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An elementary proof of Frank's constructive characterization of the graphs having k edge disjoint spanning trees

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Abstract. We give an elementary proof of Frank's Theorem stating that a (finite, undirected, nonempty) multigraph has k edge disjoint spanning trees if and only if it can be obtained from K_1 by repeatedly (i) adding an edge or (ii) chosing a sequence σ of k vertices and pairwise distinct non-loop edges, deleting the edges of σ, and adding a new vertex plus one edge to each vertex of σ plus one edge to each end of every edge of σ.

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All graphs and hypergraphs considered here are supposed to be finite and undirected and may contain loops and multiple edges. For terminology not defined here I refer to [1] and [2].

One of the classic results in graph theory, Tutte's and Nash–Williams's base packing theorem, states that a graph admits k edge disjoint spanning trees if and only if for every partition $\mathcal{P}$ of its vertex set there are at least $k \cdot (|\mathcal{P}| - 1)$ many edges connecting distinct classes of $\mathcal{P}$ [7] [5].

There is a constructive characterization of the graphs having k edge disjoint spanning trees in terms of the following *Henneberg operation.* Let $\sigma := x_1, \ldots, x_k$ be a sequence of vertices and pairwise distinct non-loop edges of some graph G. Let $S := E(\sigma) := \{x_1, \ldots, x_k\} \cap E(G)$. Let G^+ be obtained from $G - S$ by adding a new vertex z, adding a new edge from x_i to z for each vertex x_i in σ and adding a new edge from each $y \in V(e)$ to z, for every $e \in S$. We then say that G^+ is obtained from G by a *k-lifting according to* σ. The *degree* of the lifting is defined to be $d_{G^+}(z)$, which equals $k + |E(\sigma)|$. Observe that $|E(G^+)| = |E(G)| + k$.

Theorem 1 ([3]). *Let $k \geq 1$. Then a graph has k edge disjoint spanning trees if and only if it can be obtained from K_1 by a sequence of edge additions or k-liftings of degree less than $2k$.*

Frank observed in [3], without proof, that it is possible to deduce Theorem 1 by combining the base packing theorem with another fundamental result of graph connectivity theory, namely Mader's constructive characterization of the k-edge-connected digraphs [4]. Earlier, Tay gave a proof of Theorem 1 for all integers $k \geq 1$ of the form $k = n \cdot (n-1)/2$ [6], relying on some results on bar and body frameworks and on the core lemma from [5]. Here, we give an elementary proof of Theorem 1.

We start with a lemma on a separation property of spanning tree factorizations.

Lemma 1. *Let G be a graph which admits a factorization into k spanning trees. Then for distinct edges $e_1, \ldots, e_k$ there exists a factorization $S_1, \ldots, S_k$ of G into spanning trees such that $e_1 \in E(S_1), \ldots, e_k \in E(S_k)$.*

Proof. Let $S_1, \ldots, S_k$ be a factorization into k spanning trees such that as many of its factors as possible intersect $X := \{e_1, \ldots, e_k\}$. It clearly suffices to prove that they all do. Without loss of generality we may assume, to the contrary, that S_1 does not intersect X and that S_2 contains two edges $e \neq f$ from X. Let C be the edge set of the unique cycle in $S_1 + e$ and let C^* be the set of edges in $S_1 + S_2$ whose end vertices are in distinct components of $S_2 - e$. Since C intersects the cut C^* in e, there must be a $g \in C \cap C^* - \{e\}$. Now $T_1 := (S_1 - g) + e$, $T_2 := (S_2 - e) + g$, $S_3, \ldots, S_k$ is a factorization of G into k spanning trees with $e \in E(T_1)$, $f \in E(T_2)$; so one more factor, compared to $S_1, \ldots, S_k$, intersects X, contradicting our choice. □

Lemma 2. *Suppose that G has a factorization into k edge disjoint spanning trees. Then every graph obtained from G by a k-lifting has a factorization into k edge disjoint spanning trees.*

Proof. Suppose G^+ has been obtained from a k-lifting according to $\sigma = x_1, \ldots, x_k$. Without loss of generality, x_i is an edge for every $i \leq s := |E(\sigma)|$. By Lemma 1, there exists a factorization into spanning trees $S_1, \ldots, S_k$ such that $x_1 \in E(S_1), \ldots, x_s \in E(S_s)$. For $i \leq s$, let T_i be obtained from $S_i - x_i$ by adding the two edges added to the endvertices of the edge x_i in the lifting, and for $i > s$, let T_i be obtained from S_i by adding the edge added to the vertex x_i in the lifting. Then $T_1, \ldots, T_k$ is a factorization of G^+ into spanning trees. □

Hence, by repeatedly applying k-liftings, starting with K_1, we generate *only* graphs which admit a factorization into k spanning trees. To prove that *all* these graphs arise in that way, we shall apply the following Lemma. It states that when k finite partitions of some nonempty set together with an SDR each are given then they can be made connected (considered as hypergraphs) simultaneously by adding edges of size 2 such that each element is at most as often incident with a new edge as it occurs in the SDRs if and only if there are at most $2k$ partition classes in total.

Lemma 3. *Let $N \neq \emptyset$ be a set. For each $i \in \{1, \dots, k\}$, let $\mathcal{P}_i$ be a finite partition of N and $X_i \subseteq N$ with $|X_i \cap P| = 1$ for all $P \in \mathcal{P}_i$.*

Then there exist graphs $G_1, \dots, G_k$ on N with

(i) $|E(G_i)| = |\mathcal{P}_i| - 1$ *for every* $i \in \{1, \dots, k\}$,

(ii) *the hypergraph* $G_i + \mathcal{P}_i$ *is connected for every* $i \in \{1, \dots, k\}$, *and*

(iii) $\sum_{i=1}^k d_{G_i}(x) \leq \sum_{i=1}^k |X_i \cap \{x\}|$ *for every* $x \in N$

if and only if $\sum_{i=1}^k |\mathcal{P}_i| \leq 2k$.[1]

Proof. Let $a_i := |\mathcal{P}_i| \geq 1$ and $b := \sum_{i=1}^k a_i$. For the "only if" part we estimate $2b - 2k \overset{\text{(i)}}{=} 2\sum_{i=1}^k |E(G_i)| = \sum_{i=1}^k \sum_{x \in N} d_{G_i}(x) = \sum_{x \in N} \sum_{i=1}^k d_{G_i}(x) \overset{\text{(iii)}}{\leq} \sum_{x \in N} \sum_{i=1}^k |X_i \cap \{x\}| = \sum_{i=1}^k \sum_{x \in N} |X_i \cap \{x\}| = \sum_{i=1}^k a_i = b$, so $b \leq 2k$.

The remaining statement is proved by induction on k. If $k = 0$ then there is nothing to show. For $k > 0$, suppose $b \leq 2k$. If $b < 2k$ then $a_i = 1$ for some i, and $i = k$ without loss of generality, so $\sum_{i=1}^{k-1} a_i \leq 2(k-1)$. By induction, we find graphs $G_1, \dots, G_{k-1}$ with the desired properties, and, taking the edgeless graph on N for G_k, the statement follows. So we may assume $b = 2k$. If $a_i = 2$ for all i then we let $E(G_i)$ consist of a single edge connecting the two vertices in X_i; it is easy to check that this choice satisfies all the conditions. Otherwise, $a_i > 2$ for some i and, at the same time, $a_j = 1$ for some j. Without loss of generality, $i = k-1$ and $j = k$. Let p be the vertex in X_k, let P be the unique class in $\mathcal{P}_{k-1}$ which contains p, let $Q \in \mathcal{P}_{k-1} - \{P\}$, and let q be the vertex in $X_{k-1} \cap Q$. Set $\mathcal{Q} := (\mathcal{P}_{k-1} - \{P, Q\}) \cup \{P \cup Q\}$ and $Y := X_{k-1} - \{q\}$. By induction, applied to $\mathcal{P}_1, \dots, \mathcal{P}_{k-2}$ and $\mathcal{Q}$ for $\mathcal{P}_{k-1}$ and $X_1, \dots, X_{k-2}$ and Y for X_{k-1}, we find subgraphs $G_1, \dots, G_{k-2}$ and H for G_{k-1} with the following properties: $|E(G_i)| = a_i - 1$ and $G_i + \mathcal{P}_i$ is connected for $i \in \{1, \dots, k-2\}$, and $|E(H)| = a_{k-1} - 2$ and $H + \mathcal{Q}$ is connected, and

$$\sum_{i=1}^{k-2} d_{G_i}(x) + d_H(x) \leq \sum_{i=1}^{k-2} |X_i \cap \{x\}| + |Y \cap \{x\}| \tag{0.1}$$

[1] As it is easy to see, (i) can be ommitted in the statement — but the proofs of Lemma 3 and Lemma 4 run slightly smoother if we prove it "en passant".

for all $x \in N$. Now let G_{k-1} be obtained from H by adding an edge connecting p and q, and let G_k be the edgeless graph on N. Then both $G_{k-1} + \mathcal{P}_{k-1}$ and $G_k + \mathcal{P}_k$ are connected. For $x \in N - \{p,q\}$, $d_H(x) = d_{G_{k-1}}(z) + d_{G_k}(x)$ and $|Y \cap \{x\}| = |X_{k-1} \cap \{x\}| + |X_k \cap \{x\}|$, whereas for $x \in \{p,q\}$, $d_H(x) = d_{G_{k-1}}(x) + d_{G_k}(x) - 1$ and $|Y \cap \{x\}| = |X_{k-1} \cap \{x\}| + |X_k \cap \{x\}| - 1$, and so (0.1) implies (iii), which accomplishes the induction. □

Lemma 4. *Suppose that $G^+ \not\cong K_1$ has a factorization into k edge disjoint spanning trees. Then there exists a graph G which admits a factorization into k edge disjoint spanning trees such that G^+ is obtained from G by a k-lifting of degree less than $2k$.*

Proof. Let $S_1, \ldots, S_k$ be a factorization of G^+ into k edge disjoint spanning trees. As the average degree of a tree is less than 2, the average degree of G^+ is less than $2k$. So let z be a vertex of degree less than $2k$ in G^+, and let $N := N_{G^+}(z)$. For each i, let $\mathcal{P}_i$ be the partition of N formed by its intersections with the components of $S_i - z$, and let $X_\ell := N_{S_i}(z)$. We apply Lemma 3 and find subgraphs $G_1, \ldots, G_k$ with (i), (ii), (iii) as there, which we choose pairwise edge disjoint and edge disjoint from $G^+ - z$. Hence $|E(G_i)| \overset{\text{(i)}}{=} |N_{S_i}(z)| - 1 = d_{S_i}(z) - 1$, and $G := (G^+ - z) + (G_1 + \cdots + G_k)$ has k edges less than G^+. By (ii), $T_1 := (S_1 - z) + G_1, \ldots, T_k := (S_k - z) + G_k$ is a factorization of G into connected spanning subgraphs, and, as $|E(G)| = |E(G^+)| - k = k|V(G^+)| - k - k = k|V(G)| - k$, they must be trees. Note that $d_G(x) = d_{G^+}(x)$ for all $x \in V(G) - N$. By (iii), $d_G(x) \leq d_{G^+}(x)$ for all $x \in N$. Let σ be a sequence in which every edge of $E(G) - E(G^+ - z)$ occurs exactly once and every $x \in N$ occurs exactly $d_{G^+}(x) - d_G(x) \geq 0$ many times. Then G^+ is obtained from G by a lifting of degree $d_{G^+}(z)$ according to σ; the length of σ is $|E(G^+)| - |E(G)| = k$. □

We thus obtain the following, from which Theorem 1 follows immediately.

Theorem 2. *Let $k \geq 1$. Then a graph admits a factorization into k edge disjoint spanning trees if and only if it can be obtained from K_1 by a sequence of k-liftings of order less than $2k$.*

Proof. The if part is Lemma 2, the only if part is Lemma 4. □

Note that in both Theorem 1 and Theorem 2 the restriction to the degree of the k-lifting can be omitted.

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