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Symmetry, integrability and solutions of the Kawahara equation

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Abstract. Integrability and symmetry reductions of the Kawahara equation are investigated. Exact traveling wave solutions, soliton solutions and Galilean invariant solutions are constructed. Auto-Bäcklund transformations and multiplication law of solutions are deduced.

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§1. Introduction

Recently [2], Benitez and Kaikina investigated the initial-boundary value problem on a half-line for the nonlinear equation

$$\begin{cases} u_t + \mathcal{N}(u, u_x) + \mathcal{K}(u) = f, & t > 0, x > 0, \\ u(x, 0) = u_0(x), & x > 0; \\ \partial_x^{j-1} u(0, t) = h_j(t), & t > 0, j = 1, \dots, M, \end{cases} \tag{1.1}$$

where the nonlinear term $\mathcal{N}(u, u_x)$ depends on the unknown function u and its derivative u_x and satisfies the estimate

$$|\mathcal{N}(u, v)| \leq C\, |u|^\rho\, |v|^\sigma\,,$$

with $\sigma \geq 0, \rho \geq 2$. The linear operator $\mathcal{K}(u)$ is defined by

$$\mathcal{K}u = a_n \partial_x^n + a_m \partial_x^m,$$

where the constants $a_n,\ a_m \in \mathbb{R},\ n, m$ are integers. The number M of the boundary data depends essentially on the operator $\mathcal{K}$. Equation (1.1) is a simple universal model, arising in the description of the dispersive dissipative nonlinear waves [8], and describing in a unified way wide classes of nonlinear equations which are of great interest for physical applications. To mention a few, belong to these classes:

- the Korteweg-de Vries-Burgers equation

$$u_t + uu_x + \alpha u_{xxx} - \nu u_{xx} = 0, \tag{1.2}$$

  which appears in the theory of nonlinear acoustics for fluids with gas bubbles,

- the Kuramoto-Sivashinski equation

$$u_t + \frac{1}{2}u_x^2 + u_{xx} + \alpha u_{xxx} = 0, \tag{1.3}$$

  which is applied, for instance, in the theory of combustion to model a flame front and also in the study of two-dimensional turbulence and

- the Kawahara equation

$$u_t = u_{xxxxx} - uu_x - \beta u_{xxx}, \tag{1.4}$$

  where $u = u(x,t)$ and $\beta \in \mathbb{R}$ which describes propagation of signals in transmission lines, propagation of long waves under ice cover in liquid depth, and also gravity waves on the surface of liquid with surface tension.

The authors [2] studied the global existence of solutions and find the main term of the asymptotic representation of solutions to (1.1). Previously, some results on the decay estimates of the solutions in different norms to the Cauchy problems for Korteweg-de Vries-Burgers type equations were obtained, while a general theory of nonlinear nonlocal equations on a half line was developed (see [2] and references therein). Zhibin and Mingliang [12] discussed the structure of traveling wave solutions and studied the similarity solutions and transformations of the two-dimensional version

$$(u_t + uu_x - mu_{xx} + nu_{xxx})_x + su_{yy} = 0$$

of (1.2). A. Nuseir [9] constructed several traveling wave solutions to the local version

$$u_t + uu_x + au_{xx} + bu_{xxxx} = 0$$

of the Kuramoto-Sivashinsky equation.

As far as we know, neither the truncated Painlevé expansion method nor the symmetry analysis were implemented before to the Kawahara equation.

The aim of this contribution is to investigate the complete integrability, identify some of the properties such as the Bäcklund transformation, carry out symmetry reductions and construct exact solutions, including soliton solutions of the Kawahara equation [2]. For this purpose, the main tools used in this work are the Painlevé test, the truncated Painlevé expansion method and the classical Lie symmetry group method for analysis of nonlinear partial differential equations (PDEs).

The plan of this work is as follows. In section 2, the integrability of the (1.4) is investigated using the Painlevé test. In section 3, the truncated Painlevé expansion method is used to construct traveling waves and especially soliton solutions of (1.4). An auto -Bäcklund transformation admitted by this PDE is also derived. In section 4, the classical Lie group methods [10] are implemented to find the invariance algebra of (1.4) and to reduce it to ODE's. Then group invariant solutions and group multiplication law of solutions are deduced.

§2. Integrability of the Kawahara equation

Among various approaches followed to study the behavior of nonlinear partial differential equations (PDE's), Painlevé analysis has proved to be one of the most fruitful, providing an algorithmic procedure that affords a systematic way to deal with nonlinear PDE's. Besides, it has been often merely used as a test of integrability while other methods, as Hirota's method or inverse scattering have been used to obtain explicit solutions. However, the truncated Painlevé expansion method [11] reveals to be a powerful method for solving integrable nonlinear PDE's that admit soliton solutions, as well as some of the non-integrable ones with solitary wave solutions, provided these equations have Painlevé property. A PDE has the Painlevé property when the solutions of the PDE are "single-valued" about a singularity manifold and have no worse singularities than movable poles. More precisely, the singularity manifold is given by

$$\phi(z_1, \dots, z_n) = 0, \tag{2.1}$$

where ϕ is an analytic function of the independent variables $\underline{z} = (z_1, \dots, z_n)$. Then, one assumes that a solution $u = u(z_1, \dots, z_n)$ of the PDE can be represented in a Laurent series in ϕ:

$$u(z_1, \dots, z_n) = \phi^{-p}(z_1, \dots, z_n) \sum_{k=0}^{\infty} u_k(z_1, \dots, z_n) \phi^k(z_1, \dots, z_n), \tag{2.2}$$

where p is a non negative integer, $\phi = \phi(z_1, \dots, z_n)$ and $u = u_k(z_1, \dots, z_n)$ are analytic functions of $\underline{z} = (z_1, \dots, z_n)$ in the neighborhood of the singularity manifold $\phi(z_1, \ldots, z_n) = 0$. Substituting (2.2) into the equation and equating coefficients of like powers of ϕ determine the possible value(s) of p and define recursion relations for u_n, for $n \geq 1$ of the form

$$(n-\alpha_1)(n-\alpha_2)\cdots(n-\alpha_N)u_n = F_n(u_0, u_1, \ldots, u_{n-1}, \phi, \underline{z}), \tag{2.3}$$

where N is the order of the equation, for some functional F_n. This defines u_n unless $n = \alpha_j$ for some j, $1 \leq j \leq N$, $n = \alpha_1, \alpha_2, \ldots, \alpha_N$ are resonances. For each positive integer resonance, there is a compatibility condition (i.e $F_\alpha = 0$) which must be identically satisfied for the PDE to have a solution of the form (2.2) and $u_\alpha(\underline{z})$ is an arbitrary function. Essentially, in order for a given PDE to pass the Painlevé PDE test, it is required that p is a positive integer and there are $N-1$ consistent recursion relations, so that the series (2.2) contains the requisite number of arbitrary functions as required by the Cauchy-Kowalevski theorem [1] ($\phi(\underline{z})$ is the N-th arbitrary function) and thus corresponds to the general solution of the equation. One should emphasize that the Painlevé property is a sufficient condition for the complete integrability of PDE's. Counter examples, such as the Dym-Kruskal equation [6], show that it is not a necessary condition.

To check the Painlevé property for (1.4), we use the Kruskal simplifying ansatz [7] by requiring a generalized Laurent expansion of the solution $u = u(x,t)$ in the form

$$u(x,t) = \phi^{-p}(x,t)\sum_{j=0}^{\infty} u_j(t)\phi^j(x,t), \quad \phi(x,t) = x + \psi(t), \tag{2.4}$$

where $\psi = \psi(t)$ is an arbitrary function and $u_j(t)$, $j = 0, 1, \ldots$, are analytic functions with $u_0(t) \neq 0$, in the neighborhood of a non-characteristic movable singularity manifold defined by $\phi(x,t) = 0$. To determine the dominant behavior, we substitute $u \longrightarrow u_0\phi^{-p}$ into (1.4) to get

$$\begin{aligned} &pu_0^2\phi^{-2p-1} + (-50p^2 - p^5 - 24p - 10p^4 - 35p^3)u_0\phi^{-p-5} \\ &\quad +(2p+p^3)\beta u_0\phi^{-p-3} + p\psi_t u_0\phi^{-p-1} - \tfrac{du_0}{dt}\phi^{-p} = 0. \end{aligned} \tag{2.5}$$

From the above equation, we can see that the most singular powers of ϕ are $-2p-1$ and $-p-5$. By equating these powers, one gets

$$p = 4. \tag{2.6}$$

Hence, the most singular term, i.e. the term in ϕ^{-9} in (2.5):

$$(-6720u_0 + 4u_0^2)\phi^{-9}$$

will vanish if

$$u_0 = 1680. \tag{2.7}$$

Substituting (2.6) and (2.7) into the first equation of (2.4), then inserting u into the Kawahara equation, we have

$$
\begin{aligned}
&9240u_1\phi^{-8} + (9360u_2 + 3u_1^2 + 120\beta u_0)\phi^{-7} + (8280u_3 + 5u_1u_2 + 60\beta u_1)\phi^{-6} \\
&+(4u_0\psi_t + 6720u_4 + 4u_1u_3 + 2u_2^2 + 24\beta u_2)\phi^{-5} + \sum_{j=5}^{\infty}\{-\frac{du_{j-5}}{dt} - (j-8) \\
&\times u_{j-4}\psi_t + (j-4)(j-5)(j-6)(j-7)(j-8)u_j - \sum_{k=0}^{j}(k-4)u_{j-k}u_k - \beta \\
&\times(j-6)(j-7)(j-8)u_{j-2}\}\phi^{j-9} = 0.
\end{aligned}
$$

This requires the coefficients of the various powers of ϕ to vanish. We then obtain

$$
u_1 = u_3 = 0;\ u_2 = -\frac{280\beta}{13},\ u_4 = -\frac{17360\beta^2}{284089} - \frac{1680}{1681}\psi_t \tag{2.8}
$$

and for $j \geq 5$

$$
\begin{aligned}
&-\frac{du_{j-5}}{dt} - (j-8)u_{j-4}\psi_t + (j-4)(j-5)(j-6)(j-7)(j-8)u_j \\
&-\sum_{k=0}^{j}(k-4)u_{j-k}u_k - \beta(j-6)(j-7)(j-8)u_{j-2} = 0,
\end{aligned}
$$

i.e.

$$
\begin{aligned}
&-\frac{du_{j-5}}{dt} - (j-8)u_{j-4}\psi_t + (j-4)(j-5)(j-6)(j-7)(j-8)u_j \\
&-\left(-4u_ju_0 + \sum_{k=1}^{j-1}(k-4)u_{j-k}u_k + (j-4)u_0u_j\right) \\
&-\beta(j-6)(j-7)(j-8)u_{j-2} = 0,
\end{aligned}
$$

or equivalently

$$
(j-8)[(j-4)(j-5)(j-6)(j-7) - u_0]u_j = F_j \tag{2.9}
$$

where

$$
\begin{aligned}
F_j \;=\; &\frac{du_{j-5}}{dt} + (j-8)u_{j-4}\psi_t + \sum_{k=1}^{j-1}(k-4)u_ku_{j-k} \\
&+\beta(j-6)(j-7)(j-8)u_{j-2}.
\end{aligned} \tag{2.10}
$$

Since $u_0 = 1680$, then $(j-4)(j-5)(j-6)(j-7) - u_0 = j^4 - 22j^3 + 179j^2 - 638j - 840 = (j-12)(j+1)(j^2 - 11j + 70)$. Hence, (2.9) is equivalent to

$$
(j-8)(j-12)(j+1)(j^2 - 11j + 70)u_j = F_j. \tag{2.11}
$$

Therefore, the resonances are $j=-1,8,12$ and the compatibility conditions are

$$F_8 = F_{12} = 0. \tag{2.12}$$

Taking $j=5,6,7,9,10,11$ in (2.11), we find that

$$u_5 = u_6 = u_7 = 0,\ u_9 = -\frac{14}{21853}\psi_{tt},\ u_{10} = \frac{7\beta}{4290}u_8,\ \text{and } u_{11} = -\frac{14\beta}{852267}\psi_{tt}.$$

For $j=8$, $F_8=0$ and the compatibility condition at this resonance is identically satisfied, showing that $u_8=u_8(t)$ is an arbitrary function. Turning to the resonance $j=12$, we have

$$F_{12} = 0 \Longleftrightarrow \psi_t = \frac{11767\beta^2}{5570},$$

i.e. ψ must satisfy the condition

$$\psi(t) = \frac{11767\beta^2}{5570}t + C, \tag{2.13}$$

C being a constant, leading thus to $u_5=u_6=u_7=u_9=u_{11}=0$, $u_2=\frac{-280\beta}{13}$, $u_4=-\frac{343758184\beta^2}{284089}$, and u_8, u_{10} and u_{12} being arbitrary functions of t, so that the requisite number of arbitrary functions is fully obtained at the resonances. Summing up, we conclude that the Kawahara equation is conditionally integrable for any $\beta\in\mathbb{R}$.

§3. Truncated Painlevé expansion and solutions of the Kawahara equation

The principle of the truncated Painlevé expansion method consists in that a special truncated Painlevé expansion is obtained by cutting the series (2.2) at the constant level term in ϕ. The term retained in the truncated expansion will then define a transformation of the dependent variable. This transformation will allow one to homogenize the equation; once the equation has been homogenized, it can be readily solved. The truncated Painlevé expansion method is a fairly systematic method that can be programmed using any symbolic computer package such as MACSYMA, MATHEMATICA, MAPLE and REDUCE. In this paper, for sake of accurateness, most of cumbersome computations have been performed using such symbolic computation packages.

For the truncated Painlevé expansion method to be fully efficient, this PDE under study must have the (conditional) Painlevé property. For the purpose, if the candidate equation has independent variables x, t and dependent variable

u, the method consists in, first, writing the solution $u(x,t)$ as a Laurent series in the complex plane

$$u(x,t) = \phi^{-p}(x,t)\sum_{k=0}^{\infty} u_k(x,t)\phi^k(x,t), \tag{3.1}$$

where $\phi(x,t)$ is the non-characteristic manifold for the poles, and $p \in \mathbb{N} - \{0\}$. Second, by substituting the series into the equation and requiring that the most singular term vanishes, one obtains the values for p and $u_0(x,t)$. If the next most singular terms are required to vanish, one gets the expressions for $u_1(x,t)$, $u_2(x,t)$, etc. After that, the series is truncated at the constant level term. The truncated series defines a transformation of the dependent variable, which turns out to be crucial in the process of determining exact closed-form solutions.

Provided the conditional integrability of the Kawahara equation, let us solve it using the above-described method. In order to find the leading order p, we truncate (3.1) at $k = 0$

$$u \sim u_0\phi^{-p}. \tag{3.2}$$

Remarking that this leading order is nothing but that obtained using the first equation of the simplified Kruskal ansatz (2.4), i.e. $p = 4$, then substituting (3.2) into (1.4) yields the most singular term, (here the term in ϕ^{-9}),

$$(-6720u_0\phi_x^5 + 4u_0^2\phi_x)\phi^{-9} \tag{3.3}$$

which vanishes if

$$u_0(x,t) = 1680\phi_x^4(x,t).$$

Next, to find u_1, let $u = u_0\phi^{-4} + u_1\phi^{-3}$, i.e

$$u = 1680\phi_x^4\phi^{-4} + u_1\phi^{-3}.$$

Then, substituting u into (1.4), we compute again the coefficient of the most singular term (here, ϕ^{-8}) and require that this coefficient vanishes. This gives

$$u_1 = -3360\phi_{xx}\phi_x^2.$$

Iterating this process, we set

$$u = 1680\phi_x^4\phi^{-4} - 3360\phi_{xx}\phi_x^2\phi^{-3} + u_2\phi^{-2},$$

that we substitute into (1.4), require the most singular term in ϕ (here the term in ϕ^{-7}) to vanish and find

$$u_2 = -\frac{280}{13}\beta\phi_x^2 + 1120\phi_x\phi_{xxx} + 840\phi_{xx}^2.$$

Next, letting

$$\begin{aligned} u &= 1680\phi_x^4\phi^{-4} - 3360\phi_{xx}\phi_x^2\phi^{-3} + \Big(-\frac{280}{13}\beta\phi_x^2 + 1120\phi_x\phi_{xxx} + 840\phi_{xx}^2\Big)\phi^{-2} \\ &\quad + u_3\phi^{-1} \end{aligned}$$

and then, substituting u into (1.4), we find, vanishing the most singular term in ϕ (term in ϕ^{-6}), that

$$u_3 = -280\phi_{xxxx} + \frac{280\beta}{13}\phi_{xx}.$$

Finally, we truncate the solution at the constant level term in ϕ by setting

$$\begin{aligned} u(x,t) &= 1680\phi_x^4\phi^{-4} - 3360\phi_{xx}\phi_x^2\phi^{-3} + \Big(-\frac{280}{13}\beta\phi_x^2 + 1120\phi_x\phi_{xxx} \\ &\quad + 840\phi_{xx}^2\Big)\phi^{-2} + \Big(-280\phi_{xxxx} + \frac{280\beta}{13}\phi_{xx}\Big)\phi^{-1} + u_4(x,t). \end{aligned} \tag{3.4}$$

Substituting now u into (1.4) and requiring that the most singular term, i.e. the term in ϕ^{-5}, vanishes, we find that $u_4 = u_4(x,t)$ satisfies the PDE

$$\frac{\partial}{\partial t}u_4 = \frac{\partial^5}{\partial x^5}u_4 - u_4\frac{\partial}{\partial x}u_4 - \beta\frac{\partial^3}{\partial x^3}u_4 \tag{3.5}$$

which is nothing but the Kawahara equation. Thus, (3.4) defines an auto-Bäcklund transformation for the Kawahara equation: it relates two solutions of (1.4). We find that (3.4) can be rewritten in terms of Schwartzian derivatives

$$u(x,t) = 280\left[\frac{\beta}{13}\frac{\partial^2}{\partial x^2}\ln\phi(x,t) - \frac{\partial^4}{\partial x^4}\ln\phi(x,t)\right] + u_4(x,t).$$

Hence, we truncate the Laurent expansion (3.1) at the constant level term $u_4(x,t)$ in the form (3.4) and set $u_4(x,t) = 0$. Now, we seek $\phi(x,t)$ in the form

$$\phi(x,t) = 1 + \exp(kx - wt + \delta), \tag{3.6}$$

so that (3.4) becomes the ansatz

$$\begin{aligned} u(x,t) &= \frac{280}{13}k^2\Big\{(\beta - 13k^2)[e^{3kx-3wt-3\delta} + e^{kx-wt+\delta}] \\ &\quad + (2\beta + 52k^2)e^{2kx-2wt+2\delta}\Big\}\Big(1 + e^{kx-wt+\delta}\Big)^{-4}. \end{aligned} \tag{3.7}$$

Substituting (3.7) into (1.4) and canceling out the coefficients of independent powers of $\exp(kx - wt + \delta)$, we find that the pair (k,w) must necessarily satisfy the system of algebraic equations

$$-169wk^2 + 13w\beta - 13k^3\beta^2 + 182\beta k^5 - 169k^7 = 0$$

$$-4901k^7 + 13\beta w + 1859k^2 w - 13k^3\beta^2 - 3458k^5\beta = 0 \tag{3.8}$$

whose solutions can be expressed as

$$\begin{aligned}
(k, w) &= \left(\varepsilon\frac{\sqrt{13\beta}}{13}, \frac{36\varepsilon\beta^2\sqrt{13\beta}}{2197}\right), \quad \varepsilon = \pm 1, \\
(k, w) &= (0, 0), \\
(k, w) &= \left(r, \frac{\beta r(2093r^2 + 31\beta)}{1690}\right),
\end{aligned} \tag{3.9}$$

where r is any of the roots of the quartic equation

$$1690r^4 + 403\beta r^2 + 31\beta^2 = 0,$$

i.e.

$$\begin{aligned}
r \in &\left\{\pm\frac{1}{130}\left[\beta\left(-2015 + 195i\sqrt{31}\right)\right]^{\frac{1}{2}},\right. \\
&\left.\pm\frac{1}{130}\left[\beta\left(-2015 - 195i\sqrt{31}\right)\right]^{\frac{1}{2}}\right\},
\end{aligned} \tag{3.10}$$

where $i^2 = -1$ is the standard complex number. Note that, no restriction is set on δ. Thus we can state the following.

Proposition 1. *δ-families of solutions to the Kawahara equation are defined by the functions*

$$\begin{aligned}
u_\delta(x, t) = &\frac{280}{13}k^2\left\{(\beta - 13k^2)[e^{3kx-3wt-3\delta} + e^{kx-wt-\delta}]\right. \\
&\left. + (2\beta + 52k^2)e^{2kx-2wt+2\delta}\right\}\left(1 + e^{kx-wt+\delta}\right)^{-4},
\end{aligned} \tag{3.11}$$

where δ is an arbitrary constant and (k, w) satisfy (3.9) *and* (3.10).

Note that, for $\beta > 0$, and if $(k, w) = \left(\varepsilon\frac{\sqrt{13\beta}}{13}, \frac{36\varepsilon\beta^2\sqrt{13\beta}}{2197}\right)$, $\varepsilon = \pm 1$, the solution (3.11) takes the form

$$u_\delta(x, t) = \frac{1680}{169}\beta^2\frac{e^{2\varepsilon\left(\frac{x\sqrt{13\beta}}{13} - \frac{36\beta^{\frac{5}{2}}\sqrt{13}t}{2197}\right)+2\delta}}{\left(1 + e^{\varepsilon\left(\frac{x\sqrt{13\beta}}{13} - \frac{36\beta^{\frac{5}{2}}\sqrt{13}t}{2197}\right)+\delta}\right)^4}$$

or equivalently

$$u_\delta(x, t) = \frac{1680}{2704}\beta^2 sech^4\left\{\frac{\varepsilon}{2}\left(\frac{x\sqrt{13\beta}}{13} - \frac{36\beta^{\frac{5}{2}}\sqrt{13}t}{2197}\right) + \frac{\delta}{2}\right\} \tag{3.12}$$

$\varepsilon = \pm 1,\ \delta \in \mathbb{R}$. Clearly (3.12) describes a family of soliton solutions to the Kawahara equation. In this case, δ is the phase shift of the soliton. We illustrate this behavior of the solution (3.12) for $\beta = 0.5$, $\varepsilon = 1$ in Figures 1 and 2 for $\delta = 0,\ 1$, respectively.

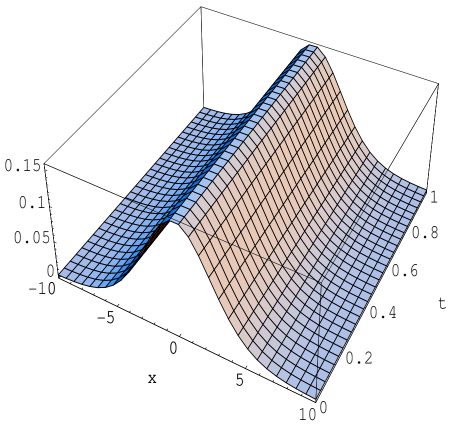

Figure 1: Function u versus x and t for $\beta = 0.5,\ \varepsilon = 1,\ \delta = 0$.

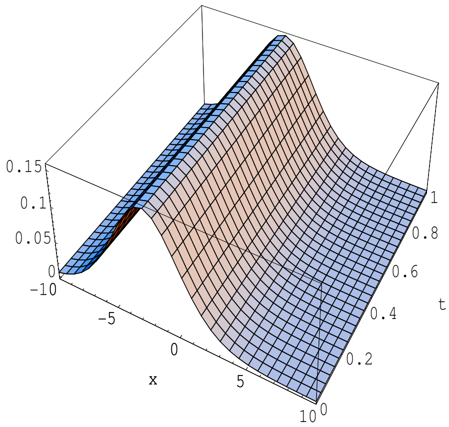

Figure 2: Function v versus x and t for $\beta = 0.5,\ \varepsilon = 1,\ \delta = 1$.

Note that the pair $(k, w) = (0, 0)$ leads to the trivial solution $u = 0$, while the pairs $(k, w) = \left(r, \frac{\beta r(2093r^2+31\beta)}{1690}\right)$ with $\beta > 0$ and r satisfying (3.10) yield the complex-valued functions

$$\begin{aligned} u_\delta(x,t) \quad = \quad & \frac{280}{13} r^2 \Big\{ (-13r^2 + \beta) \Big(e^{3rx - 3\frac{\beta r(2093r^2+31\beta)}{1690}t + 3\delta} \\ & + e^{rx - \frac{\beta r(2093r^2+31\beta)}{1690}t+\delta} \Big) + (2\beta + 52r^2) e^{2rx - \frac{\beta r(2093r^2+31\beta)}{845}t + 2\delta} \Big\} \\ & \times \Big(1 + e^{rx - \frac{\beta r(2093r^2+31\beta)}{1690}t+\delta} \Big)^{-4}. \end{aligned}$$

§4. Symmetry reduction of the Kawahara equation

Quite often, solutions obtained via Painlevé analysis can be also deduced from other techniques such as the symmetry group methods and Hirota methods [5]. Symmetry groups have been used in several different applications in the context of differential equations [10]. The symmetry group techniques lead to solutions in special forms, obtained by exploiting the symmetries of the original equation. At the same time, group theoretical techniques are used to reduce the total number of dependent and independent variables of a PDE.

An advantage of these techniques is that they are applicable to all PDEs, irrespective whether or not the equations are integrable. In this section, we use the Lie group techniques [10] to obtain exact solutions of (1.4). The Lie group method of infinitesimal transformations is the classical method used to find symmetry reductions of PDEs. To apply this method to a n-th order PDE

$$\Delta(x, t, u^{(n)}) = 0 \tag{4.1}$$

where $u^{(n)}$ denotes all the partial derivatives of u with respect to x and t up to order n, one considers the one-parameter Lie groups of infinitesimal transformations in (x, t, u), given by

$$\begin{aligned} \tilde{x} &= x + \varepsilon\xi(x, t, u) + O(\varepsilon^2), \\ \tilde{t} &= t + \varepsilon\tau(x, t, u) + O(\varepsilon^2), \\ \tilde{x} &= u + \varepsilon\phi(x, t, u) + O(\varepsilon^2), \end{aligned} \tag{4.2}$$

where ε is the group parameter. This requires that this transformation leaves invariant the solution manifold

$$S_\Delta = \{u(x, t) : \Delta = 0\} \tag{4.3}$$

of this PDE. This yields an overdetermined linear system of equations for the infinitesimals $\xi(x, t, u)$, $\tau(x, t, u)$ and $\phi(x, t, u)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$Q = \xi(x, t, u)\frac{\partial}{\partial x} + \tau(x, t, u)\frac{\partial}{\partial t} + \phi(x, t, u)\frac{\partial}{\partial u}. \tag{4.4}$$

Having determined the infinitesimals, the symmetry variables are found by solving the characteristic equations

$$\frac{dx}{\xi(x, t, u)} = \frac{dt}{\tau(x, t, u)} = \frac{du}{\phi(x, t, u)} \tag{4.5}$$

which is equivalent to solving the invariant surface condition

$$\Psi \equiv \xi(x, t, u)u_x + \tau(x, t, u)u_t - \phi(x, t, u) = 0.$$

The set S_Δ is invariant under the transformation (4.2), provided that

$$pr^{(n)}Q(\Delta) \mid_{\Delta=0} = 0, \tag{4.6}$$

where $pr^{(n)}Q$ is the n-th prolongation of the vector field (4.4), which is given in terms of ξ, τ and ϕ (cf [10]). In the case of equation (1.4), this yields a system of eighty one equations, as calculated using the MACSYMA package "symmgrp.max" [3]. A triangulation or standard form [4] of these equations is the following system of seven equations

$$\phi = \xi_t, \quad \xi_{tt} = 0, \quad \xi_x = 0, \quad \xi_u = 0, \quad \tau_x = 0, \quad \tau_t = 0, \quad \tau_u = 0$$

from which we easily obtain the following infinitesimals

$$\xi = c_1 t + c_2, \quad \tau = c_3, \quad \phi = c_1,$$

where c_1, c_2 and c_3 are arbitrary constants. The associated Lie algebra is spanned by the vector fields

$$Q_1 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad Q_2 = \frac{\partial}{\partial x}, \quad Q_3 = \frac{\partial}{\partial t}. \tag{4.7}$$

Operator Q_1 generates a Galilean symmetry, while Q_2 and Q_3 generate space and time translational invariance, respectively. Hence $wQ_2 + kQ_3$ generates traveling wave solutions to (1.4) for some constants $w,\ k \in R - \{0\}$. Solving the characteristic system (4.5) with the infinitesimals $\xi = t,\ \tau = 0,\ \phi = 1$, corresponding to Q_1, we obtain the following canonical symmetry reduction

$$u(x,t) = \frac{x + v(t)}{t},\ t \neq 0, \tag{4.8}$$

where $v = v(t)$ satisfies $v'(t) = 0$, i.e. $v(t) = const$. Therefore, the Galilean invariant solution of the Kawahara equation is given by

$$u(x,t) = \frac{x + c}{t},\ c \in \mathbb{R}, t \neq 0. \tag{4.9}$$

Turning to the operator $kQ_3 + wQ_2 = k\frac{\partial}{\partial t} + w\frac{\partial}{\partial x}$, its infinitesimals $\xi = w,\ \tau = k,\ \phi = 0$, lead to the following canonical symmetry reduction

$$u(x,t) = v(y), \quad y = kx - wt, \tag{4.10}$$

where $v(y)$ satisfies the fifth order PDE

$$-wv'(y) = k^5 v''''' - kvv' - \beta k^3 v'''. \tag{4.11}$$

Solutions of (4.11) yield all the traveling wave solutions of the Kawahara equation. In particular, using the ansatz

$$v(y) = \frac{280}{13}k^2 \left\{(\beta - 13k^2)[e^{3y} + e^y] + (2\beta + 52k^2)e^{2y}\right\}(1 + e^y)^{-4} \tag{4.12}$$

which is equivalent to the ansatz (3.7), where $kx - wt + \delta$ is substituted by y, inserting (4.14) into the reduced equation (4.11), we get

$$(-169k^7 + 182\beta k^5 - 169wk^2 - 13\beta^2 k^3 + 13w\beta)(e^{3y} - 1)$$

$$+(-4901k^7 - 3458\beta k^5 - 137\beta^2 k^3 + 13w\beta + 1859wk^2)(e^{2y} - e^y) = 0$$

which must hold identically for any $y \in \mathbb{R}$. Thus, necessarily k and w must satisfy the system of algebraic equations

$$\begin{aligned} -169wk^2 + 13w\beta - 13k^3\beta^2 + 182\beta k^5 - 169k^7 &= 0 \\ -4901k^7 + 13\beta w + 1859k^2 w - 13k^3\beta^2 - 3458k^5\beta &= 0 \end{aligned} \tag{4.13}$$

corresponding to the system (3.8). Thus k and w satisfy (3.9), and substituting them into (4.14) with the use of (4.10) we recover the solutions constructed in the previous section in the particular case of $\delta = 0$, i.e.

$$\begin{aligned} u_0(x,t) &= \frac{280}{13}k^2 \Big\{(\beta - 13k^2)[e^{3kx-3wt} + e^{kx-wt}] \\ &\qquad + (2\beta + 52k^2)e^{2kx-2wt}\Big\} \left(1 + e^{kx-wt}\right)^{-4}. \end{aligned} \tag{4.14}$$

Besides, exponentiating the operators Q_1, Q_2 and Q_3 by solving their respective flow equations, we easily find that their corresponding one-parameter Lie point transformations $\exp(\varepsilon Q_i)$, $i = 1, 2$ and 3, operate on the Kawahara fields (x, t, u) according to

$$\begin{aligned} \exp(\varepsilon Q_1)(x,t,u) &= (\varepsilon t + x, t, u + \varepsilon), \\ \exp(\varepsilon Q_2)(x,t,u) &= (x + \varepsilon, t, u), \\ \exp(\varepsilon Q_3)(x,t,u) &= (x, t + \varepsilon, u), \end{aligned} \tag{4.15}$$

respectively. Hence, the most general Lie point transformation g leaving invariant the solution manifold S_Δ of the Kawahara equation depends on three parameters, say $\alpha, \lambda, \sigma \in \mathbb{R}$, i.e.

$$g = \exp(\alpha Q_1)\exp(\lambda Q_2)\exp(\sigma Q_3), \tag{4.16}$$

operating according to

$$g.(x,t,u) := (\tilde{x}, \tilde{t}, \tilde{u}) = (x + \alpha t + \lambda + \alpha\sigma, t + \sigma, u + \alpha). \tag{4.17}$$

Thus, if $(x,t,u) \in S_\Delta$, then $(\tilde{x}, \tilde{t}, \tilde{u}) := g.(x,t,u) \in S_\Delta$ and $\tilde{u}(\tilde{x}, \tilde{t}) = u(x,t) + \alpha$ solves (1.4). More precisely, if $u = f(x,t)$ is a solution of (1.4), so is $\tilde{u}(\tilde{x}, \tilde{t}) = f(\tilde{x} - \alpha\tilde{t} - \lambda, \tilde{t} - \sigma) + \alpha$. Dropping the tilde for the sake of simplicity, we state the following.

Proposition 2. *The most general solution of the Kawahara equation, obtained from a given solution $u = f(x,t)$ by Lie point symmetry, is of the form*

$$u(x,t) = f(x - \alpha t - \lambda, t - \sigma) + \alpha,$$

where α, λ, σ are arbitrary constants.

Hence, using this latter statement, we generalize the solution (4.14) to get the three parameter-family of solutions as

$$
\begin{aligned}
u_{\alpha,\lambda,\sigma}(x,t) \quad = \quad & \frac{280}{13}k^2 \Big\{ (\beta - 13k^2)[e^{3kx-3(k\alpha+w)t+3w\sigma-3k\lambda} \\
& + e^{kx-(k\alpha+w)t+w\sigma-k\lambda}] + (2\beta + 52k^2) \\
& \times e^{2kx-2(k\alpha+w)t+2w\sigma-2k\lambda} \Big\} \\
& \times \Big(1 + e^{kx-(k\alpha+w)t+w\sigma-k\lambda}\Big)^{-4} + \alpha,
\end{aligned} \tag{4.18}
$$

where α, λ, σ are arbitrary constants. Remarking that if we make $\alpha = 0$ and set $\delta = w\sigma - k\lambda$ in (4.18), we recover the δ-family solutions (3.7), (3.9), (3.10), derived in the previous section, we deduce that the arbitrariness of δ parameterizing these solutions is justified by the space-time translational invariance property of (1.4). Moreover, as postulated in [11, 5], the solutions obtained via the truncated Painlevé expansion can be also constructed using the symmetry reduction techniques.

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