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The cycle-complete graph Ramsey number $r(C_8, K_8)$

M.M.M. Jaradat and B.M.N. Alzaleq

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Abstract. The cycle-complete graph Ramsey number $r(C_m, K_n)$ is the smallest integer N such that every graph G of order N contains a cycle C_m on m vertices or has independent number $\alpha(G) \geq n$. It has been conjectured by Erdős, Faudree, Rousseau and Schelp that $r(C_m, K_n) = (m-1)(n-1)+1$ for all $m \geq n \geq 3$ (except $r(C_3, K_3) = 6$). In this paper we will present a proof for the conjecture in the case $n = m = 8$.

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§1. Introduction

Through out this paper a cycle on m vertices will be denoted by C_m, the complete graph on n vertices by K_n, a star graph on n vertices by S_n and a path on n vertices by P_n. The graph $K_1 + P_n$ is obtained by adding an additional vertex to the path P_n and connecting this new vertex to each vertex of P_n. The minimum degree of a graph G is denoted by $\delta(G)$. An independent set of vertices of a graph G is a subset of $V(G)$ in which no two vertices are adjacent. The independence number of a graph G, $\alpha(G)$, is the size of the largest independent set. The neighbor of the vertex u is the set of all vertices of G that are adjacent to u, denoted by $N(u)$. $N[u]$ denote to $N(u) \cup \{u\}$. Let H be a subgraph of the graph G and $U \subseteq V(G)$, $N_H(U)$ is defined as $(\cup_{u \in U} N(u)) \cap V(H)$. Suppose that $V_1 \subseteq V(G)$ and V_1 is a non empty, the subgraph of G whose vertex set is V_1 and whose edge set is the set of those edges of G that have both ends in V_1 is called the subgraph of G induced by V_1, denoted by $\langle V_1 \rangle_G$.

The cycle-complete graph Ramsey number $r(C_m, K_n)$ is the smallest integer N such that for every graph G of order N contains C_m or $\alpha(G) \geq n$. The graph $(n-1)K_{m-1}$ shows that $r(C_m, K_n) \geq (m-1)(n-1)+1$. In one of the earliest contributions to graphical Ramsey theory, Bondy and Erdős [3] proved the following result:

Theorem 1.1 (Bondy and Erdős)**.** *For all $m \geq n^2 - 2$, $r(C_m, K_n) = (m-1)(n-1)+1$.*

It has been thought from the beginning that the conclusion is likely to hold under a rather less restriction hypothesis. The restriction in Theorem 1.1 was improved by Nikiforov [9] when he proved the equality for $m \geq 4n+2$. Erdős eta al. [4] gave the following conjecture:

Conjecture: $r(C_m, K_n) = (m-1)(n-1)+1$, for all $m \geq n \geq 3$ except $r(C_3, K_3) = 6$.

The conjecture was confirmed by Faudree and Schepl [5] and Rosta [11] for $n = 3$ in early work on Ramsey theory. Sheng et al. [14] and Bollobás et al. [2] proved the conjecture for $n = 4$ and $n = 5$, respectively. Recently, the conjecture was proved by Schiermeyer [12] for $n = 6$. Most recently, Baniabedalruhman [1] proved that $r(C_7, K_7) = 37$. In a related work, Radziszowski and Tse [10] showed that $r(C_4, K_7) = 22$ and $r(C_4, K_8) = 26$. In [8] Jayawardene and Rousseau proved that $r(C_5, K_6) = 21$. Also, Schiermeyer [13] proved that $r(C_5, K_7) = 25$. Jaradat and Baniabedalruhman [7] proved that $r(C_8, K_7) = 43$. In this article we will prove the conjecture for the case $n = m = 8$ which is the first step to show that $r(C_m, K_8) = 7m - 6$.

§2. Main Result

It is known, by taking $G = (n-1)K_{m-1}$, that $r(C_m, K_n) \geq (m-1)(n-1)+1$. In this section we prove that this bound is exact in the case $m = n = 8$. Our proof depends on a sequence of 8 lemmas.

Lemma 2.1. *Let G be a graph of order ≥ 50 that contains neither C_8 nor an 8-element independent set. Then $\delta(G) \geq 7$.*

Proof. Suppose that G contains a vertex of degree less than 7, say u. Then $|V(G - N[u])| \geq 43$. Since $r(C_8, K_7) = 43$, as a result $G - N[u]$ has an independent set consists of 7 vertices. This set with the vertex u is an 8-element independent set of vertices of G. This is a contradiction. □

Throughout all Lemmas 2.2 to 2.8, we let G be a graph with minimum degree $\delta(G) \geq 7$ that contains neither C_8 nor an 8-element independent set.

Lemma 2.2. *If G contains K_7, then $|V(G)| \geq 56$.*

Proof. Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ be the vertex set of K_7. Let $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \leq i \leq 7$. Since $\delta(G) \geq 7$, $U_i \neq \varnothing$ for all $1 \leq i \leq 7$. Since there is a path of order 7 joining any two vertices of U, as a result $U_i \cap U_j = \varnothing$ for all $1 \leq i < j \leq 7$ (otherwise, if $w \in U_i \cap U_j$ for some $1 \leq i < j \leq 7$, then the concatenation of the u_iu_j-path of order 7 with u_iwu_j, is a cycle of order 8, a contradiction). Similarly, since there is a path of order 6 joining any two vertices of U, as a result for all $1 \leq i < j \leq 7$ and for all $x \in U_i$ and $y \in U_j$ we have that $xy \notin E(G)$ (otherwise, if there are $1 \leq i < j \leq 7$ such that $x \in U_i$, $y \in U_j$ and $xy \in E(G)$, then the concatenation of the u_iu_j-path of order 6 with u_ixyu_j is a cycle of order 8, a contradiction). Also, since there is a path of order 5 joining any two vertices of U, as a result, $N_R(U_i) \cap N_R(U_j) = \varnothing$, $1 \leq i < j \leq 7$ (otherwise, if there are $1 \leq i < j \leq 7$ such that $w \in N_R(U_i) \cap N_R(U_j)$, then the concatenation of the u_iu_j-path of order 5 with u_ixwyu_j, is a cycle of order 8 where $x \in U_i$, $y \in U_j$ and $xw,wy \in E(G)$, a contradiction). Therefore $|U_i \cup N_R(U_i) \cup \{u_i\}| \geq \delta(G) + 1$. Thus, $|V(G)| \geq 7(\delta(G) + 1) \geq (7)(8) = 56$. □

Lemma 2.3. *If G contains $K_7 - S_5$, then G contains K_7.*

Proof. Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ be the vertex set of $K_7 - S_5$ where the induced subgraph of $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ is isomorphic to K_6. Without loss of generality we may assume that u_1u_7, $u_2u_7 \in E(G)$. Let $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \leq i \leq 7$. Then, as in Lemma 2.2, $U_i \neq \varnothing$ for all $1 \leq i \leq 7$. Also, using the same arguments as in Lemma 2.2, we have the following: (1) $U_i \cap U_j = \varnothing$ for all $1 \leq i < j \leq 7$ except possibly for $i = 1$ and $j = 2$. (2) For all $1 \leq i < j \leq 7$ and for any $x \in U_i$ and $y \in U_j$, we have $xy \notin E(G)$. (3) $N_R(U_i) \cap N_R(U_j) = \varnothing$, $1 \leq i < j \leq 7$ for all $1 \leq i < j \leq 7$. (4) For all $1 \leq i < j \leq 7$ and for any $x \in N_R(U_i)$ and $y \in N_R(U_j)$, we have $xy \notin E(G)$.

Since, $\alpha(G) \leq 7$, as a result at least three of the induced subgraphs $\langle U_i \cup N_R(U_i) \rangle_G$, $3 \leq i \leq 7$ are complete. Since $\delta(G) \geq 7$, it implies that these complete graphs contains K_7. □

Lemma 2.4. *If G contains K_6, then G contains $K_7 - S_5$ or K_7.*

Proof. Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ be the vertex set of K_6. Let $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \leq i \leq 6$. Since $\delta(G) \geq 7$, $U_i \neq \varnothing$ for all $1 \leq i \leq 6$. Now we consider the following two cases:

Case 1: $U_i \cap U_j \neq \varnothing$ for some $1 \leq i < j \leq 6$, say $w \in U_i \cap U_j$. Then it is clear that G contains $K_7 - S_5$. In fact, the induced subgraph $\langle U \cup \{w\} \rangle_G$ contains $K_7 - S_5$.

Case 2: $U_i \cap U_j = \varnothing$ for each $1 \leq i < j \leq 6$. Note that between any two vertices of U there are paths of order 4, 5 and 6. Thus, as in Lemma 2.2 for all $1 \leq i < j \leq 6$, we have the following: (1) for all $x \in U_i$ and $y \in U_j$ we have that $xy \notin E(G)$. (2) $N_R(U_i) \cap N_R(U_j) = \varnothing$. (3) for all $x \in N_R(U_i)$ and $y \in N_R(U_j)$, $xy \notin E(G)$.

Since $\alpha(G) \leq 7$, we have that at least 5 of the induced subgraphs $\langle U_i \cup N_R(U_i)\rangle_G, 1 \leq i \leq 6$ are complete graphs. Since $\delta(G) \geq 7$, as a result these complete graphs contain K_7. Hence, G contains K_7. □

Lemma 2.5. *If G contains $K_1 + P_6$, then Gcontains K_6.*

Proof. Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ be the vertex set of $K_1 + P_6$ where $K_1 = u_1$ and $P_6 = u_2u_3u_4u_5u_6u_7$. Let $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \leq i \leq 7$. Then $U_i \neq \varnothing$ for all $1 \leq i \leq 7$ because $\delta(G) \geq 7$. Now we have the following:
(1) $U_i \cap U_j = \varnothing$ for each $2 \leq i < j \leq 7$ except possibly for $i = 3$ and $j = 5$, 6, and for $i = 4$ and $j = 6$. To see that, suppose that there are $2 \leq i < j \leq 7$ such that $x \in U_i \cap U_j$ and $(i, j) \neq (3, 5)$, $(3, 6)$, $(4, 6)$. Then we have the following:

(i) $i = 2$ and $j = 3$. Then $xu_2u_1u_7u_6u_5u_4u_3x$ is a C_8, a contradiction.

(ii) $i = 2$ and $j = 4$. Then $xu_2u_3u_1u_7u_6u_5u_4x$ is a C_8, a contradiction.

(iii) $i = 2$ and $j = 5$. Then $xu_2u_3u_4u_1u_7u_6u_5x$ is a C_8, a contradiction.

(iv) $i = 2$ and $j = 6$. Then $xu_2u_3u_4u_5u_1u_7u_6x$ is a C_8, a contradiction.

(v) $i = 2$ and $j = 7$. Then $xu_2u_3u_4u_5u_6u_1u_7x$ is a C_8, a contradiction.

(vi) $i = 3$ and $j = 4$. Then $xu_3u_2u_1u_7u_6u_5u_4x$ is a C_8, a contradiction.

(vii) $i = 4$ and $j = 5$. Then $xu_4u_3u_2u_1u_7u_6u_5x$ is a C_8, a contradiction.

(viii) i, j are not as in the above cases. Then we use the symmetry in the subgraph $K_1 + P_6$ and we argue as in (i)–(vii).

(2) For all $2 \leq i < j \leq 7$ and for any $x \in U_i$ and $y \in U_j$ we have that $xy \notin E(G)$. (3) $N_R(U_i) \cap N_R(U_j) = \varnothing$ for all $2 \leq i < j \leq 7$. (4) For all $2 \leq i < j \leq 7$ and for all $x \in N_R(U_i)$ and $y \in N_R(U_j)$ we have that $xy \notin E(G)$ ((2), (3), and (4) follows easily from being that $K_1 + P_6$ contains paths of order $6, 5$ and 4 between any two vertices u_i and u_j, $2 \leq i < j \leq 7$).

Now, since $\alpha(G) \leq 7$, at least one induced subgraph of $\langle U_i \cup N_R(U_i)\rangle_G, i =$ 2, 4, 5, 7 is a complete graph. Since $\delta(G) \geq 7$, then $|N_R(U_i)| \geq 5$ and so $|U_i \cup N_R(U_i)| \geq 6$ for each $2 \leq i \leq 7$. Therefore at least one induced subgraph of $\langle U_i \cup N_R(U_i)\rangle_G, i =$ 2, 4, 5, 7 contains K_6. Thus, G contains K_6. □

Lemma 2.6. *If G contains $K_1 + P_5$, then G contains $K_1 + P_6$ or K_6.*

Proof. Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ be the vertex set of $K_1 + P_5$ where $K_1 = u_1$ and $P_6 = u_2u_3u_4u_5u_6$. Let $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \leq i \leq 6$. Then $|U_i| \geq 2$ for all $1 \leq i \leq 6$ because $\delta(G) \geq 7$. Now we have the following cases:

Case 1: $U_i \cap U_j = \varnothing$ for all $2 \leq i < j \leq 6$. Then we have the following: (1) For any $x \in U_i$ and $y \in U_j$ we have that $xy \notin E(G)$, $2 \leq i \leq j \leq 6$ except possibly for $i = 3$ and $j = 5$ because otherwise G contains C_8. (2) $N_R(U_i) \cap N_R(U_j) = \varnothing$ for all $2 \leq i \leq j \leq 6$ because otherwise G contains C_8. (3) For all $2 \leq i \leq j \leq 6$ and for all $x \in N_R(U_i)$ and $y \in N_R(U_j)$ we have $xy \notin E(G)$ because otherwise G contains C_8. Therefore as in Lemma 2.5 at least three of $\langle U_i \cup N_R(U_i)\rangle_G$, $2 \leq i \leq 6$ are complete graphs. Since $\delta(G) \geq 7$, these three complete graphs contains K_6. And so, G contains K_6.

Case 2. $U_i \cap U_j \neq \varnothing$ for some $2 \leq i < j \leq 6$, say $u_7 \in U_r \cap U_s$. Then we have the following:

Subcase 2.a. $r = 5$ and $s = 6$. Then (i) if $x \in U_6 - \{u_7\}$, then $xu_7 \notin E(G)$ (otherwise, $xu_7u_5u_4u_3u_2u_1u_6x$ is a C_8, a contradiction). (ii) if $x \in U_5 - \{u_7\}$, then $xu_7 \notin E(G)$ (otherwise, $xu_5u_4u_3u_2u_1u_6u_7x$ is a C_8, a contradiction). To this end, we have the following claims:

Claim 1. $(U_i \cap U_6) - \{u_7\} = \varnothing$ *for all* $2 \leq i \leq 4$.
Proof of Claim 1. Suppose that there is $2 \leq i \leq 4$ such that $x \in (U_i \cap U_6) - \{u_7\}$. Then (1) if $i = 2$, then $u_2xu_6u_7u_5u_4u_3u_1u_2$ is a C_8, a contradiction. (2) if $i = 3$, then $u_3xu_6u_7u_5u_4u_1u_2u_3$ is a C_8, a contradiction. (3) if $i = 4$, then $u_4xu_6u_7u_5u_1u_2u_3u_4$ is a C_8, a contradiction. The proof of the Claim is complete.

Claim 2. $(U_i \cap U_j) - \{u_7\} = \varnothing$ *for all* $2 \leq i < j \leq 5$ *except for* $i = 3$ *and* $j = 5$.
Proof of Claim 2. By using the same arguments as in Claim 1, we obtain the same contradiction. The proof of the claim is complete.

Now we split this subcase into two subsubcases:
Subsubcases 2.a.1. $|U_5 - \{u_7\}| = 1$. Since $d(u_5) \geq 7$, we have u_5u_2, $u_5u_3 \in E(G)$. Then G contains $K_1 + P_6$ where $K_1 = u_5$ and $P_6 = u_7u_6u_1u_2u_3u_4$.
Subsubcases 2.a.2. $|U_5 - \{u_7\}| \geq 2$. Then we have the following:
Subsubsubcases 2.a.2.i. $U_3 \cap U_5 = \varnothing$. Then, by Claims 1 and 2, for any $x \in U_i - \{u_7\}$, x is adjacent to at most u_i and u_1 where $i = 2, 3, 4$. Since $\delta(G) \geq 7$, we have $|\,(U_i - \{u_7\}) \cup N_R(U_i - \{u_7\})| \geq 6$ for each $i = 2, 3, 4$. Note that, from (ii) above, $\alpha(U_5) \geq 2$. Thus, at least one induced subgraph of $\langle (U_i - \{u_7\}) \cup N_R(U_i - \{u_7\})\rangle_G$, $2 \leq i \leq 4$ is complete. Thus, this complete graph contains K_6.

Subsubsubcases 2.a.2.ii. $U_3 \cap U_5 \neq \varnothing$, say $u_8 \in U_3 \cap U_5$. Then for all $2 \leq i < j \leq 6$ and for all $x \in N_R(U_i)$ and $y \in N_R(U_j)$ we have that $xy \notin E(G)$ because otherwise G contains C_8.

Claim 3. $\alpha(U_3 \cup U_5 \cup U_6) \geq 4$.

Proof of Claim 3. Since $|U_5 - \{u_7\}| \geq 2$, we can assume that $v_1 \in U_5 - \{u_7\}$ with $u_8 \neq v_1$. Also, since $d(u_3), d(u_6) \geq 7$, there are two vertices of $U_3 - \{u_8\}$ and $U_6 - \{u_7\}$, say $v_2 \in U_3 - \{u_8\}$ and $v_3 \in U_6 - \{u_7\}$. Note that, by Claim 1, $u_8 \neq u_7, u_8 \neq v_3, u_7 \neq v_2$ and $v_3 \neq v_2$. Also, from (i) above, we have that $v_3u_7 \notin E(G)$. Therefor, either $v_1 \neq v_2$, and so $\{v_1, v_2, u_7, u_8\}$ is an independent set of 4 vertices or $v_1 \neq v_3$, and so $\{v_1, v_3, u_7, u_8\}$ is an independent set of 4 vertices (otherwise, if both $v_1 = v_2$ and $v_1 = v_3$, then $v_2 = v_3$ which contradicts Claim 1). The proof of Claim 3 is complete.

Now, by Claims 1 and 2, for any $x \in U_i - \{u_7\}$, x is adjacent to at most u_i and u_1 where $i = 2$, 4. Thus, $|\,(U_i - \{u_7\}) \cup N_R(U_i - \{u_7\})| \geq 6$ for each $i = 2$, 4. Since $\alpha(G) \leq 7$, at least one of $\langle (U_i - \{u_7\}) \cup N_R(U_i - \{u_7\}) \rangle_G$, $i =$ 2, 4 is complete graph. Therefore, G contains K_6.

Subcase 2.b. $r = 4$ and $s = 6$. Then we have the following:
(1) We may assume that $U_5 \cap U_6 = \varnothing$ (if $U_5 \cap U_6 \neq \varnothing$, then we get Subcase 2.a, and so G contains either $K_1 + P_6$ or K_6).
(2) By a similar argument as in Subcase 2.a, for any $x \in U_i$, we have $xu_7 \notin E(G)$ where $i = 4$, 6.
(3) Since between any two vertices of $\{u_2, u_3, u_4, u_5, u_6\}$ there is a path of order 6, as a result for all $2 \leq i < j \leq 6$ and for any $x \in U_i$ and $y \in U_j$ we have that $xy \notin E(G)$.
(4) If $w \in U_4 \cap U_5$, then for any $x \in U_4 \cap U_5$ we have that $xw \notin E(G)$ (to see this, assume that $xw \in E(G)$. Then if $x \in U_4$, then $xwu_5u_6u_1u_2u_3u_4x$ is a C_8, which is a contradiction. If $x \in U_4$, then $wxu_5u_6u_1u_2u_3u_4w$ is a C_8, which is a contradiction).

Now, we consider the following two subsubcases:
Subsubcase 2.b.1. $|U_4 - \{u_7\}| = 1$. Then u_4u_2, $u_4u_6 \in E(G)$ and so G contains $K_1 + P_6$ where $K_1 = u_4$ and $P_6 = u_1u_2u_3u_5u_6u_7$.
Subsubcase 2.b.2. $|U_4 - \{u_7\}| \geq 2$. By (1) of Subcase 2.b, $U_5 \cap U_6 = \varnothing$ and so $u_5u_7 \notin E(G)$. Thus, $|U_i - \{u_7\}| \geq 2$ for each $i = 4$, 5. Now we have the following claim:

Claim 4. $\alpha(U_4 \cup U_5 \cup U_6) \geq 4$.

Proof of Claim 4. To prove the claim we consider the following cases:

Case I. $|U_4 \cap U_5| = 1$, say $v \in U_4 \cap U_5$. Then, by (1), $v \neq u_7$ and, by (4), $U_4 \cup U_5 - \{u_7\}$ contains an independent set of 3 vertices, say $\{v_1, v_2, v\}$ where $v_1 \in U_4$ and $v_2 \in U_5$. Thus, $\{v_1, v_2, v, u_7\}$ is an independent set of 4 vertices.

Case II. $|U_4 \cap U_5| \geq 2$. Let $u, v \in U_4 \cap U_5$. Then, by (3) of Subcase 2.b, $\{u, v\}$ is an independent set. Since $d(u_6) \geq 7$, there is $w \in U_6 - \{u_7\}$. By (1), (2) and (3), $\{u, v, w, u_7\}$ is an independent set of 4 vertices.

Case III. $|U_4 \cap U_5| = 0$. Then we have the following subcases:

Subcase III.1. $U_4 \cap U_6 - \{u_7\} = \varnothing$. Then $U_4 - \{u_7\}$, U_5, $U_6 - \{u_7\}$ are mutually disjoint sets. Therefore, by (2) and (3), $\{v_1, v_2, v_3, u_7\}$ is an independent set of 4 vertices where $v_1 \in U_4 - \{u_7\}, v_2 \in U_5$ and $v_3 \in U_6 - \{u_7\}$.

Subcase III.2. $U_4 \cap U_6 - \{u_7\} \neq \varnothing$, say $u \in U_4 \cap U_6 - \{u_7\}$. Since $d(u_5) \geq 7$ and $U_5 \cap U_4 = U_5 \cap U_6 = \varnothing$, as a result $|U_5 - \{u, u_7\}| \geq 2$. Let $v_1, v_2 \in U_5 - \{u, u_7\}$. Then, by (2) and (3), $\{u, v_1, v_2, u_7\}$ is an independent set of 4 vertices. The proof of the Claim is complete.

Now, since between any two vertices of $\{u_2, u_3, u_4, u_5, u_6\}$ there is a path of order 5, as a result $N_R(U_i - \{u_7\}) \cap N_R(U_j - \{u_7\}) = \varnothing$ for each $2 \leq i < j \leq 6$, and so no vertex of the independent set of $U_4 \cup U_5 \cup U_6$ adjacent to any vertex of $N_R(U_2) \cup N_R(U_3)$. Also, since between any two vertices of $\{u_2, u_3, u_4, u_5, u_6\}$ there is a path of order 7 except possibly for u_4 and u_5 and for u_4 and u_6, as a result $U_i \cap U_j - \{u_7\} = \varnothing$ for each $2 \leq i < j \leq 6$ except possibly for $i = 4$ and $j = 5, 6$. Moreover, since there is a path of order 4 between u_2 and u_3, it implies that for each $x \in N_R(U_2)$ and $u \in N_R(U_3)$ we have that $xy \notin E(G)$. Hence, by (3) and being $\alpha(G) \leq 7$, at least one of the induced subgraphs $\langle U_i \cup N_R(U_i) \rangle_G, i = 2, 3$ is complete. Since $\delta(G) \geq 7$, it implies that this complete subgraph contains K_6.

Subcase 2.c. $r = 2$ and $s = 6$. Then we have the following:
(1) $U_i \cap U_j = \varnothing$ for all $1 \leq i < j \leq 6$ except $i = 2$ and $j = 6$ otherwise G contains C_8.
(2) Since between any two vertices of $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ there is a path of order 6, as a result for all $x \in U_i$ and $y \in U_j$ we have that $xy \notin E(G)$ for any $1 \leq i < j \leq 6$.

Claim 5. $\alpha(U_1 \cup U_2 \cup U_4 \cup U_6) \geq 4$.

Proof of Claim 5. By (1), U_1, U_2, U_4 are mutually disjoint sets. Since $\delta(G) \geq 7, |U_i - \{u_7\}| \geq 2$ for each $i = 1, 4$. Therefore, by (2), $\{v_1, v_2, v_3, u_7\}$ is an independent set of 4 vertices where $v_1 \in U_1, v_2 \in U_4$, and $v_3 \in U_2$ with $v_3 \neq u_7$. The proof of the claim is complete.

Now, $N_R(U_i) \cap N_R(U_j) = \varnothing$ for each $1 \leq i < j \leq 6$ except $i = 1$ and $j = 4$ otherwise G contains C_8. Hence, there is no vertex of $\{v_1, v_2, v_3, u_7\}$ adjacent to any vertex of $N_R(U_3) \cup N_R(U_5)$. Now, since there is a path of order 4 between u_3 and u_5, it implies that for each $x \in N_R(U_3)$ and $y \in N_R(U_5)$, we have that $xy \notin E(G)$. Hence, at least one of $\langle U_i \cup N_R(U_i) \rangle_G, i = 3, 5$ is complete (otherwise, $\{v_1, v_2, v_3, u_7\}$ with two independent vertices of $\langle U_3 \cup N_R(U_3) \rangle_G$ and two independent vertices of $\langle U_5 \cup N_R(U_5) \rangle_G$ form an independent set of

8 vertices, a contradiction). Since $\delta(G) \geq 7$, it implies that this complete subgraph contains K_6.

Subcase 2.d. $r=2$ and $s=3$. Then we have a subcase similar to Subcase 2.a.

Subcase 2.e. $r=2$ and $s=4$. Then we have a subcase similar to Subcase 2.b.

Subcase 2.f. $r=4$ and $s=6$. Then we have the following:
(1) Since between any two vertices of $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ there is a path of order 6, as a result for all $x \in U_i$ and $y \in U_j$ we have that $xy \notin E(G)$ for any $1 \leq i < j \leq 6$.
(2) Since between any two vertices of $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ there is a path of order 5, as a result $N_R(U_i) \cap N_R(U_j) = \varnothing$ for any $1 \leq i < j \leq 6$.
(3) Since between any two vertices of $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ there is a path of order 4, as a result for all $x \in N_R(U_i)$ and $y \in N_R(U_j)$ we have that $xy \notin E(G)$ for any $1 \leq i < j \leq 6$.
(4) If $U_2 \cap U_4 \neq \varnothing$ or $U_2 \cap U_3 \neq \varnothing$ or $U_4 \cap U_6 \neq \varnothing$ or $U_5 \cap U_6 \neq \varnothing$ or $U_2 \cap U_6 \neq \varnothing$, then we get Subcase 2a, or 2b, or 2c, or 2d, or 2e. Also, if $x \in U_2 \cap U_5$, then $u_5xu_2u_1u_4u_3u_7u_6u_5$ is a C_8, a contradiction. If $x \in U_2 \cap U_6$, then $u_6xu_2u_1u_5u_4u_3u_7u_5$ is a C_8, a contradiction. If $x \in U_4 \cap U_5$, then $u_5xu_4u_1u_2u_3u_7u_6u_5$ is a C_8, a contradiction. If $x \in U_1 \cap U_4$, then $u_4xu_1u_2u_3u_7u_6u_5u_4$ is a C_8, a contradiction. Moreover, if $w \in U_1 \cap U_2$, then G contains $K_1 + P_6$ where $K_1 = u_1$ and $P_6 = wu_2u_3u_4u_5u_6$, and if $w \in U_1 \cap U_6$, then G contains $K_1 + P_6$ where $K_1 = u_1$ and $P_6 = u_2u_3u_4u_5u_6w$. Therefore, in the rest of this subcase we may assume that $U_i \cap U_j = \varnothing$ for any $1 \leq i < j \leq 6$, except possibly for $i = 3$ and $j = 4$, 5, 6 and for $i = 1$ and $j = 3$, 5.

Claim 6. $\alpha(U_1 \cup U_5 \cup U_6) \geq 4$.

Proof of Claim 6. Since $u_7 \in U_3 \cap U_6$ and since $d(u_6) \geq 7$, as a result, $|U_6 - \{u_7\}| \geq 1$ and so there is $v_1 \in U_6 - \{u_7\}$. By (4) $U_5 \cap U_6 = \varnothing$, so there is $v_2 \in U_5$ with $v_2 \neq v_1$ and $v_2 \neq u_7$. Also, by (4), $U_1 \cap U_6 = \varnothing$, so $|U_1 - \{v_2\}| \geq 2$. Hence, there is $v_3 \in U_1 - \{v_2\}$. Thus, by (1), $\{u_7, v_1, v_2, v_3\}$ is an independent set of 4 vertices in $U_1 \cup U_5 \cup U_6$. The proof of the claim is complete.

Now, by (2), there is no vertex of $\{u_7, v_1, v_2, v_3\}$ adjacent to any vertex of $N_R(U_2) \cup N_R(U_4)$. Also, by (3), for each $x \in N_R(U_2)$ and $y \in N_R(U_4)$ we have that $xy \notin E(G)$. Hence, at least one of $\langle U_i \cup N_R(U_i)\rangle_G$, $i = 2$, 4 is complete (otherwise $\{u_7, v_1, v_2, v_3\}$ with two independent vertices of $\langle U_2 \cup N_R(U_2)\rangle_G$ and two independent vertices of $\langle U_4 \cup N_R(U_4)\rangle_G$ forms an independent set of 8 vertices, a contradiction). Since $\delta(G) \geq 7$, this complete graph contains K_6.

Subcase 2.g. $r=2$ and $s=5$. Then we have a subcase similar to Subcase 2.f.

Subcase 2.h. $r = 4$ and $s = 5$. Then we have the following:

(1) Since between any two vertices of $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ there is a path of order 6 except for u_1 and u_4, as a result for all $x \in U_i$ and $y \in U_j$ we have that $xy \notin E(G)$ for any $1 \le i < j \le 6$ except possibly for $i = 1$ and $j = 4$.
(2) Since between any two vertices of $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ there is a path of order 5, as a result $N_R(U_i) \cap N_R(U_j) = \varnothing$ for any $1 \le i < j \le 6$.
(3) Since between any two vertices of $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ there is a path of order 4, as a result for all $x \in N_R(U_i)$ and $y \in N_R(U_j)$ we have that $xy \notin E(G)$ for any $1 \le i < j \le 6$. Now, we have the following claim:

Claim 7. *$U_i \cap U_j = \varnothing$ for any $1 \le i < j \le 6$ except possibly for $i = 5$ and $j = 3$, 4 and for $i = 1$ and $j = 3$, 4, 5.*

Proof of Claim 7. Suppose that there is $1 \le i < j \le 6$ such that $x \in U_i \cap U_j$ and $(i, j) \ne (5, p)$ or $(i, j) \ne (1, q)$ where $p = 3$, 4 and $q = 3$, 4, 5. Then we have the following:

(i) $i = 2$ and $j = 4$. Then we get Subcase 2.e and so G contains $K_1 + P_6$ or K_6, a contradiction.

(ii) $i = 2$ and $j = 3$. Then we get Subcase 2.d and so G contains $K_1 + P_6$ or K_6, a contradiction.

(iii) $i = 4$ and $j = 6$. Then we get Subcase 2.b and so G contains $K_1 + P_6$ or K_6, a contradiction.

(iv) $i = 5$ and $j = 6$. Then we get Subcase 2.a and so G contains $K_1 + P_6$ or K_6, a contradiction.

(v) $i = 2$ and $j = 6$. Then we get Subcase 2.c and so G contains $K_1 + P_6$ or K_6, a contradiction.

(vi) $i = 1$ and $j = 6$. Then G contains $K_1 + P_6$ where $K_1 = u_1$ and $P_6 = u_2u_3u_4u_5u_6x$, a contradiction.

(vii) $i = 3$ and $j = 6$. Then we get Subcase 2.f and so G contains $K_1 + P_6$ or K_6, a contradiction.

(viii) $i = 2$ and $j = 5$. Then we get Subcase 2.g and so G contains $K_1 + P_6$ or K_6, a contradiction.

(ix) $i = 3$ and $j = 4$. Then $u_4xu_3u_2u_1u_6u_5u_7u_4$ is a C_8, a contradiction.

(x) $i = 1$ and $j = 2$. Then $u_2xu_1u_6u_5u_7u_4u_3u_2$ is a C_8, a contradiction. The proof of the claim is complete.

Now, we split our work into the following subsubcases:
Subsubcase 2.h.i. $|U_4 - \{u_7\}| = 1$. Then $u_2u_4, u_4u_6 \in E(G)$. Therefore, G contains $K_1 + P_6$ where $K_1 = u_4$ and $P_6 = u_7u_5u_6u_1u_2u_3$.
Subsubcase 2.h.ii. $|U_4 - \{u_7\}| \geq 2$. Note that $|U_5 - \{u_7\}| \geq 1$. Hence there are $v_1 \in U_4 - \{u_7\}$ and $v_2 \in U_5 - \{u_7\}$ such that $v_1 \neq v_2$. Now, by Claim 7, $U_3 \cap U_4 = \varnothing$. Since $d(u_3) \geq 7$, $|U_3 - \{u_7\}| \geq 2$. Hence, there is $v_3 \in U_3 - \{u_7\}$ with $v_3 \neq v_1$ and $v_3 \neq v_2$. Then, by (1), $\{u_7, v_1, v_2, v_3\}$ is an independent set of 4 vertices of $U_3 \cup U_4 \cup U_5$. Thus, $\alpha(U_3 \cup U_4 \cup U_5) \geq 4$. Now, by (2), there is no vertex of $\{u_7, v_1, v_2, v_3\}$ adjacent to any vertex of $N_R(U_2) \cap N_R(U_6)$. Also, by (3), for each $x \in N_R(U_2)$ and $y \in N_R(U_6)$, we have that $xy \notin E(G)$. Hence at least one of $\langle U_i \cup N_R(U_i)\rangle_G, i = 2, 6$ is complete (otherwise, $\{u_7, v_1, v_2, v_3\}$ with two independent vertices of $\langle U_2 \cup N_R(U_2)\rangle_G$ and two independent vertices of $\langle U_6 \cup N_R(U_6)\rangle_G$ is an 8 independent vertices, a contradiction). Since, $\delta(G) \geq 7$, this complete graph contains K_6.

Subcase 2.i. $r=3$ and $s=4$. Then we have a subcase similar to Subcase 2.h.

Subcase 2.j. $r = 3$ and $s = 5$. Then we have the following:
(1) Since between any two vertices of $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ there is a path of order 5, as a result $N_R(U_i) \cap N_R(U_j) = \varnothing$ for any $1 \leq i < j \leq 6$.
(2) Since between any two vertices of $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ there is a path of order 4, as a result for all $x \in N_R(U_i)$ and $y \in N_R(U_j)$ we have that $xy \notin E(G)$ for any $1 \leq i < j \leq 6$.
(3) If $x \in U_i$ and $y \in U_j$, we have that $xy \notin E(G)$ for any $1 \leq i < j \leq 6$ except possibly for $i = 3$ and $j = 5$ and for $i = 1$ and $j = 3$, 5 because otherwise G contains C_8, a contradiction.
(4) If $U_2 \cap U_4 \neq \varnothing$ or $U_2 \cap U_3 \neq \varnothing$ or $U_4 \cap U_6 \neq \varnothing$ or $U_5 \cap U_6 \neq \varnothing$ or $U_2 \cap U_6 \neq \varnothing$ or $U_2 \cap U_5 \neq \varnothing$ or $U_3 \cap U_6 \neq \varnothing$ or $U_3 \cap U_4 \neq \varnothing$ or $U_4 \cap U_5 \neq \varnothing$, then we get Subcase 2a, or 2c, or 2g, or 2f, or 2i or 2h. Also, if $x \in U_1 \cap U_2$, then G contains $K_1 + P_6$ where $K_1 = u_1$ and $P_6 = wu_2u_3u_4u_5u_6$, a contradiction. If $w \in U_1 \cap U_6$, then G contains $K_1 + P_6$ where $K_1 = u_1$ and $P_6 = u_2u_3u_4u_5u_6w$, a contradiction. Therefore, in the rest of this subcase we may assume that $U_i \cap U_j = \varnothing$ for any $1 \leq i < j \leq 6$, except possibly for $i = 3$ and $j = 5$ and for $i = 1$ and $j = 3$, 4, 5. Now we have the following two subsubcases:
Subsubcase 2.j.I. There is a u_3u_5-path of order 4 in $G - U$, say u_3xyu_5 where x, $y \in G - U$. Then $U_1 \cap U_4 = \varnothing$ (otherwise, if $w \in U_1 \cap U_4$, then $wu_1u_2u_3xyu_5u_4w$ is a C_8, which is a contradiction). It implies that, by (4), $U_i \cap U_j = \varnothing$, for any $i, j \in \{1, 2, 4, 5\}$ with $i \neq j$. And so, by (1), (2), and (3), at least one of $\langle U_i \cup N_R(U_i)\rangle_G$, $i = 1, 2, 4, 6$ is a complete graph (otherwise two independent vertices of each of $\langle U_i \cup N_R(U_i)\rangle_G$, $i = 1, 2, 4, 6$ form an independent set of 8 elements). Since $\delta(G) \geq 7$, this complete graph contains K_6.
Subsubcase 2.j.II. There is no a u_3u_5-path of order 4 in $G - U$. That is for

any $x \in U_3$ and $y \in U_5$, we have that $xy \notin E(G)$.

Claim 8. *$\alpha(U_3 \cup U_5) \geq 2$.*

Proof of the Claim 8. Since $d(u_3) \geq 7$, $|U_3 - \{u_7\}| \geq 1$ and so there is $v \in U_3 - \{u_7\}$. Then $\{u_7, v\}$ is an independent set of two vertices. The proof of the claim is complete.

Now, by (2), there is no vertex of $\{u_7, v\}$ adjacent to any vertex of $N_R(U_2) \cup N_R(U_4) \cup N_R(U_6)$. Also, from (1), for any $x \in N_R(U_i)$ and $y \in N_R(U_j)$, we have $xy \notin E(G)$ for each $i, j \in \{2, 4, 6\}$ with $i \neq j$. Hence at least one of $\langle U_i \cup N_R(U_i)\rangle_G$, $i = 2, 4, 6$ is a complete graph (otherwise, $\{u_7, v\}$ with two independent vertices of each of $\langle U_i \cup N_R(U_i)\rangle_G$, $i = 2, 4, 6$ form an independent set of 8 elements). Since $\delta(G) \geq 7$, this complete graph contains K_6. □

Lemma 2.7. *If G contains K_5, then G contains $K_1 + P_5$ or K_6.*

Proof. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be the vertex set of K_5. Let $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \leq i \leq 5$. Then $|U_i| \geq 3$ for all $1 \leq i \leq 5$ because $\delta(G) \geq 7$. Now we split our work into the following two cases:

Case 1. There are $1 \leq i < j \leq 5$ such that $U_i \cap U_j \neq \varnothing$. Then G contains $K_1 + P_5$.

Case 2. $U_i \cap U_j = \varnothing$ for all $1 \leq i < j \leq 5$. Then we consider the following subcases:

Subcase 2.a. There is no u_iu_j-path of order 4 in $G - U$, that is for any $x \in U_i$ and $y \in U_j$, we have that $xy \notin E(G)$ for each $1 \leq i < j \leq 5$. Since between any two vertices of U there are paths of order 4 and 5, as a result $N_R(U_i) \cap N_R(U_j) = \varnothing$, and for any $x \in N_R(U_i)$ and $y \in N_R(U_j)$, we have that $xy \notin E(G)$ for each $1 \leq i < j \leq 5$. Therefore, since $\delta(G) \geq 7$, at least three of $\langle U_i \cup N_R(U_i)\rangle_G$, $1 \leq i \leq 5$ are complete graph. Since $\delta(G) \geq 7$, this complete graph contains K_6.

Subcase 2.b. There is a u_iu_j-path of order 4 in $G - U$, say $i = 1$ and $j = 2$ and $u_1u_6u_7u_2$ is a path. For simplicity, in the rest of this subcase we consider $U_i^{'} = N(u_i) \cap V(R^{'})$ where $R^{'} = G - U \cup \{u_6, u_7\}$. Then $U_6^{'} \cap U_i^{'} = \varnothing$ for $i = 1, 3, 4, 5, 7$, and also $U_7^{'} \cap U_i^{'} = \varnothing$ for $2 \leq i \leq 6$ because otherwise G contains C_8, a contradiction. Since between any two vertices of $\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ there is a path of order 6 except possibly between u_1 and u_2, as a result for all $x \in U_i^{'}$ and $y \in U_j^{'}$ we have that $xy \notin E(G)$ for any $1 \leq i < j \leq 7$ except possibly for $i = 1$ and $j = 2$. Now we split this subcase into two subsubcases:

Subsubcase 2.b.I. At least one of $U_6^{'}$ and $U_7^{'}$ is complete graph. Since u_6 is adjacent only to u_1 and u_7 of $U \cup \{u_6, u_7\}$ and u_7 is adjacent only to u_2 and u_6 of $U \cup \{u_6, u_7\}$ and since $d(u_6), d(u_7) \geq 7$, as a result G contains K_6.

Subsubcase 2.b.II. Non of $U_6^{'}$ and $U_7^{'}$ is complete. Then we have the following
(1) $\alpha(\langle U_6^{'}\rangle_G) \geq 2$ and $\alpha(\langle U_7^{'}\rangle_G) \geq 2$.
(2) $\alpha(\langle U_6^{'} \cup U_2^{'}\rangle_G) \geq 3$. To see this we argue as follows:
a) If $U_6^{'} \cap U_2^{'} = \varnothing$, then, by (1), the result is obtained.
b) If $U_6^{'} \cap U_2^{'} \neq \varnothing$, say $w \in U_6^{'} \cap U_2^{'}$, then for any $x \in U_i^{'}, i = 2, 6$ we have that $xw \notin E(G)$ (otherwise, if $x \in U_2^{'}$, then $xwu_6u_1u_5u_4u_3u_2x$ is a C_8, a contradiction. If $x \in U_6^{'}$, then $xu_6u_1u_5u_4u_3u_2wx$ is a C_8, a contradiction). Since $|U_2^{'}| \geq 2$ and $|U_6^{'}| \geq 4$, $\{v_1, v_2, w\}$ is an independent set of $U_6^{'} \cup U_2^{'}$ where $v_1 \in U_2^{'}$ and $v_2 \in U_6^{'}$. Thus, $\alpha(\langle U_6^{'} \cup U_2^{'}\rangle_G) \geq 3$.
(3) $\alpha(\langle U_3^{'} \cup U_7^{'}\rangle_G) \geq 3$. This follows from (1) and being $U_3^{'} \cap U_7^{'} = \varnothing$.
(4) $\alpha(\langle U_4^{'} \cup U_5^{'}\rangle_G) \geq 2$. This follows from being that $U_4^{'} \cap U_5^{'} = \varnothing$.

Note that there is no edge connecting two partitions among $U_6^{'} \cup U_2^{'}, U_3^{'} \cup U_7^{'}$ and $U_4^{'} \cup U_5^{'}$. Hence, we have that $\alpha(\langle U_6^{'} \cup U_2^{'} \cup U_3^{'} \cup U_7^{'} \cup U_4^{'} \cup U_5^{'}\rangle_G) = \alpha(\langle U_6^{'} \cup U_2^{'}\rangle_G) + \alpha(\langle U_3^{'} \cup U_7^{'}\rangle_G) + \alpha(\langle U_4^{'} \cup U_5^{'}\rangle_G) \geq 3 + 3 + 2 = 8$. Which is a contradiction. $\square$

Lemma 2.8. *If G be a graph of order ≥ 50, then G contains $K_1 + P_5$ or K_5.*

Proof. Suppose that G contains neither $K_1 + P_5$ nor K_5. Then we have the following claims:

Claim 1. $|N(u)| \leq 21$ *for any* $u \in V(G)$.

Proof of Claim 1. Suppose that u is a vertex with $|N(u)| \geq 22$. Since G contains neither $K_1 + P_5$ nor K_5, the induced subgraph $\langle N(u)\rangle_G$ contains neither P_5 nor K_4. Thus the best case to have for $\langle N(u)\rangle_G$, regarding the number of independent vertices, is a subgraph consisting of 7 non adjacent triangles and an isolated vertex (see Figure 1), Which implies that $\alpha(\langle N(u)\rangle_G) = 8$. The other options for $\langle N(u)\rangle_G$ gives that $\alpha(\langle N(u)\rangle_G) \geq 8$. And so $\alpha(G) \geq 8$. This is a contradiction. The proof of the Claim is complete.

Claim 2. $\alpha(G) = 7$.

Proof of the Claim 2. Since $|V(G)| \geq 50$ and G contains no C_8 and since $r(C_8, K_7) = 43$, $\alpha(G) \geq 7$. But G has no 8-elements independent set, so $\alpha(G) \leq 7$. Thus, $\alpha(G) = 7$. The proof of the claim is complete.

Now, for any seven independent vertices $u_1, u_2, u_3, u_4, u_5, u_6$, and u_7, set $N_i[u_{i+1}] = N[u_{i+1}] - \left(\cup_{j=1}^{i} N[u_j]\right)$, $1 \leq i \leq 6$. Analogously, we set $N_i(u_{i+1})$, $1 \leq i \leq 6$. Let $A = \cup_{i=1}^{6} N_i[u_{i+1}]$, $B = \cup_{i=1}^{6} N_i(u_{i+1})$ and $\beta = \alpha(\langle B\rangle_G)$.

Claim 3. $|N(u_1) \cup B| \geq 43$.

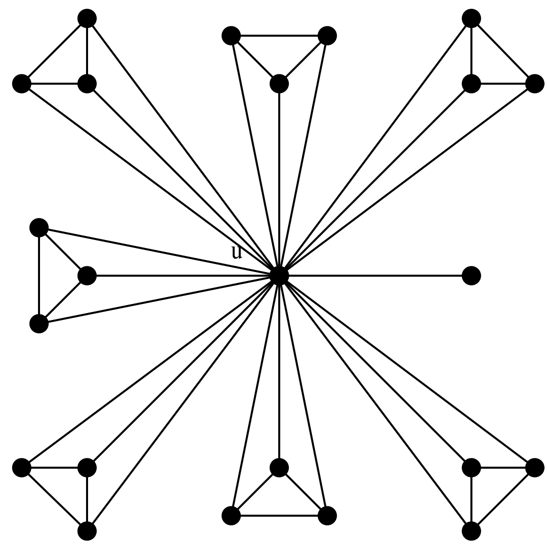

Figure 1:

Proof of Claim 3. Suppose that $|N(u_1) \cup B| \leq 42$. Then $|N[u_1] \cup A| \leq 49$. And so $|G-(N[u_1] \cup A)| \geq 50-49 = 1$. But $r(C_8, K_1) = 1$, so $G-(N[u_1] \cup A)$ contains a vertex, say u_8, which is not adjacent to any of $u_1, u_2, u_3, u_4, u_5, u_6$, and u_7. Thus, $\{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ is a an independent set of vertices. A contradiction. The proof of the claim is complete.

Now, by Lemma 1.1, $\delta(G) \geq 7$ and so by Claim 1, we have that $7 \leq |N(u_1)| \leq 21$. Thus, If $|N(u_1)| = r$, then $|B| \geq 43-r$. By a similar argument as in Claim 1, we have that $\alpha(\langle N(u_1)\rangle_G) \geq \lceil \frac{r}{3} \rceil$ and $\beta \geq \lceil \frac{43-r}{3} \rceil$. Note that for any $7 \leq r \leq 21$ either $\lceil \frac{r}{3} \rceil$ or $\lceil \frac{43-r}{3} \rceil$ is greater than or equal to 8. And so $\alpha(G) \geq 8$. That is a contradiction. □

Theorem 2.1. $r(C_8, K_8) = 50$.

Proof. Suppose that there exist a graph G of order 50 that contains neither C_8 nor an 8-elements independent set. Then by Lemma 2.1, $\delta(G) \geq 7$ and by Lemma 2.8, G contains $K_1 + P_5$ or K_5. Thus, by Lemmas 2.7, 2.6, 2.5, 2.4, 2.3 and 2.2, we have that $|V(G)| \geq 56$. This is a contradiction. Thus, The proof is complete. □

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M.M.M. Jaradat
Department of Mathematics
Yarmouk University
Irbid-Jordan
mmjst4@yu.edu.jo

Department of Mathematics and Physics
Qatar University
Doha-Qatar
mmjst4@qu.edu.qa

B.M.N. Alzaleq
Department of Mathematics
Yarmouk University
Irbid-Jordan
alzaleq@yahoo.com