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Some constructions of supermagic graphs using antimagic graphs

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Abstract. A graph G is called supermagic if it admits a labelling of the edges by pairwise different consecutive integers such that the sum of the labels of the edges incident with a vertex, the weight of vertex, is independent of the particular vertex. A graph G is called $(a,1)$-antimagic if it admits a labelling of the edges by the integers $\{1,\dots,|E(G)|\}$ such that the set of weights of the vertices consists of different consecutive integers. In this paper we will deal with the $(a,1)$-antimagic graphs and their connection to the supermagic graphs. We will introduce three constructions of supermagic graphs using some $(a,1)$-antimagic graphs.

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§1. Introduction

We consider finite undirected graphs without loops, multiple edges and isolated vertices. If G is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and edge set of G, respectively.

Let a graph G and a mapping f from $E(G)$ into positive integers be given. The *index-mapping* of f is the mapping $f^\star$ from $V(G)$ into positive integers defined by

$$f^\star(v) = \sum_{e\in E(G)} \eta(v,e)f(e) \quad \text{for every } v \in V(G),$$

where $\eta(v,e)$ is equal to 1 when e is an edge incident with a vertex v, and 0 otherwise. An injective mapping f from $E(G)$ into positive integers is called a *magic labelling* of G for an *index* λ if its index-mapping $f^\star$ satisfies

$$f^\star(v) = \lambda \quad \text{for all } v \in V(G).$$

A magic labelling f of G is called a *supermagic labelling* of G if the set $\{f(e) : e \in E(G)\}$ consists of consecutive positive integers. We say that a graph G is *supermagic* (*magic*) if and only if there exists a supermagic (magic) labelling of G.

The concept of magic graphs was introduced by Sedláček [17]. The regular magic graphs are characterized in [4]. Two different characterizations of all magic graphs are given in [14] and [13]. Supermagic graphs were introduced by Stewart [19]. It is easy to see that the classical concept of a magic square of n^2 boxes corresponds to the fact that the complete bipartite graph $K_{n,n}$ is supermagic for every positive integer $n \neq 2$ (see also [19]). Stewart [20] characterized supermagic complete graphs. In [10] supermagic regular complete multipartite graphs and supermagic cubes are characterized. In [11] there are given characterizations of magic line graphs of general graphs and supermagic line graphs of regular bipartite graphs. In [16] and [1] supermagic labellings of the Möbius ladders and two special classes of 4-regular graphs are constructed. Some constructions of supermagic labellings of various classes of regular graphs are described in [9] and [10]. In [5] there are established some bounds for number of edges in supermagic graph. More comprehensive information on magic and supermagic graphs can be found in [8].

Let G be a graph. A bijective mapping f from $E(G)$ into the set of integers $\{1, 2, \dots, |E(G)|\}$ is called an *antimagic labelling* of G if the index-mapping $f^\star$ is injective, i.e., it satisfies

$$f^\star(v) \neq f^\star(u) \quad \text{for every} \quad u, v \in V(G),\ u \neq v.$$

The concept of an antimagic labelling was introduced by Hartsfield and Ringel [9]. Bodendiek and Walther [2] introduced the special case of antimagic graphs. For positive integers a, d, a graph G is said to be (a, d)-*antimagic*, if it admits an antimagic labelling f such that

$$\{f^\star(v) : v \in V(G)\} = \{a, a + d, \dots, a + (|V(G)| - 1)d\}.$$

Obviously, $a = \frac{|E(G)|(|E(G)|+1)}{|V(G)|} - \frac{(|V(G)|-1)d}{2}$ in this case.

In this paper we will deal with the $(a, 1)$-antimagic graphs and their connection to the supermagic graphs. We will introduce three constructions of supermagic graphs using some $(a, 1)$-antimagic graphs.

§2. $(a, 1)$-antimagic graphs

It is known that the cycle C_n and the path P_n on n vertices are $(a, 1)$-antimagic if and only if n is odd, see [3]. To find other $(a, 1)$-antimagic graphs we use the edge-magic graphs which were introduced by Kotzig and Rosa [15].

Let G be a graph. A bijection $f : E(G) \cup V(G) \longrightarrow \{1, 2, \dots, |E(G)| + |V(G)|\}$ is called an *edge-magic total labelling* of G if there is a constant σ such that

$$f(u) + f(uv) + f(v) = \sigma,$$

for every edge $uv \in E(G)$. Moreover, if the vertices are labelled with the values from the set $\{1, 2, \dots, |V(G)|\}$ we say that G is a *super edge-magic* graph.

Theorem 2.1. *Let G be a 2-regular graph. Then G is super edge-magic if and only if it is $(a, 1)$-antimagic.*

Proof. Evidently, there is a digraph $\vec{G}$ which we get from G by an orientation of its edges such that the outdegree of every vertex of $\vec{G}$ is equal to 1. Let $[u, v]$ denote an arc of $\vec{G}$.

Suppose that f is a super edge-magic labelling of G. Then the labelling g, defined by $g(uv) = f(u)$ for every arc $[u, v]$ of $\vec{G}$, is $(a, 1)$-antimagic.

Assume that g is an $(a, 1)$-antimagic labelling of G. Then the labelling f, defined by $f(u) = g(uv)$ for every arc $[u, v]$ of $\vec{G}$ and $f(uv) = (5|V(G)| + 3)/2 - f(u) - f(v)$, is super edge-magic. □

According to the previous theorem and a corresponding result for super edge-magic graphs proved in [12] we have the following statement.

Corollary 2.2. *Let kG be a disjoint union of k copies of a graph G. If G is a 2-regular $(a_1, 1)$-antimagic graph, then kG is $(a_2, 1)$-antimagic for every odd positive integer k.*

Using the previous assertions and results on super edge-magic unions of two cycles (see [6]) we have

Corollary 2.3. *Let k, n and m be positive integers. For k odd each of the following graphs is $(a, 1)$-antimagic*

(i) kC_n *if* $3 \le n \equiv 1 \pmod 2$,

(ii) $k(C_3 \cup C_n)$ *if* $6 \le n \equiv 0 \pmod 2$,

(iii) $k(C_4 \cup C_n)$ *if* $5 \le n \equiv 1 \pmod 2$,

(iv) $k(C_5 \cup C_n)$ *if* $4 \le n \equiv 0 \pmod 2$,

(v) $k(C_m \cup C_n)$ *if* $6 \le m \equiv 0 \pmod 2$, $n \equiv 1 \pmod 2$, $n \ge m/2 + 2$.

Graphs G_1, G_2 form a decomposition of a graph G if $V(G_1) = V(G_2) = V(G)$, $E(G_1) \cap E(G_2) = \emptyset$ and $E(G_1) \cup E(G_2) = E(G)$. If G_2 is an r-regular graph then we say that the graph G arose from G_1 by adding the r-factor G_2. At IWOGL held in Herľany 2005 Petr Kovář presented an interesting method of construction of vertex-magic and antimagic total labellings of graphs (for definitions see [7]). However, this idea can be also used for (a,d)-antimagic graphs.

Theorem 2.4. *Let k be a positive integer and let H be a graph which arose from a graph G by adding an arbitrary $2k$-factor. If G is an $(a_1,1)$-antimagic graph, then H is also $(a_2,1)$-antimagic.*

Proof. As every $2k$-regular graph is decomposable into k edge-disjoint 2–factors, it is sufficient to consider that H arose from G by adding a 2-factor F. Let $\vec{F}$ be a digraph which we get from F by an orientation of its edges such that the outdegree of every vertex of $\vec{F}$ is equal to 1. Let $[u,v]$ denote an arc of $\vec{F}$.

The graph G is $(a_1,1)$-antimagic and so there is its $(a_1,1)$-antimagic labelling f, where $a_1 = \min\{f^\star(v) : v \in V(G)\}$. Consider a mapping $h : E(H) \longrightarrow \{1,2,\dots,|E(H)|\}$ defined by

$$h(e) = \begin{cases} f(e) & \text{if } e \in E(G), \\ a_1 + |E(H)| - f^\star(u) & \text{if } e = uv \in E(F) \text{ and } [u,v] \text{ is an arc of } \vec{F}. \end{cases}$$

It is easy to see that h is a bijection and $h^\star(v) = a_1 + |E(H)| + h(uv)$, where $[u,v]$ is an arc of $\vec{F}$. As $\{h(e) : e \in E(F)\} = \{|E(G)|+1, |E(G)|+2,\dots,|E(H)|\}$, the labelling h is $(a_2,1)$-antimagic, where $a_2 = a_1 + |E(H)| + |E(G)| + 1$. □

Let n, m and $1 \le a_1 < \cdots < a_m \le \left\lfloor \frac{n}{2} \right\rfloor$ be positive integers. A graph $C_n(a_1,\dots,a_m)$ with the vertex set $\{v_1,\dots,v_n\}$ and the edge set $\{v_iv_{i+a_j} : 1 \le i \le n, 1 \le j \le m\}$, the indices are being taken modulo n, is called a *circulant graph*. Clearly, $C_n(a_1,\dots,a_m)$ arose from $C_n(a_m)$ by adding a $2(m-1)$-factor. Moreover, if n is odd, then $C_n(a_m)$ is an $(a,1)$-antimagic graph because it is isomorphic to kC_r, where k and r are odd. Therefore, we have immediately

Corollary 2.5. *Every circulant graph of odd order is $(a,1)$-antimagic.*

The cycle of odd order is $(a,1)$-antimagic and every regular Hamiltonian graph arose from its Hamilton cycle by adding a factor, so

Corollary 2.6. *Every $2r$-regular Hamiltonian graph of odd order is $(a,1)$-antimagic.*

Any graph of order n with minimum degree at least $n/2$ is Hamiltonian, thus we get

Corollary 2.7. *Let G be a $2r$-regular graph of odd order n. If $n < 4r$, then G is $(a,1)$-antimagic.*

§3. Supermagic graphs

For any graph G we define a graph $G^{\bowtie}$ by $V(G^{\bowtie}) = \bigcup_{v\in V(G)}\{v^0, v^1\}$ and $E(G^{\bowtie}) = E_1(G^{\bowtie}) \cup E_2(G^{\bowtie})$, where $E_1(G^{\bowtie}) = \bigcup_{vu\in E(G)}\{v^0u^1, v^1u^0\}$ and $E_2(G^{\bowtie}) = \bigcup_{v\in V(G)}\{v^0v^1\}$.

Theorem 3.1. *Let G be an $(a,1)$-antimagic $2r$-regular graph. Then $G^{\bowtie}$ is a supermagic graph.*

Proof. Put $n := |V(G)|$. As G is a $2r$-regular graph, every its component is Eulerian. Therefore, there is a digraph $\vec{G}$ which we get from G by an orientation of its edges such that the outdegree (and also the indegree) of every vertex of $\vec{G}$ is equal to r. By $[u,v]$ we denote an arc of $\vec{G}$ and by $N^+(v)$, $N^-(v)$ the outneighbourhood, inneighbourhood of a vertex v in $\vec{G}$, respectively.

Let $f : E(G) \longrightarrow \{1, 2, \dots, rn\}$ be an $(a,1)$-antimagic labelling of G. Consider the bijection $g : E_1(G^{\bowtie}) \longrightarrow \{1, 2, \dots, 2rn\}$ given by

$$g(u^iv^j) = \begin{cases} f(uv) & \text{if } i=0,\ j=1,\\ f(uv)+rn & \text{if } i=1,\ j=0, \end{cases}$$

for every arc $[u,v]$ of $\vec{G}$.

For its index-mapping we have

$$\begin{aligned} g^\star(v^0) &= \sum_{w\in N^+(v)} g(v^0w^1) + \sum_{u\in N^-(v)} g(u^1v^0) \\ &= \sum_{w\in N^+(v)} f(vw) + \sum_{u\in N^-(v)} (f(uv)+rn) = f^\star(v) + r^2n \end{aligned}$$

for every vertex $v^0 \in V(G^{\bowtie})$. Similarly, we have $g^\star(v^1) = f^\star(v) + r^2n$ for every vertex $v^1 \in V(G^{\bowtie})$. Thus $g^\star(v^0) = g^\star(v^1) = f^\star(v) + r^2n$ for every vertex $v \in V(G)$. As f is an $(a,1)$-antimagic labelling, the set $\{f^\star(v) : v \in V(G)\}$ consists of consecutive integers. It means that the bijection $h : E(G^{\bowtie}) \longrightarrow \{1, 2, \dots, (2r+1)n\}$, given by

$$\begin{aligned} h(u^iv^j) &= g(u^iv^j) \quad \text{for } u^iv^j \in E_1(G^{\bowtie}),\\ h(v^0v^1) &= \frac{2rn(r+1)+(2r+1)(n+1)}{2} - f^\star(v) \quad \text{for } v\in V(G), \end{aligned}$$

is a supermagic labelling of $G^{\bowtie}$. □

Note, that $C_n^{\bowtie}$ is a graph isomorphic to either the Möbius ladder M_{2n}, for n odd, or the graph of n-side prism S_n, for n even. Moreover, for the disjoint union of graphs G_1 and G_2 it holds $(G_1 \cup G_2)^{\bowtie} = G_1^{\bowtie} \cup G_2^{\bowtie}$. According to Theorem 3.1 and Corollary 2.3 we have

Corollary 3.2. *Let k, n and m be positive integers. For k odd the following graphs are supermagic*

(i) kM_{2n} *when* $3 \leq n \equiv 1 \pmod{2}$,

(ii) $k(M_6 \cup S_n)$ *when* $6 \leq n \equiv 0 \pmod{2}$,

(iii) $k(S_4 \cup M_{2n})$ *when* $5 \leq n \equiv 1 \pmod{2}$,

(iv) $k(M_{10} \cup S_n)$ *when* $4 \leq n \equiv 0 \pmod{2}$,

(v) $k(S_m \cup M_{2n})$ *when* $6 \leq m \equiv 0 \pmod{2}$, $n \equiv 1 \pmod{2}$, $n \geq m/2 + 2$.

Similarly, using Theorem 3.1 and Corollaries 2.5, 2.6 and 2.7 we get

Corollary 3.3. *Let G be a $2r$-regular graph of odd order n. If G is circulant, Hamiltonian or $n < 4r$, then $G^{\bowtie}$ is a supermagic graph.*

One can see that $G^{\bowtie}$ is isomorphic to the Cartesian product $G \times K_2$ whenever G is a bipartite graph. However, a regular bipartite graph of even degree is never $(a, 1)$-antimagic. So, in the next theorem we describe another construction of supermagic Cartesian products.

Theorem 3.4. *Let G be an $(a, 1)$-antimagic graph decomposable into two edge-disjoint r-factors. Then $G \times K_2$ is a supermagic graph.*

Proof. Suppose that F^1, F^2 are edge-disjoint r-factors which form a decomposition of G and $f : E(G) \longrightarrow \{1, 2, \ldots, rn\}$, where $n = |V(G)|$, is an $(a, 1)$-antimagic labelling of G.

We can denote the vertices of $G \times K_2$ by v_i, $i \in \{1, 2\}$, $v \in V(G)$, in such a way that the vertices $\{v_i : v \in V(G)\}$ induce a subgraph G_i isomorphic to G. So, $G \times K_2$ consists of subgraphs G_1, G_2 and n edges v_1v_2 for all $v \in V(G)$. By F_i^j, $i \in \{1, 2\}$, $j \in \{1, 2\}$, we denote the factor of G_i corresponding to F^j.

Consider the bijection $g : E(G_1 \cup G_2) \longrightarrow \{1, 2, \ldots, 2rn\}$ given by

$$g(e) = \begin{cases} f(e) & \text{if } e \in F_1^1 \text{ or } e \in F_2^2, \\ f(e) + rn & \text{if } e \in F_2^1 \text{ or } e \in F_1^2. \end{cases}$$

For its index-mapping we have

$$\begin{aligned} g^\star(v_1) &= \sum_{v_1u_1\in E(G_1)} g(v_1u_1) = \sum_{v_1u_1\in E(F_1^1)} g(v_1u_1) + \sum_{v_1w_1\in E(F_1^2)} g(v_1w_1) \\ &= \sum_{vu\in E(F^1)} f(vu) + \sum_{vw\in E(F^2)} (f(vw)+rn) \\ &= \sum_{vu\in E(G)} f(uv) + r^2n = f^\star(v) + r^2n \end{aligned}$$

for every vertex $v_1 \in V(G_1)$. Similarly, $g^\star(v_2) = f^\star(v) + r^2n$ for every vertex $v_2 \in V(G_2)$. Thus $g^\star(v_1) = g^\star(v_2) = f^\star(v) + r^2n$ for every vertex $v \in V(G)$. As f is an $(a, 1)$-antimagic labelling, the set $\{f^\star(v) : v \in V(G)\}$ consists of consecutive integers.

It means that the bijection $h : E(G \times K_2) \longrightarrow \{1, 2, \ldots, (2r+1)n\}$ given by

$$h(e) = g(e) \quad \text{for every } e \in E(G_1 \cup G_2),$$
$$h(v_1v_2) = \frac{2rn(r+1) + (2r+1)(n+1)}{2} - f^\star(v) \quad \text{for every } v \in V(G)$$

is a supermagic labelling of $G \times K_2$. □

As every $4r$-regular graph is decomposable into two edge-disjoint $2r$-factors, immediately from Theorem 3.4 and Corollaries 2.5, 2.6 and 2.7 we get

Corollary 3.5. *Let G be a $4r$-regular graph of odd order n. If G is circulant, Hamiltonian or $n < 8r$, then $G \times K_2$ is a supermagic graph.*

Finally we describe a construction of supermagic joins $G \oplus K_1$. In [18] there are given some conditions for the existence of such graphs.

Theorem 3.6. *Let G be an $(a, 1)$-antimagic r-regular graph of order n. If $(n-r-1)$ is a divisor of the non-negative integer $a + n(1 + r - \frac{n+1}{2})$, then the join $G \oplus K_1$ is a supermagic graph.*

Proof. Put $\lambda_1 := a + n(1+r)$ and $\lambda_2 := \frac{n(n+1)}{2}$. According to the assumption there is a non-negative integer p such that $\lambda_1 - \lambda_2 = p(n-r-1)$ (thus $(r+1)p + \lambda_1 = np + \lambda_2$). Let f be an $(a, 1)$-antimagic labelling of G. The join $G \oplus K_1$ is obtained from G by adding the vertex w and the edges wv for all $v \in V(G)$.

Consider the mapping h from $E(G \oplus K_1)$ into positive integers given by

$$h(e) = \begin{cases} p + n + f(e) & \text{if } e \in E(G), \\ p + n + a - f^\star(v) & \text{if } e = wv \text{ for } v \in V(G). \end{cases}$$

Evidently, $\{h(wv) : v \in V(G)\} = \{p+1, p+2, \dots, p+n\}$ and $\{h(e) : e \in E(G)\} = \{p+n+1, p+n+2, \dots, p+n+|E(G)|\}$. Thus, the set $\{h(e) : e \in E(G \oplus K_1)\}$ consists of consecutive positive integers. Moreover, $h^\star(w) = np+\lambda_2$ and $h^\star(v) = (r+1)p+\lambda_1$ for all $v \in V(G)$. Therefore, h is a supermagic labelling of $G \oplus K_1$. □

Using the divisibility it is not difficult to check the assumptions of Theorem 3.6 for given values n and r. Thus we have

Corollary 3.7. *Let n and r be positive integers such that one of the following conditions is satisfied:*

(i) $5 \le n \equiv 1 \pmod 2$ *and* $r = n-3$,

(ii) $11 \le n \equiv 1 \pmod 2$ *and* $r = n-7$,

(iii) $8 \le n \equiv 0 \pmod 4$ *and* $r = \frac{n}{2} - 1$,

(iv) $11 \le n \equiv 3 \pmod 8$ *and* $r = n-5$,

(v) $12 \le n \equiv 4 \pmod 8$ *and* $r = n-3$,

(vi) $12 \le n \equiv 4 \pmod 8$ *and* $r = n-7$,

(vii) $13 \le n \equiv 5 \pmod 8$ *and* $r = n-5$.

If G is an $(a,1)$-antimagic r-regular graph of order n, then the join $G \oplus K_1$ is supermagic.

Immediately from Corollaries 2.7 and 3.7 we get

Corollary 3.8. *Let G be any $(n-3)$-regular ($(n-7)$-regular) graph of odd order $n \ge 7$ ($n \ge 15$). Then $G \oplus K_1$ is a supermagic graph.*

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