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A curvature form for pseudoconnections

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Abstract. We obtain the curvature form $F^\nabla = P \circ d^\nabla \circ \nabla - d^\nabla \circ P \circ \nabla + d^\nabla \circ \nabla \circ P$ for a vector bundle pseudoconnection ∇, where d^∇ is the exterior derivative associated to ∇. We use F^∇ to obtain the curvature of ∇. We also prove that $F^\nabla = 0$ is a necessary (but not sufficient) condition for d^∇ to be a chain complex. Instead we prove that $F^\nabla = 0$ and $d^\nabla \circ d^\nabla \circ \nabla = 0$ are necessary and sufficient conditions for d^∇ to be a *chain* 2*-complex*, i.e., $d^\nabla \circ d^\nabla \circ d^\nabla = 0$.

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§1. Introduction

Let M be a differentiable manifold. Denote by $\Omega^0(M)$ the ring of C^∞ real valued maps in M. Denote by $\chi^\infty(M)$ and $\Omega^k(M)$ respectively the $\Omega^0(M)$-modules of C^∞ vector fields and k-forms defined on M, $k \geq 0$. Denote by $d : \Omega^k(M) \to \Omega^{k+1}(M)$ the standard exterior derivation of k-forms of M. We denote by $Hom(A, B)$ the set of homomorphisms from the modulus A to the modulus B. If ξ is a real smooth vector bundle over M we denote by $\Omega^k(\xi)$ the set of ξ-valued k-forms of M, namely, $\Omega^0(\xi)$ is the $\Omega^0(M)$-module of smooth sections of ξ and $\Omega^k(\xi) = \Omega^k(M) \otimes \Omega^0(\xi)$ for all $k \geq 1$. If η is another vector bundle over M we denote by $HOM(\xi, \eta)$ the set of bundle homomorphims from ξ to η over the identity. Every $P \in HOM(\xi, \eta)$ induces a homomorphism $P \in Hom(\Omega^0(\xi), \Omega^0(\eta))$ of $\Omega^0(M)$ modules in the usual way. It also defines a homomorphism $P \in Hom(\Omega^k(\xi), \Omega^k(\eta))$ for all $k \geq 1$ by setting $P(\omega \otimes s) = \omega \otimes P(s)$ at every generator $\omega \otimes s \in \Omega^k(\xi)$.

A *pseudoconnection* of a vector bundle ξ over M is an $\mathbb{R}$-linear map $\nabla : \Omega^0(\xi) \to \Omega^0(\xi)$ for which there is a bundle homomorphism $P \in HOM(\xi, \xi)$

called the principal homomorphism of ∇ such that the Leibnitz rule below holds:

$$\nabla(f \cdot s) = df \otimes P(s) + f \cdot \nabla(s), \qquad \forall (f, s) \in \Omega^0(M) \times \Omega^0(\xi).$$

(See [2].) An *ordinary connection* is a pseudoconnection whose principal homomorphism is the identity. By classical arguments we shall associate to any pseudoconnection ∇ an *exterior derivative*, that is, a sequence of linear maps $d^\nabla : \Omega^k(\xi) \to \Omega^{k+1}(\xi)$ which reduces to ∇ when $k = 0$ and satisfies a Leibnitz rule. We shall use it to define the curvature form $F^\nabla : \Omega^0(\xi) \to \Omega^2(\xi)$ as the alternating sum

$$F^\nabla = P \circ d^\nabla \circ \nabla - d^\nabla \circ P \circ \nabla + d^\nabla \circ \nabla \circ P.$$

Notice that F^∇ above reduces to the classical curvature form if ∇ were an ordinary connection. We shall prove that F^∇ is a tensor, that is, $F^\nabla \in Hom(\Omega^0(\xi), \Omega^2(\xi))$, and explain how to obtain the Abe's curvature [2] from F^∇. We also prove that $F^\nabla = 0$ is a necessary (but not sufficient) condition for d^∇ to be a chain complex. Instead we prove that $F^\nabla = 0$ and $d^\nabla \circ d^\nabla \circ \nabla = 0$ are necessary and sufficient conditions for d^∇ to be a *chain* 2-*complex*, i.e., $d^\nabla \circ d^\nabla \circ d^\nabla = 0$.

§2. Results

Let ξ, η be real smooth vector bundles over a differentible manifold M. An *O-derivative operator from ξ to η with principal homomorphism $P \in HOM(\xi, \eta)$* is an $I\!R$-linear map $\nabla : \Omega^0(\xi) \to \Omega^0(\eta)$ satisfying the Leibnitz rule

$$\nabla(f \cdot s) = df \otimes P(s) + f \cdot \nabla(s), \qquad \forall (f, s) \in \Omega^0(M) \times \Omega^0(\xi).$$

(See [1].) Notice that a pseudoconnection of ξ is nothing but an O-derivative operator from ξ to itself.

As in [1] we denote by $O(\xi, \eta; P)$ the set whose elements are the O-derivative operators with principal homomorphism P from ξ to η. We even write $O(\xi; P)$ instead of $O(\xi, \xi; P)$ and define

$$O(\xi, \eta) = \bigcup_P O(\xi, \eta; P) \qquad \text{and} \qquad O(\xi) = \bigcup_P O(\xi; P).$$

Every $\alpha \in HOM(\xi, \eta)$ induces an alternating product

$$\wedge_\alpha : \Omega^k(M) \times \Omega^l(\xi) \to \Omega^{k+l}(\eta)$$

defined at the generators by

$$\beta \wedge_\alpha (\omega \otimes s) = (\beta \wedge \omega) \otimes \alpha(s).$$

If $\eta = \xi$ and α is the identity, then $\wedge_\alpha$ reduced to the ordinary alternating product $\wedge$ ([3]).

Lemma 2.1. *For every $\nabla \in O(\xi, \eta; P)$ there is a unique sequence of linear maps $d^\nabla : \Omega^k(\xi) \to \Omega^{k+1}(\eta)$, $k \geq 0$, satisfying the following properties:*

1. *If $k = 0$, then*
$$d^\nabla = \nabla.$$
2. *If $k, l \geq 0$, $\omega \in \Omega^k(M)$ and $S \in \Omega^l(\xi)$, then*
$$d^\nabla(\omega \wedge S) = d\omega \wedge_P S + (-1)^k \omega \wedge d^\nabla S.$$

Proof. First define the map $D^\nabla : \Omega^k(M) \times \Omega^0(\xi) \to \Omega^{k+1}(\eta)$ by

$$D^\nabla(\omega, s) = d\omega \otimes P(s) + (-1)^k \omega \wedge \nabla s, \qquad \forall (\omega, s) \in \Omega^k(M) \times \Omega^0(\xi).$$

Clearly D^∇ is linear and satisfies

$$D^\nabla(f \cdot \omega, s) = D^\nabla(\omega, f \cdot s)$$

for all $f \in \Omega^0(M)$ and $(\omega, s) \in \Omega^k(M) \times \Omega^0(\xi)$. Therefore D^∇ induces a linear map $d^\nabla : \Omega^k(\xi) \to \Omega^{k+1}(\eta)$ whose value at the generator $\omega \otimes s$ of $\Omega^k(\xi)$ is given by

$$d^\nabla(\omega \otimes s) = d\omega \otimes P(s) + (-1)^k \omega \wedge \nabla s.$$

It follows that d^∇ and ∇ coincide at the generators (for $k = 0$) by the Leibnitz rule of ∇. Therefore (1) holds. The proof of (2) follows as in [3]. This ends the proof. □

The sequence d^∇ in the lemma above will be refered to as the *exterior derivative* of $\nabla \in O(\xi, \eta)$. Next we state the following definition.

Definition 2.2. Let ∇ be a pseudoconnection with principal homomorphism P on a vector bundle ξ. We define the following maps

$$E^\nabla, F^\nabla : \Omega^0(\xi) \to \Omega^2(\xi), \quad L^\nabla : \Omega^0(\xi) \to \Omega^1(\xi) \quad \text{and} \quad G^\nabla : \Omega^0(\xi) \to \Omega^3(\xi)$$

by

- $E^\nabla = d^\nabla \circ \nabla$;
- $F^\nabla = P \circ d^\nabla \circ \nabla - d^\nabla \circ P \circ \nabla + d^\nabla \circ \nabla \circ P$;
- $L^\nabla = P \circ \nabla - \nabla \circ P$;
- $G^\nabla = d^\nabla \circ d^\nabla \circ \nabla$.

The map F^∇ will be refered to as the *curvature form* of ∇.

The maps in the definition above are related by the expressions

$$F^\nabla = P \circ E^\nabla - d^\nabla \circ L^\nabla, \quad G^\nabla = d^\nabla \circ E^\nabla. \tag{2.1}$$

As already observed the curvature form F^∇ of a pseudoconnection ∇ reduces to the classical curvature form when ∇ is an ordinary connection [3].

Theorem 2.3. *If ∇ is a pseudoconnection of ξ, then $F^\nabla \in Hom(\Omega^0(\xi), \Omega^2(\xi))$ and $L^\nabla \in Hom(\Omega^0(\xi), \Omega^1(\xi))$.*

Proof. It is not difficult to see that $L^\nabla \in Hom(\Omega^0(\xi), \Omega^1(\xi))$. On the other hand,

$$P(\omega \wedge S) = \omega \wedge_P S, \quad \forall \omega \in \Omega^1(M), \forall S \in \Omega^1(\xi)$$

so

$$E^\nabla(f \cdot s) = df \wedge L^\nabla(s) + f \cdot E^\nabla(s), \quad \forall f \in \Omega^0(M), \forall s \in \Omega^0(\xi).$$

Then, (2.1) implies

$$F^\nabla(f \cdot s) = P(E^\nabla(f \cdot s))) - d^\nabla(L^\nabla(f \cdot s)) = P(df \wedge L^\nabla(s) + f \cdot E^\nabla(s)) - d^\nabla(f \cdot L^\nabla(s))$$

$$= df \wedge_P L^\nabla(s) - df \wedge_P L^\nabla(s) + f \cdot F^\nabla(s) = f \cdot F^\nabla(s)$$

$\forall f \in \Omega^0(M)$, $\forall s \in \Omega^0(\xi)$. Therefore $F^\nabla \in Hom(\Omega^0(\xi), \Omega^2(\xi))$ and we are done. This ends the proof. □

Lemma 2.4. *If ∇ is a pseudoconnection of a vector bundle ξ and $i \geq 0$, then*

$$d^\nabla \circ d^\nabla \circ d^\nabla(\omega \otimes s) = d\omega \wedge F^\nabla(s) + (-1)^i \omega \wedge G^\nabla(s) \tag{2.2}$$

for every generator $\omega \otimes s \in \Omega^i(\xi)$.

Proof. First notice that

$$d^\nabla \circ d^\nabla(\omega \otimes s) = d^\nabla(d^\nabla(\omega \otimes s)) = d^\nabla(d\omega \otimes P(s) + (-1)^i \omega \wedge \nabla s) =$$

$$= d^2\omega \otimes P^2(s) + (-1)^{i+1} d\omega \wedge \nabla P(s) + (-1)^i (dw \wedge_P \nabla s +$$

$$+ (-1)^i \omega \wedge d^\nabla(\nabla s)) = (-1)^i [d\omega \wedge (P\nabla s - \nabla P s) + (-1)^i \omega \wedge d^\nabla(\nabla s)].$$

Therefore

$$d^\nabla \circ d^\nabla(\omega \otimes s) = \omega \wedge E^\nabla(s) + (-1)^i d\omega \wedge L^\nabla(s).$$

Applying d^∇ to this expression we get (2.2). The proof follows. □

As is well known the curvature form F^∇ of an ordinary connection ∇ measures how the exterior derivative d^∇ of ∇ deviates from being a *chain complex*, i.e., $d^\nabla \circ d^\nabla = 0$. Indeed, d^∇ is a chain complex if and only if $F^\nabla = 0$. However, the analogous result for pseudoconnections is false in general by Proposition 2.8 below. Despite we shall obtain a pseudoconnection version of this result based on the following definition.

Definition 2.5. A pseudoconnection ∇ is called:

1. *strongly flat* if $E^\nabla = 0$ and $L^\nabla = 0$,
2. *weakly flat* if $F^\nabla = 0$ and $G^\nabla = 0$.

For ordinary connections one has $F^\nabla = E^\nabla$, $L^\nabla = 0$ and then the notions of flatness above coincide with the classical flatness [3]. The exterior derivative d^∇ of a pseudoconnection ∇ is said to be a *chain* 2*-complex* if $d^\nabla \circ d^\nabla \circ d^\nabla = 0$. With these definitions we have the following result.

Theorem 2.6. *A pseudoconnection ∇ is weakly flat (resp. strongly flat) if and only if d^∇ is a chain* 2*-complex (resp. chain complex).*

Proof. We only prove the result for weakly flat since the proof for strongly flat is analogous.

Fix a pseudoconnection ∇ with principal homomorphism P on a vector bundle ξ. If ∇ is weakly flat then d^∇ is a chain 2-complex by (2.2) in Lemma 2.4. Conversely, if d^∇ is a chain 2-complex, then both $d^\nabla \circ d^\nabla \circ d^\nabla$ and $d^\nabla \circ d^\nabla \circ \nabla$ vanish hence $\omega \wedge F^\nabla(s) = 0$ for all exact form ω of M and all $s \in \Omega^0(\xi)$ by (2.2) in Lemma 2.4. Since every form in M is locally a $\Omega^0(M)$-linear combination of alternating product of exact forms we obtain that $\omega \wedge F^\nabla(s) = 0$ for all k-form ω of M $(k \geq 1)$ and all $s \in \Omega^0(\xi)$. From this we obtain that $F^\nabla = 0$ so ∇ is weakly flat. The proof follows. □

The following is a direct corollary of the above theorem.

Corollary 2.7. *If the exterior derivative d^∇ of a pseudoconnection ∇ is a chain complex, then $F^\nabla = 0$.*

The converse of the above corollary is false by the following proposition.

Proposition 2.8. *There is a pseudoconnection ∇ with $F^\nabla = 0$ such that d^∇ is not a chain complex.*

Proof. Choose a suitable vector bundle ξ over $M = I\!R^3$, $\Phi_2, \Phi_3 \in HOM(\xi, \xi)$ such that $\Phi_3 \circ \Phi_2 \neq \Phi_2 \circ \Phi_3$ and three 1-forms $\omega_1, \omega_2, \omega_3 \in \Omega^1(M)$ such that $\omega_1 \wedge \omega_2 \wedge \omega_3$ never vanishes. Define the map $\nabla : \Omega^0(\xi) \to \Omega^1(\xi)$ by

$$\nabla s = \omega_1 \otimes s + \omega_2 \otimes \Phi_2(s) + \omega_3 \otimes \Phi_3(s).$$

We have that $\nabla \in Hom(\Omega^0(\xi), \Omega^1(\xi))$ therefore ∇ is a pseudoconnection with zero principal homomorphism so $F^\nabla = 0$. On the other hand, an straightforward computation yields

$$G^\nabla(s) = (\omega_1 \wedge \omega_2 \wedge \omega_3) \otimes (\Phi_3 \circ \Phi_2 - \Phi_2 \circ \Phi_3)(s), \quad \forall s \in \Omega^0(\xi)$$

therefore $G^\nabla \neq 0$ and so ∇ is not weakly flat. Then, d^∇ cannot be a chain complex by Theorem 2.6 since a chain complex is necessarily a chain 2-complex. This ends the proof. □

To finish we explain how the Abe's curvature [2] can be obtained from the curvature form F^∇. For this we need some short definitions (see [3]).

Given a vector bundle ξ over M and k vector fields $X_1, \cdots, X_k \in \chi^\infty(M)$ we define the evaluation map $Ev_{X_1,\cdots,X_k} : \Omega^k(\xi) \to \Omega^0(\xi)$ by defining

$$Ev_{X_1,\cdots,X_k}(\omega \otimes s) = w(X_1, \cdots, X_k) \cdot s$$

at each generator $\omega \otimes s \in \Omega^k(\xi)$. If $\nabla \in O(\xi)$ and $X, Y \in \chi^\infty(M)$ we define $\nabla_X : \Omega^0(\xi) \to \Omega^0(\xi)$ by

$$\nabla_X s = Ev_X(\nabla s)$$

and $F^\nabla_{X,Y} : \Omega^0(\xi) \to \Omega^0(\xi)$ by

$$F^\nabla_{X,Y} = Ev_{X,Y}(F^\nabla(s)), \quad \forall s \in \Omega^0(\xi).$$

Theorem 2.9. *If $\nabla \in O(\xi; P)$ then*

$$F^\nabla_{X,Y}(s) = \nabla_X \nabla_Y (Ps) - \nabla_Y \nabla_X (Ps) - \nabla_X P(\nabla_Y s) + P \nabla_X \nabla_Y s +$$
$$+ \nabla_Y P(\nabla_X s) - P \nabla_Y \nabla_X s - P\left(\nabla_{[X,Y]} P(s)\right),$$

for all $X, Y \in \chi^\infty(M)$ and all $s \in \Omega^0(\xi)$.

Proof. By definition we have

$$F^\nabla_{X,Y}(s) = \tag{2.3}$$
$$= Ev_{X,Y}(P(d^\nabla(\nabla s))) - Ev_{X,Y}(d^\nabla(P(\nabla s))) + Ev_{X,Y}(d^\nabla(\nabla(Ps))).$$

Let us compute the three sumands separated way. First of all if $\omega \otimes s \in \Omega^1(\xi)$ is a generator then

$$Ev_{X,Y}(d^\nabla(\omega \otimes s)) = dw(X,Y) \cdot P(s) - \omega(X) \cdot \nabla_Y s + \omega(Y) \cdot \nabla_X s. \tag{2.4}$$

Now, as $\nabla s \in \Omega^1(\xi)$ and $\{\omega \otimes s' : (\omega, s') \in \Omega^1(M) \times \Omega^0(\xi)\}$ is a generating set of $\Omega^1(\xi)$ we obtain

$$\nabla s = \sum_{r=1}^{k} \omega_r \otimes s_r, \tag{2.5}$$

for some $(\omega_r, s_r) \in \Omega^1(M) \times \Omega^0(\xi)$, $r = 1, \cdots, k$. Then (2.4) implies

(2.6) $$Ev_{X,Y}(d^{\nabla}(\nabla s)) =$$

$$= \sum_{r=1}^{k} \{dw_r(X,Y) \cdot P(s_r) - \omega_r(X) \cdot \nabla_Y s_r + \omega_r(Y) \cdot \nabla_X s_r\}.$$

On the other hand, (2.5) yields

$$\nabla_X s = \sum_{r=1}^{k} \omega_r(X) \cdot s_r$$

therefore

$$\nabla_Y \nabla_X s = \sum_{r=1}^{k} \{d[\omega_r(X)](Y) \cdot P(s_r) + \omega_r(Y) \cdot \omega_r(X) \cdot \nabla_Y s_s\}.$$

But $\nabla_{[X,Y]} s = \sum_{r=1}^{k} \omega_r([X,Y]) \cdot s_r$, so

$$P\left(\nabla_{[X,Y]} s\right) = \sum_{r=1}^{k} \omega_r([X,Y]) \cdot P(s_r)$$

and then

(2.7) $$Ev_{X,Y}(d^{\nabla}(\nabla s)) = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - P\left(\nabla_{[X,Y]} s\right)$$

because of (2.6). Replacing s by $P(s)$ in (2.7) we obtain

(2.8) $$Ev_{X,Y}(d^{\nabla}(\nabla P(s))) = \nabla_X \nabla_Y P(s) - \nabla_Y \nabla_X P(s) - P\left(\nabla_{[X,Y]} P(s)\right).$$

Besides (2.5) implies

$$P(\nabla s) = \sum_{r=1}^{k} \omega_r \otimes P(s_r)$$

thus

$$Ev_{X,Y}(d^{\nabla}(P\nabla s)) = \sum_{r=1}^{k} Ev_{X,Y}(d^{\nabla}(\omega_r \otimes P(s_r))) =$$

$$= \sum_{r=1}^{k} [dw_r(X,Y) \cdot P^2(s_r) - \omega_r(X) \nabla_X P(s_r) + +\omega_r(Y) \cdot \nabla_X P(s_r)]$$

and then

$$\nabla_Y P(\nabla_X s) = \sum_{r=1}^{k} \left\{d[w_r(X)](Y) \cdot P^2(s_r) + w_r(X) \cdot \nabla_Y P(s_r)\right\}.$$

As $P^2\left(\nabla_{[X,Y]}s\right) = \sum_{r=1}^{k} \omega_r([X,Y]) \cdot P^2(s_r)$ we obtain

$$(2.9)\quad Ev_{X,Y}(d^{\nabla}(P(\nabla s))) = \nabla_X P(\nabla_Y s) - \nabla_Y P(\nabla_X s) - P^2\left(\nabla_{[X,Y]}s\right).$$

As the maps P and $Ev_{X,Y}$ commute we can apply P to (2.7) and use (2.3), (2.8), (2.9) to obtain the result. □

Remark 2.10. $F^{\nabla}_{X,Y}(s)$ in Theorem 2.9 is the curvature $K(\nabla)_{X,Y}(s)$ defined in [2] p. 328.

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