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On λ-large subgroups of n-summable C_{ω_1}-groups

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Abstract. For any ordinal $\omega \leq \lambda \leq \omega_1$ and any natural $1 \leq n < \omega$ we prove that a λ-large subgroup L of a primary C_{ω_1}-group A is n-summable if and only if A is n-summable. This strengthens a classical result due to Linton (Pac. J. Math., 1978) and a recent author's result (Algebra Colloq., 2009) as well.

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§1. Preliminaries

Let all our groups here considered be additively written, abelian, p-primary groups for some arbitrary prime p, fixed for the duration. The most part of the terminology and notation which we will use in the sequel is standard and essentially follow the cited at the end of this paper bibliography. Nevertheless, for the readers' convenience and for completeness of the exposition, we shall recollect some basic concepts and facts.

Definition 1.1. A group A is said to be *totally projective* if it is reduced and has a nice composition series, i.e., a smooth well-ordered chain consisting of nice subgroups with cyclic quotients (for more details, see [9]).

Definition 1.2. A group A is called *a C_λ-group* whenever λ is a limit ordinal if $A/p^\alpha A$ is totally projective for all $\alpha < \lambda$.

Definition 1.3. ([9]) A subgroup B of a group A is said to be *a λ-basic subgroup* for some ordinal λ if the following three conditions hold:
1) B is totally projective of length strictly less than λ;
2) B is p^λ-pure in A;

3) A/B is divisible.

We specify that all fully invariant subgroups of A in what follows are of the type $A(\mathbf{u})$, where $\mathbf{u}$ is an increasing sequence of ordinals and symbols ∞.

Definition 1.4. If A is a C_λ-group for some ordinal λ and L is a fully invariant subgroup of A, then L is called *a λ-large subgroup* whenever $A = B + L$ for all λ-basic subgroups B of A.

It follows from [8] that $p^\alpha A$ is always a λ-large subgroup of a group A provided that $\alpha < \lambda$ as well as $p^\lambda A \subseteq L$ whenever L is λ-large in A.

Likewise, a theorem of [9] states that the group A contains a proper λ-basic subgroup if and only if A is a C_λ-group and λ is cofinal with ω. Since ω_1, the first uncountable limit ordinal, is not cofinal with ω, some additional clarifications are necessary. In fact, B is an ω_1-basic subgroup of A only when $B = 1$ or $B = A$, and so L is an ω_1-large subgroup of A uniquely when $L = A$ and either $L = 1$ or $L \neq 1$ and it can take different forms; for instance $L = p^\alpha A$ where $\alpha < \text{length}(A) \leq \omega_1$.

Linton showed in [8] an ingenious example that the properties of λ-large subgroups for $\lambda > \omega$ are not preserved in general by these of the whole group and conversely; for instance the direct sums of countable groups.

However, this is not the case for totally projective groups.

Theorem 1.5. ([7], [8]) *Let L be a λ-large subgroup of a group A. Then L is totally projective if and only if A is totally projective.*

§2. Main statements

The aim of the present paper is to strengthen the above assertion to a very exotic class of groups, called n-summable groups.

Definition 2.1. ([5]) A group A is said to be *an n-summable group*, $n \in \mathbb{N}$, if $A[p^n] = \bigoplus_{i\in I} A_i$ where, for each $i \in I$, $|A_i| \leq \aleph_0$ and, for each ordinal α, $\left(\bigoplus_{i\in I} A_i\right) \cap p^\alpha A = \bigoplus_{i\in I}(A_i \cap p^\alpha A)$.

Note that such a direct sum is called *valuated*, that is,

$$ht_A(a_{i_1} + \cdots + a_{i_s}) = min\{ht_A(a_{i_1}), \cdots, ht_A(a_{i_s})\},$$

where $a_{i_1}, \cdots, a_{i_s}$ belong to $A_{i_1}, \cdots, A_{i_s}$ and all indices are different, respectively.

Under this new dispensation, direct sums of countable groups are themselves n-summable for any positive integer n. It is self-evident that summable groups encompass n-summable groups for every natural n and 1-summable groups are precisely summable groups. Thus each n-summable group has length not exceeding ω_1. The following affirmation demonstrates that we must restrict our further attention only on n-summable groups of length ω_1.

Theorem 2.2. ([6]) *Suppose λ is a countable limit ordinal. If A is a summable C_λ-group of length λ, then A is a direct sum of countable groups and visa versa.*

In [6] and [1], respectively, were constructed summable C_{ω_1}-groups which are not direct sums of countable groups. In virtue of [5] it can be refined these constructions by finding for any $n \geq 1$ an n-summable C_{ω_1}-group that is not necessarily a direct sum of countable groups. So, the investigation of the discussed above theme for λ-large subgroups of n-summable C_{ω_1}-groups will be of interest. We also established there the validity of the following criterion.

Theorem 2.3. ([5]) *A is an n-summable group if and only if $p^\omega A$ is an n-summable group and some $p^{\omega+n-1}$-high subgroup of A is a direct sum of countable groups.*

It is worthwhile noticing that for $n = 1$ (i.e., for summable groups) this was obtained in [2]. Moreover, an immediate consequence is that A is an n-summable group if and only if $p^m A$ is an n-summable group, where m is a natural number.

And so, we have laid most of the groundwork necessary for proving the following.

Theorem 2.4. *Suppose that A is a C_{ω_1}-group with a λ-large subgroup L for some ordinal λ such that $\omega \leq \lambda \leq \omega_1$ and $n < \omega$ is a natural. Then A is n-summable if and only if L is n-summable.*

Proof. "$\Rightarrow$". In virtue of ([8], p. 484, Theorem 3) there is a countable limit ordinal $\nu \leq \lambda$ such that $p^\nu A = p^\omega L$. Moreover, $L/p^\omega L = L/p^\nu A$ is a λ-large subgroup of $A/p^\nu A$ (see also [8]), where the latter quotient is totally projective by assumption. Therefore, Theorem 1.5 applies to deduce that $L/p^\omega L$ is totally projective, in fact, a direct sum of cyclic groups. That is why, some $p^{\omega+n-1}$-high subgroup H of L is a direct sum of countable groups. Indeed, what suffices to show is that $H/p^\omega H$ is a direct sum of cyclic groups because $p^\omega H$ is bounded by p^{n-1}. In order to do that, we observe that $(H + p^\omega L)/p^\omega L \subseteq L/p^\omega L$ is also a direct sum of cyclic groups as a subgroup. But H is isotype in L, whence $(H + p^\omega L)/p^\omega L \cong H/(H \cap p^\omega L) = H/p^\omega H$ which substantiates our claim.

On the other hand, A being n-summable yields that $p^\omega L = p^\nu A$ is n-summable (see, for more details, [5]). Consequently, Theorem 2.3 is applicable to infer the claim.

"$\Leftarrow$". Same as above, $p^{\nu}A = p^{\omega}L$ for some countable limit ordinal $\nu \leq \lambda$. But L being n-summable implies that $p^{\omega}L = p^{\nu}A$ is n-summable (see, e.g., [5]). Likewise, $A/p^{\nu+n-1}A$ is totally projective of countable length, hence a direct sum of countable groups. Let H be a $p^{\nu+n-1}$-high subgroup of A. Since $p^{\nu}H$ is p^{n-1}-high in $p^{\nu}A$ one may write $p^{\nu}A = p^{\nu}H \oplus X$ for some subgroup X, whence $p^{\nu+n-1}A = p^{n-1}X$. Moreover, $A[p] = H[p] \oplus (p^{\nu+n-1}A)[p] = H[p] \oplus X[p]$. In fact, $H[p] \cap X[p] \subseteq H \cap X = H \cap (p^{\nu}A \cap X) = (H \cap p^{\nu}A) \cap X = p^{\nu}H \cap X = 0$ because H is isotype in A (i.e., heights computed in H and A agree). Consequently, there is a valuated direct sum $(p^{\nu}A)[p^n] = (p^{\nu}H) \oplus X[p^n]$. Even more, $A[p^n] = H[p^n] \oplus X[p^n]$ is a valuated direct sum, where X is a valuated subgroup of $p^{\nu}A$ with $X[p] = (p^{\nu+n-1}A)[p]$. Indeed, if $a \in A[p^n]$ then $p^{n-1}a \in A[p] = H[p] \oplus X[p]$. Since $X[p] = (p^{\nu+n-1}A)[p]$ and H is pure in A, it easily follows that $a \in H + p^{\nu}A + A[p^{n-1}] = H \oplus X + A[p^{n-1}]$ because $p^{\nu}A = p^{\nu}H \oplus X$, and by induction the desired decomposition now follows. That this sum is valuated follows like this: If $z \in H[p^n]$ and $x \in X[p^n]$, then $\mathrm{ht}_A(z + x) = \min\{\mathrm{ht}_A(z), \mathrm{ht}_A(x)\}$ since either $\mathrm{ht}_A(z) < \lambda \leq \mathrm{ht}_A(x)$ or $p^{\nu}A = p^{\nu}H \oplus X$ when $\mathrm{ht}_A(z) \geq \lambda$. Therefore, $p^{\nu}A$ is n-summable if and only if X has this property (see, for example, [5]) because $p^{\nu}H$ is bounded by p^{n-1}.

Next, observe that $H \cong H/\{0\} = H/p^{\nu+n-1}H = H/(H \cap p^{\nu+n-1}A) \cong (H + p^{\nu+n-1}A)/p^{\nu+n-1}A$, where the last factor-group is obviously isotype in $A/p^{\nu+n-1}A$, and thus it is a direct sum of countable groups as well. It follows that H is a direct sum of countable groups. Furthermore, both X and H are n-summable. But by what we have demonstrated above $A[p^n] = H[p^n] \oplus X[p^n]$ is a valuated direct sum and, from this, our assertion follows directly by Definition 2.1. $\square$

As an immediate consequence for $n = 1$, we derive the following.

Corollary 2.5. ([2], [3], [4]) *Suppose A is a C_{ω_1}-group with a λ-large subgroup L for some ordinal number λ such that $\omega \leq \lambda \leq \omega_1$. Then A is summable if and only if L is summable.*

It is worth noting also that our proof of Theorem 2.4, and hence of Corollary 2.5, is at all different from these in [2], [3] and [4], respectively.

We close the study with

Problem 2.6. Does it follow that if both $p^{\alpha}A$ and $A/p^{\alpha}A$ are n-summable groups for some $n \geq 1$ and some ordinal α, then A is n-summable?

For $n = 1$ (i.e., for summable groups) we refer to [4]. Notice also that in view of [5] it can be obtained some results in this aspect under certain limitations on α which depend on n.

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