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6-Shredders in 6-Connected Graphs

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Abstract. For a graph G, a subset S of $V(G)$ is called a shredder if $G - S$ consists of three or more components. We show that if G is a 6-connected graph of order at least 325, then the number of shredders of cardinality 6 of G is less than or equal to $(2|V(G)| - 9)/3$.

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§1. INTRODUCTION

In this paper, we consider only finite, undirected, simple graphs with no loops and no multiple edges. Let $G = (V(G), E(G))$ be a graph. As is introduced by Cheriyan and Thurimella in [1], a subset S of $V(G)$ is called a *shredder* if $G - S$ consists of three or more components. A shredder of cardinality k is referred to as a k-shredder. In [2; Theorem 1], it is proved that if $k \geq 5$ and G is a k-connected graph, then the number of k-shredders of G is less than $2|V(G)|/3$, and it is shown that for each fixed $k \geq 5$, the coefficient $2/3$ in the upper bound is best possible. For $k = 5$, it is shown in [3; Theorem 3] that if G is a 5-connected graph of order at least 135, then the number of 5-shredders of G is less than or equal to $(2|V(G)| - 10)/3$, and it is shown that this bound is attained by infinitely many graphs (for results concerning the case where $1 \leq k \leq 4$, the reader is referred to [4] and [2; Theorem 2]). In this paper, we prove:

Theorem *Let G be a 6-connected graph of order at least 325. Then the number of 6-shredders of G is less than or equal to*

$$(2|V(G)| - 9)/3.$$

We conclude this section by constructing an infinite family of graphs G which attain the bound $(2|V(G)|-9)/3$ in the Theorem. Let $m \geq 3$. First define a graph H of order $4m$ by

$$\begin{array}{rcl} V(H) & = & \{y_{i,j} | 1 \leq i \leq 2m,\ j=1,2\}, \\ E(H) & = & \{y_{i,j}y_{i+2,k} | 1 \leq i \leq 2m-2,\ j=1,2,\ k=1,2\} \\ & \cup & \{y_{1,j}y_{2,k},\ y_{2m-1,j}y_{2m,k} | j=1,2,\ k=1,2\}. \end{array}$$

Thus H is the graph obtained from the cycle of length $2m$ by "splitting" each vertex into two independent vertices, where $\{y_{1,1}, y_{1,2}\}$, $\{y_{3,1}, y_{3,2}\}$, $\{y_{5,1},\ y_{5,2}\}$, $\cdots$, $\{y_{2m-1,1},\ y_{2m-1,2}\}$, $\{y_{2m,1},\ y_{2m,2}\}$, $\{y_{2m-2,1},\ y_{2m-2,2}\}$, $\{y_{2m-4,1}, y_{2m-4,2}\}$, $\cdots$, $\{y_{2,1}, y_{2,2}\}$ occur in this order along the cycle. Now define a graph G of order $6m-3$ by

$$\begin{array}{rcl} V(G) = V(H) & \cup & \{x_i | 3 \leq i \leq 2m-3\} \cup \{a, b\}, \\ E(G) = E(H) & \cup & \{x_i y_{i,j},\ x_i y_{i+1,j} | 3 \leq i \leq 2m-3,\ j=1,2\} \\ & \cup & \{a x_i,\ b x_i | 3 \leq i \leq 2m-3\} \\ & \cup & \{a y_{3,j},\ b y_{2m-2,j} | j=1,2\} \\ & \cup & \{a y_{i,j},\ b y_{i,j} | i=1,2,2m-1,2m,\ j=1,2\}. \end{array}$$

Then G is 6-connected, and has $4m-5$ 6-shredders

$\{y_{i,1}, y_{i,2}, y_{i+4,1}, y_{i+4,2}, x_{i+1}, x_{i+2}\}$ $(2 \leq i \leq 2m-5)$,

$\{y_{2m-4,1}, y_{2m-4,2}, y_{2m,1}, y_{2m,2}, x_{2m-3}, b\}$,

$\{y_{2m-3,1}, y_{2m-3,2}, y_{2m,1}, y_{2m,2}, a, b\}$,

$\{y_{2m-2,1}, y_{2m-2,2}, y_{2m-1,1}, y_{2m-1,2}, a, b\}$,

$\{y_{1,1}, y_{1,2}, y_{5,1}, y_{5,2}, x_3, a\}$,

$\{y_{1,1}, y_{1,2}, y_{4,1}, y_{4,2}, a, b\}$,

$\{y_{2,1}, y_{2,2}, y_{3,1}, y_{3,2}, a, b\}$,

$\{y_{i,1}, y_{i,2}, y_{i+1,1}, y_{i+1,2}, a, b\}$ $(3 \leq i \leq 2m-3)$.

Thus the number of 6-shredders of G is $4m-5 = (2(6m-3)-9)/3 = (2|V(G)|-9)/3$.

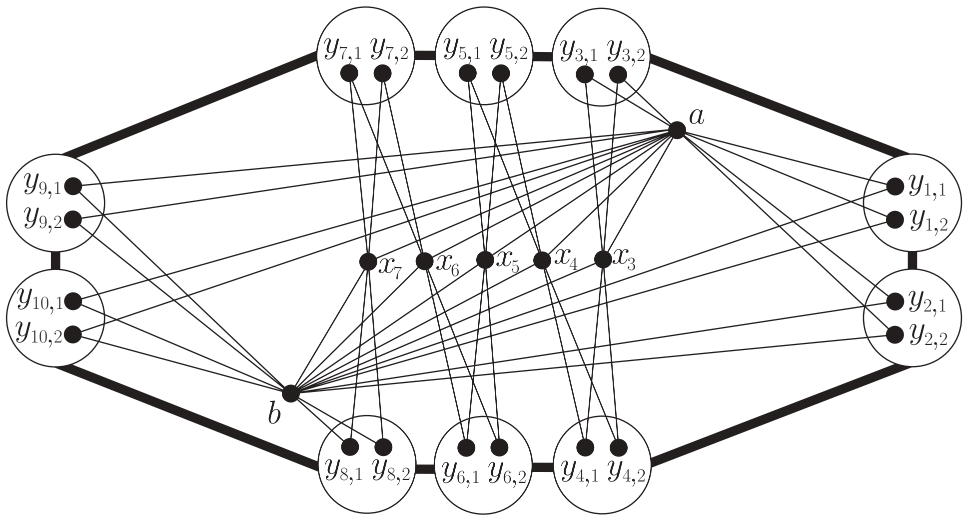

Figure 1: $m = 5$

§2. PRELIMINARY RESULTS

Throughout the rest of this paper, let G be a 6-connected graph, and let $\mathscr{S}$ denote the set of 6-shredders of G. For each $S \in \mathscr{S}$, we define $\mathscr{K}(S)$, $\mathscr{L}(S)$ and $L(S)$ as follows. Let $S \in \mathscr{S}$. We let $\mathscr{K}(S)$ denote the set of components of $G - S$. Write $\mathscr{K}(S) = \{H_1, \cdots, H_s\}$ ($s = |\mathscr{K}(S)|$). We may assume $|V(H_1)| \geq |V(H_2)| \geq \cdots \geq |V(H_s)|$ (any such labeling will do). Under this notation, we let $\mathscr{L}(S) = \mathscr{K}(S) - \{H_1\}$ and $L(S) = \cup_{2 \leq i \leq s} V(H_i)$; thus $L(S) = \cup_{C \in \mathscr{L}(S)} V(C)$. Now let $\mathscr{L} = \cup_{S \in \mathscr{S}} \mathscr{L}(S)$. A member F of $\mathscr{L}$ is said to be *saturated* if there exists a subset $\mathscr{C}$ of $\mathscr{L} - \{F\}$ such that $V(F) = \cup_{C \in \mathscr{C}} V(C)$.

Let $S, T \in \mathscr{S}$ with $S \neq T$. We say that S *meshes* with T if S intersects with at least two members of $\mathscr{K}(T)$. It is easy to see that if S meshes with T, then T intersects with all members of $\mathscr{K}(S)$, and hence T meshes with S and S intersects with all members of $\mathscr{K}(T)$ (see [1; Lemma 4.3 (1)]).

The following three lemmas are proved in [4; Lemma 2.1 and Claims 2.3 and 3.3] (see also [2; Lemmas 3.2, 3.4 and 3.5]).

Lemma 2.1. *Let $S, T \in \mathscr{S}$ with $S \neq T$, and suppose that S does not mesh with T. Then one of the following holds:*

(i) *$L(S) \cap L(T) = \emptyset$, $(L(S) \cup L(T)) \cap (S \cup T) = \emptyset$, and no edge of G joins a vertex in $L(S)$ and a vertex in $L(T)$;*

(ii) *there exists $C \in \mathscr{L}(S)$ such that $V(C) \supseteq L(T)$ (so $L(S) \supseteq L(T)$); or*

(iii) *there exists* $D \in \mathscr{L}(T)$ *such that* $V(D) \supseteq L(S)$ *(so* $L(T) \supseteq L(S)$*).*

Lemma 2.2. *Let* $S, T \in \mathscr{S}$ *with* $S \neq T$*, and suppose that* S *meshes with* T*. Then the following hold.*

(i) $S \supseteq L(T)$ *or* $T \supseteq L(S)$.

(ii) $L(S) \cap L(T) = \emptyset$.

Lemma 2.3. *Let* $C, D \in \mathscr{L}$*. Then one of the following holds:*

(i) $V(C) \cap V(D) = \emptyset$;

(ii) $V(C) \supseteq V(D)$; *or*

(iii) $V(D) \supseteq V(C)$.

The following lemma is proved in [2; Lemma 3.6].

Lemma 2.4. *Let* $F \in \mathscr{L}$*. Suppose that* F *is saturated, and let* $\mathscr{C}$ *be a subset of* $\mathscr{L} - \{F\}$ *with minimum cardinality such that* $V(F) = \cup_{C \in \mathscr{C}} V(C)$*. Then the following hold.*

(i) $V(C) \cap V(D) = \emptyset$ *for all* $C, D \in \mathscr{C}$ *with* $C \neq D$.

(ii) $\mathscr{C} = \cup_{T \in \mathscr{T}} \mathscr{L}(T)$ *for some subset* $\mathscr{T}$ *of* $\mathscr{S}$ *(so* $V(F) = \cup_{T \in \mathscr{T}} L(T)$*).*

(iii) $L(S) \cap L(T) = \emptyset$ *for all* $S, T \in \mathscr{T}$ *with* $S \neq T$.

(iv) $|\mathscr{T}| \geq 2$.

(v) $|\mathscr{C}| \geq 4$.

(vi) *If we define a graph* $\mathscr{G}$ *on* $\mathscr{T}$ *by joining* S *and* T *(*$S, T \in \mathscr{T}$, $S \neq T$*) if and only if* S *meshes with* T*, then* $\mathscr{G}$ *is connected.*

The following lemma is essentially proved in [2; Lemma 3.7].

Lemma 2.5. *Let* $V \neq \emptyset$ *be a finite set, and let* $\mathscr{M}$ *be a family of subsets of* V *which satisfies the following three properties:*

(a) $\emptyset \notin \mathscr{M}$;

(b) *if* $C, D \in \mathscr{M}$*, then* $C \cap D = \emptyset$ *or* $C \supseteq D$ *or* $D \supseteq C$*; and*

(c) *if* $F \in \mathscr{M}$, $\mathscr{C} \subseteq \mathscr{M} - \{F\}$ *and* $F = \cup_{C \in \mathscr{C}} C$*, then* $|\mathscr{C}| \geq 4$.

Then the following hold.

(I) $|\mathscr{M}| \leq (4|V|-1)/3$.

(II) *If* $|\mathscr{M}| = (4|V|-1)/3$, *then* $V \in \mathscr{M}$ *and one of the following holds:*

(i) $|V| = 1$*; or*

(ii) *there exists* $\mathscr{C} \subseteq \mathscr{M} - \{V\}$ *with* $|\mathscr{C}| = 4$ *such that* $V = \cup_{C\in\mathscr{C}} C$, *and such that for each* $C \in \mathscr{C}$, $|\{X \in \mathscr{M} | X \subseteq C\}| = (4|C|-1)/3$.

(III) *If* $V \in \mathscr{M}$ *and* $(4|V|-3)/3 \leq |\mathscr{M}| \leq (4|V|-2)/3$, *then one of the following holds:*

(i) *there exists* $\mathscr{C} \subseteq \mathscr{M} - \{V\}$ *with* $4 \leq |\mathscr{C}| \leq 5$ *such that* $V = \cup_{C\in\mathscr{C}} C$*;*

(ii) *there exists* $\mathscr{C} \subseteq \mathscr{M} - \{V\}$ *with* $|\mathscr{C}| = 6$ *such that* $V = \cup_{C\in\mathscr{C}} C$, *and such that for each* $C \in \mathscr{C}$, $|\{X \in \mathscr{M} | X \subseteq C\}| = (4|C|-1)/3$*;*

(iii) *there exists* $C \in \mathscr{M}$ *such that* $|C| = |V| - 1$*; or*

(iv) *there exist* $C, D \in \mathscr{M}$ *with* $C \cap D = \emptyset$ *such that* $|C \cup D| = |V| - 1$, $|\{X \in \mathscr{M} | X \subseteq C\}| = (4|C|-1)/3$, *and* $|\{X \in \mathscr{M} | X \subseteq D\}| = (4|D|-1)/3$.

The following two lemmas follow from Lemma 2.5, and are essentially proved in [4; Lemmas 2.8 and 2.9].

Lemma 2.6. *Let* $F \in \mathscr{L}$, *and set* $\mathscr{T} = \{T \in \mathscr{S} | L(T) \subseteq V(F)\}$. *Then the following hold.*

(I) $|\mathscr{T}| \leq (2|V(F)|-2)/3$.

(II) *If* $|\mathscr{T}| = (2|V(F)|-2)/3$, *then one of the following holds:*

(i) F *is trivial (i.e.,* $|V(F)| = 1$*); or*

(ii) F *is saturated, and there exist* $T_1, T_2 \in \mathscr{T}$ *such that* $V(F) = L(T_1) \cup L(T_2)$, T_1 *meshes with* T_2, $|\mathscr{L}(T_1)| = |\mathscr{L}(T_2)| = 2$, *and* $|\{T \in \mathscr{S} | L(T) \subseteq L(T_i)\}| = (2|L(T_i)|-1)/3$ *for each* $i = 1, 2$.

(III) *If* $|\mathscr{T}| = (2|V(F)|-3)/3$, *then one of the following holds:*

(i) F *is saturated, and there exist* $T_1, T_2 \in \mathscr{T}$ *such that* $V(F) = L(T_1) \cup L(T_2)$, T_1 *meshes with* T_2, *and* $|\mathscr{L}(T_1)| = 2$ *and* $2 \leq |\mathscr{L}(T_2)| \leq 3$*;*

(ii) F *is saturated, and there exist* $T_1, T_2, T_3 \in \mathscr{T}$ *such that* $V(F) = L(T_1) \cup L(T_2) \cup L(T_3)$, T_3 *meshes with* T_1 *and* T_2, $|\mathscr{L}(T_1)| = |\mathscr{L}(T_2)| = |\mathscr{L}(T_3)| = 2$, *and* $|\{T \in \mathscr{S} | L(T) \subseteq L(T_i)\}| = (2|L(T_i)|-1)/3$ *for each* $i = 1, 2, 3$*; or*

(iii) *F is not saturated, and there exists $T_0 \in \mathscr{T}$ such that $|L(T_0)| = |V(F)|-1$, $|\mathscr{L}(T_0)| = 2$, and $|\{T \in \mathscr{S} | L(T) \subseteq L(T_0)\}| = (2|L(T_0)|-1)/3$.*

Lemma 2.7. *Let $S \in \mathscr{S}$, and write $\mathscr{L}(S) = \{F_1, \cdots, F_p\}$ ($p = |\mathscr{L}(S)|$). Set $\mathscr{T} = \{T \in \mathscr{S} | L(T) \subseteq L(S)\}$, and set $\mathscr{T}_i = \{T \in \mathscr{S} | L(T) \subseteq V(F_i)\}$. Then the following hold.*

(I) $|\mathscr{T}| \leq (2|L(S)| - 2p + 3)/3 \leq (2|L(S)| - 1)/3$.

(II) *If $|\mathscr{T}| = (2|L(S)| - 1)/3$, then $p = 2$ and $|\mathscr{T}_i| = (2|V(F_i)| - 2)/3$ for each i.*

(III) *If $|\mathscr{T}| = (2|L(S)|-2)/3$, then $p = 2$, and either $|\mathscr{T}_1| = (2|V(F_1)|-2)/3$ and $|\mathscr{T}_2| = (2|V(F_2)| - 3)/3$, or $|\mathscr{T}_1| = (2|V(F_1)| - 3)/3$ and $|\mathscr{T}_2| = (2|V(F_2)| - 2)/3$.*

The following two lemmas are proved in [3; Lemmas 2.11 and 2.12].

Lemma 2.8. *Let $S, T \in \mathscr{S}$, and suppose that S meshes with T and $L(S) \not\subseteq T$. Then $L(T) \subseteq S$ and $|L(T)| = 2$.*

Lemma 2.9. *Suppose that $|V(G)| \geq 13$. Let $S, T \in \mathscr{S}$, and suppose that S meshes with T, $L(S) \subseteq T$ and $L(T) \subseteq S$. Then $|L(S)| + |L(T)| \leq 6$.*

The following lemma follows from Lemmas 2.8 and 2.9.

Lemma 2.10. *Suppose that $|V(G)| \geq 13$. Let $S, T \in \mathscr{S}$, and suppose that S meshes with T and $|L(S)| \geq 4$. Then $L(T) \subseteq S$ and $|L(T)| = 2$.*

As an immediate corollary of Lemma 2.10, we obtain the following lemma.

Lemma 2.11. *Suppose that $|V(G)| \geq 13$. Let $S, T \in \mathscr{S}$ with $S \neq T$, and suppose that $|L(S)|, |L(T)| \geq 4$. Then S does not mesh with T.*

We now proceed to prove a refinement of Lemma 2.8 (see Lemma 2.13).

Lemma 2.12. *Let $S, T \in \mathscr{S}$, and suppose that S meshes with T and $L(S) \not\subseteq T$. Let $F \in \mathscr{K}(S)$, and suppose that $|V(F)| \geq 2$. Then $|T \cap V(F)| \geq 2$.*

Proof. If $V(F) \subseteq T$, then we clearly have $|T \cap V(F)| = |V(F)| \geq 2$. Thus we may assume $V(F) \not\subseteq T$. Since $L(S) \not\subseteq T$, we have $L(T) \subseteq S$ and $|L(T)| = 2$ by Lemma 2.8. Set $R = (T \cap V(F)) \cup (S - L(T))$. Then R separates $V(F) - (T \cap V(F))$ from the rest. This implies $|R| \geq 6$, and hence $|T \cap V(F)| = |R| - |S - L(T)| \geq 6 - |S - L(T)| = |L(T)| = 2$. □

Lemma 2.13. *Let $S, T \in \mathscr{S}$, and suppose that S meshes with T and $L(S) \not\subseteq T$. Write $\mathscr{L}(S) = \{F_1, \cdots, F_p\}$ ($p = |\mathscr{L}(S)|$) with $|V(F_1)| \leq |V(F_2)| \leq \cdots \leq |V(F_p)|$. Then $|L(T)| = 2$ and $3 \leq |T \cap L(S)| \leq 4$, and one of the following holds:*

(i) $p = 2$, $|V(F_1)| = 1$, $|V(F_2)| \geq 3$, $V(F_1) \subseteq T$, *and* $|T \cap V(F_2)| = 2$;

(ii) $p = 2$, $|V(F_1)| = 1$, $|V(F_2)| \geq 4$, $V(F_1) \subseteq T$, *and* $|T \cap V(F_2)| = 3$;

(iii) $p = 3$, $|V(F_1)| = |V(F_2)| = 1$, $|V(F_3)| \geq 3$, $V(F_1) \cup V(F_2) \subseteq T$, *and* $|T \cap V(F_3)| = 2$; *or*

(iv) $p = 2$, $|V(F_1)| \geq 2$, $|V(F_2)| \geq 3$, *and* $|T \cap V(F_1)| = |T \cap V(F_2)| = 2$.

Proof. By Lemma 2.8, $|L(T)| = 2$. Let $q = \max\{i | 1 \leq i \leq p,\ |V(F_i)| = 1\}$ (if $|V(F_1)| = 2$, we let $q = 0$). Then $V(F_i) \subseteq T$ for each $1 \leq i \leq q$ by the assumption that S meshes with T, and $|T \cap V(F_i)| \geq 2$ for each $q+1 \leq i \leq p$ by Lemma 2.12. Since $L(S) \not\subseteq T$, we have $p \geq q+1$, i.e., $|V(F_p)| \geq 2$. Write $\mathscr{K}(S) - \mathscr{L}(S) = \{C\}$. Then $|V(C)| \geq |V(F_p)| \geq 2$ by the definition of $\mathscr{L}(S)$, and hence $|T \cap V(C)| \geq 2$ by Lemma 2.12. Since $(\sum_{1 \leq i \leq p} |T \cap V(F_i)|) + |T \cap V(C)| \leq |T| = 6$, we obtain

$$q + 2(p-q) \leq q + \sum_{q+1 \leq i \leq p} |T \cap V(F_i)| \leq 4. \tag{2.1}$$

Now if $q \geq 2$, then since $p \geq q+1$, it follows from (2.1) that $q = 2$, $p = 3$ and $|T \cap V(F_3)| = 2$, and hence (iii) holds because $L(S) \not\subseteq T$; if $q = 0$, then since $p \geq 2$, it follows from (2.1) that $p = 2$ and $|T \cap V(F_1)| = |T \cap V(F_2)| = 2$, and hence (iv) holds because $L(S) \not\subseteq T$; if $q = 1$, then it follows from (2.1) that $p = 2$ and $|T \cap V(F_2)| = 2$ or 3, and hence (i) or (ii) holds because $L(S) \not\subseteq T$. □

Lemma 2.14. *Let $S, T \in \mathscr{S}$, and suppose that S meshes with T and $|L(S)| \geq 3$. Then $|T \cap L(S)| \geq 3$.*

Proof. If $L(S) \subseteq T$, then clearly $|T \cap L(S)| = |L(S)| \geq 3$; if $L(S) \not\subseteq T$, then $|T \cap L(S)| \geq 3$ by Lemma 2.13. □

We define an order relation $\leq$ in $\mathscr{S}$ as follows:

$$S \leq T \Longleftrightarrow L(S) \subseteq L(T) (S, T \in \mathscr{S}).$$

Lemma 2.15. *Let $S \in \mathscr{S}$ and $F \in \mathscr{L}(S)$, and suppose that $|V(F)| \geq 4$. Let $\mathscr{T} = \{T \in \mathscr{S} | L(T) \subseteq V(F)\}$. Let $T_1, \cdots, T_s$ be the maximal members of $\mathscr{T}$ (with respect to the order relation defined above), and suppose that $|V(F) - (L(T_1) \cup \cdots \cup L(T_s))| \leq 1$.*

(i) (a) *Let* $P \in \mathscr{S}$*, and suppose that* P *meshes with* S*. Then there exists* i $(1 \leq i \leq s)$ *such that* P *meshes with* T_i *and such that* $P \cap L(T_j) = \emptyset$ *for each* $1 \leq j \leq s$ *with* $j \neq i$.
(b) *If* $|P \cap V(F)| = 2$*, then* $|L(T_i)| = 2$.

(ii) *Let* $1 \leq i \leq s$*, and suppose that* $|L(T_i)| = 2$*. Then there exists at most one member of* $\mathscr{S}$ *which meshes with both* S *and* T_i.

(iii) *Let* $\mathscr{S}_0$ *be the set of those members* P *of* $\mathscr{S}$ *such that* P *meshes with* S *and* $|P \cap V(F)| = 2$*. Then* $|\mathscr{S}_0| \leq |\{i | 1 \leq i \leq s,\ |L(T_i)| = 2\}|$.

Proof. Set $X = V(F) - (L(T_1) \cup \cdots \cup L(T_s))$ (so $|X| = 0$ or 1 by assumption). Let $P \in \mathscr{S}$, and suppose that P meshes with S. Since $|L(S)| \geq |V(F)| + 1 \geq 5$, $|L(S)| + |L(P)| \geq 7$, and hence $L(P) \subseteq S$, $L(S) \not\subseteq P$ and $|L(P)| = 2$ by Lemmas 2.9 and 2.8. Consequently

$$2 \leq |P \cap V(F)| \leq 3 \tag{2.2}$$

by Lemma 2.13. Since $|X| \leq 1$, (2.2) implies that there exists i such that $P \cap L(T_i) \neq \emptyset$. Since $L(P) \cap L(T_i) = \emptyset$ (recall that $L(P) \subseteq S$), this together with Lemma 2.1 implies that P meshes with T_i. Suppose that there exists $j \neq i$ such that $P \cap L(T_j) \neq \emptyset$. Then as above, P meshes with T_j. Consequently, we have $|P \cap L(T_i)| \geq 2$ and $|P \cap L(T_j)| \geq 2$, and hence $|P \cap V(F)| \geq 4$, which contradicts (2.2). Thus $P \cap L(T_j) = \emptyset$ for each $j \neq i$. This proves (i) (a). Now if $|L(T_i)| \geq 3$, then $|P \cap V(F)| \geq |P \cap L(T_i)| \geq 3$ by Lemma 2.14, which proves (i) (b). To prove (ii), let now $1 \leq i \leq s$ with $|L(T_i)| = 2$, and suppose that there exist two members P, Q of $\mathscr{S}$ which mesh with S and T_i. Set $U = (N_G(L(P) \cup L(Q)) \cap V(F)) \cup (S - (L(P) \cup L(Q)))$. Since $N_G(L(P)) - L(P) = P$, it follows from (i) (a) that $N_G(L(P)) \cap V(F) = P \cap V(F) \subseteq L(T_i) \cup X$ and, similarly $N_G(L(Q)) \cap V(F) \subseteq L(T_i) \cup X$. Also since $|L(P)| = |L(Q)| = 2$, it follows from Lemma 2.1 and 2.2 that $L(P) \cap L(Q) = \emptyset$. Consequently $|U| \leq |L(T_i) \cup X| + 2 \leq 5$. On the other hand, since S separates $V(F)$ from the rest, U separates $V(F) - (N_G(L(P) \cup L(Q)) \cap V(F))$ from the rest. Therefore we get a contradiction to the assumption that G is 6-connected. Thus (ii) is proved. Finally we prove (iii). For each $P \in \mathscr{S}_0$, let i_P denote the unique index such that P meshes with T_{i_P}. Then by (i) (b), $|L(T_{i_P})| = 2$ for every $P \in \mathscr{S}_0$. Further by (ii), $i_P \neq i_Q$ for any $P, Q \in \mathscr{S}_0$ with $P \neq Q$. Hence $|\mathscr{S}_0| = |\{i_P | P \in \mathscr{S}_0\}| \leq |\{i|\ |L(T_i)| = 2\}|$, as desired. □

Lemma 2.16. *Let* $S \in \mathscr{S}$*, and suppose that* $|L(S)| \geq 9$*, and* $|\{T \in \mathscr{S} | L(T) \subseteq L(S)\}| = (2|L(S)| - 1)/3$*. Suppose further that there exist two members* P_1, P_2 *of* $\mathscr{S}$ *which mesh with* S*. Then* $|\mathscr{L}(S)| = 2$*, one of the components in* $\mathscr{L}(S)$ *is trivial, and we have* $|P_1 \cap L(S)| = 4$ *or* $|P_2 \cap L(S)| = 4$.

Proof. By Lemma 2.7 (II), $|\mathscr{L}(S)| = 2$. Write $\mathscr{L}(S) = \{F_1, F_2\}$ with $|V(F_1)| \le |V(F_2)|$. By Lemma 2.13, $2 \le |P_j \cap V(F_2)| \le 3$ for each $j = 1, 2$. Since $|L(S)| \ge 9$, we have $|V(F_2)| \ge 5$. Again by Lemma 2.7 (II), $|\{T \in \mathscr{S} | L(T) \subseteq V(F_2)\}| = (2|V(F_2)| - 2)/3$. By Lemma 2.6 (II), this implies that there exist $T_1, T_2 \in \mathscr{S}$ such that $V(F_2) = L(T_1) \cup L(T_2)$. Since $|V(F_2)| \ge 5$, we clearly have $|\{i | 1 \le i \le 2,\ |L(T_i)| = 2\}| \le 1$. By Lemma 2.15 (iii), this implies that we have $|P_1 \cap V(F_2)| = 3$ or $|P_2 \cap V(F_2)| = 3$. We may assume $|P_1 \cap V(F_2)| = 3$. Then by Lemma 2.13, $|V(F_1)| = 1$ and $|P_1 \cap L(S)| = 4$, as desired. □

Lemma 2.17. *Let $S \in \mathscr{S}$, and suppose that $|L(S)| \ge 12$. Suppose further that there exist three members of $\mathscr{S}$ which mesh with S. Then $|\{T \in \mathscr{S} | L(T) \subseteq L(S)\}| \le (2|L(S)| - 2)/3$.*

Proof. Let P_1, P_2, P_3 be members of $\mathscr{S}$ which mesh with S. By Lemma 2.7 (I), $|\{T \in \mathscr{S} | L(T) \subseteq L(S)\}| \le (2|L(S)| - 1)/3$. Suppose that $|\{T \in \mathscr{S} | L(T) \subseteq L(S)\}| = (2|L(S)| - 1)/3$. We argue as in Lemma 2.16. By Lemma 2.7 (II), $|\mathscr{L}(S)| = 2$. Write $\mathscr{L}(S) = \{F_1, F_2\}$ with $|V(F_1)| \le |V(F_2)|$. By Lemma 2.16, $|V(F_1)| = 1$, and hence $|V(F_2)| \ge 11$. By Lemma 2.7 (II), $|\{T \in \mathscr{S} | L(T) \subseteq V(F_2)\}| = (2|V(F_2)| - 2)/3$. By Lemma 2.6 (II), there exist $T_1, T_2 \in \mathscr{S}$ such that $V(F_2) = L(T_1) \cup L(T_2)$, T_1 meshes with T_2, and

$$|\{T \in \mathscr{S} | L(T) \subseteq L(T_i)\}| = (2|L(T_i)| - 1)/3 \text{ for each } i = 1, 2. \tag{2.3}$$

We may assume $|L(T_1)| \le |L(T_2)|$. Since $|L(T_1)| + |L(T_2)| = |V(F_2)| \ge 11$, it follows from Lemma 2.10 that $|L(T_1)| = 2$, and hence

$$|L(T_2)| \ge 9. \tag{2.4}$$

By (i) (a) and (ii) of Lemma 2.15,

$$\text{at least two of } P_1,\ P_2 \text{ and } P_3 \text{ mesh with } T_2. \tag{2.5}$$

On the other hand, since $|P_j \cap V(F_2)| \le 3$ for each $1 \le j \le 3$ by Lemma 2.13, we clearly have

$$|P_j \cap L(T_2)| \le 3 \text{ for each } 1 \le j \le 3. \tag{2.6}$$

Now in view of (2.3) through (2.6), we get a contradiction by applying Lemma 2.16 with S replaced by T_2. □

§3. PROOF OF THE THEOREM

We continue with the notation of the preceeding section, and prove the Theorem. Thus let $|V(G)| \ge 325$ and, by way of contradiction, suppose that

$$(3.1) \qquad |\mathscr{S}| \geq (2|V(G)| - 8)/3.$$

Let $S_1, \cdots, S_m$ be the maximal members of $\mathscr{S}$ with respect to the order relation defined immediately before Lemma 2.15. We may assume $|L(S_1)| \geq \cdots \geq |L(S_m)|$. Let $p_i = |\mathscr{L}(S_i)|$ for each i, and let $W = V(G) - (L(S_1) \cup \cdots \cup L(S_m))$. Arguing as in [3; Claims 3.2 through 3.4], we obtain the following three claims. We include sketches of their proofs for the convenience of the reader.

Claim 3.1.

(i) $m + 2|W| \leq 8$.

(ii) $2p_1 + (m-1) + 2|W| \leq 11$.

Sketch of Proof. By (3.1) and Lemma 2.7 (I), $(2|V(G)|-8)/3 \leq \sum_{1\leq i\leq m} (2|L(S_i)| - 2p_i + 3)/3$, and hence $2(p_1 + \cdots + p_m) - 3m + 2|W| \leq 8$. Since $p_i \geq 2$ for all i, both (i) and (ii) follow from this. □

Claim 3.2. $|L(S_1)| \geq 17$.

Sketch of Proof. If $|L(S_1)| \leq 16$, then by Claim 3.1 (i), $|V(G)| \leq 16m + |W| \leq 128$, which contradicts the assumption that $|V(G)| \geq 325$. □

Claim 3.3. *$m \geq 2$ and $|L(S_2)| \geq 17$.*

Sketch of Proof. Suppopse that $m = 1$ or $|L(S_2)| \leq 16$. Then by Claim 3.1 (ii), $|V(G) - L(S_1)| \leq 16(m-1) + |W| \leq 176 - 32p_1$, and hence $|V(G) - (S_1 \cup L(S_1))| \leq 170 - 32p_1$, which implies $|L(S_1)| \leq p_1(170 - 32p_1)$. Consequently $|V(G)| \leq p_1(170 - 32p_1) + 176 - 32p_1 \leq 324$, which contradicts the assumption that $|V(G)| \geq 325$. □

In what follows, we do not make use of the inequality $|L(S_1)| \geq |L(S_2)|$; thus the roles of S_1 and S_2 are symmetric. By Lemma 2.11, Claims 3.2 and 3.3 imply that S_1 dose not mesh with S_2. Since $L(S_1) \cap L(S_2) = \emptyset$ by the maximality of $L(S_1)$ and $L(S_2)$, $L(S_1) \cap S_2 = L(S_2) \cap S_1 = \emptyset$ by Lemma 2.1. Write $\mathscr{K}(S_1) - \mathscr{L}(S_1) = \{C_1\}$ and $\mathscr{K}(S_2) - \mathscr{L}(S_2) = \{C_2\}$; thus $C_1 = G - S_1 - L(S_1)$ and $C_2 = G - S_2 - L(S_2)$. We define $\mathscr{T}_1$, $\mathscr{T}_2$, $\mathscr{T}_{1,1}$, $\mathscr{T}_{1,2}$, $\mathscr{T}_{1,3}$, $\mathscr{T}_{2,1}$, $\mathscr{T}_{2,2}$, $\mathscr{T}_{2,3}$ as follows:

$$\begin{aligned}
\mathscr{T}_1 &= \{T \in \mathscr{S} | L(T) \cap (S_1 \cup S_2) = \emptyset\},\\
\mathscr{T}_2 &= \{T \in \mathscr{S} | L(T) \subseteq S_1 \cup S_2\},\\
\mathscr{T}_{1,1} &= \{T \in \mathscr{S} | L(T) \subseteq L(S_1)\},\\
\mathscr{T}_{1,2} &= \{T \in \mathscr{S} | L(T) \subseteq L(S_2)\},\\
\mathscr{T}_{1,3} &= \{T \in \mathscr{S} | L(T) \subseteq V(C_1) \cap V(C_2)\},\\
\mathscr{T}_{2,1} &= \{T \in \mathscr{T}_2 | L(T) \subseteq S_1 - S_2\},\\
\mathscr{T}_{2,2} &= \{T \in \mathscr{T}_2 | L(T) \subseteq S_2 - S_1\},\\
\mathscr{T}_{2,3} &= \{T \in \mathscr{T}_2 | L(T) \subseteq S_1 \cap S_2\}.
\end{aligned}$$

In view of the maximality of $L(S_1)$ and $L(S_2)$ and Claims 3.2 and 3.3, it follows from Lemmas 2.1 and 2.10 that $\mathscr{T}_1$ is the set of those members of $\mathscr{S}$ which mesh with neither S_1 nor S_2, and $\mathscr{T}_2$ is the set of those members of $\mathscr{S}$ which mesh with S_1 or S_2. Thus $\mathscr{S} = \mathscr{T}_1 \cup \mathscr{T}_2$ (disjoint union). Further by Lemma 2.1, $\mathscr{T}_1 = \mathscr{T}_{1,1} \cup \mathscr{T}_{1,2} \cup \mathscr{T}_{1,3}$ (disjoint union) and, by Lemma 2.10, $\mathscr{T}_2 = \mathscr{T}_{2,1} \cup \mathscr{T}_{2,2} \cup \mathscr{T}_{2,3}$ (disjoint union).

The following two claims immediately follow from Lemma 2.7 (I) (see also [3; Claim 3.6]).

Claim 3.4. $|\mathscr{T}_{1,i}| \leq (2|L(S_i)| - 1)/3$ $(i = 1, 2)$.

Claim 3.5. $|\mathscr{T}_{1,3}| \leq 2|V(C_1) \cap V(C_2)|/3$.

The following claim is proved in [3; Claim 3.8].

Claim 3.6.

(i) $|\mathscr{T}_{2,1}| \leq |S_1 - S_2|/2$.

(ii) $|\mathscr{T}_{2,2}| \leq |S_2 - S_1|/2$.

(iii) $|\mathscr{T}_{2,3}| \leq |S_1 \cap S_2|/2$.

Claim 3.7. $|S_1 \cap S_2|$ *is even.*

Proof. Suppose that $|S_1 \cap S_2|$ is odd. Then it follows from Claim 3.6 that $|\mathscr{T}_2| \leq (|S_1 \cup S_2| - 3)/2$, and it follows from Claims 3.4 and 3.5 that $|\mathscr{T}_1| \leq (2(|V(G)| - |S_1 \cup S_2|) - 2)/3$. Hence $|\mathscr{S}| \leq (2|V(G)| - (|S_1 \cup S_2| + 13)/2)/3$. Since $|S_1 \cup S_2| \geq 7$, this contradicts (3.1). □

Write $|S_1 \cap S_2| = 2x$. Then $|S_1 \cup S_2| = 12 - 2x$. Hence it follows from Claim 3.6 that

(3.2) $$|\mathscr{T}_2| \leq 6 - x,$$

and it follows from Claims 3.4 and 3.5 that

$$|\mathscr{T}_1| \le (2|V(G)| - 26 + 4x)/3. \tag{3.3}$$

By (3.2) and (3.3), $|\mathscr{S}| \le (2|V(G)| - 8 + x)/3$. In view of (3.1), this implies that equality holds in (3.2) (note that $x \le 2$). Thus it follows from Claim 3.6 that

$$|\mathscr{T}_{2,1}| = 3 - x, |\mathscr{T}_{2,2}| = 3 - x, |\mathscr{T}_{2,3}| = x. \tag{3.4}$$

By Lemma 2.17, this implies that

$$|\mathscr{T}_{1,i}| \le (2|L(S_i)| - 2)/3 \text{ for each } i = 1, 2. \tag{3.5}$$

Now it follows from (3.2), (3.5) and Claim 3.5 that $|\mathscr{S}| \le (2|V(G)| - 10 + x)/3$. In view of (3.1), this implies that $x = 2$ and equality holds in (3.5), i.e.,

$$|\mathscr{T}_{1,i}| = (2|L(S_i)| - 2)/3 \text{ for each } i = 1, 2. \tag{3.6}$$

Having (3.4) in mind, write $\mathscr{T}_{2,1} = \{P_1\}$, $\mathscr{T}_{2,2} = \{P_2\}$ and $\mathscr{T}_{2,3} = \{P_3, P_4\}$. It follows from Lemma 2.10 and Claims 3.2 and 3.3 that $|L(P_j)| = 2$ for each $1 \le j \le 4$.

Claim 3.8. *Let $j = 3$ or 4. Then $|P_j \cap L(S_1)| = |P_j \cap L(S_2)| = 3$.*

Proof. By Lemma 2.14, $|P_j \cap L(S_1)|, |P_j \cap L(S_2)| \ge 3$. Since $|P_j| = 6$ and $L(S_1) \cap L(S_2) = \emptyset$, this implies $|P_j \cap L(S_1)| = |P_j \cap L(S_2)| = 3$. □

In what follows, we mainly consider S_1. As in Lemma 2.13, write $\mathscr{L}(S_1) = \{F_1, \cdots, F_p\}$ ($p = |\mathscr{L}(S_1)|$) with $|V(F_1)| \le |V(F_2)| \le \cdots \le |V(F_p)|$.

Claim 3.9. *$p = 2$, $|V(F_1)| = 1$, and $|P_3 \cap V(F_2)| = |P_4 \cap V(F_2)| = 2$.*

Proof. In view of Claim 3.8, this follows from Lemma 2.13. □

Since $|L(S_1)| \ge 17$, it follows from Claim 3.9 that

$$|V(F_2)| \ge 16. \tag{3.7}$$

Set $\mathscr{T} = \{T \in \mathscr{S} | L(T) \subseteq V(F_2)\}$. Since $|V(F_1)| = 1$ by Claim 3.9, we clearly have $|\{T \in \mathscr{S} | L(T) \subseteq V(F_1)\}| = 0 = (2|V(F_1)| - 2)/3$. Hence by (3.6) and Lemma 2.7 (III),

$$|\mathscr{T}| = (2|V(F_2)| - 3)/3. \tag{3.8}$$

As in Lemma 2.15, let $T_1, \cdots, T_s$ be the maximal members of $\mathscr{T}$.

Claim 3.10. *F_2 is saturated.*

Proof. Suppose that F_2 is not saturated. Then by (3.8) and Lemma 2.6 (III), $s = 1$ and $|V(F_2) - L(T_1)| = 1$. Let $\mathscr{S}_0$ be as in Lemma 2.15 (iii) with $S = S_1$ and $F = F_2$. Then by Claim 3.9, $P_3, P_4 \in \mathscr{S}_0$, and hence $|\mathscr{S}_0| \geq 2$. But since we clearly have $|\{i|1 \leq i \leq s,\ |L(T_i)| = 2\}| \leq s = 1$, this contradicts Lemma 2.15 (iii). □

We are now in a position to complete the proof of the Theorem. By Claim 3.10, $V(F_2) = L(T_1) \cup \cdots \cup L(T_s)$. By (3.8) and Lemma 2.6 (III), $s \leq 3$. Set $I = \{i|\ |L(T_i)| = 2\}$. By (3.7), $|I| \leq s - 1$. Let $\mathscr{S}_0$ be again as in Lemma 2.15 (iii) with $S = S_1$ and $F = F_2$. Then $P_3, P_4 \in \mathscr{S}_0$ by Claim 3.9, and hence $|I| \geq |\mathscr{S}_0| \geq 2$ by Lemma 2.15 (iii). This forces $s = 3$, $|I| = 2$ and $\mathscr{S}_0 = \{P_3, P_4\}$. We may assume $|L(T_1)| = |L(T_2)| = 2$. We have $|L(T_3)| \geq 12$ by (3.7), and

$$|\{T \in \mathscr{S}|L(T) \subseteq L(T_3)\}| = (2|L(T_3)| - 1)/3 \tag{3.9}$$

by Lemma 2.6 (III). By (i) (b) and (ii) of Lemma 2.15, we may assume that P_3 meshes with T_1, and P_4 meshes with T_2. By (i) (a) and (ii) of Lemma 2.15, P_1 meshes with T_3. If T_1 meshes with T_2 and T_3, then we have $T_1 \supseteq L(P_3), L(T_2)$ because $|L(P_3)| = |L(T_2)| = 2$, and we also have $|T_1 \cap L(T_3)| \geq 3$ by Lemma 2.14, and hence $6 = |T_1| \geq |L(P_3)| + |L(T_2)| + |T_1 \cap L(T_3)| \geq 7$, which is absurd. Thus T_1 does not mesh with at least one of T_2 and T_3. Similarly T_2 does not mesh with at least one of T_1 and T_3. In view of Lemma 2.6 (III), this implies that T_3 meshes with T_1 and T_2; that is to say, T_3 meshes with P_1, T_1 and T_2. Therefore applying Lemma 2.17 with S replaced by T_3, we obtain $|\{T \in \mathscr{S}|L(T) \subseteq L(T_3)\}| \leq (2|L(T_3)| - 2)/3$, which contradicts (3.9). This completes the proof of the Theorem.

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