SUT Journal of Mathematics
Vol. 45, No. 1 (2009), 59–71

On three-dimensional quasi-Sasakian manifolds

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(Received September 9, 2008; Revised December 1, 2008)

Abstract. The object of the present paper is to study locally ϕ-symmetric three-dimensional quasi-Sasakian manifolds and such manifolds with η-parallel Ricci tensor and cyclic parallel Ricci tensor. An example of a locally ϕ-symmetric three-dimensional quasi-Sasakian manifold is also given.

AMS 2000 *Mathematics Subject Classification.* 53C15, 53C40

Key words and phrases. quasi-Sasakian manifold, structure function, locally ϕ-symmetric, η-parallel Ricci tensor, cyclic parallel Ricci tensor.

§1. Introduction

On a three-dimensional quasi-Sasakian manifold the structure function β was defined by Z. Olszak [8] and with the help of this function he has obtained necessary and sufficient conditions for the manifold to be conformally flat [9]. Next he has proved that if the manifold is additionally conformally flat with $\beta =$ constant, then (a) the manifold is locally a product of R and a two-dimensional Kaehlerian space of constant Gauss curvature (the cosymplectic case), or, (b) the manifold is of constant positive curvature (the non-cosymplectic case, here the quasi-Sasakian structure is homothetic to a Sasakian structure).

The object of the present paper is to study three-dimensional quasi-Sasakian manifolds. Section 2 of the paper is concerned with preliminaries. In section 3, we recall the notion of three-dimensional quasi-Sasakian structures. In section 4, we study a three-dimensional locally ϕ-symmetric quasi-Sasakian manifold and prove that a three-dimensional non-cosymplectic quasi-Sasakian manifold with constant structure function is locally ϕ-symmetric if and only if the scalar curvature of the manifold is constant. Section 5 of our paper deals with a three-dimensional quasi-Sasakian manifold with η-parallel Ricci tensor. In this section we also prove that in a non-cosymplectic quasi-Sasakian manifold of dimension three the Ricci tensor is η-parallel if and only if the manifold is η-Einstein. Section 6 is devoted to study a three-dimensional non-cosymplectic quasi-Sasakian manifold with cyclic parallel Ricci tensor. The last section contains an illustrative example of a three-dimensional locally ϕ-symmetric quasi-Sasakian manifold with constant scalar curvature and constant structure function.

§2. Preliminaries

Let M be a $(2n+1)$-dimensional connected differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is an 1-form and g is the Riemannian metric on M such that [1], [2]

$$\phi^2 X = -X + \eta(X)\xi, \;\; \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \;\; X, Y \in TM. \tag{2.2}$$

Then also

$$\phi\xi = 0, \;\; \eta(\phi X) = 0, \;\; \eta(X) = g(X, \xi). \tag{2.3}$$

Let Φ be the fundamental 2-form of M defined by $\Phi(X, Y) = g(X, \phi Y)$, $X, Y \in TM$. Then $\Phi(X, \xi) = 0$, $X \in TM$. M is said to be quasi-Sasakian if the almost contact structure (ϕ, ξ, η, g) is normal and the fundamental 2-form Φ is closed ($d\Phi = 0$), which was first introduced by Blair [3]. The normality condition gives that the induced almost contact structure of $M \times R$ is integrable or equivalently, the torsion tensor field $N = [\phi, \phi] + 2\xi \otimes d\eta$ vanishes identically on M. The rank of the quasi-Sasakian structure is always odd [3], it is equal to 1 if the structure is cosymplectic and it is equal to $2n+1$ if the structure is Sasakian.

§3. Quasi-Sasakian structure of dimension three

An almost contact metric manifold of dimension three is quasi-Sasakian if and only if [8]

$$\nabla_X \xi = -\beta\phi X, \;\; X \in TM, \tag{3.1}$$

for a function β defined on the manifold, ∇ being the operator of the covariant differentiation with respect to the Levi-Civita connection of the manifold. Also we note that if there is a function β on the manifold satisfying $\nabla_X \xi = -\beta\phi X$, then $\xi\beta = 0$, because, from (3.1), we find

$$\nabla_X(\nabla_Y \xi) = -(X\beta)\phi Y - \beta^2\{g(X,Y)\xi - \eta(Y)X\} - \beta\phi\nabla_X Y,$$

which implies that

$$R(X,Y)\xi = -(X\beta)\phi Y + (Y\beta)\phi X + \beta^2\{\eta(Y)X - \eta(X)Y\},$$

where R is the Riemannian curvature tensor of the manifold. Thus we get

$$\begin{aligned} R(X,Y,Z,\xi) &= (X\beta)g(\phi Y,Z) - (Y\beta)g(\phi X,Z) \\ &\quad -\beta^2\{\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\}. \end{aligned}$$

Putting $X = \xi$, we obtain

$$R(\xi,Y,Z,\xi) = \beta^2\{g(Y,Z) - \eta(Y)\eta(Z)\} + g(\phi Y,Z)\xi\beta.$$

Therefore, taking the skew symmetric part, we can easily verify that $\xi\beta = 0$. Clearly, such a quasi-Sasakian manifold is cosymplectic if and only if $\beta = 0$. As a consequence of (3.1), we have [8]

$$(\nabla_X\phi)Y = \beta(g(X,Y)\xi - \eta(Y)X), X, Y \in TM, \tag{3.2}$$

$$(\nabla_X\eta)Y = g(\nabla_X\xi, Y) = -\beta g(\phi X, Y), \tag{3.3}$$

and

$$(\nabla_X\eta)\xi = -\beta\eta(\phi X) = 0. \tag{3.4}$$

In three-dimensional Riemannian manifolds, the Weyl conformal curvature tensor vanishes, that is,

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ &\quad -S(X,Z)Y - \frac{r}{2}(g(Y,Z)X - g(X,Z)Y), \end{aligned} \tag{3.5}$$

where Q is the Ricci operator, that is, $g(QX,Y) = S(X,Y)$ and r is the scalar curvature of the manifold.

Let M^3 be a three-dimensional quasi-Sasakian manifold. The Ricci tensor S of M^3 is given by [9]

$$\begin{aligned} S(Y,Z) &= (\frac{r}{2} - \beta^2)g(Y,Z) + (3\beta^2 - \frac{r}{2})\eta(Y)\eta(Z) \\ &\quad -\eta(Y)d\beta(\phi Z) - \eta(Z)d\beta(\phi Y), \end{aligned} \tag{3.6}$$

where r is the scalar curvature of M^3.
From the above equation we obtain

$$
\begin{aligned}
(3.7)\qquad (\nabla_X S)(Y,Z) &= (\frac{1}{2}Xr - 2\beta X\beta)g(Y,Z) \\
&+(8\beta X\beta - \frac{1}{2}Xr)\eta(Y)\eta(Z) \\
&-\beta(3\beta^2 - \frac{r}{2})\{\eta(Y)g(\phi X,Z) \\
&+\eta(Z)g(\phi X,Y)\} \\
&+\eta\{g(\phi X,Y)d\beta(\phi Z) \\
&+g(\phi X,Z)d\beta(\phi Y)\} \\
&+\eta(Y)g(\nabla_X \mathrm{grad}\beta, \phi Z) \\
&-\eta(Z)g(\nabla_X \mathrm{grad}\beta, \phi Y),
\end{aligned}
$$

where the gradient of a function f is related to the exterior derivative df by the formula

$$
(3.8)\qquad df(X) = g(\mathrm{grad}f, X).
$$

From (3.5) and (3.6) we get

$$
\begin{aligned}
(3.9)\qquad R(X,Y)Z &= g(Y,Z)[(\frac{r}{2} - \beta^2)X \\
&+(3\beta^2 - \frac{r}{2})\eta(X)\xi \\
&+\eta(X)(\phi\mathrm{grad}\beta) - d\beta(\phi X)\xi] \\
&-g(X,Z)[(\frac{r}{2} - \beta^2)Y \\
&+(3\beta^2 - \frac{r}{2})\eta(Y)\xi \\
&+\eta(Y)(\phi\mathrm{grad}\beta) - d\beta(\phi Y)\xi] \\
&+[(\frac{r}{2} - \beta^2)g(Y,Z) \\
&+(3\beta^2 - \frac{r}{2})\eta(Y)\eta(Z) \\
&-\eta(Y)d\beta(\phi Z) - \eta(Z)d\beta(\phi Y)]X \\
&-[(\frac{r}{2} - \beta^2)g(X,Z) \\
&+(3\beta^2 - \frac{r}{2})\eta(X)\eta(Z) \\
&-\eta(X)d\beta(\phi Z) - \eta(Z)d\beta(\phi X)]Y \\
&-\frac{r}{2}[g(Y,Z)X - g(X,Z)Y].
\end{aligned}
$$

§4. Locally ϕ-symmetric quasi-Sasakian manifolds

Definition 4.1. A quasi-Sasakian manifold is said to be locally ϕ-symmetric if

$$\phi^2(\nabla_W R)(X,Y)Z = 0,$$

for all vector fields W, X, Y, Z orthogonal to ξ. This notion was introduced for Sasakian manifolds by Takahashi [10].

Differentiating (3.9) with respect to W and using (3.1) we obtain

$$\begin{aligned}
(4.1)\quad (\nabla_W R)(X,Y)Z \;=\; & g(Y,Z)[(\frac{1}{2}dr(W) - 2\beta(W\beta))X \\
& +(6\beta(W\beta) - \frac{1}{2}dr(W))\eta(X)\xi \\
& +(3\beta^2 - \frac{r}{2})((\nabla_W\eta)(X)\xi + \eta(X)(-\beta\phi W)) \\
& +(\nabla_W\eta)(X)(\phi\mathrm{grad}\beta) \\
& +\eta(X)(\nabla_W\phi)\mathrm{grad}\beta \\
& +\eta(X)\phi(\nabla_W\mathrm{grad}\beta) - (\nabla_W d\beta)(\phi X)\xi \\
& -d\beta(\nabla_W\phi)(X)\xi - d\beta(\phi X)(-\beta\phi W)] \\
& -g(X,Z)[(\frac{1}{2}dr(W) - 2\beta(W\beta))Y \\
& +(6\beta(W\beta) - \frac{1}{2}dr(W))\eta(Y)\xi \\
& +(3\beta^2 - \frac{r}{2})((\nabla_W\eta)(Y)\xi + \eta(Y)(-\beta\phi W)) \\
& +(\nabla_W\eta)(Y)(\phi\mathrm{grad}\beta) \\
& +\eta(Y)(\nabla_W\phi)(\mathrm{grad}\ \beta) \\
& +\eta(Y)\phi(\nabla_W\mathrm{grad}\beta) - (\nabla_W d\beta)(\phi Y)\xi \\
& -d\beta(\nabla_W\phi)(Y)\xi - d\beta(\phi Y)(-\beta\phi W)] \\
& +[(\frac{1}{2}dr(W) - 2\beta(W\beta))g(Y,Z) + (6\beta(W\beta) \\
& -\frac{1}{2}dr(W))\eta(Y)\eta(Z) \\
& +(3\beta^2 - \frac{r}{2})((\nabla_W\eta)(Y)\eta(Z) \\
& +\eta(Y)(\nabla_W\eta)(Z)) \\
& -(\nabla_W\eta)(Y)d\beta(\phi Z) - \eta(Y)(\nabla_W d\beta)(\phi Z) \\
& -\eta(Y)d\beta(\nabla_W\phi Z) \\
& -(\nabla_W\eta)(Z)d\beta(\phi Y) - \eta(Z)(\nabla_W d\beta)\phi(Y) \\
& -\eta(Z)d\beta(\nabla_W\phi)(Y)]X
\end{aligned}$$

$$
\begin{aligned}
&-[(\frac{1}{2}dr(W) - 2\beta(W\beta))g(X,Z) + (6\beta(W\beta)\\
&-\frac{1}{2}dr(W))\eta(X)\eta(Z)\\
&+(3\beta^2 - \frac{r}{2})((\nabla_W\eta)(X)\eta(Z)\\
&+\eta(X)(\nabla_W\eta)(Z))\\
&-(\nabla_W\eta)(X)d\beta(\phi Z) - \eta(X)(\nabla_W d\beta)(\phi Z)\\
&-\eta(X)d\beta(\nabla_W\phi)(Z)\\
&-(\nabla_W\eta)(Z)d\beta(\phi X) - \eta(Z)(\nabla_W d\beta)\phi(X)\\
&-\eta(Z)d\beta(\nabla_W\phi)(X)]Y\\
&-\frac{1}{2}dr(W)[g(Y,Z)X - g(X,Z)Y].
\end{aligned}
$$

Taking W, X, Y, Z orthogonal to ξ and using (2.1) and (2.3) we get from (4.1)

$$
\begin{aligned}
(4.2)\ \phi^2(\nabla_W R)(X,Y)Z \quad = \quad & g(Y,Z)[(2\beta(W\beta) - \frac{1}{2}dr(W))X\\
&+(\nabla_W\eta)(X)(\phi^3\text{grad}\beta)\\
&+\beta d\beta(\phi X)(\phi^3 W)]\\
&-g(X,Z)[(2\beta(W\beta) - \frac{1}{2}dr(W))Y\\
&+(\nabla_W\eta)(Y)(\phi^3\text{grad}\beta)\\
&+\beta d\beta(\phi Y)(\phi^3 W)]\\
&+[(2\beta(W\beta) - \frac{1}{2}dr(W))g(Y,Z)\\
&+(\nabla_W\eta)(Y)d\beta(\phi Z)\\
&+(\nabla_W\eta)(Z)d\beta(\phi Y)]X\\
&-[(2\beta(W\beta) - \frac{1}{2}dr(W))g(Y,Z)\\
&+(\nabla_W\eta)(X)d\beta(\phi Z)\\
&+(\nabla_W\eta)(Z)d\beta(\phi X)]Y\\
&+\frac{1}{2}dr(W)[g(Y,Z)X - g(X,Z)Y]\\
= \quad & 2[2\beta(W\beta) - \frac{1}{2}dr(W)][g(Y,Z)X - g(X,Z)Y]\\
&+\beta\{g(Y,Z)d\beta(\phi X)\\
&-g(X,Z)d\beta(\phi Y)\}\phi^3 W\\
&+(\nabla_W\eta)(X)[g(Y,Z)\phi^3\text{grad}\beta
\end{aligned}
$$

$$\begin{aligned}
&-d\beta(\phi Z)Y] \\
&-(\nabla_W\eta)(Y)[g(X,Z)\phi^3\text{grad}\beta \\
&-d\beta(\phi Z)X] \\
&+(\nabla_W\eta)(Z)[d\beta(\phi Y)X \\
&-d\beta(\phi X)Y] \\
&+\frac{1}{2}dr(W)[g(Y,Z)X-g(X,Z)Y].
\end{aligned}$$

If we take β as a constant then from (4.2) we obtain

$$\phi^2(\nabla_W R)(X,Y)Z = \frac{1}{2}dr(W)[g(X,Z)Y - g(Y,Z)X].$$

From above we can conclude the following :

Theorem 4.1. *A three-dimensional non-cosymplectic quasi-Sasakian manifold with constant structure function β is locally ϕ-symmetric if and only if the scalar curvature r is constant.*

We know that [4], in a Ricci-semisymmetric three-dimensional non-cosymplectic quasi-Sasakian manifold the structure function β is constant. Hence from Theorem 4.1 we can state the following:

Corollary 4.1. *A Ricci-semisymmetric three-dimensional non-cosymplectic quasi-Sasakian manifold is locally ϕ-symmetric if and only if the scalar curvature is constant.*

§5. η-parallel Ricci tensor

Definition 5.1. The Ricci tensor S of a quasi-Sasakian manifold is called η-parallel if it satisfies

$$(\nabla_X S)(\phi Y, \phi Z) = 0,$$

for all vector fields X, Y, Z. The notion of η-parallel Ricci tensor for Sasakian manifolds was introduced by Kon[7].

From (3.7) we get

$$\begin{aligned}
(5.1) \qquad (\nabla_X S)(\phi Y, \phi Z) \;=\;& (\frac{1}{2}Xr - 2\beta X\beta)[g(Y,Z) - \eta(Y)\eta(Z)] \\
&- \beta\{g(X,Y) - \eta(X)\eta(Y)\}d\beta(Z) \\
&- \beta\{g(X,Z) - \eta(X)\eta(Z)\}d\beta(Y).
\end{aligned}$$

If the Ricci tensor is η-parallel, then

$$(\frac{1}{2}Xr - 2\beta X\beta)[g(Y,Z) - \eta(Y)\eta(Z)] \\ - \beta\{g(X,Y) - \eta(X)\eta(Y)\}d\beta(Z) \\ - \beta\{g(X,Z) - \eta(X)\eta(Z)\}d\beta(Y) = 0. \tag{5.2}$$

In the above equation putting $Y = Z = e_i$, where $\{e_i\}$ is an orthonormal basis such that $e_3 = \xi$, and taking summation over i, $1 \leq i \leq 3$, we get

$$Xr - 6\beta X\beta = 0. \tag{5.3}$$

Also, we have $Yr - 10\beta Y\beta = 0$ from (5.2) and $\xi r = 0$. By virtue of these equations, we find the scalar curvature is constant. Moreover, we get β is constant if $\beta \neq 0$. Thus a non cosymplectic quasi-Sasakian manifold M^3 with η-parallel Ricci tensor is an η-Einstein manifold.

Conversely, if the quasi-Sasakian manifold M^3 is an η-Einstein, then

$$(\nabla_X S)(\phi Y, \phi Z) = 0.$$

Thus we can state the following:

Theorem 5.1. *In a non-cosymplectic quasi-Sasakian manifold M^3, the Ricci tensor is η-parallel if and only if M^3 is η-Einstein.*

From Theorems 4.1 and 5.1, we can state the following:

Corollary 5.1. *In a non-cosymplectic quasi-Sasakian manifold M^3, if the Ricci tensor is η-parallel, then it is locally ϕ-symmetric.*

§6. Cyclic parallel Ricci tensor

A Gray [5] introduced two classes of Riemannian manifolds determined by the covariant derivative of the Ricci tensor. The first one is the class $\mathcal{A}$ consisting of all Riemannian manifolds whose Ricci tensor S is a Codazzi tensor, that is,

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z).$$

The second one is the class $\mathcal{B}$ consisting of all Riemannian manifolds whose Ricci tensor is cyclic parallel, that is,

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0.$$

Again it is known that the Ricci tensor of Cartan hypersurface [6] is cyclic parallel. We find

$$\begin{aligned}
&(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) \\
= &(\frac{1}{2}Xr - 2\beta X\beta)g(Y,Z) + (8\beta X\beta - \frac{1}{2}Xr)\eta(Y)\eta(Z) \\
&-\beta(3\beta^2 - \frac{r}{2})\{\eta(Y)g(\phi X,Z) + \eta(Z)g(\phi X,Y)\} \\
&+\beta\{g(\phi X,Y)d\beta(\phi Z) + g(\phi X,Z)d\beta(\phi Y)\} \\
&-\eta(Y)g(\nabla_X \mathrm{grad}\beta, \phi Z) - \eta(Z)g(\nabla_X \mathrm{grad}\beta, \phi Y) \\
&+(\frac{1}{2}Yr - 2\beta Y\beta)g(Z,X) + (8\beta Y\beta - \frac{1}{2}Yr)\eta(Z)\eta(X) \\
&-\beta(3\beta^2 - \frac{r}{2})\{\eta(Z)g(\phi Y,X) + \eta(X)g(\phi Y,Z)\} \\
&+\beta\{g(\phi Y,Z)d\beta(\phi X) + g(\phi Y,X)d\beta(\phi Z)\} \\
&-\eta(Z)g(\nabla_Y \mathrm{grad}\beta, \phi X) - \eta(X)g(\nabla_Y \mathrm{grad}\beta, \phi Z) \\
&+(12Zr - 2\beta Z\beta)g(X,Y) + (8\beta Z\beta - \frac{1}{2}Zr)\eta(X)\eta(Y) \\
&-\beta(3\beta^2 - \frac{r}{2})\{\eta(X)g(\phi Z,Y) + \eta(Y)g(\phi Z,X)\} \\
&+\beta\{g(\phi Z,X)d\beta(\phi Y) + g(\phi Z,Y)d\beta(\phi X)\} \\
&-\eta(X)g(\nabla_Z \mathrm{grad}\beta, \phi Y) - \eta(Y)g(\nabla_Z \mathrm{grad}\beta, \phi X).
\end{aligned}$$

If the Ricci tensor is cyclic parallel, then we obtain

$$\begin{aligned}
(6.1)\qquad &(\frac{1}{2}Xr - 2\beta X\beta)g(Y,Z) + (8\beta X\beta - \frac{1}{2}Xr)\eta(Y)\eta(Z) \\
- &\beta(3\beta^2 - \frac{r}{2})\{\eta(Y)g(\phi X,Z) + \eta(Z)g(\phi X,Y)\} \\
+ &\beta\{g(\phi X,Y)d\beta(\phi Z) + g(\phi X,Z)d\beta(\phi Y)\} \\
- &\eta(Y)g(\nabla_X \mathrm{grad}\beta, \phi Z) - \eta(Z)g(\nabla_X \mathrm{grad}\beta, \phi Y) \\
+ &(\frac{1}{2}Yr - 2\beta Y\beta)g(Z,X) + (8\beta Y\beta - \frac{1}{2}Yr)\eta(Z)\eta(X) \\
- &\beta(3\beta^2 - \frac{r}{2})\{\eta(Z)g(\phi Y,X) + \eta(X)g(\phi Y,Z)\} \\
+ &\beta\{g(\phi Y,Z)d\beta(\phi X) + g(\phi Y,X)d\beta(\phi Z)\} \\
- &\eta(Z)g(\nabla_Y \mathrm{grad}\beta, \phi X) - \eta(X)g(\nabla_Y \mathrm{grad}\beta, \phi Z) \\
+ &(12Zr - 2\beta Z\beta)g(X,Y) + (8\beta Z\beta - \frac{1}{2}Zr)\eta(X)\eta(Y) \\
- &\beta(3\beta^2 - \frac{r}{2})\{\eta(X)g(\phi Z,Y) + \eta(Y)g(\phi Z,X)\} \\
+ &\beta\{g(\phi Z,X)d\beta(\phi Y) + g(\phi Z,Y)d\beta(\phi X)\} \\
- &\eta(X)g(\nabla_Z \mathrm{grad}\beta, \phi Y) - \eta(Y)g(\nabla_Z \mathrm{grad}\beta, \phi X) = 0.
\end{aligned}$$

Putting $Z=\xi$, we get from above

$$
\begin{aligned}
(6.2)\qquad & 6\beta\{(X\beta)\eta(Y)+(Y\beta)\eta(X)\}+\frac{1}{2}(\xi r)\{g(X,Y)-\eta(X)\eta(Y)\}\\
- \ & g(\nabla_X \mathrm{grad}\beta,\phi Y)-g(\nabla_Y \mathrm{grad}\beta,\phi X)\\
- \ & \eta(X)g(\nabla_\xi \mathrm{grad}\beta,\phi Y)-\eta(Y)g(\nabla_\xi \mathrm{grad}\beta,\phi X)=0.
\end{aligned}
$$

In the above equation putting $X=Y=e_i$ and taking summation over i, we get

$$
\xi r-2\sum_{i=1}^{3} g(\nabla_{e_i}\mathrm{grad}\beta,\phi e_i)=0. \tag{6.3}
$$

Also putting $Y=\xi$ in (6.2), we have

$$
3\beta X\beta-g(\nabla_\xi \mathrm{grad}\beta,\phi X)=0. \tag{6.4}
$$

In (6.1), putting $Y=Z=e_i$ and taking summation over i, we get from (6.3) and (6.4)

$$
Xr-\eta(X)\xi r-4\beta X\beta=0.
$$

When the scalar curvature r is constant, the structure function β so is, if $\beta\neq 0$. Conversely, if β is constant, then r is constant from (6.3). Thus we are in a position to state the following:

Theorem 6.1. *In a non-cosymplectic quasi-Sasakian manifold M^3 with cyclic parallel Ricci tensor, the scalar curvature r is constant if and only if the structure function β is constant.*

§7. Example

In this section we like to construct an example of a three-dimensional locally ϕ-symmetric quasi-Sasakian manifold.

Let us consider the three-dimensional manifold $M=\{(x,y,z)\in R^3,(x,y,z)\neq(0,0,0)\}$, where (x,y,z) are the standard coordinates in R^3. The vector fields

$$
e_1=\frac{\partial}{\partial x}-y\frac{\partial}{\partial z},\quad e_2=\frac{\partial}{\partial y},\quad e_3=\frac{\partial}{\partial z}
$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$
g(e_1,e_3)=g(e_2,e_3)=g(e_1,e_2)=0,\quad g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_3)=1.
$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any Z belongs to $\chi(M)$. Let ϕ be the $(1,1)$ tensor field defined by $\phi e_1 = -e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0$. Then using the linearity of ϕ and g we have

$$\eta(e_3) = 1, \quad \phi^2 Z = -Z + \eta(Z)e_3, \quad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$. Thus for $e_3 = \xi$, $M(\phi, \xi, \eta, g)$ defines an almost contact metric manifold.

Let ∇ be the Levi-Civita connection with respect to the Riemannian metric g and R be the curvature tensor of the manifold. Then we have

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = 0.$$

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad + g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y), \end{aligned}$$

which is known as Koszul's formula. Taking $e_3 = \xi$ and using the above formula for Riemannian metric g, it can be easily calculated that

$$\begin{array}{lll} \nabla_{e_1} e_3 = -\frac{1}{2} e_2, & \nabla_{e_1} e_2 = \frac{1}{2} e_3, & \nabla_{e_1} e_1 = 0, \\ \nabla_{e_2} e_3 = \frac{1}{2} e_1, & \nabla_{e_2} e_2 = 0, & \nabla_{e_2} e_1 = -\frac{1}{2} e_3, \\ \nabla_{e_3} e_3 = 0, & \nabla_{e_3} e_2 = \frac{1}{2} e_1, & \nabla_{e_3} e_1 = -\frac{1}{2} e_2. \end{array}$$

We see that the (ϕ, ξ, η, g) structure satisfies the formula $\nabla_X \xi = -\beta \phi X$. Hence $M(\phi, \xi, \eta, g)$ is a three-dimensional quasi-Sasakian manifold with the structure function $\beta = -\frac{1}{2}$. Using the above relations we obtain the components of the curvature tensor as follows.

$$\begin{array}{lll} R(e_1, e_2)e_3 = 0, & R(e_2, e_3)e_3 = \frac{1}{4} e_2, & R(e_1, e_3)e_3 = \frac{1}{4} e_1, \\ R(e_1, e_2)e_2 = -\frac{3}{4} e_1, & R(e_2, e_3)e_2 = -\frac{1}{4} e_3, & R(e_1, e_3)e_2 = 0, \\ R(e_1, e_2)e_1 = \frac{3}{4} e_2, & R(e_2, e_3)e_1 = 0, & R(e_1, e_3)e_1 = -\frac{1}{4} e_3. \end{array}$$

From

$$(\nabla_{e_1} R)(e_1, e_2)e_1 = (\nabla_{e_2} R)(e_1, e_2)e_2 = \frac{1}{2} e_3,$$

and

$$(\nabla_{e_2} R)(e_1, e_2)e_1 = (\nabla_{e_1} R)(e_1, e_2)e_2 = 0,$$

it follows that M is locally ϕ-symmetric.

Now we see that

$$S(e_1, e_1) = g(R(e_1, e_2)e_2, e_1) + g(R(e_1, e_3)e_3, e_1) = -\frac{1}{2},$$

$$S(e_2, e_2) = g(R(e_2, e_1)e_1, e_2) + g(R(e_2, e_3)e_3, e_2) = -\frac{1}{2},$$

$$S(e_3, e_3) = g(R(e_3, e_1)e_1, e_3) + g(R(e_3, e_2)e_2, e_3) = \frac{1}{2},$$

and

$$S(e_i, e_j) = 0, (i \neq j).$$

Therefore the scalar curvature $r = -\frac{1}{2}$.

Also, because of $(\nabla_{e_2} S)(e_1, e_3) = -(\nabla_{e_1} S)(e_2, e_3) = -\frac{1}{2}$ and otherwise is zero, the Ricci tensor of M is η-parallel and cyclic parallel.

Acknowledgement. The authors are thankful to the referee for his valuable suggestions in the improvement of the paper.

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