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Vertex disjoint cycles containing specified paths of order 3 in a bipartite graph

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Abstract. Let k, n be integers with $k \geq 3$ and $n \geq 3k$, and let G be a bipartite graph having partite sets V_1, V_2 with $|V_1| = |V_2| = n$. We show that if $d_G(u)+d_G(v) \geq n+2k-1$ for any $u \in V_1$ and $v \in V_2$ with $uv \notin E(G)$, then for any vertex disjoint paths $P_1, P_2, \dots, P_k$ of order 3, G contains vertex disjoint cycles $H_1, H_2, \dots, H_k$ such that $\bigcup_{1 \leq i \leq k} V(H_i) = V(G)$ and H_i passes through P_i for each i with $1 \leq i \leq k$.

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§1. Introduction

In this paper, all graphs considered are finite, undirected and simple graphs with no loops and no multiple edges. Let $G = (V(G), E(G))$ be a graph. The order $|V(G)|$ of G is often denoted by $|G|$ for short. For $v \in V(G)$, we let $d_G(v)$ denote the degree of v in G, and we define $\delta(G)$ by $\delta(G) = \min\{d_G(v) \mid v \in V(G)\}$. When G is a bipartite graph with partite sets V_1 and V_2, we further define

$$\sigma_{1,1}(G) = \min\{d_G(u) + d_G(v) \mid u \in V_1, v \in V_2, uv \notin E(G)\}$$

(if G is a complete bipartite graph, we define $\sigma_{1,1}(G) = \infty$).

When G contains vertex disjoint subgraphs $H_1, H_2, \dots, H_k$ such that $\bigcup_{i=1}^{k} V(H_i) = V(G)$, we say that G is partitioned into $H_1, H_2, \dots, H_k$. If a path P is contained in a cycle C (resp. a path Q) as a subgraph, then we write $P \subset C$ (resp. $P \subset Q$).

In this paper, we are concerned with the existence of a partition of a bipartite graph into cycles. A sufficient condition for the existence of a partition into a specified number of cycles in a bipartite graph was given by Wang.

Theorem 1.1 (Wang [3])**.** *Let k, n be integers with $k \geq 1$ and $n \geq 2k+1$. Let G be a bipartite graph having partite sets with equal cardinality n, and suppose that $\delta(G) \geq \frac{n}{2}+1$. Then G can be partitioned into k cycles.*

Wang and Chen et al. independently considered a partition into cycles each of which contains a specified edge, and proved the following theorem.

Theorem 1.2 (Chen et al. [1]; Wang [2, 4])**.** *Let k, n be integers with $k \geq 2$ and $n \geq 3k$. Let G be a bipartite graph having partite sets with equal cardinality n, and suppose that $\sigma_{1,1}(G) \geq n+k$. Then for any independent edges $e_1, e_2, \ldots, e_k$, G can be partitioned into k cycles $H_1, H_2, \ldots, H_k$ such that $e_i \in E(H_i)$ for each i with $1 \leq i \leq k$.*

In this paper, we consider a situation in which vertex disjoint paths of order 3 are specified instead of independent edges. The main result of this paper is the following.

Theorem 1.3. *Let k, n be integers with $k \geq 3$ and $n \geq 3k$. Let G be a bipartite graph having partite sets with equal cardinality n, and suppose that $\sigma_{1,1}(G) \geq n+2k-1$. Then for any vertex disjoint paths $P_1, P_2, \ldots, P_k$ of order 3, G can be partitioned into k cycles $H_1, H_2, \ldots, H_k$ such that $P_i \subset H_i$ for each i with $1 \leq i \leq k$.*

Theorem 1.3 does not hold for $k=2$. Let $n \geq 5$, and let H be a complete bipartite graph of order $2n-2$ with partite sets $W_1 = \{a_i \,|\, 1 \leq i \leq n-1\}$ and $W_2 = \{b_j \,|\, 1 \leq j \leq n-1\}$. Let L be a complete graph of order 2 with $V(L) \cap V(H) = \emptyset$, and write $V(L) = \{c_1, c_2\}$. Define G by $V(G) = V(H) \cup V(L)$ and $E(G) = E(H) \cup E(L) \cup \{c_1 a_i \,|\, 1 \leq i \leq 3\} \cup \{c_2 b_i \,|\, 1 \leq i \leq 3\}$. Let $P_1 = a_1 b_1 a_2$ and $P_2 = b_2 a_3 b_3$ (see Figure 1). Then $c_1 a_1 b_1 a_2 c_1$ is the only cycle which contains c_1, passes through one of P_1 and P_2 and is disjoint from the other, and similarly for c_2. Hence G can not be partitioned into two cycles H_1, H_2 such that $P_i \subset H_i$ for each i with $1 \leq i \leq 2$, while $\sigma_{1,1}(G) = n+2k-1 = n+3$.

Also the degree sum condition in Theorem 1.3 is best possible in the following sense. Define a bipartite graph G of order $2n$ by letting $V(G) = A_1 \cup A_2 \cup A_3 \cup A_4$ with $|A_1| = 1$, $|A_2| = 2k-1$, $|A_3| = n-1$, $|A_4| = n-2k+1$, and $E(G) = \bigcup_{i=1}^{3} \{xy \,|\, x \in A_i, y \in A_{i+1}\}$. Write $V(A_1) = \{a\}$, $V(A_2) = \{b_1, b_2, \ldots, b_{2k-1}\}$, and $V(A_3) = \{c_1, c_2, \ldots, c_{n-1}\}$. Let $P_1 = a b_1 c_1$ and $P_i = b_i c_i b_{i+k-1}$ for each i with $2 \leq i \leq k$ (see Figure 2). Then we can not take a cycle passing through P_1 without using vertices of other specified paths. Consequently, G can not be partitioned into k cycles $H_1, H_2, \ldots, H_k$ such that $P_i \subset H_i$ for each i with $1 \leq i \leq k$, while $\sigma_{1,1}(G) = n+2k-2$.

The first step in the proof of Theorem 1.3 is to show the existence of vertex disjoint cycles that contain the specified paths of order 3.

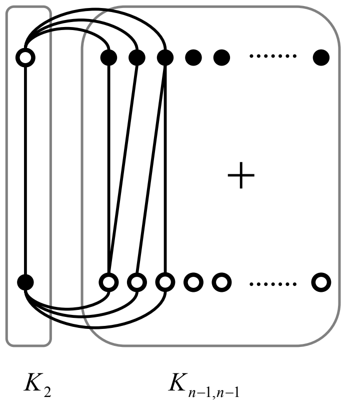

Figure 1.

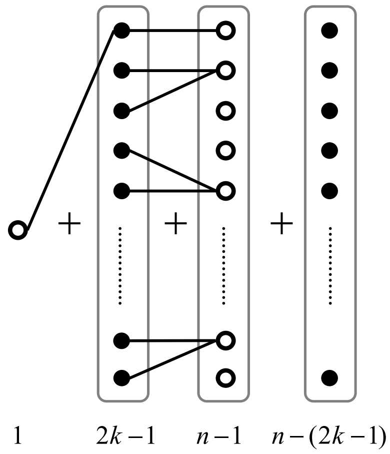

Figure 2.

Theorem 1.4. *Let k, n be integers with $k \geq 2$ and $n \geq 3k$. Let G be a bipartite graph having partite sets with equal cardinality n, and suppose that $\sigma_{1,1}(G) \geq n + 2k - 1$. Then for any vertex disjoint paths $P_1, P_2, \ldots, P_k$ of order 3, G contains k vertex disjoint cycles $C_1, C_2, \ldots, C_k$ such that $P_i \subset C_i$ and $|C_i| \leq 6$ for each i with $1 \leq i \leq k$.*

The next step is to show that this collection of cycles can be transformed into a collection of cycles that form a partition of G.

Theorem 1.5. *Let k, n be integers with $k \geq 3$ and $n \geq 2k$. Let G be a bipartite graph having partite sets with equal cardinality n, and suppose that $\sigma_{1,1}(G) \geq n + 2k - 1$. Let $P_1, P_2, \ldots, P_k$ be vertex disjoint paths of order 3, and suppose that there exist vertex disjoint cycles $C_1, C_2, \ldots, C_k$ such that $P_i \subset C_i$ for each i with $1 \leq i \leq k$. Then there exist vertex disjoint cycles $H_1, H_2, \ldots, H_k$ such that $\bigcup_{i=1}^{k} V(H_i) = V(G)$ and $P_i \subset H_i$ for each i with $1 \leq i \leq k$.*

Our notation is standard except possibly for the following. For a vertex v of a graph G, the neighborhood of v in G is denoted by $N_G(v)$; thus $d_G(v) = |N_G(v)|$. For a subset S of $V(G)$, we let $N_G(S) = \bigcup_{v \in S} N_G(v)$ and $d_G(S) = \sum_{v \in S} d_G(v)$. For a subgraph H of G and a vertex v of G with $v \in V(G) - V(H)$, let $N_G(v) \cap V(H)$ be denoted by $N_H(v)$, and let $d_H(v) = |N_H(v)|$. For a subgraph H of G and a subset S of $V(G)-V(H)$, we let $N_H(S) = \bigcup_{x\in S} N_H(x)$ and $d_H(S) = \sum_{x\in S} d_H(x)$. For a subset S of $V(G)$, we let $\langle S\rangle_G$ denote the subgraph induced by S in G, and let $G - S = \langle V(G) - S\rangle_G$. For a subgraph H of G, we write $G - H$ for $G - V(H)$.

A cycle is considered to have a fixed orientation. For a cycle $C = x_1x_2 \cdots x_nx_1$ and for two vertices $x_i, x_j \in V(C)$ with $i < j < i + n$, we define segments $C[x_i, x_j]$, $C^-[x_i, x_j]$ and $C(x_i, x_j)$ of C by $C[x_i, x_j] = x_ix_{i+1} \cdots x_{j-1}x_j$, $C^-[x_i, x_j] = x_ix_{i-1} \cdots x_{j+1}x_j$ and $C(x_i, x_j) = C[x_i, x_j]-\{x_i, x_j\}$, respectively (here indices are to be read modulo n). We let v^+ (resp. v^-) denote the successor (resp. the predecessor) of v along C, and define $v^{++} = (v^+)^+$ (resp. $v^{--} = (v^-)^-$); thus if $v = x_i$, then $v^+ = x_{i+1}$, $v^- = x_{i-1}$, $v^{++} = x_{i+2}$ and $v^{--} = x_{i-2}$. For a path $P = y_1y_2 \cdots y_m$ and for two vertices $y_i, y_j \in V(P)$ with $1 \le i < j \le m$, we define segment $P[y_i, y_j]$ of P by $P[y_i, y_j] = y_iy_{i+1} \cdots y_{j-1}y_j$. When $P_1, P_2, \ldots, P_k$ are vertex disjoint paths of order 3, a cycle C is said to be *admissible* with respect to $\{P_1, P_2, \ldots, P_k\}$ if there exists i with $1 \le i \le k$ such that $P_i \subset C$ and $V(C)\cap V(P_j) = \emptyset$ for every j with $1 \le j \le k$ and $j \ne i$.

§2. Proof of Theorem 1.4

Throughout the rest of this paper, we let G denote a bipartite graph having partite sets V_1, V_2 with $|V_1| = |V_2|$. Our proof of Theorem 1.4 requires the following lemmas.

Lemma 2.1. *Let $P = xyz$ be a path in G with $x \in V_1$, and let C be a cycle in G such that $P \subset C$. Further let $u \in V(G - C) \cap V_1$ and $v \in V(G - C) \cap V_2$.*

(i) *If $d_C(u) \ge 4$, then $\langle V(C) \cup \{u\}\rangle_G$ contains a cycle which is shorter than C and passes through P.*

(ii) *If $d_C(v) \ge 3$, then $\langle V(C) \cup \{v\}\rangle_G$ contains a cycle which is shorter than C and passes through P.*

Proof. (i) Since $x \in V_1$ and $d_C(u) \ge 4$, there exist $x_1, x_2 \in N_C(u)$ such that $|C[x_1, x_2]| \ge 5$ and $P \subset C(x_2, x_1)$. Then $uC[x_2, x_1]u$ is a cycle shorter than C and $P \subset uC[x_2, x_1]u$.
(ii) Since $x \in V_1$ and $d_C(v) \ge 3$, there exist $x_1, x_2 \in N_C(v)$ such that $|C[x_1, x_2]| \ge 5$ and $P \subset C[x_2, x_1]$. Then $vC[x_2, x_1]v$ is a cycle shorter than C and $P \subset vC[x_2, x_1]v$. □

Lemma 2.2. *Let P be a path of order 3, and let C be a cycle in G with $P \subset C$. Let $u \in V(G - C) \cap V_1$, $v \in V(G - C) \cap V_2$, and suppose that $d_C(u) + d_C(v) \ge \frac{|C|}{2} + 3$. Then either $\langle V(C) \cup \{v\}\rangle_G$ contains a cycle which* *is shorter than C and passes through P, or there exists $w \in N_C(u)$ such that $\langle V(C) \cup \{v\} - \{w\}\rangle_G$ contains a cycle passing through P.*

Proof. If $d_C(v) \geq 4$, then by Lemma 2.1, $\langle V(C) \cup \{v\}\rangle_G$ contains a cycle which is shorter than C and passes through P. Thus we may assume that $d_C(v) \leq 3$. Then $d_C(v) = 3$ and $d_C(u) = \frac{|C|}{2}$, that is, $N_C(u) = V(C) \cap V_2$. Since $d_C(v) = 3$, there exist $a, b \in N_C(v)$ with $P \subset C[b, a]$. Take $w \in V(C(a, b)) \cap V_2$. Then $w \in N_C(u)$, and $vC[b, a]v$ is a cycle in $\langle V(C) \cup \{v\} - \{w\}\rangle_G$ passing through P. □

In the rest of this section, we let G be an edge-maximal counterexample to Theorem 1.4, and write $P_i = x_i y_i z_i$ for each i with $1 \leq i \leq k$. The term "*admissible*" means "admissible with respect to $\{P_1, P_2, \ldots, P_k\}$," and a cycle is called *short* if its length is at most 6. Note that G is not a complete bipartite graph, because otherwise there exist k vertex disjoint admissible cycles of length 4. Let $x \in V_1$ and $y \in V_2$ be nonadjacent vertices of G. Then the graph obtained from G by adding the edge xy is not a counterexample by the maximality of G, which implies that G contains $k-1$ vertex disjoint admissible short cycles $C_1, C_2, \ldots, C_{k-1}$. We choose admissible short cycles $C_1, C_2, \ldots, C_{k-1}$ so that $|\bigcup_{i=1}^{k-1} V(C_i)|$ is as small as possible. Without loss of generality, we may assume that $P_i \subset C_i$ for each i with $1 \leq i \leq k-1$, and we may also assume that $x_k \in V_1$. Let $L = \langle \bigcup_{i=1}^{k-1} V(C_i)\rangle_G$, $M = G - L$ and $D_1 = M - V(P_k)$, and write $|M| = 2m$. Since $n \geq 3k$, we have $m \geq 3$. If possible, we choose $C_1, C_2, \ldots, C_{k-1}$ so that $d_{D_1}(z_k) > 0$ and $d_{D_1}(x_k) > 0$.

Claim 2.1. *We have $d_{D_1}(z_k) > 0$ and $d_{D_1}(x_k) > 0$.*

Proof. We first remark that we can choose $C_1, C_2, \ldots, C_{k-1}$ so that $d_{D_1}(z_k) > 0$. To see this, suppose that $d_{D_1}(z_k) = 0$ and take any $x \in V(D_1) \cap V_2$. Then

$$d_M(z_k) + d_M(x) \leq 1 + (m-1) = m.$$

This implies that

$$\begin{aligned} d_L(z_k) + d_L(x) &\geq n + 2k - 1 - m \\ &= \frac{|L|}{2} + 2k - 1 \\ &> \sum_{i=1}^{k-1} \Big(\frac{|C_i|}{2} + 2 \Big). \end{aligned}$$

Therefore there exists i with $1 \leq i \leq k-1$ such that $d_{C_i}(z_k) + d_{C_i}(x) \geq \frac{|C_i|}{2} + 3$. Hence it follows Lemma 2.2 and the minimality of $|L|$ that there exists $u \in N_{C_i}(z_k)$ such that $\langle V(C_i) \cup \{x\} - \{u\}\rangle_G$ contains a cycle C_i' passing through P_i. Consequently, replacing C_i by C_i', we may assume $d_{D_1}(z_k) > 0$.

Now suppose that the claim is false. In view of the remark made at the beginning of the proof, we may assume $d_{D_1}(z_k) > 0$ and $d_{D_1}(x_k) = 0$. Take $y \in N_{D_1}(z_k)$ and $v \in (V(D_1) \cap V_2) - \{y\}$. Arguing as above, we see that there exists j such that $d_{C_j}(x_k) + d_{C_j}(v) \geq \frac{|C_j|}{2} + 3$, and there exists $w \in N_{C_j}(x_k)$ such that $\langle V(C_j) \cup \{v\} - \{w\} \rangle_G$ contains a cycle C'_j passing through P_j. Now replacing C_j by C'_j, we obtain a contradiction to the choice of $C_1, C_2, \ldots, C_{k-1}$ mentioned immediately before the statement of Claim 2.1. This completes the proof of the claim. □

Take $y \in N_{D_1}(z_k)$ and $z \in N_{D_1}(x_k)$, and let $D_2 = D_1 - \{y, z\}$. Since G is a counterexample, $x_k y, z_k z \notin E(G)$, and hence $y \neq z$.

Claim 2.2. $d_{D_2}(y) > 0$.

Proof. Suppose that $d_{D_2}(y) = 0$ and take any $u \in V(D_2) \cap V_1$. Then since $yx_k \notin E(G)$,

$$d_M(u) + d_M(y) \leq (m-1) + 1 = m.$$

This implies that

$$\begin{aligned} d_L(u) + d_L(y) &\geq n + 2k - 1 - m. \\ &> \sum_{i=1}^{k-1} \Big(\frac{|C_i|}{2} + 2 \Big). \end{aligned}$$

Therefore there exists i with $1 \leq i \leq k-1$ such that $d_{C_i}(u) + d_{C_i}(y) \geq \frac{|C_i|}{2} + 3$. Since C_i is short, this forces $|C_i| = 6$ and $d_{C_i}(u) = d_{C_i}(y) = 3$. Hence by Lemma 2.1(ii), $\langle V(C_i) \cup \{y\} \rangle_G$ or $\langle V(C_i) \cup \{u\} \rangle_G$ contains a cycle of length 4 passing through P_i. This contradicts the minimality of $|L|$. □

Now take $z' \in N_{D_2}(y)$. Since G is a counterexample, $zz' \notin E(G)$. Let $D_3 = D_2 - \{z'\}$ and $S = \{z, x_k, y, z'\}$. Then again since G is a counterexample, we have $N_{D_3}(z) \cap N_{D_3}(y) = \emptyset$ and $N_{D_3}(x_k) \cap N_{D_3}(z') = \emptyset$. Hence

$$\begin{aligned} d_M(S) &= d_{\langle S \cup \{y_k, z_k\} \rangle_G}(S) + d_{D_3}(S) \\ &\leq 7 + 2m - 6 \\ &= 2m + 1. \end{aligned}$$

This implies that

$$\begin{aligned} d_L(S) &\geq 2(n + 2k - 1) - (2m + 1) \\ &= 2(n - m) + 4(k - 1) + 1 \\ &> \sum_{i=1}^{k-1} (|C_i| + 4). \end{aligned}$$

Therefore there exists i with $1 \le i \le k-1$ such that $d_{C_i}(S) \ge |C_i| + 5$. This implies that $|C_i| = 6$, and we have

$$11 \le d_{C_i}(S) \le 12. \tag{2.1}$$

Write $C_i = x_i y_i z_i abcx_i$.

CASE 1. $x_i \in V_1$.

By (2.1), $5 \le d_{C_i}(\{z, y\}) \le 6$, and hence there exists $u \in \{z, y\}$ such that $d_{C_i}(u) = 3$. Then $C'_i = ux_i y_i z_i u$ is an admissible cycle shorter than C_i, which contradicts the minimality of $|L|$.

CASE 2. $x_i \in V_2$.

If $\{x_i, z_i\} \subseteq N_{C_i}(z')$, then $z' x_i y_i z_i z'$ is an admissible cycle shorter than C_i, a contradiction. Thus $\{x_i, z_i\} \not\subseteq N_{C_i}(z')$, and hence

$$d_{C_i}(z') \le 2. \tag{2.2}$$

By (2.1) and (2.2), we obtain $d_{C_i}(z) = 3$, $d_{C_i}(x_k) = 3$ and $d_{C_i}(z') = 2$, and hence $b \in N_{C_i}(z')$. Now if we let $C'_i = zcx_i y_i z_i az$ and $C'_k = bx_k y_k z_k yz'b$, then C'_i and C'_k together with $\{C_1, C_2, \ldots, C_{k-1}\} - \{C_i\}$ form k vertex disjoint short admissible cycles. But this contradicts the assumption that G is a counterexample.

This completes the proof of Theorem 1.4.

§3. Proof of Theorem 1.5

Recall that G denotes a bipartite graph having partite sets V_1, V_2 with $|V_1| = |V_2|$. We prepare the following lemmas before proving Theorem 1.5.

Lemma 3.1. *Let P be a path in G having order 3, and let C be a cycle in G such that $P \subset C$. Let R be a path in G with $V(C) \cap V(R) = \emptyset$ such that the endvertices of R, u and v, belong to different partite sets. Suppose further that $d_C(u) + d_C(v) \ge \frac{|C|}{2} + 2$. Then there exists a cycle C' such that $V(C') = V(C) \cup V(R)$ and $P \subset C'$.*

Proof. Write $C = w_1 w_2 \cdots w_r w_1$ with $P = w_1 w_2 w_3$. By the symmetry of the roles of u and v, we may assume that w_1 and u belong to the same partite set. Then there exists i with $3 \le i \le r-1$ such that $uw_{i+1}, vw_i \in E(G)$. Now the cycle C' obtained by joining $C[w_{i+1}, w_i]$ and R satisfies $V(C') = V(C) \cup V(R)$ and $P \subset C'$. □

Lemma 3.2. *Let P be a path in G having order 3, and let Q be a path in G such that $P \subset Q$. Let R be a path with $V(Q) \cap V(R) = \emptyset$ such that the endvertices of R, u and v, belong to different partite sets. Suppose further that $d_Q(u)+d_Q(v) \geq \frac{|Q|+4}{2}$, Then there exists a path Q' having the same endvertices as Q such that $V(Q') = V(Q) \cup V(R)$ and $P \subset Q'$.*

Proof. Write $Q = w_1w_2 \cdots w_r$ with $P = w_jw_{j+1}w_{j+2}$. We may assume that w_1 and u belong to the same partite set. Then, regardless of the partite sets which w_r and w_j belong to, there exists i with $1 \leq i \leq j-1$ or $j+2 \leq i \leq r-1$ such that $uw_{i+1}, vw_i \in E(G)$. Now the path Q' obtained by joining $Q[w_1, w_i]$, R and $Q[w_{i+1}, w_r]$ satisfies $V(Q') = V(Q) \cup V(R)$ and $P \subset Q'$. $\square$

Lemma 3.3. *Let P be a path of order 3 in G. Let C be a cycle in G with $P \subset C$, and suppose that G contains no cycle D satisfying $P \subset D$ and $V(C) \subsetneq V(D)$. Further let $u \in V(G-C) \cap V_1$ and $v \in V(G-C) \cap V_2$. Then $d_C(u) + d_C(v) \leq \frac{|C|}{2} + 2$.*

Proof. Write $C = w_1w_2 \cdots w_rw_1$ with $P = w_1w_2w_3$. We may assume that $w_1 \in V_1$. Suppose that $d_C(u) + d_C(v) \geq \frac{|C|}{2} + 3$. Then there exist i and j $(3 \leq i < j \leq r-1)$ with $uw_{i+1}, vw_i, uw_{j+1}, vw_j \in E(G)$. Now if we let $D = uC[w_{i+1}, w_j]vC^-[w_i, w_{j+1}]u$, then we have $P \subset D$ and $V(C) \subsetneq V(D)$. But this contradicts the assumption that there is no such cycle. $\square$

Throughout the rest of this paper, we let k, G, $P_1, P_2, \ldots, P_k$ be as in Theorem 1.5, and write $P_i = x_iy_iz_i$ for each i with $1 \leq i \leq k$. The term "*admissible*" means "admissible with respect to $\{P_1, P_2, \ldots, P_k\}$". Choose k vertex disjoint cycles $C_1, C_2, \ldots, C_k$ with $P_i \subset C_i$ for each i with $1 \leq i \leq k$ so that $\sum_{i=1}^{k} |C_i|$ is as large as possible, and set $L = \langle \bigcup_{i=1}^{k} V(C_i) \rangle_G$. By way of contradiction, suppose that $L \neq G$, and set $M = G - L$. Let M_0 be a connected component of M.

Claim 3.1. *Let $1 \leq i \leq k$, suppose that $x_i \in V_1$. Then either $N_{C_i}(M_0) \cap V_1 = \emptyset$ or $N_{C_i - \{y_i\}}(M_0) \cap V_2 = \emptyset$.*

Proof. Without loss of generality, we may assume that $i = 1$. If $|M_0| = 1$, then the claim clearly holds. We may assume that $|M_0| \geq 2$. Suppose that $N_{C_1}(M_0) \cap V_1 \neq \emptyset$ and $N_{C_1 - \{y_1\}}(M_0) \cap V_2 \neq \emptyset$. Reversing the orientation of C_1 if necessary, we may assume that there exist $uw, vz \in E(G)$ with $u \in V(M_0) \cap V_1$, $v \in V(M_0) \cap V_2$ and $w, z \in V(C_1) - \{y_1\}$ satisfying $P_1 \subset C_1[z, w]$ and $N_G(M_0) \cap V(C_1(w, z)) = \emptyset$. If $z = w^+$, then in $\langle V(C_1) \cup V(M_0) \rangle_G$ there exists an admissible cycle longer than C_1, a contradiction. Hence we may assume that $|C_1(w, z)| \geq 2$. Let D be the cycle obtained by joining $C_1[z, w]$ and a path Q connecting u and v in M_0. Let $S = \{u, v, w^+, z^-\}$. Suppose that $d_{C_1[z,w]}(w^+) + d_{C_1[z,w]}(z^-) \geq \frac{|C_1[z,w]|+4}{2}$. Then by Lemma 3.2, there exists a path Q' with endvertices z, w such that $V(Q') = V(C_1[z,w]) \cup V(C_1[w^+, z^-])$ and $P_1 \subset Q'$. Since $C_1[z,w]$ is also a segment of D, this contradicts the maximality of $|L|$. Thus

$$d_{C_1[z,w]}(w^+) + d_{C_1[z,w]}(z^-) \leq \frac{|C_1[z,w]|+3}{2}.$$

Similarly it follows from Lemma 3.1 that

$$d_{C_i}(w^+) + d_{C_i}(z^-) \leq \frac{|C_i|}{2} + 1 \text{ for each } i \text{ with } 2 \leq i \leq k.$$

Also

$$d_{C_1[z,w]}(u) + d_{C_1[z,w]}(v) \leq \frac{|C_1[z,w]|+3}{2}$$

by Lemma 3.2, and we have $d_{C_i}(u) + d_{C_i}(v) \leq \frac{|C_i|}{2} + 1$ for each i with $2 \leq i \leq k$ by Lemma 3.1. Since $d_{C_1(w,z)}(u) + d_{C_1(w,z)}(v) = 0$, $d_{\langle V(C_1(w,z))\rangle_G}(w^+) + d_{\langle V(C_1(w,z))\rangle_G}(z^-) \leq |C_1(w,z)|$, $d_M(u)+d_M(v) \leq |M_0|$ and $d_M(w^+)+d_M(z^-) \leq |M - M_0|$, we now obtain

$$\begin{aligned} d_G(S) &\leq |M| + |C_1(w,z)| + 2\Big(\frac{|C_1[z,w]|+3}{2}\Big) + \sum_{i=2}^{k} 2\Big(\frac{|C_i|}{2}+1\Big) \\ &= |M| + \sum_{i=1}^{k}(|C_i|+2) + 1 \\ &= 2n + 2k + 1. \end{aligned}$$

On the other hand, since $uz^-, vw^+ \notin E(G)$, $d_G(S) \geq 2(n + 2k - 1)$. But this is a contradiction because $k \geq 3$. $\square$

Using an argument similar to the proof of Claim 3.1, we also obtain the following claim.

Claim 3.2. *Let $1 \leq i \leq k$, and suppose that $x_i \in V_2$. Then either $N_{C_i - \{y_i\}}(M_0) \cap V_1 = \emptyset$ or $N_{C_i}(M_0) \cap V_2 = \emptyset$.*

Hereafter, we divide the proof into two cases according to the order of M.

3.1. $|M| = 2$

Write $V(M) = \{u, v\}$. By symmetry, we may assume that $u \in V_1$ and $v \in V_2$.

Claim 3.3. *Suppose that $uv \notin E(G)$. Then $d_{C_i}(u) + d_{C_i}(v) = \frac{|C_i|}{2} + 2$ for each i with $1 \leq i \leq k$.*

Proof. By Lemma 3.3, we have

$$d_G(u) + d_G(v) = \sum_{i=1}^{k}(d_{C_i}(u) + d_{C_i}(v)) \leq \sum_{i=1}^{k}\Big(\frac{|C_i|}{2} + 2\Big) = n + 2k - 1.$$

On the other hand, since $uv \notin E(G)$, $d_G(u) + d_G(v) \geq n + 2k - 1$. Therefore

$$d_{C_i}(u) + d_{C_i}(v) = \frac{|C_i|}{2} + 2 \text{ for each } i \text{ with } 1 \leq i \leq k.$$

□

Claim 3.4. *Let $1 \leq i \leq k$, and suppose that $|C_i| \geq 6$ and $x_i \in V_1$. Then it is not possible that we have both $d_{C_i}(u) = 2$ and $d_{C_i}(v) = \frac{|C_i|}{2}$.*

Proof. Without loss of generality, we may assume that $i = 1$. Suppose that $d_{C_1}(u) = 2$ and $d_{C_1}(v) = \frac{|C_1|}{2}$. By Lemma 3.1, $uv \notin E(G)$. Write $C_1 = a_1b_1a_2b_2 \cdots a_rb_ra_1$ with $P_1 = x_1y_1z_1 = a_1b_1a_2$. If $N_{C_1}(u) = \{b_p, b_q\}$ ($2 \leq p < q \leq r$), then $\{vC_1^-[a_q, b_p]uC_1[b_q, a_p]v,\ C_2, \ldots, C_k\}$ is a required partition of G, a contradiction. Thus $N_{C_1}(u) = \{b_1, b_h\}$ ($1 < h$). Since $|C_1| \geq 6$, we have $b_{h-1} \neq y_1$ or $b_{h+1} \neq y_1$. By symmetry, we may assume that $b_{h-1} \neq y_1$. Let $C_1' = vC_1[a_h, a_{h-1}]v$ and $C_i' = C_i$ for each i with $2 \leq i \leq k$. Then $C_1', C_2', \ldots, C_k'$ are admissible and $|\bigcup_{i=1}^{k} V(C_i')| = |L|$. Hence applying Claim 3.3 with the C_i replaced by the C_i', we obtain

$$d_{C_1'}(u) + d_{C_1'}(b_{h-1}) = \frac{|C_1'|}{2} + 2.$$

Since we get $N_{C_1'}(u) = \{b_1, b_h\}$ from $N_{C_1}(u) = \{b_1, b_h\}$, this implies $d_{C_1'}(b_{h-1}) = \frac{|C_1'|}{2}$; in particular, $b_{h-1}a_{h+1} \in E(G)$. On the other hand, we have $d_{C_2'}(u) + d_{C_2'}(v) = \frac{|C_2'|}{2} + 2$ by Claim 3.3. Hence it follows Lemma 3.1 that there exists a cycle C_2'' such that $P_2 \subset C_2''$ and $V(C_2'') = V(C_2) \cup \{u, b_h, a_h, v\}$. Now if we let $C_1'' = C_1[a_{h+1}, b_{h-1}]a_{h+1}$ and let $C_i'' = C_i$ for each i with $3 \leq i \leq k$, then $\{C_1'', C_2'', \ldots, C_k''\}$ is a required partition of G, a contradiction. □

CASE I. M is connected.

Claim 3.5. *Let $1 \leq i \leq k$, and suppose that $x_i \in V_1$. Then $|N_{C_i}(v)| \leq 2$.*

Proof. Without loss of generality, we may assume that $i = 1$. Suppose that $|N_{C_1}(v)| \geq 3$. By Claim 3.1, we have

$$N_{C_1}(u) \subseteq \{y_1\}. \tag{3.1}$$

Since $|N_{C_i}(v)| \geq 3$, we can choose two vertices $w, z \in N_{C_1}(v)$ such that $P_1 \subset C_1(z, w]$ and $N_{C_1}(v) \cap V(C_1(w, z)) = \emptyset$.

Case 1. $|C_1(w,z)| \geq 3$.

Let $S = \{u, v, w^+, z^{--}\}$. Suppose that $d_{C_1[z,w]}(w^+) + d_{C_1[z,w]}(z^{--}) \geq \frac{|C_1[z,w]|+4}{2}$. Then by Lemma 3.2, there exists a path Q' with endvertices z, w such that $V(Q') = V(C_1[z,w]) \cup V(C_1[w^+, z^{--}])$ and $P_1 \subset Q'$. Let $D = vQ'v$. Then $|D| \geq 6$ and $|V(D) \cup \bigcup_{i=2}^{k} V(C_i)| = |L|$. Note that $uz^- \notin E(G)$ by (3.1). Hence applying Claim 3.3 to $D, C_2, C_3, \ldots, C_k$, we obtain

$$d_D(u) + d_D(z^-) = \frac{|D|}{2} + 2. \tag{3.2}$$

Note that (3.1) implies $N_D(u) \subseteq \{v, y_1\}$. Hence it follows from (3.2) that $N_D(u) = \{v, y_1\}$ and $d_D(z^-) = \frac{|D|}{2}$. But applying Claim 3.4 to $D, C_2, C_3, \ldots, C_k$, we see that this is impossible. Thus

$$d_{C_1[z,w]}(w^+) + d_{C_1[z,w]}(z^{--}) \leq \frac{|C_1[z,w]|+3}{2}.$$

Suppose now that there exists i with $2 \leq i \leq k$ such that $d_{C_i}(w^+) + d_{C_i}(z^{--}) \geq \frac{|C_i|}{2} + 2$. Then by Lemma 3.1, there exists a cycle C_i' such that $V(C_i') = V(C_i) \cup V(C_1[w^+, z^{--}])$ and $P_i \subset C_i'$. Let $C_1' = vC_1[z,w]v$ and $C_j' = C_j$ for each j with $2 \leq j \leq k$ and $j \neq i$. Then $|C_1'| \geq 6$ and $|\bigcup_{j=1}^{k} V(C_j')| = |L|$. Hence applying Claim 3.3 to $C_1', C_2', \ldots, C_k'$, we obtain

$$d_{C_1'}(u) + d_{C_1'}(z^-) = \frac{|C_1'|}{2} + 2,$$

which again contradicts Claim 3.4. Thus $d_{C_i}(w^+) + d_{C_i}(z^{--}) \leq \frac{|C_i|}{2} + 1$ for each i with $2 \leq i \leq k$. Also $d_{C_i}(u) + d_{C_i}(v) \leq \frac{|C_i|}{2} + 1$ for each i with $2 \leq i \leq k$ by Lemma 3.1 and, since $N_{C_1(w,z)}(v) = \emptyset$, it follows from (3.1) that $d_{C_1[z,w]}(u) + d_{C_1[z,w]}(v) \leq \frac{|C_1[z,w]|+1}{2} + 1 = \frac{|C_1[z,w]|+3}{2}$. Since $d_{C_1(w,z)}(u) + d_{C_1(w,z)}(v) = 0$, $d_{\langle V(C_1(w,z))\rangle_G}(w^+) + d_{\langle V(C_1(w,z))\rangle_G}(z^{--}) \leq |C_1(w,z)|$, $d_M(u) + d_M(v) = 2$ and $d_M(w^+) + d_M(z^{--}) = 0$, we now obtain

$$\begin{aligned} d_G(S) &\leq 2 + |C_1(w,z)| + 2\Big(\frac{|C_1[z,w]|+3}{2}\Big) + \sum_{i=2}^{k} 2\Big(\frac{|C_i|}{2} + 1\Big) \\ &= 2n + 2k + 1. \end{aligned}$$

On the other hand, since $w^+u, z^{--}v \notin E(G)$, $d_G(S) \geq 2(n + 2k - 1)$. But this is a contradiction because $k \geq 3$.

Case 2. $|C_1(w,z)| = 1$.

Write $V(C_1(w,z)) = \{a\}$. Let $C_1' = vC_1[z,w]v$ and $C_i' = C_i$ for each i with $2 \leq i \leq k$. Then $|C_1'| \geq 6$. $C_1', C_2', \ldots, C_k'$ are admissible, and $|\bigcup_{i=1}^{k} V(C_i')| =$

$|L|$. By (3.1), we also have $au \notin E(G)$. Hence by Claim 3.3, we obtain

$$d_{C'_1}(u) + d_{C'_1}(a) = \frac{|C'_1|}{2} + 2$$

In view of (3.1), this forces $N_{C'_1}(u) = \{v, y_1\}$ and $|N_{C'_1}(a)| = \frac{|C'_1|}{2}$. But then we get a contradiction by applying Claim 3.4 to $C'_1, C'_2, \ldots, C'_k$. □

Claim 3.6. *Let $1 \le i \le k$, and suppose that $x_i \in V_1$. Then $|N_{C_i - \{y_i\}}(u)| \le 1$.*

Proof. Without loss of generality, we may assume that $i = 1$. Suppose that $|N_{C_1 - \{y_1\}}(u)| \ge 2$. We can choose two vertices $w, z \in N_{C_1}(u)$ such that $P_1 \subset C_1(z, w)$ and $N_{C_1}(u) \cap V(C_i(w, z)) = \emptyset$. Since $N_{C_1}(v) = \emptyset$ by Claim 3.1, the rest of the proof is similar to and easier than that of Claim 3.5. □

Using an argument similar to the proof of Claims 3.5 and 3.6, we obtain the following two claims.

Claim 3.7. *Let $1 \le i \le k$, and suppose that $x_i \in V_2$. Then $|N_{C_i - \{y_i\}}(v)| \le 1$.*

Claim 3.8. *Let $1 \le i \le k$, and suppose that $x_i \in V_2$. Then $|N_{C_i}(u)| \le 2$.*

By Claims 3.1, 3.2, 3.5, 3.6, 3.7 and 3.8, we have $d_L(u) + d_L(v) \le 3k$. Hence $d_G(u) + d_G(v) \le 3k + 2$. On the other hand, from the assumption that $\sigma_{1,1}(G) \ge n + 2k - 1$, it follows that $d_G(x) \ge 2k$ for all $x \in V(G)$, and hence $d_G(u) + d_G(v) \ge 4k$. But this is a contradiction because $k \ge 3$.

CASE II. M is disconnected.

By Claim 3.3, we have

$$d_{C_i}(u) + d_{C_i}(v) = \frac{|C_i|}{2} + 2 \text{ for each } i \text{ with } 1 \le i \le k.$$

Suppose that there exists i with $1 \le i \le k$ such that $|C_i| = 4$. Let $C'_i = vx_iy_iz_iv$ or $C'_i = ux_iy_iz_iu$ according as $x_i \in V_1$ or $x_i \in V_2$, and let $C'_j = C_j$ for each j with $1 \le j \le k$ and $j \ne i$. Then $G - \bigcup_{i=1}^{k} V(C'_i)$ is connected. Consequently we are reduced to CASE I, which leads to a contradiction. Thus $|C_i| \ge 6$ for each i with $1 \le i \le k$. Since $k \ge 3$, we have $|\{i \,|\, x_i \in V_1\}| \ge 2$ or $|\{i \,|\, x_i \in V_2\}| \ge 2$. Without loss of generality, we may assume that $|\{i \,|\, x_i \in V_1\}| \ge 2$. We may also assume that $x_1, x_2 \in V_1$. We write $C_1 = a_1b_1a_2b_2 \cdots a_rb_ra_1$ with $P_1 = x_1y_1z_1 = a_1b_1a_2$.

Claim 3.9.

(i) $|\{p \,|\, 2 \le p \le r, a_p \in N_G(v), b_p \in N_G(u)\}| \le 1$.

(ii) $|\{p \mid 2 \le p \le r, a_{p+1} \in N_G(v), b_p \in N_G(u)\}| \le 1$ (*we take* $a_{r+1} = a_1$).

Proof. By way of contradiction, suppose that there exist p, q with $2 \le p < q \le r$ such that $a_p, a_q \in N_G(v)$ and $b_p, b_q \in N_G(u)$. Let $C_1' = uC_1[b_p, a_q]vC_1^-[a_p, b_q]u$. Then $\{C_1', C_2, \dots, C_k\}$ is a desired partition of G, which contradicts the assumption that G is a counterexample. Thus (i) is proved, and (ii) can be verified in a similar way. □

Claim 3.10.

(i) (a) $(N_G(u) \cup N_G(v)) \cap \{a_p, b_p\} \ne \emptyset$ *for each* p *with* $1 \le p \le r$.
(b) $|\{p \mid 1 \le p \le r, a_p \in N_G(v), b_p \in N_G(u)\}| = 2$.

(ii) (a) $(N_G(u) \cup N_G(v)) \cap \{b_p, a_{p+1}\} \ne \emptyset$ *for each* p *with* $1 \le p \le r$.
(b) $|\{p \mid 1 \le p \le r, a_{p+1} \in N_G(v), b_p \in N_G(u)\}| = 2$.

Proof. Set $\alpha = |\{p \mid 1 \le p \le r, a_p \in N_G(v), b_p \in N_G(u)\}|$ and $\beta = |\{p \mid 1 \le p \le r, (N_G(v) \cup N_G(u)) \cap \{a_p, b_p\} = \emptyset\}|$. Since $d_{C_1}(u) + d_{C_1}(v) = \frac{|C_1|}{2} + 2$ by Claim 3.3, we have $\alpha = \beta + 2$. Since $\alpha \le 2$ by Claim 3.9, this implies $\alpha = 2$ and $\beta = 0$. Thus (i) is proved, and (ii) can be verified in a similar way. □

Claim 3.11. *There exist* s, t *with* $2 \le s < t \le r$ *such that* $N_{C_1}(u) = \{b_1\} \cup \{b_p \mid s \le p \le t\}$ *and* $N_{C_1}(v) = \{a_p \mid 1 \le p \le s\} \cup \{a_p \mid t+1 \le p \le r\}$.

Proof. By Claims 3.9(i) and 3.10(i)(b),

$$a_1 \in N_G(v), b_1 \in N_G(u),$$
$$|\{p \mid 2 \le p \le r, a_p \in N_G(v), b_p \in N_G(u)\}| = 1.$$

Similarly by Claims 3.9(ii) and 3.10(ii)(b),

$$a_2 \in N_G(v), b_1 \in N_G(u),$$
$$|\{p \mid 2 \le p \le r, a_{p+1} \in N_G(v), b_p \in N_G(u)\}| = 1.$$

Write $\{p \mid 2 \le p \le r, a_p \in N_G(v), b_p \in N_G(u)\} = \{s\}$ and $\{p \mid 2 \le p \le r, a_{p+1} \in N_G(v), b_p \in N_G(u)\} = \{t\}$. Suppose that $s > t$. Then since $b_s \in N_G(u)$ and $a_{r+1} = a_1 \in N_G(v)$, it follows from Claim 3.10(i)(a) that there exists p' with $s \le p' \le r$ such that $b_{p'} \in N_G(u)$ and $a_{p'+1} \in N_G(v)$. But since $t < p'$, this contradicts Claim 3.9(ii). Thus $s \le t$. If there exists p' with $2 \le p' \le s-1$ such that $b_{p'} \in N_G(u)$, then by Claim 3.10(i)(a), there exists q' with $p' \le q' \le s-1$ such that $b_{q'} \in N_G(u)$ and $a_{q'+1} \in N_G(v)$, which again contradicts Claim 3.9(ii). Thus $(N_G(u) \cup N_G(v)) \cap \{a_p, b_p\} = \{a_p\}$ for each p with $2 \le p \le s-1$. Similarly $(N_G(u) \cup N_G(v)) \cap \{a_p, b_p\} = \{a_p\}$ for each p with $t+1 \le p \le r$. If $s = t$, then it follows that $N_{C_1}(u) = \{b_1, b_s\}$ and $N_{C_1}(v) = \{a_p \mid 1 \le p \le r\}$, which contradicts Claim 3.4. Thus $s < t$. Now since $b_s \in N_{C_1}(u)$, we see from Claims 3.10(i)(a) and 3.9(ii) that $(N_G(u) \cup N_G(v)) \cap \{a_p, b_p\} = \{b_p\}$ for each p with $s+1 \le p \le t-1$. □

We can now complete the proof for the case where $|M| = 2$. We write $C_2 = c_1d_1c_2d_2 \cdots c_hd_hc_1$ with $P_2 = x_2y_2z_2 = c_1d_1c_2$. Applying Claim 3.11 to C_2, we see that there exists q with $3 \le q \le h$ such that $c_q \notin N_{C_2}(v)$ and $d_{q-1}, d_q \in N_{C_2}(u)$. Let $C_1' = C_1$, $C_2' = uC_2[d_q, d_{q-1}]u$ and $C_i' = C_i$ for each i with $3 \le i \le k$. We apply Claim 3.11 with the C_i replaced by the C_i' (so u is replaced by c_q). Note that $C_1 = C_1'$, and hence $N_{C_1}(v) = N_{C_1'}(v)$. Consequently it follows that $N_{C_1}(c_q) = N_{C_1}(u) = \{b_1\} \cup \{b_p \mid s \le p \le t\}$. Now if we let $C_1'' = vC_1[a_{t+1}, a_s]v$, $C_2'' = uC_2[d_q, d_{q-1}]c_qC_1[b_s, b_t]u$ and $C_i'' = C_i$ for each i with $3 \le i \le k$, then $\{C_1'', C_2'', \dots, C_k''\}$ is a required partition of G. This is a contradiction.

This concludes the discussion for the case $|M| = 2$.

3.2. $|M| \ge 4$

Claim 3.12. *Let $u \in V(M) \cap V_1$ and $v \in V(M) \cap V_2$ with $uv \notin E(G)$. Then $d_M(u) + d_M(v) \ge \frac{|M|}{2} - 1$.*

Proof. By Lemma 3.3, we have

$$d_L(u) + d_L(v) \le \sum_{i=1}^{k} \Big(\frac{|C_i|}{2} + 2 \Big) = \frac{|L|}{2} + 2k.$$

Hence

$$d_M(u) + d_M(v) \ge n + 2k - 1 - \Big(\frac{|L|}{2} + 2k \Big) = \frac{|M|}{2} - 1.$$

□

CASE A. M is disconnected.

Let M_0 be a connected component of M which has the smallest order (among all connected components of M). Let $M_1 = M - M_0$. We may assume that $|V(M_0) \cap V_1| \ge |V(M_0) \cap V_2|$. Then $|V(M_1) \cap V_1| \le |V(M_1) \cap V_2|$. Take $u \in V(M_0) \cap V_1$ and $v \in V(M_1) \cap V_2$. We divide the proof into the following two subcases according to the value of $|M_0|$.

Subcase A.1. $|M_0| \ge 2$.

In this case, the component containing v also has order at least 2. Thus there exists $v' \in V(M_0)$ with $uv' \in E(G)$, and there exists $u' \in V(M_1)$ with

$u'v \in E(G)$. Let $S = \{u, v, u', v'\}$. By Lemma 3.1,

$$\begin{aligned} d_L(u) + d_L(v') &\leq \sum_{i=1}^{k} \Big(\frac{|C_i|}{2} + 1\Big) = \frac{|L|}{2} + k, \\ d_L(u') + d_L(v) &\leq \sum_{i=1}^{k} \Big(\frac{|C_i|}{2} + 1\Big) = \frac{|L|}{2} + k. \end{aligned}$$

Since we clearly have $d_M(u)+d_M(v') \leq |M_0|$ and $d_M(u')+d_M(v) \leq |M_1|$, this implies

$$\begin{aligned} d_G(S) &\leq |M_0| + |M_1| + 2\Big(\frac{|L|}{2} + k\Big) \\ &= |M| + |L| + 2k \\ &= 2n + 2k. \end{aligned}$$

On the other hand, since $uv, u'v' \notin E(G)$, we have $d_G(S) \geq 2(n + 2k - 1)$, which is a contradiction.

Subcase A.2. $|M_0| = 1$.

Since $d_M(u) = 0$, it follows from Claim 3.12 that $d_M(v) = \frac{|M|}{2} - 1$. Since this holds for any $v \in V(M_1) \cap V_2$, M_1 is a complete bipartite graph. From the proof of Claim 3.12, we also see that $d_{C_i}(u) + d_{C_i}(v) = \frac{|C_i|}{2} + 2$ for each i with $1 \leq i \leq k$, which in particular implies that

(3.3) $$d_{C_i}(v) \geq 2 \text{ for each } i \text{ with } 1 \leq i \leq k.$$

Take $u' \in V(M_1) \cap V_1$. By (3.3) and Claims 3.1 and 3.2,

(3.4) $$N_{C_i}(u') \subseteq \{y_i\} \text{ for each } i \text{ with } 1 \leq i \leq k,$$

and hence $d_G(u') \leq \frac{|M|}{2}+k$. Now take $b \in (V(C_1)\cap V_2)-\{y_1\}$. Since (3.4) holds for any $u' \in V(M_1) \cap V_1$, we have $N_M(b) \subseteq \{u\}$, and hence $d_G(b) \leq \frac{|L|}{2} + 1$. Therefore $d_G(u') + d_G(b) \leq (\frac{|M|}{2} + k) + (\frac{|L|}{2} + 1) = n + k + 1$. On the other hand, since $u'b \notin E(G)$, we have $d_G(u') + d_G(b) \geq n + 2k - 1$, which is a contradiction.

CASE B. M is connected.

Claim 3.13. *Let $1 \leq i \leq k$, and suppose that $x_i \in V_1$. Then $|N_{C_i}(M) \cap V_1| \leq 1$.*

Proof. Without loss of generality, we may assume that $i = 1$. Suppose that $|N_{C_1}(M) \cap V_1| \geq 2$. Then $N_{C_1}(M) \cap V_2 \subseteq \{y_1\}$ by Claim 3.1. Choose two vertices $w, z \in N_{C_1}(M) \cap V_1$ so that $P_1 \subset C_1[z, w]$ and $N_G(M) \cap V(C_1(w, z)) = \emptyset$.

Take $v \in N_M(w)$ and $v' \in N_M(z)$, and let Q be a path joining v and v' in M. Then $\langle V(C_1[z,w]) \cup V(M)\rangle_G$ contains an admissible cycle D such that $V(D) = V(C_1[z,w]) \cup V(Q)$. Suppose that $|C_1(w,z)| = 1$. Write $V(C_1(w,z)) = \{b\}$. If $v \neq v'$, then we get a contradiction to the maximality of $|L|$; if $v = v'$, then since $N_M(b) = \emptyset$, $G - V(D) - \bigcup_{i=2}^{k} V(C_i)$ is disconnected, and hence we are reduced to CASE A, which also leads to a contradiction. Thus $|C_1(w,z)| \geq 3$. Take $u \in V(M) \cap V_1$. Let $S = \{w^+, z^{--}, u, v\}$. Suppose that $d_{C_1[z,w]}(w^+) + d_{C_1[z,w]}(z^{--}) \geq \frac{|C_1[z,w]|+4}{2}$. Then it follows Lemma 3.2 that there exists a cycle D' such that $P_1 \subset D'$ and $V(D') = V(D) \cup V(C_1[w^+, z^{--}]) = (V(C_1) - \{z^-\}) \cup V(Q)$. If $v \neq v'$, then this contradicts the maximality of $|L|$. If $v = v'$, then since $N_M(z^-) = \emptyset$, we are reduced to CASE A. Thus $d_{C_1[z,w]}(w^+) + d_{C_1[z,w]}(z^{--}) \leq \frac{|C_1[z,w]|+3}{2}$. Similarly it follows from Lemma 3.1 that $d_{C_i}(w^+) + d_{C_i}(z^{--}) \leq \frac{|C_i|}{2} + 1$ for each i with $2 \leq i \leq k$. Also $d_{C_i}(u) + d_{C_i}(v) \leq \frac{|C_i|}{2} + 1$ for each i with $2 \leq i \leq k$ by Lemma 3.1, and we have $d_{C_1[z,w]}(u) + d_{C_1[z,w]}(v) \leq \frac{|C_1[z,w]|+3}{2}$ because $N_{C_1}(u) \subseteq \{y_1\}$ by Claim 3.1. Since $d_{C_1(w,z)}(u) + d_{C_1(w,z)}(v) = 0$, $d_{\langle V(C_1(w,z))\rangle_G}(w^+) + d_{\langle V(C_1(w,z))\rangle_G}(z^{--}) \leq |C_1(w,z)|$, $d_M(u) + d_M(v) \leq |M|$ and $d_M(w^+) + d_M(z^{--}) = 0$, we now obtain

$$\begin{aligned} d_G(S) &\leq |M| + |C_1(w,z)| + |C_1[z,w]| + 3 + \sum_{i=2}^{k} 2\Big(\frac{|C_i|}{2} + 1\Big) \\ &= 2n + 2k + 1. \end{aligned}$$

On the other hand, since $w^+u, z^{--}v \notin E(G)$, we have $d_G(S) \geq 2(n + 2k - 1)$, which is a contradiction. $\square$

Using an argument similar to the proof of Claim 3.13, we obtain the following claim.

Claim 3.14. *Let $1 \leq i \leq k$, and suppose that $x_i \in V_2$. Then $|N_{C_i - \{y_i\}}(M) \cap V_1| \leq 1$.*

We are now in a position to complete the proof of Theorem 1.5. By symmetry, we may assume that $|\{i \mid x_i \in V_1\}| \geq |\{i \mid x_i \in V_2\}|$. Let $t = |\{i \mid x_i \in V_1\}|$. Since $k \geq 3$, we have $t \geq 2$. At the cost of relabeling, we may assume that $x_1 \in V_1$. Take $a \in (V(C_1) \cap V_1) - N_{C_1}(M)$ and $v \in V(M) \cap V_2$. Then since $d_M(a) = 0$ and $d_L(a) \leq \frac{|L|}{2}$, we have $d_G(a) \leq \frac{|L|}{2}$. By Claims 3.13 and 3.14, $d_L(v) \leq t + 2(k - t) = 2k - t$. Hence $d_G(v) = d_M(v) + d_L(v) \leq \frac{|M|}{2} + 2k - t$. Therefore $d_G(a) + d_G(v) \leq n + 2k - t \leq n + 2k - 2$. But since $av \notin E(G)$, this contradicts the assumption that $\sigma_{1,1}(G) \geq n + 2k - 1$.

This completes the proof of Theorem 1.5.

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References

[1] G.Chen, H.Enomoto, K.Kawarabayashi, D.Lou, K.Ota, A.Saito, Vertex-disjoint cycles containing specified edges in a bipartite graph, *Australas. J. Combin.* 23 (2001), 37-48.

[2] H.Wang, Covering a bipartite graph with cycles passing through given edges, *Australas. J. Combin.* 19 (1999), 115-121.

[3] H.Wang, On 2-factors of a bipartite graph, *J. Graph Theory* 31 (1999), 101-106.

[4] H.Wang, Proof of a conjecture on cycles in a bipartite graph, *J. Graph Theory* 31 (1999), 333-343.

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