2\psi_j'(0)\Lambda^{(j)}$, if they exist. Throughout this paper, we assume $\Sigma = \Sigma^{(1)} = \cdots = \Sigma^{(k)}$, $\mathrm{E}[||\boldsymbol{x}^{(j)}||^8] < \infty$ and Cramér's condition is satisfied. We define the kurtosis parameter as $\kappa_j = \{\psi_j''(0)/(\psi_j'(0))^2\} - 1$. Elliptical distributions include the multivariate normal, the ε-contaminated normal, the multivariate t, the symmetric Kotz type and many other distributions, see e.g. Muirhead [4], Fang, Kotz and Ng [1], Kotz and Nadarajah [3].

§2. A first order Bonferroni approximation

Consider simultaneous confidence intervals for pairwise multiple comparisons among k independent p-dimensional mean vectors under elliptical populations. Let $\boldsymbol{x}_1^{(j)}, \ldots, \boldsymbol{x}_{N_j}^{(j)}$ $(j = 1, \ldots, k)$ be N_j independent observations on $\boldsymbol{x}^{(j)}$ that has an elliptical distribution with mean vector $\boldsymbol{\mu}^{(j)}$ and common covariance matrix Σ. Let the j-th sample mean vector, the j-th sample covariance matrix and the pooled sample covariance matrix be

$$\begin{aligned}\overline{\boldsymbol{x}}^{(j)} &= \frac{1}{N_j}\sum_{i=1}^{N_j}\boldsymbol{x}_i^{(j)},\\ S^{(j)} &= \frac{1}{N_j-1}\sum_{i=1}^{N_j}(\boldsymbol{x}_i^{(j)}-\overline{\boldsymbol{x}}^{(j)})(\boldsymbol{x}_i^{(j)}-\overline{\boldsymbol{x}}^{(j)})',\\ S &= \frac{1}{\nu}\sum_{j=1}^{k}(N_j-1)S^{(j)},\end{aligned}$$

respectively, where $\nu = \sum_{j=1}^{k} N_j - k$.

The simultaneous confidence intervals for pairwise multiple comparisons among mean vectors are given by

$$\boldsymbol{a}'(\boldsymbol{\mu}^{(l)} - \boldsymbol{\mu}^{(m)}) \in \left[\boldsymbol{a}'(\overline{\boldsymbol{x}}^{(l)} - \overline{\boldsymbol{x}}^{(m)}) \pm t\sqrt{d_{lm}\boldsymbol{a}'S\boldsymbol{a}}\right], \quad \forall \boldsymbol{a} \in \mathbf{R}^p - \{\mathbf{0}\}\,,\ 1 \le l < m \le k, \tag{2.1}$$

where $d_{lm} = 1/N_l + 1/N_m$, $\mathbf{R}^p - \{\mathbf{0}\}$ is the set of any nonnull real p-dimensional vectors and the value t (> 0) satisfies as follows:

$$\Pr\left\{T^2_{\max} > t^2\right\} = \alpha,$$

where

$$\begin{aligned} T^2_{\max} &= \max_{1 \le l < m \le k} \left\{T^2_{lm}\right\}, \\ T^2_{lm} &= d^{-1}_{lm}\left(\boldsymbol{y}^{(l)} - \boldsymbol{y}^{(m)}\right)' S^{-1}\left(\boldsymbol{y}^{(l)} - \boldsymbol{y}^{(m)}\right), \\ \boldsymbol{y}^{(j)} &= \overline{\boldsymbol{x}}^{(j)} - \boldsymbol{\mu}^{(j)},\ j = 1, \ldots, k. \end{aligned}$$

By Bonferroni's inequality for $\Pr\left\{T^2_{\max} > t^2\right\}$:

$$\Pr\left\{T^2_{\max} > t^2\right\} < \sum_{l=1}^{k-1} \sum_{m=l+1}^{k} \Pr\left\{T^2_{lm} > t^2\right\},$$

the approximate upper percentile t_1^2 of $T^2_{\max}$ is given by

$$\sum_{l=1}^{k-1} \sum_{m=l+1}^{k} \Pr\left\{T^2_{lm} > t_1^2\right\} = \alpha. \tag{2.2}$$

Without a loss of generality, we assume $\Sigma = I_p$ and $N_j \le N_1 = N$ for $j = 2, \ldots, k$. Put $r_j = N_j/N$ for $j = 1, \ldots, k$, $s = 1/(\sum_{j=1}^{k} r_j)$ and $w_{lm} = \sqrt{r_m/(r_l + r_m)}$.

Letting

$$\begin{aligned} \overline{\boldsymbol{x}}^{(j)} &= \boldsymbol{\mu}^{(j)} + \frac{1}{\sqrt{N_j}}\boldsymbol{z}^{(j)}, \\ W^{(j)} &= \frac{1}{N_j}\sum_{i=1}^{N_j}(\boldsymbol{x}_i^{(j)} - \boldsymbol{\mu}^{(j)})(\boldsymbol{x}_i^{(j)} - \boldsymbol{\mu}^{(j)})' \\ &= I_p + \frac{1}{\sqrt{N_j}}Z^{(j)}, \end{aligned}$$

we have

$$T^2_{lm} = \boldsymbol{\tau}'_{lm} S^{-1} \boldsymbol{\tau}_{lm},$$

where

$$\begin{aligned}
\boldsymbol{\tau}_{lm} &= w_{lm}\boldsymbol{z}^{(l)} - w_{ml}\boldsymbol{z}^{(m)}, \\
S^{-1} &= I_p - \frac{1}{\sqrt{N}} s \sum_{j=1}^{k} \sqrt{r_j} Z^{(j)} + \frac{1}{N}\left[s\sum_{j=1}^{k} \boldsymbol{z}^{(j)}\boldsymbol{z}^{(j)\prime} + s^2 \sum_{j=1}^{k} r_j Z^{(j)2} \right. \\
&\quad \left. + s^2 \left\{ \sum_{i=1}^{k-1}\sum_{j=i+1}^{k} \sqrt{r_i r_j}\left(Z^{(i)}Z^{(j)} + Z^{(j)}Z^{(i)} \right) \right\} - skI_p \right] + o_p(N^{-1}).
\end{aligned}$$

Expanding and inverting the characteristic function, we have the first order Bonferroni approximate upper 100α percentiles of $T^2_{\max}$, that is, $t^2_{1\cdot\chi^2} \equiv t^2_{1\cdot\chi^2}(\alpha)$ and $t^2_{1\cdot F} \equiv t^2_{1\cdot F}(\alpha)$, which are derived by Okamoto and Seo [5] as follows:

$$\begin{aligned}
(2.3)\quad t^2_{1\cdot\chi^2} &= \chi^2_p\left(\frac{\alpha}{K}\right) - \frac{1}{2NK}\chi^2_p\left(\frac{\alpha}{K}\right) \\
&\quad \times \sum_{l=1}^{k-1}\sum_{m=l+1}^{k} \left\{ \frac{1}{p}c^{(0)}_{lm} - \frac{1}{p(p+2)} c^{(2)}_{lm}\chi^2_p\left(\frac{\alpha}{K}\right) \right\}, \\
(2.4)\quad t^2_{1\cdot F} &= \frac{\nu p}{\nu - p + 1} F_{p,\nu-p+1}\left(\frac{\alpha}{K}\right) - \frac{1}{2NK}\chi^2_p\left(\frac{\alpha}{K}\right) \\
&\quad \times \sum_{l=1}^{k-1}\sum_{m=l+1}^{k} \left\{ \left(\frac{1}{p}c^{(0)}_{lm} + sp \right) - \left(\frac{1}{p(p+2)} c^{(2)}_{lm} - s \right)\chi^2_p\left(\frac{\alpha}{K}\right) \right\},
\end{aligned}$$

where $K = k(k-1)/2$, $\chi^2_p(\alpha/K)$ and $F_{p,\nu-p+1}(\alpha/K)$ are the upper $100(\alpha/K)$ percentile of the χ^2 distribution with p degrees of freedom and that of the F-distribution with p and $\nu - p + 1$ degrees of freedom, respectively, and

$$\begin{aligned}
c^{(0)}_{lm} &= -sp^2 + \frac{1}{2}p(p+2) \\
&\quad \times \left[\left(\frac{1}{r_l} w^4_{lm} - 2sw^2_{lm} \right)\kappa_l + \left(\frac{1}{r_m} w^4_{ml} - 2sw^2_{ml} \right)\kappa_m - s^2 \sum_{j=1}^{k} r_j\kappa_j \right], \\
c^{(2)}_{lm} &= sp(p+2) + \frac{1}{2}p(p+2) \\
&\quad \times \left[\left(\frac{1}{r_l} w^4_{lm} - 6sw^2_{lm} \right)\kappa_l + \left(\frac{1}{r_m} w^4_{ml} - 6sw^2_{ml} \right)\kappa_m + 3s^2 \sum_{j=1}^{k} r_j\kappa_j \right].
\end{aligned}$$

§3. A modified second order Bonferroni approximation

In this section, a modified second order Bonferroni procedure is described to improve the first order Bonferroni approximation. Let $\boldsymbol{y}_1 = w_{12}\boldsymbol{z}^{(1)} - w_{21}\boldsymbol{z}^{(2)}$, $\boldsymbol{y}_2 = w_{13}\boldsymbol{z}^{(1)} - w_{31}\boldsymbol{z}^{(3)}, \ldots, \boldsymbol{y}_K = w_{k-1,k}\boldsymbol{z}^{(k-1)} - w_{k,k-1}\boldsymbol{z}^{(k)}$, $w_{lm} = \sqrt{r_m/(r_l + r_m)}$. Then Bonferroni's inequality for $\Pr\{T^2_{\max} > t^2\}$ is given by

$$\sum_{i=1}^{K} \Pr\left\{\boldsymbol{y}_i' S^{-1}\boldsymbol{y}_i > t^2\right\} - \beta(t^2) < \Pr\{T^2_{\max} > t^2\} < \sum_{i=1}^{K} \Pr\left\{\boldsymbol{y}_i' S^{-1}\boldsymbol{y}_i > t^2\right\},$$

where

$$\beta(t^2) = \sum_{i=1}^{K-1}\sum_{j=i+1}^{K} \Pr\left\{\boldsymbol{y}_i' S^{-1}\boldsymbol{y}_i > t^2,\ \boldsymbol{y}_j' S^{-1}\boldsymbol{y}_j > t^2\right\}.$$

The first order Bonferroni approximation t_1^2, which uses the first term of Bonferroni's inequality, is defined as a critical value that satisfies the equality

$$T_1^2(t_1^2) \equiv \sum_{i=1}^{K} \Pr\left\{\boldsymbol{y}_i' S^{-1}\boldsymbol{y}_i > t_1^2\right\} = \alpha.$$

Note that $T_1^2(t_1^2)$ is equal to the left side in (2.2). The second order Bonferroni approximation t_2^2, which uses the first and the second terms of Bonferroni's inequality, is defined as a critical value that satisfies the equality

$$T_2^2(t_2^2) \equiv \sum_{i=1}^{K} \Pr\left\{\boldsymbol{y}_i' S^{-1}\boldsymbol{y}_i > t_2^2\right\} - \beta(t_2^2) = \alpha.$$

The modified second order Bonferroni approximation t_M^2 is defined as a critical value that satisfies the equality

$$\sum_{i=1}^{K} \Pr\left\{\boldsymbol{y}_i' S^{-1}\boldsymbol{y}_i > t_M^2\right\} = \alpha + \beta(t_1^2).$$

Approximate values t_1^2, t_2^2, t_M^2 and an exact upper percentile t^2 are shown as Figure 1. If an error due to the asymptotic expansion is not taken into consideration, then $t_2^2 < t^2 < t_1^2$ and $t_2^2 < t_M^2 < t_1^2$.

In order to obtain the modified second order Bonferroni approximation t_M^2, we discuss the evaluation of $\beta(t_1^2)$. Consider two cases of joint probabilities to evaluate the $\beta(t_1^2)$; that is, $\beta_{1\cdot ijkl}(t_1^2) = \Pr\{T_{ij}^2 > t_1^2, T_{kl}^2 > t_1^2\}$ (i, j, k, l are all distinct) and $\beta_{2\cdot ijk}(t_1^2) = \Pr\{T_{ij}^2 > t_1^2, T_{ik}^2 > t_1^2\}$ (i, j, k are all distinct) under the elliptical distribution setup.

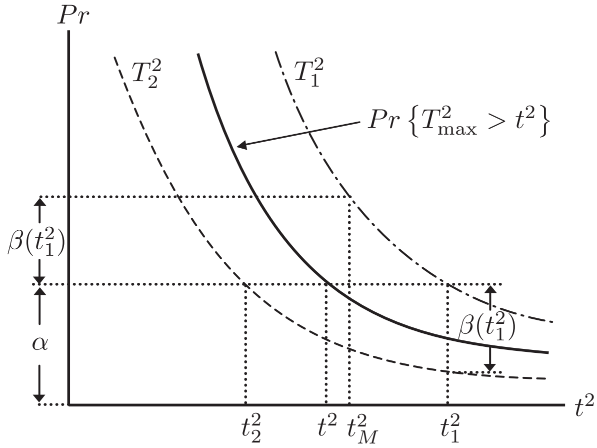

Figure 1: Exact and approximate upper percentiles.

Consider an asymptotic expansion for $\beta_{1\cdot ijkl}(t_1^2)$. For convenience, we discuss the joint characteristic function of T_{12}^2 and T_{34}^2, that is,

$$C_1(it_1, it_2) = \mathrm{E}[\exp(it_1 T_{12}^2 + it_2 T_{34}^2)].$$

Let

$$\overline{\boldsymbol{x}}^{(j)} = \boldsymbol{\mu}^{(j)} + \frac{1}{\sqrt{N_j}}\boldsymbol{z}^{(j)}, \quad W^{(j)} = I_p + \frac{1}{\sqrt{N_j}}Z^{(j)}$$

for $j = 1, 2, \ldots, k$. The joint characteristic function $C_1(it_1, it_2)$ can be written as

$$C_1(it_1, it_2) = \mathrm{E}\left[\exp(it_1 T_{12}^{(1)} + it_2 T_{34}^{(1)})\left(1 + \frac{1}{\sqrt{N}}B_1 + \frac{1}{N}B_2\right)\right] + o(N^{-1}),$$

where

$$B_1 = it_1 T_{12}^{(2)} + it_2 T_{34}^{(2)},$$
$$B_2 = it_1 T_{12}^{(3)} + \frac{(it_1)^2}{2}(T_{12}^{(2)})^2 + it_2 T_{34}^{(3)} + \frac{(it_2)^2}{2}(T_{34}^{(2)})^2 + (it_1)(it_2)T_{12}^{(2)}T_{34}^{(2)},$$

and

$$T_{12}^{(1)} = \boldsymbol{\tau}'_{12}\boldsymbol{\tau}_{12}, \quad T_{34}^{(1)} = \boldsymbol{\tau}'_{34}\boldsymbol{\tau}_{34},$$
$$T_{12}^{(2)} = -\boldsymbol{\tau}'_{12}\left(s\sum_{j=1}^{k}\sqrt{r_j}Z^{(j)}\right)\boldsymbol{\tau}_{12}, \quad T_{34}^{(2)} = -\boldsymbol{\tau}'_{34}\left(s\sum_{j=1}^{k}\sqrt{r_j}Z^{(j)}\right)\boldsymbol{\tau}_{34},$$
$$T_{12}^{(3)} = \boldsymbol{\tau}'_{12}\left(s\sum_{j=1}^{k}\boldsymbol{z}^{(j)}\boldsymbol{z}^{(j)\prime} + s^2\sum_{i=1}^{k}\sum_{j=1}^{k}\sqrt{r_i r_j}Z^{(i)}Z^{(j)} - skI_p\right)\boldsymbol{\tau}_{12},$$

$$T_{34}^{(3)} = \boldsymbol{\tau}_{34}' \left(s \sum_{j=1}^{k} \boldsymbol{z}^{(j)} \boldsymbol{z}^{(j)\prime} + s^2 \sum_{i=1}^{k} \sum_{j=1}^{k} \sqrt{r_i r_j} Z^{(i)} Z^{(j)} - skI_p \right) \boldsymbol{\tau}_{34},$$

and

$$\boldsymbol{\tau}_{12} = w_1 \boldsymbol{z}^{(1)} - w_2 \boldsymbol{z}^{(2)}, \quad w_1 \equiv w_{12} = \sqrt{\frac{r_2}{r_1 + r_2}}, \quad w_2 \equiv w_{21} = \sqrt{\frac{r_1}{r_1 + r_2}},$$
$$\boldsymbol{\tau}_{34} = w_3 \boldsymbol{z}^{(3)} - w_4 \boldsymbol{z}^{(4)}, \quad w_3 \equiv w_{34} = \sqrt{\frac{r_4}{r_3 + r_4}}, \quad w_4 \equiv w_{43} = \sqrt{\frac{r_3}{r_3 + r_4}}.$$

Some results of the expectation with respect to $Z^{(j)}$ and the joint density function of $\boldsymbol{z}^{(j)}$ and $Z^{(j)}$ are given by Iwashita [2]. Some results of the expectation with respect to $\boldsymbol{z}^{(j)}$ are given by Okamoto and Seo [5]. Using these results, the expectation $\mathrm{E}[\exp(it_1 T_{12}^2 + it_2 T_{34}^2)]$ with respect to $\boldsymbol{z}^{(j)}$ and $Z^{(j)}$, $j = 1, \ldots, k$ is calculated as follows:

$$\begin{aligned} \mathrm{E}[\exp(it_1 T_{12}^2 + it_2 T_{34}^2)] &= (u_1 u_2)^{-\frac{p}{2}} + \frac{1}{N}(u_1 u_2)^{-\frac{p}{2}} \\ &\quad \times \{a_1 + (a_{21} u_1^{-1} + a_{22} u_2^{-1}) \\ &\quad + (a_{31} u_1^{-2} + a_{32} u_2^{-2}) + a_4 (u_1 u_2)^{-1}\} + o(N^{-1}), \end{aligned}$$

where $u_1 = 1 - 2it_1$, $u_2 = 1 - 2it_2$, $i = \sqrt{-1}$,

$$\begin{aligned} a_1 &= -\frac{1}{2} sp(p-1) + \frac{1}{8} p(p+2) \left[\left(\frac{w_1^4}{r_1} - 4sw_1^2 \right) \kappa_1 + \left(\frac{w_2^4}{r_2} - 4sw_2^2 \right) \kappa_2 \right. \\ &\quad \left. + \left(\frac{w_3^4}{r_3} - 4sw_3^2 \right) \kappa_3 + \left(\frac{w_4^4}{r_4} - 4sw_4^2 \right) \kappa_4 \right], \\ a_{21} &= -sp + \frac{1}{4} p(p+2) \Bigg[-2s\kappa_r + s(w_1^2 \kappa_1 + w_2^2 \kappa_2 + w_3^2 \kappa_3 + w_4^2 \kappa_4) \\ &\quad - \left\{ \left(\frac{w_1^4}{r_1} - 4sw_1^2 \right) \kappa_1 + \left(\frac{w_2^4}{r_2} - 4sw_2^2 \right) \kappa_2 \right\} \Bigg], \\ a_{22} &= -sp + \frac{1}{4} p(p+2) \Bigg[-2s\kappa_r + s(w_1^2 \kappa_1 + w_2^2 \kappa_2 + w_3^2 \kappa_3 + w_4^2 \kappa_4) \\ &\quad - \left\{ \left(\frac{w_3^4}{r_3} - 4sw_3^2 \right) \kappa_3 + \left(\frac{w_4^4}{r_4} - 4sw_4^2 \right) \kappa_4 \right\} \Bigg], \\ a_{31} &= \frac{1}{8} p(p+2) \left[2s + 3s\kappa_r + \left(\frac{w_1^4}{r_1} - 6sw_1^2 \right) \kappa_1 + \left(\frac{w_2^4}{r_2} - 6sw_2^2 \right) \kappa_2 \right], \\ a_{32} &= \frac{1}{8} p(p+2) \left[2s + 3s\kappa_r + \left(\frac{w_3^4}{r_3} - 6sw_3^2 \right) \kappa_3 + \left(\frac{w_4^4}{r_4} - 6sw_4^2 \right) \kappa_4 \right], \\ a_4 &= \frac{1}{4} sp \left[2 + (p+2)\kappa_r - (p+2)(w_1^2 \kappa_1 + w_2^2 \kappa_2 + w_3^2 \kappa_3 + w_4^2 \kappa_4) \right], \end{aligned}$$

$\kappa_r = s\sum_{j=1}^{k} r_j\kappa_j$ and $a_1 + a_{21} + a_{22} + a_{31} + a_{32} + a_4 = 0$.

Inverting this characteristic function $C_1(it_1, it_2)$, the following results are obtained.

Theorem 1. *For large N, an asymptotic expansion for the joint probability $\beta_{1\cdot ijkl}(t_1^2)$ is given by*

$$\Pr\{T_{ij}^2 > t_1^2, T_{kl}^2 > t_1^2\} = G_{\frac{p}{2}}^2(\eta_1) + \frac{1}{N}\{c_1 g_{\frac{p}{2}}(\eta_1)G_{\frac{p}{2}}(\eta_1) + c_2 g_{\frac{p}{2}}^2(\eta_1)\} + o(N^{-1}),$$

where

$$\eta_1 = \frac{1}{2}t_1^2, \quad G_{\frac{p}{2}}(\eta_1) = \int_{\eta_1}^{\infty} g_{\frac{p}{2}}(t)dt, \quad g_{\frac{p}{2}}(t) = \frac{1}{\Gamma(\frac{p}{2})}t^{\frac{p}{2}-1}e^{-t},$$

and

$$\begin{aligned} c_1 &= \frac{s\eta_1}{4}\Bigg[4(p + 2\eta_1) + 2(p + 6\eta_1 + 2)\kappa_r \\ &\qquad - \frac{1}{s}(p - 2\eta_1 + 2)K^{(2)} + 2(p - 6\eta_1 + 2)K^{(1)}\Bigg], \\ c_2 &= \frac{s\eta_1^2}{p}[2 + (p + 2)\kappa_r - (p + 2)K^{(1)}], \\ K^{(1)} &= w_i^2\kappa_i + w_j^2\kappa_j + w_k^2\kappa_k + w_l^2\kappa_l, \\ K^{(2)} &= \frac{w_i^4}{r_i}\kappa_i + \frac{w_j^4}{r_j}\kappa_j + \frac{w_k^4}{r_k}\kappa_k + \frac{w_l^4}{r_l}\kappa_l, \\ & w_i \equiv w_{ij}, \ w_j \equiv w_{ji}, \ w_k \equiv w_{kl}, \ w_l \equiv w_{lk}. \end{aligned}$$

Secondly, consider an asymptotic expansion for $\beta_{2\cdot ijk}(t_1^2)$. The joint characteristic function $C_2(it_1, it_2) = \mathrm{E}[\exp(it_1T_{12}^2 + it_2T_{13}^2)]$ can be written as

$$C_2(it_1, it_2) = \mathrm{E}\left[\exp(it_1T_{12}^{(1)} + it_2T_{13}^{(1)})\left(1 + \frac{1}{\sqrt{N}}D_1 + \frac{1}{N}D_2\right)\right] + o(N^{-1}),$$

where

$$\begin{aligned} D_1 &= it_1T_{12}^{(2)} + it_2T_{13}^{(2)}, \\ D_2 &= it_1T_{12}^{(3)} + \frac{(it_1)^2}{2}(T_{12}^{(2)})^2 + it_2T_{13}^{(3)} + \frac{(it_2)^2}{2}(T_{13}^{(2)})^2 + (it_1)(it_2)T_{12}^{(2)}T_{13}^{(2)}, \end{aligned}$$

and

$$\begin{aligned} T_{12}^{(1)} &= \boldsymbol{\tau}'_{12}\boldsymbol{\tau}_{12}, \quad T_{13}^{(1)} = \boldsymbol{\tau}'_{13}\boldsymbol{\tau}_{13}, \\ T_{12}^{(2)} &= -\boldsymbol{\tau}'_{12}\left(s\sum_{j=1}^{k}\sqrt{r_j}Z^{(j)}\right)\boldsymbol{\tau}_{12}, \quad T_{13}^{(2)} = -\boldsymbol{\tau}'_{13}\left(s\sum_{j=1}^{k}\sqrt{r_j}Z^{(j)}\right)\boldsymbol{\tau}_{13}, \\ T_{12}^{(3)} &= \boldsymbol{\tau}'_{12}\left(s\sum_{j=1}^{k}\boldsymbol{z}^{(j)}\boldsymbol{z}^{(j)\prime} + s^2\sum_{i=1}^{k}\sum_{j=1}^{k}\sqrt{r_ir_j}Z^{(i)}Z^{(j)} - skI_p\right)\boldsymbol{\tau}_{12}, \end{aligned}$$

$$T_{13}^{(3)} = \boldsymbol{\tau}_{13}' \left(s \sum_{j=1}^{k} \boldsymbol{z}^{(j)} \boldsymbol{z}^{(j)\prime} + s^2 \sum_{i=1}^{k} \sum_{j=1}^{k} \sqrt{r_i r_j} Z^{(i)} Z^{(j)} - skI_p \right) \boldsymbol{\tau}_{13},$$

and

$$\boldsymbol{\tau}_{12} = w_1 \boldsymbol{z}^{(1)} - w_2 \boldsymbol{z}^{(2)}, \quad w_1 \equiv w_{12} = \sqrt{\frac{r_2}{r_1 + r_2}}, \quad w_2 \equiv w_{21} = \sqrt{\frac{r_1}{r_1 + r_2}},$$
$$\boldsymbol{\tau}_{13} = w_3 \boldsymbol{z}^{(1)} - w_4 \boldsymbol{z}^{(3)}, \quad w_3 \equiv w_{13} = \sqrt{\frac{r_3}{r_1 + r_3}}, \quad w_4 \equiv w_{31} = \sqrt{\frac{r_1}{r_1 + r_3}}.$$

Let

$$\begin{aligned} T_{123} &= \exp(it_1 T_{12}^{(1)} + it_2 T_{13}^{(1)}), & v_0 &= w_1^2 w_3^2, \\ A_1 &= u_1 - (u_1 - 1) w_1^2, & A_2 &= u_2 - (u_2 - 1) w_3^2, \\ A_{14} &= u_1 - (u_1 - 1) v_0, & A_{24} &= u_2 - (u_2 - 1) v_0. \end{aligned}$$

If $\boldsymbol{z}^{(1)}$, $\boldsymbol{z}^{(2)}$ and $\boldsymbol{z}^{(3)}$ are random vectors from the multivariate normal distribution, we obtain some results as follows:

$$\begin{aligned}
\mathrm{E}[T_{123}] &= U^{-\frac{p}{2}}, \\
\mathrm{E}[(\boldsymbol{\tau}_{12}'\boldsymbol{z}^{(1)})^2 T_{123}] &= pU^{-\frac{p}{2}-2} A_2[(p+2)w_1^2 A_2 + w_2^2 U], \\
\mathrm{E}[(\boldsymbol{\tau}_{13}'\boldsymbol{z}^{(1)})^2 T_{123}] &= pU^{-\frac{p}{2}-2} A_1[(p+2)w_3^2 A_1 + w_4^2 U], \\
\mathrm{E}[(\boldsymbol{\tau}_{12}'\boldsymbol{z}^{(2)})^2 T_{123}] &= pU^{-\frac{p}{2}-2}[(p+2)w_2^2 u_2^2 + w_1^2 A_2 U], \\
\mathrm{E}[(\boldsymbol{\tau}_{13}'\boldsymbol{z}^{(3)})^2 T_{123}] &= pU^{-\frac{p}{2}-2}[(p+2)w_4^2 u_1^2 + w_3^2 A_1 U], \\
\mathrm{E}[(\boldsymbol{\tau}_{13}'\boldsymbol{z}^{(2)})^2 T_{123}] &= pU^{-\frac{p}{2}-2}[(p+2)(u_1-1)^2 v_0 w_2^2 \\
&\qquad +\{1 + (u_1 - 1)w_1^2 w_4^2\}U], \\
\mathrm{E}[(\boldsymbol{\tau}_{12}'\boldsymbol{z}^{(3)})^2 T_{123}] &= pU^{-\frac{p}{2}-2}[(p+2)(u_2-1)^2 v_0 w_4^2 \\
&\qquad +\{1 + (u_2 - 1)w_3^2 w_2^2\}U], \\
\mathrm{E}[(\boldsymbol{\tau}_{12}'\boldsymbol{\tau}_{12})(\boldsymbol{z}^{(1)\prime}\boldsymbol{z}^{(1)})T_{123}] &= pU^{-\frac{p}{2}-2} A_2[(p+2)w_1^2 A_2 + p w_2^2 U], \\
\mathrm{E}[(\boldsymbol{\tau}_{13}'\boldsymbol{\tau}_{13})(\boldsymbol{z}^{(1)\prime}\boldsymbol{z}^{(1)})T_{123}] &= pU^{-\frac{p}{2}-2} A_1[(p+2)w_3^2 A_1 + p w_4^2 U], \\
\mathrm{E}[(\boldsymbol{\tau}_{12}'\boldsymbol{\tau}_{12})(\boldsymbol{z}^{(2)\prime}\boldsymbol{z}^{(2)})T_{123}] &= pU^{-\frac{p}{2}-2}[(p+2)w_2^2 u_2^2 + p w_1^2 A_2 U], \\
\mathrm{E}[(\boldsymbol{\tau}_{13}'\boldsymbol{\tau}_{13})(\boldsymbol{z}^{(3)\prime}\boldsymbol{z}^{(3)})T_{123}] &= pU^{-\frac{p}{2}-2}[(p+2)w_4^2 u_1^2 + p w_3^2 A_1 U], \\
\mathrm{E}[(\boldsymbol{\tau}_{13}'\boldsymbol{\tau}_{13})(\boldsymbol{z}^{(2)\prime}\boldsymbol{z}^{(2)})T_{123}] &= pU^{-\frac{p}{2}-2}[(p+2)(u_1-1)^2 v_0 w_2^2 \\
&\qquad +p\{1 + (u_1 - 1)w_1^2 w_4^2\}U], \\
\mathrm{E}[(\boldsymbol{\tau}_{12}'\boldsymbol{\tau}_{12})(\boldsymbol{z}^{(3)\prime}\boldsymbol{z}^{(3)})T_{123}] &= pU^{-\frac{p}{2}-2}[(p+2)(u_2-1)^2 v_0 w_4^2 \\
&\qquad +p\{1 + (u_2 - 1)w_3^2 w_2^2\}U], \\
\mathrm{E}[(\boldsymbol{\tau}_{12}'\boldsymbol{\tau}_{12})T_{123}] &= pU^{-\frac{p}{2}-2}[A_{24}U],
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}[(\boldsymbol{\tau}_{13}'\boldsymbol{\tau}_{13})T_{123}] &= pU^{-\frac{p}{2}-2}[A_{14}U],\\
\mathrm{E}[(\boldsymbol{\tau}_{12}'\boldsymbol{\tau}_{12})^2T_{123}] &= pU^{-\frac{p}{2}-2}[(p+2)A_{24}^2],\\
\mathrm{E}[(\boldsymbol{\tau}_{13}'\boldsymbol{\tau}_{13})^2T_{123}] &= pU^{-\frac{p}{2}-2}[(p+2)A_{14}^2],\\
\mathrm{E}[(\boldsymbol{\tau}_{12}'\boldsymbol{\tau}_{13})^2T_{123}] &= pU^{-\frac{p}{2}-2}[(p+2)v_0+(1-v_0)U],\\
\mathrm{E}[(\boldsymbol{\tau}_{12}'\boldsymbol{\tau}_{12})(\boldsymbol{\tau}_{13}'\boldsymbol{\tau}_{13})T_{123}] &= pU^{-\frac{p}{2}-2}[(p+2)v_0+p(1-v_0)U],
\end{aligned}$$

where $U = u_1u_2 - (u_1-1)(u_2-1)v_0$.

Therefore, we obtain the asymptotic expansion for the expectation of $\exp(it_1\, T_{12}^{(1)} + it_2T_{13}^{(1)})$ in elliptical distributions.

$$\begin{aligned}
&\mathrm{E}[\exp(it_1T_{12}^{(1)} + it_2T_{13}^{(1)})]\\
&\quad = U^{-\frac{p}{2}} + \frac{1}{8N}p(p+2)U^{-\frac{p}{2}-2}\\
&\qquad \times\left[\frac{1}{r_1}\{(u_1-1)u_2w_1^2 + (u_2-1)u_1w_3^2 - 2(u_1-1)(u_2-1)v_0\}^2\kappa_1\right.\\
&\qquad\qquad \left. +\frac{1}{r_2}(u_1-1)^2u_2^2w_2^4\kappa_2 + \frac{1}{r_3}u_1^2(u_2-1)^2w_4^4\kappa_3\right] + o(N^{-1}).
\end{aligned}$$

Let $\lambda_1 = 1-2(1-v_0)it_1$, $\lambda_2 = 1-2(1-v_0)it_2$, then

$$u_1 = \frac{\lambda_1 - v_0}{1-v_0}, \quad u_2 = \frac{\lambda_2 - v_0}{1-v_0}, \quad U = \frac{\lambda_1\lambda_2 - v_0}{1-v_0}.$$

Calculating the expectation $\mathrm{E}[\exp(it_1T_{12}^2 + it_2T_{13}^2)]$ with respect to $\boldsymbol{z}^{(j)}$ and $Z^{(j)}$, $j = 1,\ldots,k$ by using above results, we have

$$C_2(it_1, it_2) = U^{-\frac{p}{2}}\left[1 + \frac{1}{N}(b_1U^{-1} + b_2U^{-2})\right] + o(N^{-1})$$

with the coefficients b_1 and b_2 given by

$$\begin{aligned}
b_1 &= \frac{p}{128v_1^3}\left\{64(p+1)sv_1^2b_{11} + 8(p+2)sv_1b_{12} - (p+2)b_{13}\right\},\\
b_2 &= \frac{p(p+2)}{512v_1^4}(128sv_1^2b_{21} + 32sv_1b_{22} + 192sv_1^2b_{23} + 4b_{24}),
\end{aligned}$$

where $v_1 = v_0 - 1$, $v_2 = w_1^2 + w_3^2$,

$$\begin{aligned}
b_{11} &= 3\lambda_1\lambda_2 - 2(\lambda_1+\lambda_2) + 1,\\
b_{12} &= (-4\lambda_1\lambda_2 + \lambda_1 + \lambda_2 + 2)\kappa_1 + (-4\lambda_1\lambda_2 + \lambda_1 - 2\lambda_2 + 5)\kappa_2\\
&\quad +(-4\lambda_1\lambda_2 - 2\lambda_1 + \lambda_2 + 5)\kappa_3 + 4(1-v_0)(\lambda_1+\lambda_2-2)\kappa_r,\\
b_{13} &= 4(\lambda_1\lambda_2 - 1)\kappa_1 + (4\lambda_1\lambda_2 - \lambda_1 - 2\lambda_2 - 1)\kappa_2 + (4\lambda_1\lambda_2 - 2\lambda_1 - \lambda_2 - 1)\kappa_3,\\
b_{21} &= (\lambda_1 + \lambda_2 - 2\lambda_1\lambda_2)^2,
\end{aligned}$$

$$
\begin{aligned}
b_{22} \;=\; & \Big[(12w_2^4w_3^2)\lambda_1^2+(12w_1^2w_4^4)\lambda_2^2-4(4v_0-2v_2+1)\lambda_1^2\lambda_2^2\\
&+\left\{-4\left(4w_3^4-10w_3^2+5\right)w_1^2-4w_3^2+1\right\}\lambda_1\lambda_2^2\\
&+\left\{-4\left(4w_1^4-10w_1^2+5\right)w_3^2-4w_1^2+1\right\}\lambda_1^2\lambda_2\\
&+v_0\left\{8w_1^2-28v_2+v_0\left(-8w_1^2+32\right)+23\right\}\lambda_1\\
&+v_0\left\{8w_3^2-28v_2+v_0\left(-8w_3^2+32\right)+23\right\}\lambda_2\\
&+2\left\{-2v_0(8v_0-11v_2+15)+2v_2+1\right\}\lambda_1\lambda_2\\
&+2v_0\left\{4v_0(v_2-4)+4v_2-1\right\}\Big]\kappa_1\\
&+\Big[12v_0w_2^2\lambda_1^2+12w_2^2\lambda_2^2-4\left(2w_1^2-1\right)\lambda_1^2\lambda_2^2\\
&+2\left(10w_1^2-11\right)\lambda_1\lambda_2^2+\left\{4\left(w_1^2-4v_0w_2^2\right)-3\right\}\lambda_1^2\lambda_2\\
&+v_0\left\{4\left(5w_1^2-2v_0w_2^2\right)-21\right\}\lambda_1+2v_0\left(14w_1^2-13\right)\lambda_2\\
&+\left\{-4w_1^2-4v_0(11w_1^2-12)+9\right\}\lambda_1\lambda_2\\
&+v_0\left\{-8\left(v_0+w_4^2\right)w_1^2+3\right\}\Big]\kappa_2\\
&+\Big[12w_4^2\lambda_1^2+12v_0w_4^2\lambda_2^2-4\left(2w_3^2-1\right)\lambda_1^2\lambda_2^2\\
&+\left\{4\left(w_3^2-4v_0w_4^2\right)-3\right\}\lambda_1\lambda_2^2+2\left(10w_3^2-11\right)\lambda_1^2\lambda_2\\
&+2v_0\left(14w_3^2-13\right)\lambda_1+v_0\left\{4(5w_3^2-2v_0w_4^2)-21\right\}\lambda_2\\
&+\left\{-4w_3^2-4v_0\left(11w_3^2-12\right)+9\right\}\lambda_1\lambda_2\\
&+v_0\left\{-8\left(v_0+w_2^2\right)w_3^2+3\right\}\Big]\kappa_3,
\end{aligned}
$$

$$
b_{23} \;=\; (-2\lambda_1\lambda_2+\lambda_1+\lambda_2)(-2v_0+\lambda_1+\lambda_2)\kappa_r,
$$

$$
\begin{aligned}
b_{24} \;=\; & \frac{4}{r_1}\Big[4w_2^8w_3^4\lambda_1^2+4w_1^4w_4^8\lambda_2^2+\left\{4(v_2-2v_0)^2-r_1\right\}\lambda_1^2\lambda_2^2\\
&+8(2v_0-v_2)\left(v_2-w_3^2\right)w_4^4\lambda_1\lambda_2^2+8(2v_0-v_2)w_2^4w_3^2\lambda_1^2\lambda_2\\
&-8v_0(v_2-2)w_2^4w_3^2\lambda_1-8v_0(v_2-2)w_1^2w_4^4\lambda_2\\
&+\left[r_1(v_0+1)+8v_0\left\{\left(w_4^4+2w_4^2-1\right)w_2^4-2w_2^2w_3^2w_4^2-w_4^4\right\}\right]\lambda_1\lambda_2\\
&+\left\{4v_0^2(v_2-2)^2-r_1v_0\right\}\Big]\kappa_1\\
&+\frac{1}{r_2}\Big[16v_0^2w_2^4\lambda_1^2+16w_2^4\lambda_2^2+4\left(4w_2^4-r_2\right)\lambda_1^2\lambda_2^2-2\left(16w_2^4-r_2\right)\lambda_1\lambda_2^2\\
&-(32v_0w_2^4-r_2)\lambda_1^2\lambda_2-v_0\left(32v_0w_2^4+r_2\right)\lambda_1-2v_0\left(16w_2^4+r_2\right)\lambda_2\\
&+\left\{4v_0\left(16w_2^4+r_2\right)+r_2\right\}\lambda_1\lambda_2+v_0\left(16v_0w_2^4-r_2\right)\Big]\kappa_2
\end{aligned}
$$

$$+\frac{1}{r_3}\Bigg[16w_4^4\lambda_1^2+16v_0^2w_4^4\lambda_2^2+4\left(4w_4^4-r_3\right)\lambda_1^2\lambda_2^2$$
$$-(32v_0w_4^4-r_3)\lambda_1\lambda_2^2-2\left(16w_4^4-r_3\right)\lambda_1^2\lambda_2$$
$$-2v_0\left(16w_4^4+r_3\right)\lambda_1-v_0\left(32v_0w_4^4+r_3\right)\lambda_2$$
$$+\left\{4v_0\left(16w_4^4+r_3\right)+r_3\right\}\lambda_1\lambda_2+v_0\left(16v_0w_4^4-r_3\right)\Bigg]\kappa_3.$$

We also note that

$$U^{-\frac{p}{2}}=(1-v_0)^{\frac{p}{2}}\sum_{m=0}^{\infty}\frac{\left(\frac{1}{2}p\right)_m}{m!}v_0^m\lambda_1^{-\frac{p}{2}-m}\lambda_2^{-\frac{p}{2}-m},$$

where

$$\left(\frac{1}{2}p\right)_m=\frac{\Gamma\left(\frac{p}{2}+m\right)}{\Gamma\left(\frac{p}{2}\right)}=\frac{p}{2}\left(\frac{p}{2}+1\right)\cdots\left(\frac{p}{2}+m-1\right).$$

Therefore, we obtain the following theorem.

Theorem 2. *For large N, an asymptotic expansion for the joint probability $\beta_{2\cdot ijk}(t_1^2)$ is given by*

$$\Pr\left\{T_{ij}^2>t_1^2, T_{ik}^2>t_1^2\right\}$$
$$=(1-v_0)^{\frac{p}{2}}\sum_{m=0}^{\infty}\frac{\left(\frac{1}{2}p\right)_m}{m!}v_0^m$$
$$\times\left[G^2_{\frac{p}{2}+m}(\eta_2)+\frac{1}{N}\left\{d_1g_{\frac{p}{2}+m}(\eta_2)G_{\frac{p}{2}+m}(\eta_2)+d_2g^2_{\frac{p}{2}+m}(\eta_2)\right\}\right]+o(N^{-1}),$$

where

$$\eta_2=\frac{1}{2(1-v_0)}t_1^2,$$
$$G_{\frac{p}{2}+m}(\eta_2)=\int_{\eta_2}^{\infty}g_{\frac{p}{2}+m}(t)dt,\quad g_{\frac{p}{2}+m}(t)=\frac{1}{\Gamma\left(\frac{p}{2}+m\right)}t^{\frac{p}{2}+m-1}e^{-t},$$

and

$$d_1 = \frac{\eta_2}{32v_1^2}\left\{32sv_1^2(p-2m+2\eta_2)+8sv_1d_{11}+d_{12}\right\},$$
$$d_2 = \frac{\eta_2^2}{16qv_1^2(p+2m)}\left\{32sqv_1^2(2m+1)+8sv_1d_{21}+d_{22}\right\},$$
$$d_{11} = 2\left[3(m-\eta_2v_0)+v_1v_2\left\{2\eta_2(2v_1-1)+q\right\}\right]\kappa_1$$
$$+\left[2v_1w_2^2(4v_1\eta_2+q)+9m+\eta_2\left\{v_1(4w_1^2-13)-9\right\}\right]\kappa_2$$
$$+\left[2v_1w_4^2(4v_1\eta_2+q)+9m+\eta_2\left\{v_1(4w_3^2-13)-9\right\}\right]\kappa_3$$
$$+\left[2v_1\left\{p+6m-6\eta_2(2v_1+1)+2\right\}\right]\kappa_r,$$

$$\begin{aligned}
d_{12} &= 8\left[\frac{1}{r_1}(2\eta_2-q)v_1^2(v_2^2-2v_0)+m-\eta_2(v_1+1)\right]\kappa_1 \\
&\quad+\left[\frac{8}{r_2}(2\eta_2-q)v_1^2w_2^4+5m-5\eta_2(v_1+1)\right]\kappa_2 \\
&\quad+\left[\frac{8}{r_3}(2\eta_2-q)v_1^2w_4^4+5m-5\eta_2(v_1+1)\right]\kappa_3, \\
d_{21} &= \left[4v_0\eta_2^2\left\{4v_0(v_2-4)+4v_2-1\right\}+\left\{-8v_0+2(v_0+1)v_2+1\right\}q^2\right. \\
&\qquad\left.-\left\{p-2v_0\eta_2(4v_2(v_0-4)+21)+2\right\}q\right]\kappa_1 \\
&\quad+\left[2v_0\eta_2^2\left\{-8(v_0+1)w_1^2+8v_0+3\right\}\right. \\
&\qquad+v_0\eta_2\left\{-8(v_0-4)w_1^2+8v_0-41\right\}q+5m^2+2(p+2)^2(v_0+1)w_2^2 \\
&\qquad\left.+(p+m+2)m\left\{-8(v_0+1)w_1^2+8v_0+13\right\}\right]\kappa_2 \\
&\quad+\left[2v_0\eta_2^2\left\{-8(v_0+1)w_3^2+8v_0+3\right\}\right. \\
&\qquad+v_0\eta_2\left\{-8(v_0-4)w_3^2+8v_0-41\right\}q+5m^2+2(p+2)^2(v_0+1)w_4^2 \\
&\qquad\left.+(p+m+2)m\left\{-8(v_0+1)w_3^2+8v_0+13\right\}\right]\kappa_3 \\
&\quad+\left[2v_1(p+6m-12v_0\eta_2+2)q\right]\kappa_r, \\
d_{22} &= \left[4\left\{(m-2v_0\eta_2)q-2v_0\eta_2^2\right\}\right. \\
&\qquad+\frac{8v_0}{r_1}\left[\left\{(v_2-2)^2+v_1\left(2v_1-v_2^2+4\right)\right\}q^2\right. \\
&\qquad\qquad\left.\left.+4\eta_2(2v_1-v_2+2)(v_2-2)q+4v_0\eta_2^2(v_2-2)^2\right]\right]\kappa_1 \\
&\quad+\left[\frac{8v_0w_2^4}{r_2}(2\eta_2-q)\left\{2v_0\eta_2+(v_1-1)q\right\}+(m-5v_0\eta_2)q-2v_0\eta_2^2\right]\kappa_2 \\
&\quad+\left[\frac{8v_0w_4^4}{r_3}(2\eta_2-q)\left\{2v_0\eta_2+(v_1-1)q\right\}+(m-5v_0\eta_2)q-2v_0\eta_2^2\right]\kappa_3, \\
q &= p+2m+2,\ \ w_1\equiv w_{ij},\ \ w_2\equiv w_{ji},\ \ w_3\equiv w_{ik},\ \ w_4\equiv w_{ki}.
\end{aligned}$$

Therefore, the modified second order Bonferroni approximate upper 100α percentiles of $T^2_{\max}$, that is, $t^2_{M\cdot\chi^2}\equiv t^2_{M\cdot\chi^2}(\alpha)$ and $t^2_{M\cdot F}\equiv t^2_{M\cdot F}(\alpha)$, are obtained as follows:

$$\begin{aligned}
(3.1)\quad t^2_{M\cdot\chi^2} &= \chi^2_p(\gamma)-\frac{1}{2NK}\chi^2_p(\gamma) \\
&\quad\times\sum_{l=1}^{k-1}\sum_{m=l+1}^{k}\left\{\frac{1}{p}c^{(0)}_{lm}-\frac{1}{p(p+2)}c^{(2)}_{lm}\chi^2_p(\gamma)\right\}, \\
(3.2)\quad t^2_{M\cdot F} &= \frac{\nu p}{\nu-p+1}F_{p,\nu-p+1}(\gamma)-\frac{1}{2NK}\chi^2_p(\gamma) \\
&\quad\times\sum_{l=1}^{k-1}\sum_{m=l+1}^{k}\left\{\left(\frac{1}{p}c^{(0)}_{lm}+sp\right)-\left(\frac{1}{p(p+2)}c^{(2)}_{lm}-s\right)\chi^2_p(\gamma)\right\},
\end{aligned}$$

where $\gamma = \{\alpha + \beta(t_1^2)\}/K$ and

$$\beta(t_1^2) = \sum \beta_{1\cdot ijkl}(t_1^2) + \sum \beta_{2\cdot ijk}(t_1^2)$$

(i, j, k, l are all distinct for $\beta_{1\cdot ijkl}(t_1^2)$ and i, j, k are all distinct for $\beta_{2\cdot ijk}(t_1^2)$). When sample sizes are equal, these results coincide with those of Seo [7].

§4. Approximations for comparisons with a control

In this section, simultaneous confidence intervals for comparisons with a control among k independent p-dimensional mean vectors are discussed under elliptical populations. Letting the first population be a control, the simultaneous confidence intervals for comparisons with a control among mean vectors are given by

$$\boldsymbol{a}'(\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(m)}) \in \left[\boldsymbol{a}'(\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(m)}) \pm t\sqrt{d_{1m}\boldsymbol{a}'S\boldsymbol{a}}\right], \quad \forall \boldsymbol{a} \in \mathbf{R}^p - \{\mathbf{0}\},\ 2 \le m \le k.$$

The value t (> 0) satisfies as follows:

$$\Pr\left\{T^2_{\max\cdot c} > t^2\right\} = \alpha,$$

where

$$T^2_{\max\cdot c} = \max_{2\le m\le k}\left\{T^2_{1m}\right\},$$
$$T^2_{1m} = d_{1m}^{-1}\left(\boldsymbol{y}^{(1)} - \boldsymbol{y}^{(m)}\right)' S^{-1}\left(\boldsymbol{y}^{(1)} - \boldsymbol{y}^{(m)}\right).$$

By Bonferroni's inequality for $\Pr\left\{T^2_{\max\cdot c} > t^2\right\}$:

$$\Pr\left\{T^2_{\max\cdot c} > t^2\right\} < \sum_{m=2}^{k} \Pr\left\{T^2_{1m} > t^2\right\},$$

the approximate upper percentile t_1^2 of $T^2_{\max\cdot c}$ is given by

$$\sum_{m=2}^{k} \Pr\left\{T^2_{1m} > t_1^2\right\} = \alpha. \tag{4.1}$$

Let $\boldsymbol{y}_1 = w_{12}\boldsymbol{z}^{(1)} - w_{21}\boldsymbol{z}^{(2)}$, $\boldsymbol{y}_2 = w_{13}\boldsymbol{z}^{(1)} - w_{31}\boldsymbol{z}^{(3)}, \ldots, \boldsymbol{y}_{k-1} = w_{1k}\boldsymbol{z}^{(1)} - w_{k1}\boldsymbol{z}^{(k)}$, $w_{lm} = \sqrt{r_m/(r_l + r_m)}$. Bonferroni's inequality for $\Pr\{T^2_{\max\cdot c} > t^2\}$ is given by

$$\sum_{i=1}^{k-1} \Pr\left\{\boldsymbol{y}_i' S^{-1}\boldsymbol{y}_i > t^2\right\} - \beta_c(t^2) < \Pr\{T^2_{\max\cdot c} > t^2\} < \sum_{i=1}^{k-1} \Pr\left\{\boldsymbol{y}_i' S^{-1}\boldsymbol{y}_i > t^2\right\},$$

where

$$\beta_c(t^2) = \sum_{i=1}^{k-2} \sum_{j=i+1}^{k-1} \Pr\left\{ \boldsymbol{y}_i' S^{-1} \boldsymbol{y}_i > t^2,\ \boldsymbol{y}_j' S^{-1} \boldsymbol{y}_j > t^2 \right\}.$$

The first order Bonferroni approximation t_1^2 is defined as a critical value that satisfies the equality

$$T_{1\cdot c}^2(t_1^2) \equiv \sum_{i=1}^{k-1} \Pr\left\{ \boldsymbol{y}_i' S^{-1} \boldsymbol{y}_i > t_1^2 \right\} = \alpha.$$

Note that $T_{1\cdot c}^2(t_1^2)$ is equal to the left side in (4.1). The second order Bonferroni approximation t_2^2 is defined as a critical value that satisfies the equality

$$T_{2\cdot c}^2(t_2^2) \equiv \sum_{i=1}^{k-1} \Pr\left\{ \boldsymbol{y}_i' S^{-1} \boldsymbol{y}_i > t_2^2 \right\} - \beta_c(t_2^2) = \alpha.$$

The modified second order Bonferroni approximation t_{Mc}^2 is defined as a critical value that satisfies the equality

$$\sum_{i=1}^{k-1} \Pr\left\{ \boldsymbol{y}_i' S^{-1} \boldsymbol{y}_i > t_{Mc}^2 \right\} = \alpha + \beta_c(t_1^2),$$

where

$$\beta_c(t_1^2) = \sum \beta_{2\cdot ijk}(t_1^2)$$

$(i, j, k$ are all distinct).

By using the same way as pairwise multiple comparisons, the first and the modified second order Bonferroni approximate upper 100α percentiles of $T_{\max\cdot c}^2$ are obtained as follows:

$$\begin{aligned} t_{1\cdot\chi^2\cdot c}^2(\alpha) &= \chi_p^2\left(\frac{\alpha}{k-1}\right) - \frac{1}{2N(k-1)}\chi_p^2\left(\frac{\alpha}{k-1}\right) \\ &\quad \times \sum_{m=2}^{k}\left\{\frac{1}{p}c_{1m}^{(0)} - \frac{1}{p(p+2)}c_{1m}^{(2)}\chi_p^2\left(\frac{\alpha}{k-1}\right)\right\}, \\ t_{1\cdot F\cdot c}^2(\alpha) &= \frac{\nu p}{\nu-p+1}F_{p,\nu-p+1}\left(\frac{\alpha}{k-1}\right) - \frac{1}{2N(k-1)}\chi_p^2\left(\frac{\alpha}{k-1}\right) \\ &\quad \times \sum_{m=2}^{k}\left\{\left(\frac{1}{p}c_{1m}^{(0)} + sp\right) - \left(\frac{1}{p(p+2)}c_{1m}^{(2)} - s\right)\chi_p^2\left(\frac{\alpha}{k-1}\right)\right\}, \end{aligned}$$

$$\begin{aligned}
t^2_{M\cdot\chi^2\cdot c}(\alpha) &= \chi^2_p(\gamma_c) - \frac{1}{2N(k-1)}\chi^2_p(\gamma_c) \\
&\quad \times \sum_{m=2}^{k}\left\{\frac{1}{p}c^{(0)}_{1m} - \frac{1}{p(p+2)}c^{(2)}_{1m}\chi^2_p(\gamma_c)\right\}, \\
t^2_{M\cdot F\cdot c}(\alpha) &= \frac{\nu p}{\nu-p+1}F_{p,\nu-p+1}(\gamma_c) - \frac{1}{2N(k-1)}\chi^2_p(\gamma_c) \\
&\quad \times \sum_{m=2}^{k}\left\{\left(\frac{1}{p}c^{(0)}_{1m} + sp\right) - \left(\frac{1}{p(p+2)}c^{(2)}_{1m} - s\right)\chi^2_p(\gamma_c)\right\},
\end{aligned}$$

where $\gamma_c = \{\alpha + \beta_c(t_1^2)\}/(k-1)$.

§5. Accuracy and conservativeness of approximations

In order to evaluate the accuracy and conservativeness of the obtained approximations, the Monte Carlo simulation for the upper percentiles of $T^2_{\max}$ is implemented for varied parameters. The accuracy of the modified second order Bonferroni approximation is compared with that of the first order Bonferroni approximation. The accuracy of the first order Bonferroni approximation is good when the number of populations is small. However, that becomes worse as the number of populations or the kurtosis parameter increases for $k \geq 6$. Therefore, we discuss the accuracy and conservativeness for $k = 6, 10$ throughout this section. In the simulation, the k populations have the same distributions and consider three types of distributions: the multivariate normal ($\kappa = 0$), the ε-contaminated normal ($\kappa = 1.78$ with $\varepsilon = 0.1$ & $\sigma = 3$) and the ε-contaminated normal ($\kappa = 3.24$ with $\varepsilon = 0.1$ & $\sigma = 4$).

Tables 1–3 give the simulated and approximate values of the upper percentile of $T_{\max}$ ($= \sqrt{T^2_{\max}}$) for the following parameters: $p = 5$, $k = 6$, 10, N_j ($= N$) $= 10$, 20, 40, 80 ($j = 1, \ldots, k$), $r = 1$ and $\alpha = 0.05$. Tables 4–6 give them for the following parameters: $p = 5$, $k = 6$, 10, $r = 0.5$, $\alpha = 0.05$ and $N = 10$, 20, 40, 80; the sample sizes of the first $k/2$ populations are N and the rest of them are rN. For example, when $k = 6$, $N_1 = N_2 = N_3 = N$ and $N_4 = N_5 = N_6 = rN$. Values $t_{1\cdot\chi^2}$, $t_{1\cdot F}$, $t_{M\cdot\chi^2}$ and $t_{M\cdot F}$ stand for approximations $\sqrt{t^2_{1\cdot\chi^2}}$, $\sqrt{t^2_{1\cdot F}}$, $\sqrt{t^2_{M\cdot\chi^2}}$ and $\sqrt{t^2_{M\cdot F}}$ found in (2.3), (2.4), (3.1) and (3.2), respectively. $P_{1\cdot\chi^2}$, $P_{1\cdot F}$, $P_{M\cdot\chi^2}$ and $P_{M\cdot F}$ stand for lower tail probabilities $\Pr\{T^2_{\max} < t^2_{1\cdot\chi^2}\}$, $\Pr\{T^2_{\max} < t^2_{1\cdot F}\}$, $\Pr\{T^2_{\max} < t^2_{M\cdot\chi^2}\}$ and $\Pr\{T^2_{\max} < t^2_{M\cdot F}\}$, respectively. t^* is a simulated value and $\Pr\{T^2_{\max} < t^{*2}\} = 1 - \alpha$. If lower tail probability is larger than $1 - \alpha$, we can construct conservative simultaneous confidence intervals from (2.1). Figures 2–4 show the graphs of the corresponding data in Tables 1–3 for $k = 10$, $r = 1$, $\kappa = 0, 1.78, 3.24$. Figures 5–7 show them in Tables 4–6 for $k = 10$, $r = 0.5$, $\kappa = 0, 1.78, 3.24$.

In Tables 1–3 and Figures 2–4, the modified second order Bonferroni approximations $t_{M \cdot \chi^2}$ and $t_{M \cdot F}$ have highly precise for large N even if κ is large, that is, lower tail probabilities are very close to 0.95 and at the same time these are conservative. In general, the effect of nonnormality becomes small as α increases, and approximate accuracy is good, see Okamoto and Seo [5]. Therefore, the effect of nonnormality for the modified second order Bonferroni approximation is smaller than that for the first order Bonferroni approximation. Approximate errors are occurred by Bonferroni's inequality and an asymptotic expansion. The first order Bonferroni approximation always leads to overestimates when the error by using the asymptotic expansion is ignored. Underestimates occur when the error of the asymptotic expansion is larger than that of Bonferroni's inequality. It is not always true that the modified second order Bonferroni approximation leads to overestimates even if the error of the asymptotic expansion is ignored. When the first order Bonferroni approximation is not conservative, the modified second order Bonferroni approximation is not conservative, either, see e.g. Figure 5. There is a case that the modified second order Bonferroni approximation is not conservative though the first order Bonferroni approximation is conservative, see e.g. Figure 2. For such cases, it is not useful to use the modified second order Bonferroni approximation. However, both the first and the modified second order Bonferroni approximations are conservative for large N regardless of κ, p, k, and r.

In Tables 4–6 and Figures 5–7, lower tail probability of the first order Bonferroni approximation and that of the modified second order Bonferroni approximation come to hardly change as κ increases. This means that the effect of nonnormality becomes large and $\beta(t_1^2)$ becomes small. The effect of nonnormality becomes small and approximate accuracy becomes good as r increases. The probability $\beta(t_1^2)$ tends to become small as r decreases, that is, the conservativeness of the first order Bonferroni approximation gets closer to that of the modified second order Bonferroni approximation for small r. For example, in Table 2, $P_{M \cdot F} = 0.960$ for $k = 10$, $r = 1$, $N = 20$; total sample size is 200. In Table 5, $P_{M \cdot F} = 0.960$ for $k = 10$, $r = 0.5$, $N = 80 : 40$; total sample size is 600. Therefore, approximate accuracy for $r = 1$ is better even if total sample size is smaller than that for $r = 0.5$. Though the modified second order Bonferroni approximation does not always become theoretically conservative, this value is conservative for many parameters as results of simulation. It is preferable that r is close to 1 and we recommend to use the modified second order Bonferroni approximation for $k \geq 6$.

$\kappa = 0, r = 1, p = 5, \alpha = 0.05$										
k	N	$t_{1\cdot\chi^2}$	$t_{1\cdot F}$	$P_{1\cdot\chi^2}$	$P_{1\cdot F}$	$t_{M\cdot\chi^2}$	$t_{M\cdot F}$	$P_{M\cdot\chi^2}$	$P_{M\cdot F}$	t^*
6	10	4.59	4.71	.947	.960	4.56	4.68	.942	.957	4.62
	20	4.40	4.43	.955	.959	4.36	4.38	.950	.953	4.36
	40	4.31	4.31	.957	.958	4.26	4.26	.950	.951	4.25
	80	4.26	4.26	.958	.958	4.21	4.21	.951	.951	4.20
10	10	4.78	4.85	.953	.961	4.74	4.80	.947	.956	4.76
	20	4.64	4.66	.958	.960	4.59	4.60	.950	.953	4.58
	40	4.57	4.58	.959	.959	4.52	4.52	.950	.951	4.51
	80	4.54	4.54	.959	.959	4.48	4.48	.951	.951	4.48

Table 1: Approximations and lower tail probabilities for equal sample sizes.

$\kappa = 1.78, r = 1, p = 5, \alpha = 0.05$										
k	N	$t_{1\cdot\chi^2}$	$t_{1\cdot F}$	$P_{1\cdot\chi^2}$	$P_{1\cdot F}$	$t_{M\cdot\chi^2}$	$t_{M\cdot F}$	$P_{M\cdot\chi^2}$	$P_{M\cdot F}$	t^*
6	10	4.59	4.71	.950	.963	4.56	4.68	.946	.960	4.59
	20	4.40	4.43	.957	.960	4.36	4.38	.951	.954	4.35
	40	4.31	4.31	.958	.959	4.26	4.26	.951	.952	4.25
	80	4.26	4.26	.958	.958	4.21	4.21	.951	.951	4.21
10	10	5.02	5.09	.969	.974	5.00	5.06	.966	.972	4.85
	20	4.77	4.78	.964	.966	4.72	4.74	.959	.960	4.66
	40	4.64	4.64	.961	.962	4.58	4.58	.953	.954	4.56
	80	4.57	4.57	.960	.960	4.51	4.51	.951	.951	4.51

Table 2: Approximations and lower tail probabilities for equal sample sizes.

$\kappa = 3.24, r = 1, p = 5, \alpha = 0.05$										
k	N	$t_{1\cdot\chi^2}$	$t_{1\cdot F}$	$P_{1\cdot\chi^2}$	$P_{1\cdot F}$	$t_{M\cdot\chi^2}$	$t_{M\cdot F}$	$P_{M\cdot\chi^2}$	$P_{M\cdot F}$	t^*
6	10	4.59	4.71	.953	.966	4.56	4.68	.949	.963	4.56
	20	4.40	4.43	.959	.962	4.36	4.38	.953	.956	4.34
	40	4.31	4.31	.958	.959	4.26	4.26	.951	.952	4.25
	80	4.26	4.26	.957	.958	4.21	4.21	.951	.951	4.21
10	10	5.21	5.28	.979	.983	5.20	5.26	.978	.982	4.91
	20	4.87	4.89	.970	.971	4.83	4.85	.966	.967	4.71
	40	4.69	4.69	.964	.964	4.63	4.64	.956	.956	4.59
	80	4.60	4.60	.961	.961	4.54	4.54	.953	.953	4.53

Table 3: Approximations and lower tail probabilities for equal sample sizes.

$\kappa = 0, r = 0.5, p = 5, \alpha = 0.05$										
k	N	$t_{1\cdot\chi^2}$	$t_{1\cdot F}$	$P_{1\cdot\chi^2}$	$P_{1\cdot F}$	$t_{M\cdot\chi^2}$	$t_{M\cdot F}$	$P_{M\cdot\chi^2}$	$P_{M\cdot F}$	t^*
6	10:5	4.71	4.94	.937	.962	4.70	4.93	.936	.961	4.81
	20:10	4.47	4.52	.954	.960	4.44	4.49	.951	.958	4.43
	40:20	4.34	4.35	.958	.959	4.31	4.32	.954	.956	4.28
	80:40	4.27	4.28	.959	.959	4.24	4.25	.955	.955	4.21
10	10:5	4.87	4.99	.949	.963	4.85	4.97	.946	.961	4.88
	20:10	4.69	4.72	.958	.962	4.66	4.69	.954	.958	4.63
	40:20	4.60	4.60	.960	.961	4.56	4.57	.956	.957	4.53
	80:40	4.55	4.55	.961	.961	4.52	4.52	.956	.956	4.48

Table 4: Approximations and lower tail probabilities for unequal sample sizes.

$\kappa = 1.78, r = 0.5, p = 5, \alpha = 0.05$										
k	N	$t_{1\cdot\chi^2}$	$t_{1\cdot F}$	$P_{1\cdot\chi^2}$	$P_{1\cdot F}$	$t_{M\cdot\chi^2}$	$t_{M\cdot F}$	$P_{M\cdot\chi^2}$	$P_{M\cdot F}$	t^*
6	10:5	4.91	5.14	.956	.973	4.91	5.13	.956	.972	4.86
	20:10	4.57	4.62	.962	.967	4.56	4.61	.961	.966	4.47
	40:20	4.40	4.41	.961	.963	4.37	4.39	.959	.960	4.31
	80:40	4.30	4.31	.960	.960	4.28	4.28	.957	.957	4.23
10	10:5	5.43	5.55	.978	.983	5.43	5.55	.978	.983	5.09
	20:10	4.99	5.02	.972	.974	4.98	5.01	.971	.973	4.78
	40:20	4.75	4.76	.967	.967	4.73	4.74	.964	.965	4.62
	80:40	4.63	4.63	.963	.963	4.60	4.60	.960	.960	4.53

Table 5: Approximations and lower tail probabilities for unequal sample sizes.

$\kappa = 3.24, r = 0.5, p = 5, \alpha = 0.05$										
k	N	$t_{1\cdot\chi^2}$	$t_{1\cdot F}$	$P_{1\cdot\chi^2}$	$P_{1\cdot F}$	$t_{M\cdot\chi^2}$	$t_{M\cdot F}$	$P_{M\cdot\chi^2}$	$P_{M\cdot F}$	t^*
6	10:5	5.08	5.29	.967	.980	5.07	5.29	.967	.980	4.87
	20:10	4.66	4.71	.969	.972	4.65	4.70	.968	.972	4.49
	40:20	4.44	4.45	.964	.966	4.42	4.44	.963	.964	4.32
	80:40	4.33	4.33	.961	.962	4.30	4.31	.959	.959	4.24
10	10:5	5.86	5.97	.988	.991	5.86	5.96	.988	.991	5.21
	20:10	5.23	5.25	.981	.982	5.22	5.25	.980	.982	4.87
	40:20	4.88	4.88	.972	.972	4.86	4.87	.970	.971	4.69
	80:40	4.69	4.70	.965	.966	4.67	4.67	.963	.963	4.58

Table 6: Approximations and lower tail probabilities for unequal sample sizes.

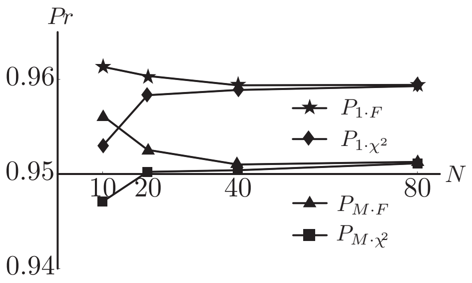

Figure 2: Conservativeness for $k = 10$, $p = 5$, $r = 1$, $\kappa = 0$.

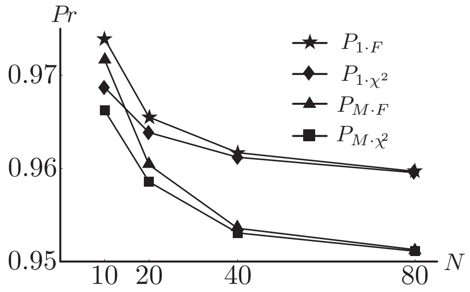

Figure 3: Conservativeness for $k = 10$, $p = 5$, $r = 1$, $\kappa = 1.78$.

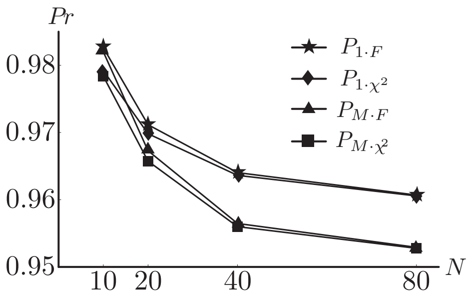

Figure 4: Conservativeness for $k = 10$, $p = 5$, $r = 1$, $\kappa = 3.24$.

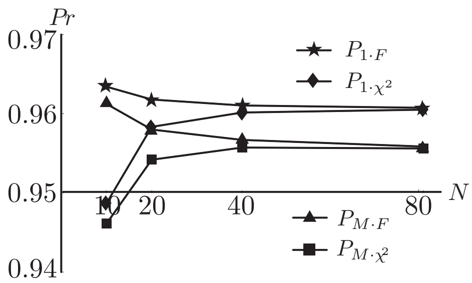

Figure 5: Conservativeness for $k = 10$, $p = 5$, $r = 0.5$, $\kappa = 0$.

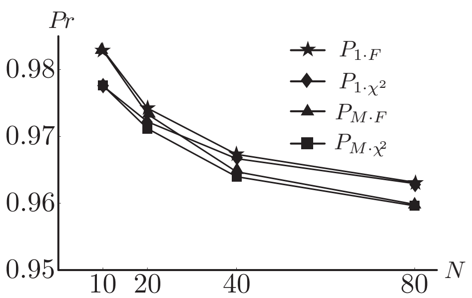

Figure 6: Conservativeness for $k = 10$, $p = 5$, $r = 0.5$, $\kappa = 1.78$.

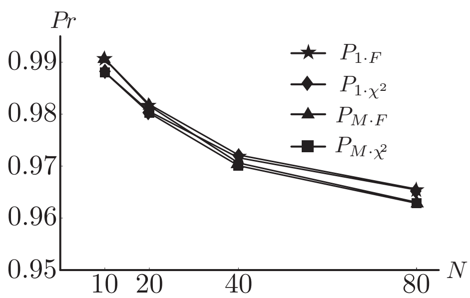

Figure 7: Conservativeness for $k = 10$, $p = 5$, $r = 0.5$, $\kappa = 3.24$.

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