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Neumann problem for a nonlinear nonlocal equation on a half-line

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Abstract. Our goal is to study the global existence and large time asymptotic behavior of solutions to the Neumann initial-boundary value problem for the nonlinear nonlocal equation on a half-line

$$\begin{cases} u_t + \mathcal{N}(u, u_x) + \mathcal{L}u = f, & (t, x) \in \mathbf{R}^+ \times \mathbf{R}^+, \\ u(0, x) = u_0(x), & x \in \mathbf{R}^+, \\ \partial_x u(t, 0) = h(t), & t \in \mathbf{R}^+, \end{cases}$$

where the nonlinear term is $\mathcal{N}(u, u_x) = u^\rho u_x^\sigma$, with $\rho, \sigma > 0$, and $\mathcal{L}$ is a pseudodifferential operator defined by the inverse Laplace transform

$$\mathcal{L}u = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{px} E p^{\frac{5}{2}} \left(\hat{u}(t, p) - \left(\frac{u(t, 0)}{p} + \frac{\partial_x u(t, 0)}{p^2} \right) \right) dp$$

where $\hat{u}(p) = \int_0^\infty e^{-px} u(x) dx$. We prove that if the initial data $u_0 \in \mathbf{L}^{1,a} \cap \mathbf{L}^\infty$ for $a \in [0, 1)$, then there exists a unique solution

$$u \in \mathbf{C}\left([0, \infty); \mathbf{L}^{1,a}\right) \cap \mathbf{C}\left((0, \infty); \mathbf{L}^\infty \cap \mathbf{W}^1_\infty\right),$$

for the inital-boundary value problem. We also obtain the large time asymptotic formulas for the solutions...

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§1. Introduction

We study the Neumann initial-boundary value problem for the nonlinear Ott-Sudan-Ostrovsky type equation on a half-line

$$\begin{cases} u_t + \mathcal{N}(u, u_x) + \mathcal{L}u = f, & t > 0, x > 0, \\ u(0, x) = u_0(x), & x > 0, \\ u_x(t, 0) = h(t), & t > 0, \end{cases} \tag{1.1}$$

where the nonlinear term is $\mathcal{N}(u, u_x) = u^\rho u_x^\sigma$ with ρ, $\sigma > 0$, $\mathcal{L}$ is a pseudodifferential operator defined by the inverse Laplace transformation as follows

$$\mathcal{L}u = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px} L(p) \left(\widehat{u}(t,p) - \textstyle\sum_{j=1}^{2} p^{-j}\, \partial_x^{j-1} u(t,x)\big|_{x=0}\right) dp,$$

where $\widehat{u}(p) = \int_0^\infty e^{-px} u(x)\, dx$ denotes the Laplace transform of u. The symbol $L(p) = -Ep^{5/2}$ has the dissipative property, i.e. the constant $E \in \mathbb{C}$ is such that $\operatorname{Re} L(p) \geq C|p|^{5/2}$ for $p \in (-i\infty, i\infty)$, where $C > 0$. For example, the constant E can be chosen as follows $E = 1$. By $p^{5/2}$ we understand the main branch of the complex analytic function in the complex half-plane $\operatorname{Re} p \geq 0$, so that $1^{5/2} = 1$ (we make a cut along the negative real axis $(-\infty, 0)$). More generally, we can fix the argument of E as follows $-\frac{\pi}{4} < \arg E < \frac{\pi}{4}$.

Note that the equation

$$u_t + uu_x + u_{xxx} + u + \int_{-\infty}^{\infty} \frac{\operatorname{sign}(x-y)}{\sqrt{|x-y|}} u_y(t,y)\, dy = 0$$

describes the ion-acoustic waves in plasma (see [11]). It also can be written in the form (1.1) if we choose $\rho = \sigma = 1$ and the symbol of the linear operator $L(p) = p^3 + 1 + \sqrt{|p|}$. This symbol is nonhomogeneous and nonanalytic so that it is difficult to investigate the Neumann initial-boundary value problem for this equation. To make the first step in the study of the Neumann initial-boundary value problem we replace the complete symbol by a homogeneous and analytic one of a higher order (if, for example, we replace the complete symbol by a single term like $\sqrt{p}$, then no boundary data are necessary, see [2].) Thus we arrive to equation (1.1) which represents a simple nonlinear model including a derivative of a fractional order and such that it is possible to pose the Neumann type boundary condition.

Recently much attention was given to the study of the global existence and asymptotic behavior of solutions to the Dirichlet problems for various nonlinear local and nonlocal equations (see papers [1], [3], [4], [5], [7], [10]). Dirichlet problem for Ott-Sudan-Ostrovsky type equations on a segment with homogeneous boundary data was studied in papers [9], [8]. A general theory of Dirichlet problems for nonlinear nonlocal equations on a half-line was developed in book [2]. This paper presents a further development of this theory, considering the Neumann type boundary conditions. We propose a general method of constructing the Green operator for problem (1.1). Also we prove the global existence of solutions and find the large time asymptotics in the case of nonhomogeneous boundary data. The main difficulty which we overcome in the present paper is to evaluate the contribution of the boundary data into the large time asymptotic behavior of solutions since it can be completely different compared with the corresponding Dirichlet problem.

Now we give some definitions. The usual Lebesgue space is $\mathbf{L}^r(\mathbf{R}^+)$ for $1 \le r \le \infty$. In what follows we write $\mathbf{L}^r$ instead of $\mathbf{L}^r(\mathbf{R}^+)$, for simplicity. The weighted Lebesgue space $\mathbf{L}^{r,\alpha}$ for $\alpha \ge 0$ has the norm $\|\phi\|_{\mathbf{L}^{r,\alpha}} = \|(1+x)^{\alpha}\phi\|_{\mathbf{L}^r}$. The Sobolev space is defined as follows

$$\mathbf{W}_r^s = \left\{\phi \in \mathbf{S}' \colon \ \|\phi\|_{\mathbf{W}_r^s} = \textstyle\sum_{k=0}^s \left\|\partial_x^k \phi\right\|_{\mathbf{L}^r} < \infty\right\}.$$

Different positive constants we denote by the same letter C.

We now introduce the space for the solution

$$\mathbf{X}^{\alpha,\beta} = \left\{\phi \in \mathbf{C}\left([0,\infty);\mathbf{L}^{1,\alpha}\right) \cap \mathbf{C}\left((0,\infty);\mathbf{W}_\infty^1\right) : \|\phi\|_{\mathbf{X}^{\alpha,\beta}} < \infty\right\}$$

with the norm

$$\|\phi\|_{\mathbf{X}^{\alpha,\beta}} = \sup_{t>0} \langle t\rangle^{-\beta}\left(\langle t\rangle^{-\frac{2}{5}\alpha}\|\phi\|_{\mathbf{L}^{1,\alpha}} + \langle t\rangle^{\frac{2}{5}}\|\phi\|_{\mathbf{L}^\infty} + \langle t\rangle^{\frac{2}{5}} t^{\frac{2}{5}}\|\partial_x\phi\|_{\mathbf{L}^\infty}\right),$$

where $\alpha \in [0,1)$, $\beta \in \mathbf{R}$, with $\langle t\rangle \equiv \sqrt{1+t^2}$. Define also the spaces

$$\mathbf{Y}_\gamma^{\alpha,\lambda} = \left\{f \in \mathbf{C}\left((0,\infty);\mathbf{L}^{1,\alpha}\right) : \|f\|_{\mathbf{Y}_\gamma^{\alpha,\lambda}} < \infty\right\}$$

for the source function in the problem (1.1) with the norm

$$\|f\|_{\mathbf{Y}_\gamma^{\alpha,\lambda}} = \sup_{t>0} \langle t\rangle^{\gamma-\lambda} t^{1-\gamma}\left(\langle t\rangle^{-\frac{2}{5}\alpha}\|f\|_{\mathbf{L}^{1,\alpha}} + \|f\|_{\mathbf{L}^1} + \langle t\rangle^{\frac{2}{5}}\|f\|_{\mathbf{L}^\infty}\right),$$

where $\alpha \in [0,1)$, $\gamma \equiv 1 - \frac{2}{5}\sigma \in \left(\frac{2}{5},1\right]$, $\lambda \in \mathbf{R}$. Also we use the space $\mathbf{Z}^\mu = \{h(t) \in \mathbf{C}(0,\infty) : \|h\|_{\mathbf{Z}^\mu} < \infty\}$ for the boundary data, where the norm

$$\|h\|_{\mathbf{Z}^\mu} = \sup_{t>0} \langle t\rangle^{\frac{4}{5}-\mu}\left(|h(t)| + \langle t\rangle \left|h'(t)\right|\right)$$

with $\mu \in \mathbf{R}$.

We introduce the function

$$B(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px+Ep^{5/2}}dp + \frac{ME}{4\pi^2}\int_{-i\infty}^{i\infty} dp\, p^{3/2}e^{px}\int_{-i\infty}^{i\infty}\frac{q^{-\frac{3}{5}}e^q dq}{q - Ep^{5/2}},$$

where $M = |E|^{-\frac{2}{5}} e^{-\frac{2}{5}i\arg E}$.

First we consider the case, when the asymptotic behavior of solutions is determined by the initial data.

Theorem 1. *Let $u_0 \in \mathbf{L}^{1,\alpha} \cap \mathbf{L}^\infty$, $f \in \mathbf{Y}_\gamma^{\alpha,\nu}$ and $h \in \mathbf{Z}^\mu$ with $\alpha \in (0,1)$, $-\frac{2}{5} < \nu < 0$, $-\frac{1}{5} < \mu < 0$, $\gamma = 1 - \frac{2}{5}\sigma$ and have sufficiently small norms. Suppose that the powers $\rho \ge 1$, $\sigma \in \left[1,\frac{3}{2}\right)$ of the nonlinearity $\mathcal{N}$ are such that*

$\rho+2\sigma>\frac{7}{2}$. *Then there exists a unique solution* $u\in\mathbf{X}^{\alpha,0}$ *of problem (1.1). Moreover the following asymptotic representation*

$$u\left(t,x\right)=t^{-\frac{2}{5}}AB\left(xt^{-\frac{2}{5}}\right)+O\left(t^{-\frac{2}{5}-\delta}\right),$$

is true for $t\to\infty$ *uniformly with respect to* $x>0$, *where the coefficient*

$$A=\int_0^\infty u_0\left(y\right)dy+\int_0^\infty\int_0^\infty\left(f\left(\tau,y\right)-\mathcal{N}\left(u,u_x\right)\right)dyd\tau$$

and $\delta>0$ *is some small constant.*

Next we consider the case, when the asymptotic behavior of solutions is defined by a more slow time decay rate of the source or the Neumann boundary data. We introduce two functions

$$\Phi\left(\chi\right)=\int_0^1 z^{\beta-1}\left(1-z\right)^{-\frac{2}{5}}B\left(\chi\left(1-z\right)^{-\frac{2}{5}}\right)dz$$

and

$$\Psi\left(\chi\right)=\int_0^1 z^{\beta-\frac{4}{5}}\left(1-z\right)^{-\frac{3}{5}}K\left(\chi\left(1-z\right)^{-\frac{2}{5}}\right)dz,$$

where the kernel $K\left(x\right)=W\left(x\right)+\Lambda\left(x\right)$ with the functions

$$W\left(x\right)=-\frac{E}{2\pi i}\int_{-i\infty}^{i\infty}e^{px+Ep^{5/2}}p^{\frac{1}{2}}dp$$

and

$$\Lambda\left(x\right)=-\frac{E}{4\pi^2M}\int_{-i\infty}^{i\infty}dpe^{xp}p^{3/2}\int_{-i\infty}^{i\infty}\frac{\xi^{-\frac{2}{5}}e^{\xi}d\xi}{\xi-Ep^{5/2}}.$$

Theorem 2. *Let* $u_0\in\mathbf{L}^1\cap\mathbf{L}^\infty$, $h\in\mathbf{Z}^\beta$ *and* $f\in\mathbf{Y}_\gamma^{0,\beta}$ *with some* $\beta\in\left(0,\frac{2}{5}\right)$, $\gamma=1-\frac{2}{5}\sigma$ *and have sufficiently small norms. Suppose that the powers* $\rho\geq1$, $\sigma\in\left[1,\frac{3}{2}\right)$ *of the nonlinearity* $\mathcal{N}$ *are such that* $\rho+\sigma>\frac{5-2\sigma}{2-5\beta}+1$. *Then there exists a unique solution* $u\in\mathbf{X}^{0,\beta}$ *of problem (1.1). Moreover, assume that* $\left\|f\left(t\right)\right\|_{\mathbf{L}^{1,\alpha}}\leq Ct^{\beta-1}$ *with* $\alpha\in\left(0,1\right)$, *and the asymptotics for the source function and the boundary data are true*

$$\int_0^\infty f\left(t,x\right)dx=at^{\beta-1}+O\left(t^{\beta-1-\delta}\right)$$

and

$$h\left(t\right)=bt^{\beta-\frac{4}{5}}+O\left(t^{\beta-\frac{4}{5}-\delta}\right)$$

for all $t\to\infty$, *where* $0<\delta<\min\left(\beta,\frac{1}{5},\frac{2}{5}\alpha\right)$. *Then the following asymptotics*

$$u\left(t,x\right)=t^{\beta-\frac{2}{5}}\left(a\Phi\left(xt^{-\frac{2}{5}}\right)+b\Psi\left(xt^{-\frac{2}{5}}\right)\right)+O\left(t^{\beta-\frac{2}{5}-\delta}\right)$$

is true for $t\to\infty$ *uniformly with respect to* $x>0$, *where* $\delta>0$ *is some small constant.*

We organize the rest of the paper as follows. In Section 2 we consider the linear initial-boundary value problem with the pseudodifferential operator $\mathcal{L}$ and find the representation of the Green operators. Also we obtain some preliminary estimates and asymptotic formulas for the Green functions. Finally, we prove Theorems 1 and 2 in Sections 3 and 4, respectively.

§2. Estimates for the linear problem

In this section we study the linear problem corresponding to (1.1)

$$\begin{cases} u_t + \mathcal{L}u = f, \quad t > 0, x > 0, \\ u\left(0,x\right) = u_0\left(x\right), \quad x > 0, \\ u_x\left(t,0\right) = h\left(t\right), \quad t > 0. \end{cases} \tag{2.1}$$

We follow the method of [2] to show that the initial-boundary value problem (2.1) is well-posed in the functional space $\mathbf{C}\left([0,\infty);\mathbf{L}^1\right) \cap \mathbf{C}^1\left((0,\infty);\mathbf{W}^1_\infty\right)$. First under the supposition that there exists a solution $u \in \mathbf{C}\left([0,\infty);\mathbf{L}^1\right) \cap \mathbf{C}^1\left((0,\infty);\mathbf{W}^1_\infty\right)$ of problem (2.1) we obtain its integral representation. Since the solution $u\left(t\right) \in \mathbf{L}^1$ for every $t > 0$ and $u\left(t,x\right) = 0$ for all $x < 0$, $t > 0$, we see that the Laplace transform $\widehat{u}(t,p) = \int_0^\infty e^{-px} u\left(t,x\right) dx$ is bounded and analytic in the complex half-plane $\{p \in \mathbb{C} : \operatorname{Re} p \geq 0\}$. Taking the Laplace transformation of (2.1) with respect to x we get

$$\begin{cases} \widehat{u}_t\left(t,p\right) + L(p)\widehat{u}\left(t,p\right) = \widehat{H}\left(t,p\right) \\ \widehat{u}\left(t,p\right)|_{t=0} = \widehat{u_0}(p), \end{cases} \tag{2.2}$$

where $\widehat{H}\left(t,p\right) = \widehat{f}\left(t,p\right) + L\left(p\right)p^{-1}u\left(t,0\right) + L\left(p\right)p^{-2}u_x\left(t,0\right)$. Integration of (2.2) with respect to t yields

$$\widehat{u}\left(t,p\right) = e^{-L(p)t}\left(\widehat{u_0}(p) + \int_0^t e^{L(p)\tau}\widehat{H}\left(\tau,p\right) d\tau\right). \tag{2.3}$$

Before applying the inverse Laplace transformation to (2.3) we must check the necessary condition

$$\left|\widehat{u}\left(t,p\right)\right| \leq C \quad \text{for all} \quad \operatorname{Re} p \geq 0 \tag{2.4}$$

with some C. Rewriting (2.3) as

$$\widehat{u}\left(t,p\right) = e^{-L(p)t}\left(\widehat{u_0}(p) + \int_0^\infty e^{L(p)\tau}\widehat{H}\left(\tau,p\right) d\tau\right) - \int_t^\infty e^{-L(p)(t-\tau)}\widehat{H}\left(\tau,p\right) d\tau,$$

we see that to satisfy (2.4) we need to impose the following necessary condition

$$\widehat{u}_0(p) + \int_0^\infty e^{L(p)\tau}\widehat{H}\left(\tau,p\right) d\tau = 0 \tag{2.5}$$

in the domain $\{p \in \mathbb{C} : \operatorname{Re} L(p) < 0, \operatorname{Re} p \geq 0\}$. Equation (2.5) helps us to find the boundary value $u(t,0)$ of the function u, which appears in the definition of the pseudodifferential operator $\mathcal{L}$. Changing the variable $L(p) \equiv -Ep^{5/2} = -\xi$, so that $p = M\xi^{\frac{2}{5}}$ with $M = |E|^{-\frac{2}{5}} e^{-\frac{2}{5} i \arg E}$, we rewrite (2.5) as

$$\widehat{u_0}\left(M\xi^{\frac{2}{5}}\right) + \widetilde{\widehat{f}}\left(\xi, M\xi^{\frac{2}{5}}\right) - M^{-2}\xi^{\frac{1}{5}}\widetilde{u_x}(\xi,0) - M^{-1}\xi^{\frac{3}{5}}\widetilde{u}(\xi,0) = 0 \tag{2.6}$$

in the complex half-plane $\operatorname{Re}\xi > 0$, where $\widetilde{u}(\xi,x) = \int_0^\infty e^{-\xi\tau}u(\tau,x)\,d\tau$ is the Laplace transform with respect to time and $\widetilde{\widehat{f}}$ is the Laplace transform with respect to both space and time variables $\widetilde{\widehat{f}}(\xi,\eta) = \int_0^\infty \int_0^\infty e^{-\xi\tau - \eta x} f(\tau,x)\,dx d\tau$. Then by applying to (2.6) the inverse Laplace transformation with respect to time we get

$$u(t,0) = \frac{M}{2\pi i}\int_{-i\infty}^{i\infty} e^{\xi t}\xi^{-\frac{3}{5}}\left(\widehat{u_0}\left(M\xi^{\frac{2}{5}}\right) + \widetilde{\widehat{f}}\left(\xi, M\xi^{\frac{2}{5}}\right) - M^{-2}\xi^{\frac{1}{5}}\widetilde{u_x}(\xi,0)\right)d\xi. \tag{2.7}$$

Substitution of (2.7) into (2.3) yields

$$\begin{aligned}
\widehat{u}(t,p) &= e^{-L(p)t}\int_0^\infty e^{-py}u_0(y)\,dy + L(p)\,p^{-2}\int_0^t d\tau e^{-L(p)(t-\tau)}h(\tau) \\
&\quad + \int_0^t d\tau e^{-L(p)(t-\tau)}\int_0^\infty e^{-py}f(\tau,y)\,dy \\
&\quad + \frac{ML(p)}{2\pi i p}\int_0^t d\tau e^{-L(p)(t-\tau)}\int_{-i\infty}^{i\infty} d\xi e^{\xi\tau}\xi^{-\frac{3}{5}}\int_0^\infty e^{-M\xi^{\frac{2}{5}}y}u_0(y)\,dy \\
&\quad - \frac{L(p)}{2\pi i p M}\int_0^t d\tau e^{-L(p)(t-\tau)}\int_{-i\infty}^{i\infty} d\xi e^{\xi\tau}\xi^{-\frac{2}{5}}\int_0^\infty e^{-\xi\tau'}h(\tau')\,d\tau' \\
&\quad + \frac{ML(p)}{2\pi i p}\int_0^t d\tau e^{-L(p)(t-\tau)}\int_{-i\infty}^{i\infty} d\xi e^{\xi\tau}\xi^{-\frac{3}{5}} \\
&\quad \times \int_0^\infty\int_0^\infty e^{-\xi\tau' - M\xi^{\frac{2}{5}}y}f(\tau',y)\,dy d\tau'.
\end{aligned}$$

Then taking the inverse Laplace transformation with respect to space, we obtain the Duhamel formula for the solutions to problem (2.1) (see also [1])

$$u = \mathcal{G}u_0 + \mathcal{I}f + \mathcal{J}h, \tag{2.8}$$

where

$$\begin{aligned}
\mathcal{G}u_0 &= \int_0^\infty dy u_0(y)\left(\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp e^{p(x-y)-L(p)t}\right. \\
&\quad \left. - \frac{M}{4\pi^2}\int_{-i\infty}^{i\infty} dp e^{px}\frac{L(p)}{p}\int_0^t d\tau e^{-L(p)(t-\tau)}\int_{-i\infty}^{i\infty} d\xi \xi^{-\frac{3}{5}}e^{\xi\tau - M\xi^{\frac{2}{5}}y}\right),
\end{aligned}$$

$$\begin{aligned}\mathcal{I}f &= \int_0^t d\tau \int_0^\infty dy f(\tau, y) \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dp e^{p(x-y)-L(p)(t-\tau)} \\ &\quad - \frac{M}{4\pi^2} \int_{-i\infty}^{i\infty} dp e^{px} \frac{L(p)}{p} \int_0^t d\tau' e^{-L(p)(t-\tau')} \int_{-i\infty}^{i\infty} d\xi \xi^{-\frac{3}{5}} \\ &\quad \times \int_0^\infty dy \int_0^\infty d\tau e^{\xi(\tau'-\tau)-M\xi^{\frac{2}{5}}y} f(\tau, y)\end{aligned}$$

and

$$\begin{aligned}\mathcal{J}h &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dp e^{px} L(p) p^{-2} \int_0^t d\tau e^{-L(p)(t-\tau)} h(\tau) \\ &\quad + \frac{1}{4\pi^2 M} \int_{-i\infty}^{i\infty} dp e^{px} \frac{L(p)}{p} \int_0^t d\tau e^{-L(p)(t-\tau)} \\ &\quad \times \int_{-i\infty}^{i\infty} d\xi \xi^{-\frac{2}{5}} \int_0^\infty d\tau' e^{\xi(\tau-\tau')} h(\tau').\end{aligned}$$

Note that

$$\int_0^t e^{-L(p)(t-\tau)+\xi\tau} d\tau = \frac{e^{\xi t} - e^{-L(p)t}}{\xi + L(p)}$$

and

$$\int_{-i\infty}^{i\infty} d\xi \xi^{-\frac{3}{5}} (\xi + L(p))^{-1} e^{-M\xi^{\frac{2}{5}}y} = 0.$$

Denote $G(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{px+Ep^{5/2}} dp$, and

$$Q(x, y) = \frac{ME}{4\pi^2} \int_{-i\infty}^{i\infty} dp e^{xp} p^{3/2} \int_{-i\infty}^{i\infty} d\xi \frac{e^{\xi - M\xi^{\frac{2}{5}}y}}{\xi - Ep^{5/2}} \xi^{-\frac{3}{5}}.$$

Therefore we can transform

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dp e^{p(x-y)-L(p)t} = t^{-\frac{2}{5}} G\left((x-y) t^{-\frac{2}{5}}\right)$$

and

$$\begin{aligned}&-\frac{M}{4\pi^2} \int_{-i\infty}^{i\infty} dp e^{px} \frac{L(p)}{p} \int_0^t d\tau e^{-L(p)(t-\tau)} \int_{-i\infty}^{i\infty} d\xi \xi^{-\frac{3}{5}} e^{\xi\tau - M\xi^{\frac{2}{5}}y} \\ &= \frac{ME}{4\pi^2} \int_{-i\infty}^{i\infty} dp e^{px} p^{3/2} \int_{-i\infty}^{i\infty} d\xi \frac{\xi^{-\frac{3}{5}} e^{\xi t - M\xi^{\frac{2}{5}}y}}{\xi - Ep^{5/2}} = t^{-\frac{2}{5}} Q\left(xt^{-\frac{2}{5}}, yt^{-\frac{2}{5}}\right).\end{aligned}$$

Then the Green operator $\mathcal{G}$ can be rewritten as

$$\mathcal{G}(t)\phi = t^{-\frac{2}{5}} \int_0^\infty \left(G\left((x-y)t^{-\frac{2}{5}}\right) + Q\left(xt^{-\frac{2}{5}}, yt^{-\frac{2}{5}}\right)\right) \phi(y) dy.$$

In the same manner since $\int_{-i\infty}^{i\infty} d\xi \frac{\xi^{-\frac{3}{5}} e^{-M\xi^{\frac{2}{5}} y+\xi(t-\tau)}}{\xi+L(p)} = 0$ for all $\tau > t$ and

$$\int_{-i\infty}^{i\infty} d\xi \xi^{-\frac{3}{5}} \left(\xi + L\left(p\right)\right) e^{-M\xi^{\frac{2}{5}} y - \xi\tau} = 0$$

for all $\tau > 0$ we get

$$\begin{aligned}
\mathcal{I}f &= \int_0^t d\tau \int_0^\infty dy f\left(\tau, y\right) \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dp e^{p(x-y)-L(p)(t-\tau)} \\
&\quad - \frac{M}{4\pi^2} \int_0^\infty d\tau \int_0^\infty dy f\left(\tau, y\right) \int_{-i\infty}^{i\infty} dp e^{px} \frac{L\left(p\right)}{p} \int_0^t d\tau' e^{-L(p)(t-\tau')} \\
&\quad \times \int_{-i\infty}^{i\infty} d\xi \xi^{-\frac{3}{5}} e^{\xi(\tau'-\tau) - M\xi^{\frac{2}{5}} y} \\
&= \int_0^t d\tau \int_0^\infty dy f\left(\tau, y\right) \left(\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dp e^{p(x-y)-L(p)(t-\tau)} \right. \\
&\quad \left. - \frac{M}{4\pi^2} \int_{-i\infty}^{i\infty} dp e^{px} \frac{L\left(p\right)}{p} \int_{-i\infty}^{i\infty} d\xi \frac{\xi^{-\frac{3}{5}} e^{\xi(t-\tau) - M\xi^{\frac{2}{5}} y}}{\xi + L\left(p\right)} \right) \\
&= \int_0^t \mathcal{G}\left(t-\tau\right) f\left(\tau\right) d\tau.
\end{aligned}$$

Finally by the equality $\int_{-i\infty}^{i\infty} d\xi \frac{\xi^{-\frac{2}{5}}}{\xi+L(p)} e^{-\xi\tau} = 0$ for all $\tau > 0$, we can rewrite the Green operator $\mathcal{J}$ as follows

$$\begin{aligned}
\mathcal{J}h &= -\frac{E}{2\pi i} \int_0^t d\tau h\left(\tau\right) \int_{-i\infty}^{i\infty} dp p^{\frac{1}{2}} e^{px + Ep^{5/2}(t-\tau)} \\
&\quad - \frac{E}{4\pi^2 M} \int_0^\infty d\tau h\left(\tau\right) \int_{-i\infty}^{i\infty} dp e^{px} p^{3/2} \\
&\quad \times \int_0^t d\tau' e^{-L(p)(t-\tau')} \int_{-i\infty}^{i\infty} d\xi \xi^{-\frac{2}{5}} e^{\xi(\tau'-\tau)} \\
&= -E \int_0^t d\tau h\left(\tau\right) \left(\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dp p^{\frac{1}{2}} e^{px + Ep^{5/2}(t-\tau)} \right. \\
&\quad \left. + \frac{1}{4\pi^2 M} \int_{-i\infty}^{i\infty} dp e^{px} p^{3/2} \int_{-i\infty}^{i\infty} d\xi \frac{\xi^{-\frac{2}{5}}}{\xi + L\left(p\right)} e^{\xi(t-\tau)} \right) \\
&= \int_0^t \left(t-\tau\right)^{-\frac{3}{5}} K\left(x\left(t-\tau\right)^{-\frac{2}{5}}\right) h\left(\tau\right) d\tau,
\end{aligned}$$

where we introduced the kernel $K\left(x\right) = W\left(x\right) + \Lambda\left(x\right)$ with the functions

$$W\left(x\right) = -\frac{E}{2\pi i} \int_{-i\infty}^{i\infty} e^{px + Ep^{5/2}} p^{\frac{1}{2}} dp$$

and

$$\Lambda(x) = -\frac{E}{4\pi^2 M}\int_{-i\infty}^{i\infty} dp e^{xp} p^{3/2} \int_{-i\infty}^{i\infty} \frac{\xi^{-\frac{2}{5}} e^{\xi} d\xi}{\xi - Ep^{5/2}}.$$

We can also represent

$$K'(x) = -\frac{E}{2\pi i}\int_{-i\infty}^{i\infty} e^{px+Ep^{5/2}} p^{3/2} dp - \frac{1}{4\pi^2 M}\int_{-i\infty}^{i\infty} dp e^{xp} \int_{\Gamma} \frac{\xi^{\frac{3}{5}} e^{\xi} d\xi}{\xi - Ep^{5/2}}$$

with

$$\Gamma = \left\{\xi = |\xi| e^{\pm\frac{i\pi}{2}\pm i\epsilon} : |\xi| \geq 1\right\} \cup \left\{\xi = e^{i\theta} : \theta \in \left[-\frac{\pi}{2} - \epsilon, \frac{\pi}{2} + \epsilon\right]\right\},$$

where $\epsilon > 0$ is small enough, from which it follows that $K'(0) = 0$. It is interesting to note that $W(x) = \partial_x^{\frac{1}{2}} G(x)$, since by our definition of the fractional derivative $\partial_x^{\frac{1}{2}} \phi(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{xp} p^{\frac{1}{2}} \widehat{\phi}(p)\, dp$.

In the next lemmas we will obtain estimates for the Green operators $\mathcal{G}$, $\mathcal{I}$ and $\mathcal{J}$, which then imply that formula (2.8) gives us a unique solution of the initial-boundary value problem (2.1) in the functional space $\mathbf{C}\left([0,\infty);\mathbf{L}^1\right) \cap \mathbf{C}^1\left((0,\infty);\mathbf{W}_\infty^1\right)$.

First we estimate the kernels $G(x)$, $Q(x,y)$ and $K(x)$.

Lemma 1. *The estimates are true*

$$\left|\partial_x^j G(x)\right| \leq C\langle x\rangle^{\varrho-2-j}$$

for all $x \in \mathbf{R}$,

$$\left|\partial_x^j K(x)\right| \leq C\langle x\rangle^{\varrho-2-j}$$

for all $x > 0$, *and*

$$\left|\partial_x^k \partial_y^l Q(x,y)\right| \leq C\langle x\rangle^{-2-k}\langle y\rangle^{-2}$$

for all $x, y > 0$, *where* $k, l = 0, 1$, $j = 0, 1, 2$, *with some small* $\varrho > 0$.

Proof. The first estimate of the lemma along with the inequality $\left|\partial_x^j W(x)\right| \leq C\langle x\rangle^{\varrho-2-j}$ for all $x \in \mathbf{R}$ are the consequences of estimate (1.45) from [6]. We next prove the estimate for the kernels $Q(x,y)$ and $\Lambda(x)$. We first consider the case of $x > 1$, $y > 0$. Changing the contour of integration with respect to p by

$$\Gamma_0 = \left\{p = |p| e^{\pm\frac{i\pi}{2}\pm i\epsilon} : |p| > 0\right\},$$

and with respect to ξ by

$$\Gamma = \left\{\xi = |\xi| e^{\pm\frac{i\pi}{2}\pm i\epsilon} : |\xi| \geq 1\right\} \cup \left\{\xi = e^{i\theta} : \theta \in \left[-\frac{\pi}{2} - \epsilon, \frac{\pi}{2} + \epsilon\right]\right\},$$

respectively, where $\epsilon > 0$ is small enough, we get

$$\partial_x^k \partial_y^l Q(x,y) = -\frac{E(-M)^{1+l}}{4\pi^2} \int_{\Gamma_0} dp e^{xp} p^{3/2+k} \int_{\Gamma} d\xi \xi^{\frac{2l-3}{5}} \frac{e^{\xi - M\xi^{2/5} y}}{\xi - Ep^{5/2}}$$

and

$$\partial_x^j \Lambda(x) = -\frac{E}{4\pi^2 M} \int_{\Gamma_0} dp e^{xp} p^{3/2+j} \int_{\Gamma} \frac{\xi^{-\frac{2}{5}} e^{\xi} d\xi}{\xi - Ep^{5/2}}.$$

For all $p \in \Gamma_0$, $\xi = |\xi| e^{i\theta}$, with $\theta \in \left[-\frac{\pi}{2} - \epsilon, 0\right]$, we have

$$\begin{aligned} & \left|\xi - Ep^{5/2}\right| = \left|\xi e^{2i\epsilon} - Ep^{5/2} e^{2i\epsilon}\right| \geq \operatorname{Re}\left(\xi e^{2i\epsilon} - Ep^{5/2} e^{2i\epsilon}\right) \\ = \; & |\xi| \cos(\theta + 2\epsilon) + |E| \, |p|^{5/2} \cos\left(\pm\frac{\pi}{4} \pm \frac{5}{2}\epsilon + 2\epsilon + \arg E\right) \\ \geq \; & C\left(|\xi| + |p|^{5/2}\right) \end{aligned}$$

and for all $p \in \Gamma_0$, $\xi = |\xi| e^{i\theta}$, with $\theta \in \left[0, \frac{\pi}{2} + \epsilon\right]$, we write

$$\begin{aligned} & \left|\xi - Ep^{5/2}\right| = \left|\xi e^{-2i\epsilon} - Ep^{5/2} e^{-2i\epsilon}\right| \geq \operatorname{Re}\left(\xi e^{-2i\epsilon} - Ep^{5/2} e^{-2i\epsilon}\right) \\ = \; & |\xi| \cos(\theta - 2\epsilon) + |E| \, |p|^{5/2} \cos\left(\pm\frac{\pi}{4} \pm \frac{5}{2}\epsilon - 2\epsilon + \arg E\right) \\ \geq \; & C\left(|\xi| + |p|^{5/2}\right), \end{aligned}$$

since ϵ is sufficiently small. Thus we get the estimate

$$\left|\xi - Ep^{5/2}\right|^{-1} \leq C\left(|\xi| + |p|^{5/2}\right)^{-1}$$

for all $p \in \Gamma_0$, $\xi \in \Gamma$. Then using the estimates $|e^{xp}| \leq Cx^{-2-k} |p|^{-2-k}$, $\left|e^{\xi}\right| \leq C(1+|\xi|)^{-3}$, $\left|\xi^{-\frac{3}{5}}\right| \leq 1$, and

$$\left|e^{-M\xi^{\frac{2}{5}} y}\right| \leq C(1+y)^{-2}$$

for all $p \in \Gamma_0$, $\xi \in \Gamma$, $x > 1$, $y > 0$, we find

$$\begin{aligned} \left|\partial_x^k \partial_y^l Q(x,y)\right| & \leq Cx^{-2-k}(1+y)^{-2} \int_{\Gamma_0} \frac{|dp| \, |p|^{-\frac{1}{2}}}{1+|p|^{5/2}} \int_{\Gamma} \frac{|d\xi|}{(1+|\xi|)^2} \\ & \leq Cx^{-2-k}(1+y)^{-2} \end{aligned}$$

and, similarly

$$\left|\partial_x^j \Lambda(x)\right| \leq Cx^{-2-j} \int_{\Gamma_0} \frac{|dp| \, |p|^{-\frac{1}{2}}}{1+|p|^{5/2}} \int_{\Gamma} \frac{|d\xi|}{(1+|\xi|)^2} \leq Cx^{-2-j}$$

for all $x > 1$, $y > 0$.

Consider now the case of $0 < x \leq 1$, $y > 0$. In view of the identity $\int_{|p|\geq 1} \frac{dp}{p} e^{xp} = 2\int_1^\infty \sin(xp) \frac{dp}{p} = \pi - 2\operatorname{Si}(x)$, we represent

$$\begin{aligned}
\partial_x^k \partial_y^l Q(x,y) &= \frac{(-M)^{1+l}}{4\pi^2} \partial_x^k (\pi - 2\operatorname{Si}(x)) \int_\Gamma d\xi \xi^{\frac{2l-3}{5}} e^{\xi - M\xi^{\frac{2}{5}} y} \\
&\quad - \frac{(-M)^{1+l}}{4\pi^2} \int_{|p|\geq 1} dp e^{xp} p^{k-1} \int_\Gamma d\xi \xi^{\frac{2l+2}{5}} \frac{e^{\xi - M\xi^{\frac{2}{5}} y}}{\xi - Ep^{5/2}} \\
&\quad - \frac{E(-M)^{1+l}}{4\pi^2} \int_{|p|<1} dp e^{xp} p^{3/2+k} \int_\Gamma d\xi \xi^{\frac{2l-3}{5}} \frac{e^{\xi - M\xi^{\frac{2}{5}} y}}{\xi - Ep^{5/2}}
\end{aligned}$$

and

$$\begin{aligned}
\partial_x^j \Lambda(x) &= \frac{1}{4\pi^2 M} \partial_x^j (\pi - 2\operatorname{Si}(x)) \int_\Gamma d\xi \xi^{-\frac{2}{5}} e^{\xi} \\
&\quad - \frac{1}{4\pi^2 M} \int_{|p|\geq 1} dp e^{xp} p^{j-1} \int_\Gamma d\xi \xi^{\frac{3}{5}} \frac{e^{\xi}}{\xi - Ep^{5/2}} \\
&\quad - \frac{E}{4\pi^2 M} \int_{|p|<1} dp e^{xp} p^{3/2+j} \int_\Gamma d\xi \xi^{-\frac{2}{5}} \frac{e^{\xi}}{\xi - Ep^{5/2}}.
\end{aligned}$$

As above we get the estimate $\left|\xi - Ep^{5/2}\right|^{-1} \leq C\left(|\xi| + |p|^{5/2}\right)^{-1}$ for all $p \in \Gamma_0$, $\xi \in \Gamma$. Hence applying the estimates $\left|e^{\xi}\right| \leq C(1+|\xi|)^{-3}$, $\left|\xi^{-\frac{3}{5}}\right| \leq 1$, and

$$\left|e^{-M\xi^{\frac{2}{5}} y}\right| \leq C(1+y)^{-2}$$

for all $p \in \Gamma_0$, $\xi \in \Gamma$, $0 < x \leq 1$, $y > 0$, we find

$$\begin{aligned}
\left|\partial_x^k \partial_y^l Q(x,y)\right| &\leq C(1+y)^{-2} \left(1 + \int_{\Gamma_0} (1+|p|)^{k-\frac{7}{2}} |dp|\right) \int_\Gamma (1+|\xi|)^{-2} |d\xi| \\
&\leq C(1+y)^{-2}.
\end{aligned}$$

In the same manner

$$\left|\partial_x^j \Lambda(x)\right| \leq C\left(1 + \int_{\Gamma_0} (1+|p|)^{j-\frac{7}{2}} |dp|\right) \int_\Gamma (1+|\xi|)^{-2} |d\xi| \leq C$$

for all $0 < x \leq 1$, $y > 0$, $k, l = 0, 1$, $j = 0, 1, 2$. Thus the second and the third estimates of the lemma are true. Lemma 1 is proved. □

In the next lemma we obtain the estimates for the Green operators $\mathcal{G}$, $\mathcal{I}$ and $\mathcal{J}$ in our basic norms

$$\|\phi\|_{\mathbf{X}^{\alpha,\beta}} = \sup_{t>0} \langle t\rangle^{-\beta} \left(\langle t\rangle^{-\frac{2}{5}\alpha} \|\phi\|_{\mathbf{L}^{1,\alpha}} + \langle t\rangle^{\frac{2}{5}} \|\phi\|_{\mathbf{L}^\infty} + \langle t\rangle^{\frac{2}{5}} t^{\frac{2}{5}} \|\partial_x \phi\|_{\mathbf{L}^\infty} \right),$$

$$\|f\|_{\mathbf{Y}_\gamma^{\alpha,\lambda}} = \sup_{t>0} \langle t\rangle^{\gamma-\lambda} t^{1-\gamma} \left(\langle t\rangle^{-\frac{2}{5}\alpha} \|f\|_{\mathbf{L}^{1,\alpha}} + \|f\|_{\mathbf{L}^1} + \langle t\rangle^{\frac{2}{5}} \|f\|_{\mathbf{L}^\infty} \right)$$

and

$$\|h\|_{\mathbf{Z}^\mu} = \sup_{t>0} \langle t\rangle^{\frac{4}{5}-\mu} \left(|h(t)| + \langle t\rangle \left| h'(t) \right| \right).$$

Lemma 2. *The estimates are valid*

$$\|\mathcal{G}\phi\|_{\mathbf{X}^{\alpha,0}} \le C \|\phi\|_{\mathbf{L}^{1,\alpha}} + C \|\phi\|_{\mathbf{L}^\infty}$$

for $\alpha \in [0,1)$,

$$\|\mathcal{I}f\|_{\mathbf{X}^{\alpha,\beta}} \le C \|f\|_{\mathbf{Y}_\gamma^{\alpha,\lambda}}$$

for $\alpha \in [0,1)$, $\beta \ge \max(0,\lambda)$, $\lambda \ne 0$, $\gamma \in \left(\frac{2}{5}, 1\right]$, *and*

$$\|\mathcal{J}h\|_{\mathbf{X}^{\alpha,\mu}} \le C \|h\|_{\mathbf{Z}^\mu}$$

for $\alpha \in [0,1)$, $\mu > -\frac{1}{5}$, *provided that the right-hand sides are finite.*

Proof. By Lemma 1 we find

$$\begin{aligned}
\|\mathcal{G}(t)\phi\|_{\mathbf{L}^{1,\alpha}} &\le Ct^{-\frac{2}{5}} \|\phi\|_{\mathbf{L}^{1,\alpha}} \sup_{y>0} \int_0^\infty \left| G\left((x-y)t^{-\frac{2}{5}}\right) \right| dx + Ct^{-\frac{2}{5}} \|\phi\|_{\mathbf{L}^1} \\
&\quad \times \sup_{y>0} \int_0^\infty \left(|x-y|^\alpha \left| G\left((x-y)t^{-\frac{2}{5}}\right) \right| + x^\alpha \left| Q\left(xt^{-\frac{2}{5}}, yt^{-\frac{2}{5}}\right) \right| \right) dx \\
&\le C \|\phi\|_{\mathbf{L}^{1,\alpha}} \sup_{y>0} \int_0^\infty \left(1 + \left|\eta - yt^{-\frac{2}{5}}\right|\right)^{\epsilon-2} d\eta + Ct^{\frac{2}{5}\alpha} \|\phi\|_{\mathbf{L}^1} \\
&\quad \times \sup_{y>0} \int_0^\infty \left(\left(1 + \left|\eta - yt^{-\frac{2}{5}}\right|\right)^{\epsilon-2} \left|\eta - yt^{-\frac{2}{5}}\right|^\alpha + (1+\eta)^{-2} |\eta|^\alpha \right) d\eta \\
&\le Ct^{\frac{2}{5}\alpha} \|\phi\|_{\mathbf{L}^1} + C \|\phi\|_{\mathbf{L}^{1,\alpha}} \qquad (2.9)
\end{aligned}$$

for all $t > 0$, $\alpha \in [0,1)$. Again by using the estimates of Lemma 1 we have

$$\begin{aligned}
\|\mathcal{G}(t)\phi\|_{\mathbf{L}^\infty} &\le t^{-\frac{2}{5}} \|\phi\|_{\mathbf{L}^\infty} \sup_{y>0} \int_0^\infty \left(\left| G\left((x-y)t^{-\frac{2}{5}}\right) + Q\left(xt^{-\frac{2}{5}}, yt^{-\frac{2}{5}}\right) \right| \right) dx \\
&\le C \|\phi\|_{\mathbf{L}^\infty} \sup_{y>0} \int_0^\infty \left(\left(1 + \left|\eta - yt^{-\frac{2}{5}}\right|\right)^{\epsilon-2} + (1+\eta)^{-2} \right) d\eta \\
&\le C \|\phi\|_{\mathbf{L}^\infty}
\end{aligned}$$

for all $0 < t < 1$ and

$$\begin{aligned}\left\|\mathcal{G}(t)\phi\right\|_{\mathbf{L}^\infty} &\leq t^{-\frac{2}{5}}\left\|\phi\right\|_{\mathbf{L}^1}\sup_{x,y>0}\left(\left|G\left((x-y)t^{-\frac{2}{5}}\right)\right|+\left|Q\left(xt^{-\frac{2}{5}},yt^{-\frac{2}{5}}\right)\right|\right)\\&\leq Ct^{-\frac{2}{5}}\left\|\phi\right\|_{\mathbf{L}^1}\end{aligned}$$

for all $t \geq 1$. Combining these estimates we obtain

$$\left\|\mathcal{G}(t)\phi\right\|_{\mathbf{L}^\infty} \leq C\langle t\rangle^{-\frac{2}{5}}\left(\left\|\phi\right\|_{\mathbf{L}^1}+\left\|\phi\right\|_{\mathbf{L}^\infty}\right) \tag{2.10}$$

for all $t > 0$. In the same manner we get

$$\begin{aligned}\left\|\partial_x\mathcal{G}(t)\phi\right\|_{\mathbf{L}^\infty} &\leq Ct^{-\frac{2}{5}}\left\|\phi\right\|_{\mathbf{L}^\infty}\sup_{x>0}\int_0^\infty\left|\partial_x G\left((x-y)t^{-\frac{2}{5}}\right)\right|dy\\&\quad+Ct^{-\frac{2}{5}}\left\|\phi\right\|_{\mathbf{L}^\infty}\left\|\int_0^\infty\left|\partial_x Q\left(xt^{-\frac{2}{5}},yt^{-\frac{2}{5}}\right)\right|dy\right\|_{\mathbf{L}^\infty}\\&\leq Ct^{-\frac{2}{5}}\left\|\phi\right\|_{\mathbf{L}^\infty}\int_0^\infty(1+\eta)^{-2}d\eta\leq Ct^{-\frac{2}{5}}\left\|\phi\right\|_{\mathbf{L}^\infty}\end{aligned}$$

for all $0 < t < 1$ and

$$\begin{aligned}\left\|\partial_x\mathcal{G}(t)\phi\right\|_{\mathbf{L}^\infty} &\leq Ct^{-\frac{2}{5}}\left\|\phi\right\|_{\mathbf{L}^1}\left\|\partial_x G\left(xt^{-\frac{2}{5}}\right)\right\|_{\mathbf{L}^\infty(\mathbf{R})}\\&\quad+Ct^{-\frac{2}{5}}\left\|\phi\right\|_{\mathbf{L}^1}\left\|\sup_{y>0}\left|\partial_x Q\left(xt^{-\frac{2}{5}},yt^{-\frac{2}{5}}\right)\right|\right\|_{\mathbf{L}^\infty}\\&\leq Ct^{-\frac{4}{5}}\left\|\phi\right\|_{\mathbf{L}^1}\left\|\left(1+xt^{-\frac{2}{5}}\right)^{-2}\right\|_{\mathbf{L}^\infty}\leq Ct^{-\frac{4}{5}}\left\|\phi\right\|_{\mathbf{L}^1}\end{aligned}$$

for all $t \geq 1$. Combining these estimates we find

$$\left\|\partial_x\mathcal{G}(t)\phi\right\|_{\mathbf{L}^\infty} \leq C\langle t\rangle^{-\frac{2}{5}}t^{-\frac{2}{5}}\left(\left\|\phi\right\|_{\mathbf{L}^1}+\left\|\phi\right\|_{\mathbf{L}^\infty}\right) \tag{2.11}$$

for all $t > 0$. By virtue of these inequalities we obtain the first estimate of the lemma.

By estimate (2.9) we get

$$\begin{aligned}\left\|\mathcal{I}f\right\|_{\mathbf{L}^{1,\alpha}} &= \left\|\int_0^t\mathcal{G}(t-\tau)f(\tau)d\tau\right\|_{\mathbf{L}^{1,\alpha}}\\&\leq C\int_0^t\left((t-\tau)^{\frac{2}{5}\alpha}\left\|f(\tau)\right\|_{\mathbf{L}^1}+\left\|f(\tau)\right\|_{\mathbf{L}^{1,\alpha}}\right)d\tau\\&\leq C\left\|f\right\|_{\mathbf{Y}_\gamma^{\alpha,\lambda}}\left(\int_0^t\left((t-\tau)^{\frac{2}{5}\alpha}\langle\tau\rangle^{\lambda-\gamma}\tau^{\gamma-1}+\langle\tau\rangle^{\frac{2}{5}\alpha+\lambda-\gamma}\tau^{\gamma-1}\right)d\tau\right)\\&\leq C\langle t\rangle^{\beta+\frac{2}{5}\alpha}\left\|f\right\|_{\mathbf{Y}_\gamma^{\alpha,\lambda}}\end{aligned}$$

for all $t>0$, if $\alpha\in[0,1)$, $\gamma\in\left(\frac{2}{5},1\right]$, $\beta\geq\max(0,\lambda)$, $\lambda\in\mathbf{R}$.

Via (2.10) we find

$$\begin{aligned}\|\mathcal{I}f\|_{\mathbf{L}^\infty} &= \left\|\int_0^t \mathcal{G}(t-\tau)f(\tau)\,d\tau\right\|_{\mathbf{L}^\infty}\\ &\leq C\int_0^t (t-\tau)^{-\frac{2}{5}}\left(\|f(\tau)\|_{\mathbf{L}^1}+\|f(\tau)\|_{\mathbf{L}^\infty}\right)d\tau\\ &\leq C\|f\|_{\mathbf{Y}_\gamma^{\alpha,\lambda}}\int_0^t (t-\tau)^{-\frac{2}{5}}\langle\tau\rangle^{\lambda-\gamma}\tau^{\gamma-1}d\tau\leq C\langle t\rangle^{\beta-\frac{2}{5}}\|f\|_{\mathbf{Y}_\gamma^{\alpha,\lambda}}\end{aligned}$$

for all $t>0$, if $\alpha\in[0,1)$, $\gamma\in\left(\frac{2}{5},1\right]$, $\beta\geq\max(0,\lambda)$, $\lambda\neq 0$. Similarly by virtue of (2.11) we have

$$\begin{aligned}\|\partial_x\mathcal{I}f\|_{\mathbf{L}^\infty} &\leq C\int_0^t (t-\tau)^{-\frac{2}{5}}\langle t-\tau\rangle^{-\frac{2}{5}}\left(\|f(\tau)\|_{\mathbf{L}^1}+\|f(\tau)\|_{\mathbf{L}^\infty}\right)d\tau\\ &\leq C\|f\|_{\mathbf{Y}_\gamma^{\alpha,\lambda}}\int_0^t (t-\tau)^{-\frac{2}{5}}\langle t-\tau\rangle^{-\frac{2}{5}}\\ &\qquad\times\left(\langle\tau\rangle^{\lambda-\gamma}\tau^{\gamma-1}+\langle\tau\rangle^{\lambda-\gamma-\frac{2}{5}}\tau^{\gamma-1}\right)d\tau\\ &\leq C\langle t\rangle^{\beta-\frac{4}{5}}\|f\|_{\mathbf{Y}_\gamma^{\alpha,\lambda}}\end{aligned}$$

for all $t>0$, if $\alpha\in[0,1)$, $\gamma\in\left(\frac{2}{5},1\right]$, $\beta\geq\max(0,\lambda)$, $\lambda\neq 0$. Hence the second estimate of the lemma follows.

Finally, we write

$$\begin{aligned}\|\mathcal{J}h\|_{\mathbf{L}^\infty} &= \left\|\int_0^t (t-\tau)^{-\frac{3}{5}}K\left(x(t-\tau)^{-\frac{2}{5}}\right)h(\tau)\,d\tau\right\|_{\mathbf{L}^\infty}\\ &\leq C\|h\|_{\mathbf{Z}^\mu}\|K\|_{\mathbf{L}^\infty}\int_0^t (t-\tau)^{-\frac{3}{5}}\langle\tau\rangle^{\mu-\frac{4}{5}}d\tau\leq C\langle t\rangle^{\mu-\frac{2}{5}}\|h\|_{\mathbf{Z}^\mu}\end{aligned}\tag{2.12}$$

and

$$\|\mathcal{J}h\|_{\mathbf{L}^{1,\alpha}}\leq C\|h\|_{\mathbf{Z}^\mu}\|K\|_{\mathbf{L}^{1,\alpha}}\int_0^t (t-\tau)^{\frac{2}{5}\alpha-\frac{1}{5}}\langle\tau\rangle^{\mu-\frac{4}{5}}d\tau\leq C\langle t\rangle^{\mu+\frac{2}{5}\alpha}\|h\|_{\mathbf{Z}^\mu}$$

if $\mu>-\frac{1}{5}$. To estimate the derivative with respect to x we represent

$$\begin{aligned}\mathcal{J}h &= \int_0^t (t-\tau)^{-\frac{3}{5}}K\left(x(t-\tau)^{-\frac{2}{5}}\right)h(\tau)\,d\tau\\ &= h(t)\int_0^t \tau^{-\frac{3}{5}}K\left(x\tau^{-\frac{2}{5}}\right)d\tau\\ &\quad+\int_0^t (t-\tau)^{-\frac{3}{5}}K\left(x(t-\tau)^{-\frac{2}{5}}\right)(h(\tau)-h(t))\,d\tau.\end{aligned}$$

For the first integral we change the variable of integration $x\tau^{-\frac{2}{5}} = z$

$$\int_0^t \tau^{-\frac{3}{5}} K\left(x\tau^{-\frac{2}{5}}\right) d\tau = \frac{5}{2} K(0)\, t^{\frac{2}{5}} + \frac{5x}{2} \int_{xt^{-\frac{2}{5}}}^{\infty} \left(K(z) - K(0)\right) \frac{dz}{z^2}.$$

Then we differentiate with respect to x

$$\begin{aligned} \partial_x \mathcal{J} h &= \frac{5}{2} h(t) \left(\int_{xt^{-\frac{2}{5}}}^{\infty} \left(K(z) - K(0)\right) \frac{dz}{z^2} - \frac{K\left(xt^{-\frac{2}{5}}\right) - K(0)}{xt^{-\frac{2}{5}}} \right) \\ &\quad + \int_0^t (t-\tau)^{-1} K'\left(x(t-\tau)^{-\frac{2}{5}}\right) \left(h(\tau) - h(t)\right) d\tau. \end{aligned}$$

Since $K'(0) = 0$, we have the estimate

$$\begin{aligned} \left\|\partial_x \mathcal{J} h\right\|_{\mathbf{L}^\infty} &\leq C\left(\|K\|_{\mathbf{L}^\infty} + \left\|K''\right\|_{\mathbf{L}^\infty}\right) |h(t)| + C\|h\|_{\mathbf{Z}^\mu} \left\|K'\right\|_{\mathbf{L}^\infty} \int_0^t d\tau \\ &\leq C\|h\|_{\mathbf{Z}^\mu} \end{aligned}$$

for all $0 < t \leq 1$, and

$$\begin{aligned} \left\|\partial_x \mathcal{J} h\right\|_{\mathbf{L}^\infty} &\leq C\left(\|K\|_{\mathbf{L}^\infty} + \left\|K''\right\|_{\mathbf{L}^\infty}\right) |h(t)| \\ &\quad + C\|h\|_{\mathbf{Z}^\mu} \left\|K'\right\|_{\mathbf{L}^\infty} \int_0^{\frac{t}{2}} (t-\tau)^{-1} \left(\langle\tau\rangle^{\mu-\frac{4}{5}} + \langle t\rangle^{\mu-\frac{4}{5}}\right) d\tau \\ &\quad + C\|h\|_{\mathbf{Z}^\mu} \left\|K'\right\|_{\mathbf{L}^\infty} \int_{\frac{t}{2}}^{t} \langle\tau\rangle^{\mu-\frac{9}{5}} d\tau \leq C\langle t\rangle^{\mu-\frac{4}{5}} \|h\|_{\mathbf{Z}^\mu} \end{aligned}$$

for all $t > 1$, if $\mu > -\frac{1}{5}$. Thus, the third estimate of the lemma is valid. Lemma 2 is proved. □

In the next lemma we obtain the fast asymptotics for the Green operators $\mathcal{G}$, $\mathcal{I}$ and $\mathcal{J}$. Denote $B(x) = G(x) + Q(x,0)$ and $\theta = \int_0^\infty \phi(x)\, dx$, $\vartheta = \int_0^\infty \int_0^\infty f(x,t)\, dx dt$.

Lemma 3. *The estimates are valid*

$$\left\|\mathcal{G}(t)\phi - \theta t^{-\frac{2}{5}} B\left(xt^{-\frac{2}{5}}\right)\right\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{2}{5}-\frac{2}{5}\alpha} \|\phi\|_{\mathbf{L}^{1,\alpha}},$$

$$\left\|\mathcal{I}f - \vartheta t^{-\frac{2}{5}} B\left(xt^{-\frac{2}{5}}\right)\right\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{2}{5}+\lambda} \|f\|_{\mathbf{Y}_\gamma^{\alpha,\lambda}}$$

and

$$\|\mathcal{J}h\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{2}{5}+\mu} \|h\|_{\mathbf{Z}^\mu}$$

for all $t \geq 1$, *where* $\alpha \in (0,1)$, $-\frac{2}{5}\alpha < \lambda < 0$, $-\frac{1}{5} < \mu < 0$, $\gamma \in (0,1]$.

Proof. Denote the operator

$$\mathcal{G}_1(t)\phi = t^{-\frac{2}{5}} \int_0^\infty G\left((x-y)\,t^{-\frac{2}{5}}\right)\phi(y)dy.$$

Applying Lemma 1.28 from [6] we have

$$\left\| |\cdot|^{\varkappa} \left(\mathcal{G}_1(t)\phi - t^{-\frac{2}{5}}\theta G\left(xt^{-\frac{2}{5}}\right)\right)\right\|_{\mathbf{L}^q} \le Ct^{-\frac{2}{5}\left(1-\frac{1}{q}+\alpha-\varkappa\right)} \|\phi\|_{\mathbf{L}^{1,\alpha}}$$

for all $t>0$, $1\le q\le\infty$, $0\le\varkappa\le\alpha\le 1$. By Lemma 1 we find

$$\left|Q\left(xt^{-\frac{2}{5}}, yt^{-\frac{2}{5}}\right) - Q\left(xt^{-\frac{2}{5}}, 0\right)\right| \le C\left(1+xt^{-\frac{2}{5}}\right)^{-2} y^{\alpha} t^{-\frac{2}{5}\alpha}$$

then we can estimate the operator

$$\mathcal{G}_2(t)\phi = t^{-\frac{2}{5}} \int_0^\infty Q\left(xt^{-\frac{2}{5}}, yt^{-\frac{2}{5}}\right)\phi(y)dy$$

as follows

$$\begin{aligned}
&\left\| |\cdot|^{\varkappa} \left(\mathcal{G}_2(t)\phi - t^{-\frac{2}{5}}\theta Q\left(xt^{-\frac{2}{5}}, 0\right)\right)\right\|_{\mathbf{L}^q} \\
= &\left\| \left(t^{-\frac{2}{5}} \int_0^\infty x^{\varkappa}\left(Q\left(xt^{-\frac{2}{5}}, yt^{-\frac{2}{5}}\right) - Q\left(xt^{-\frac{2}{5}}, 0\right)\right)\phi(y)dy\right)\right\|_{\mathbf{L}^q} \\
\le\ & Ct^{-\frac{2}{5}(1+\alpha)} \left\| x^{\varkappa}\left(1+xt^{-\frac{2}{5}}\right)^{-2}\right\|_{\mathbf{L}^q} \int_0^\infty y^{\alpha}|\phi(y)|\,dy \\
\le\ & Ct^{-\frac{2}{5}\left(1-\frac{1}{q}+\alpha-\varkappa\right)} \|\phi\|_{\mathbf{L}^{1,\alpha}},
\end{aligned}$$

for all $t>0$, $1\le q\le\infty$, $0\le\varkappa\le\alpha\le 1$. Thus we obtain the first estimate of the lemma.

Denote $\theta(t) = \int_0^\infty f(t,x)\,dx$, then by virtue of the first estimate of the lemma we have

$$\begin{aligned}
&\left\| \int_0^t \left(\mathcal{G}(t-\tau) f(\tau) - \theta(\tau)(t-\tau)^{-\frac{2}{5}} B\left(x(t-\tau)^{-\frac{2}{5}}\right)\right) d\tau \right\|_{\mathbf{L}^\infty} \\
\le\ & C\int_0^t (t-\tau)^{-\frac{2}{5}-\frac{2}{5}\alpha} \|f(\tau)\|_{\mathbf{L}^{1,\alpha}}\,d\tau \\
\le\ & C\|f\|_{\mathbf{Y}^{\alpha,\lambda}_{\gamma}} \int_0^t (t-\tau)^{-\frac{2}{5}-\frac{2}{5}\alpha} \langle\tau\rangle^{\frac{2}{5}\alpha+\lambda-\gamma} \tau^{\gamma-1} d\tau \le C\langle t\rangle^{-\frac{2}{5}+\lambda} \|f\|_{\mathbf{Y}^{\alpha,\lambda}_{\gamma}},
\end{aligned}$$

where $\alpha\in(0,1)$, $\gamma\in(0,1]$, $-\frac{2}{5}\alpha<\lambda<0$. By applying the estimate

$$\left|(t-\tau)^{-\frac{2}{5}} B\left(x(t-\tau)^{-\frac{2}{5}}\right) - t^{-\frac{2}{5}} B\left(xt^{-\frac{2}{5}}\right)\right| \le Ct^{-\frac{2}{5}}\tau^{\frac{2}{5}}(t-\tau)^{-\frac{2}{5}}$$

we obtain

$$\left\| \int_0^t \theta(\tau)(t-\tau)^{-\frac{2}{5}} B\left(x(t-\tau)^{-\frac{2}{5}}\right) d\tau - t^{-\frac{2}{5}} B\left(xt^{-\frac{2}{5}}\right) \int_0^t \vartheta(\tau)\, d\tau \right\|_{\mathbf{L}^\infty}$$
$$\leq Ct^{-\frac{2}{5}} \int_0^t |\theta(\tau)| \tau^{\frac{2}{5}} (t-\tau)^{-\frac{2}{5}} d\tau$$
$$\leq Ct^{-\frac{2}{5}} \|f\|_{\mathbf{Y}_\gamma^{0,\lambda}} \int_0^t (t-\tau)^{-\frac{2}{5}} \langle\tau\rangle^{\lambda-\gamma} \tau^{\gamma-\frac{3}{5}} d\tau \leq Ct^{-\frac{2}{5}+\lambda} \|f\|_{\mathbf{Y}_\gamma^{0,\lambda}}$$

for all $t > 0$, if $-\frac{2}{5} < \lambda < 0$, $\gamma \in (0,1]$. Note that

$$\left| \int_t^\infty \theta(\tau)\, d\tau \right| \leq C \|f\|_{\mathbf{Y}_\gamma^{0,\lambda}} \int_t^\infty \langle\tau\rangle^{\lambda-\gamma} \tau^{\gamma-1} d\tau \leq Ct^\lambda \|f\|_{\mathbf{Y}_\gamma^{0,\lambda}}$$

for all $t > 0$, if $\lambda \in (-1, 0)$, $\gamma \in (0,1]$. Hence

$$\left\| t^{-\frac{2}{5}} B\left(xt^{-\frac{2}{5}}\right) \int_t^\infty \theta(\tau)\, d\tau \right\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{2}{5}+\lambda} \|f\|_{\mathbf{Y}_\gamma^{0,\lambda}}.$$

Therefore the second estimate of the lemma follows. To prove the last estimate of the lemma we note that $|h(t)| \leq \langle t\rangle^{\mu-\frac{4}{5}} \|h\|_{\mathbf{Z}^\mu}$. Then by (2.12) we find

$$\|\mathcal{J}h\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{2}{5}+\mu} \|h\|_{\mathbf{Z}^\mu}$$

for all $t > 0$, if $-\frac{1}{5} < \mu < 0$. Lemma 3 is proved. □

In the next lemma we consider the case of the slow asymptotics of the source and the boundary data. Denote

$$\Phi(\chi) = \int_0^1 z^{\beta-1} (1-z)^{-\frac{2}{5}} B\left(\chi(1-z)^{-\frac{2}{5}}\right) dz$$

and

$$\Psi(\chi) = \int_0^1 z^{\beta-\frac{4}{5}} (1-z)^{-\frac{3}{5}} K\left(\chi(1-z)^{-\frac{2}{5}}\right) dz.$$

Lemma 4. *Let the estimate $\|f(t)\|_{\mathbf{L}^{1,\alpha}} \leq Ct^{\beta-1}$ be valid with $\alpha \in (0,1)$. Let the asymptotic formulas*

$$\theta(t) \equiv \int_0^\infty f(t,x)\, dx = at^{\beta-1} + O\left(t^{\beta-1-\delta}\right)$$

and

$$h(t) = bt^{\beta-\frac{4}{5}} + O\left(t^{\beta-\frac{4}{5}-\delta}\right)$$

be true for all $t>0$, *where* $\beta>0$, $0<\delta<\min\left(\beta,\frac{2}{5}\alpha,\frac{1}{5}\right)$. *Then the asymptotics hold*

$$\mathcal{I}f=at^{\beta-\frac{2}{5}}\Phi\left(\chi\right)+O\left(t^{\beta-\frac{2}{5}-\delta}\right)$$

and

$$\mathcal{J}h=bt^{\beta-\frac{2}{5}}\Psi\left(\chi\right)+O\left(t^{\beta-\frac{2}{5}-\delta}\right)$$

for all $t\geq 1$ *uniformly with respect to* $x>0$, *where* $\chi=xt^{-\frac{2}{5}}$.

Proof. By virtue of the first estimate of Lemma 3 we have

$$\begin{aligned}
&\left\|\int_0^t\left(\mathcal{G}\left(t-\tau\right)f\left(\tau\right)-\theta\left(\tau\right)\left(t-\tau\right)^{-\frac{2}{5}}B\left(x\left(t-\tau\right)^{-\frac{2}{5}}\right)\right)d\tau\right\|_{\mathbf{L}^\infty}\\
\leq\ & C\int_0^t\left(t-\tau\right)^{-\frac{2}{5}-\frac{2}{5}\alpha}\left\|f\left(\tau\right)\right\|_{\mathbf{L}^{1,\alpha}}d\tau\leq C\int_0^t\left(t-\tau\right)^{-\frac{2}{5}-\frac{2}{5}\alpha}\tau^{\beta-1}d\tau\\
\leq\ & Ct^{\beta-\frac{2}{5}-\frac{2}{5}\alpha}
\end{aligned}$$

for all $t>0$. Changing $\tau=zt$, $\chi=xt^{-\frac{2}{5}}$ we find

$$\begin{aligned}
&\int_0^t\theta\left(\tau\right)\left(t-\tau\right)^{-\frac{2}{5}}B\left(x\left(t-\tau\right)^{-\frac{2}{5}}\right)d\tau\\
=\ & a\int_0^t\tau^{\beta-1}\left(t-\tau\right)^{-\frac{2}{5}}B\left(x\left(t-\tau\right)^{-\frac{2}{5}}\right)d\tau\\
&+O\left(\int_0^t\tau^{\beta-1-\delta}\left(t-\tau\right)^{-\frac{2}{5}}d\tau\right)\\
=\ & at^{\beta-\frac{2}{5}}\int_0^1 z^{\beta-1}\left(1-z\right)^{-\frac{2}{5}}B\left(\chi\left(1-z\right)^{-\frac{2}{5}}\right)dz+O\left(t^{\beta-\frac{2}{5}-\delta}\right)\\
=\ & at^{\beta-\frac{2}{5}}\Phi\left(\chi\right)+O\left(t^{\beta-\frac{2}{5}-\delta}\right).
\end{aligned}$$

for all $t>0$ if $0<\delta<\beta$. Therefore the first asymptotic formula of the lemma follows. Denote $\chi=xt^{-\frac{2}{5}}$. Changing $\tau=zt$ we obtain

$$\begin{aligned}
\mathcal{J}h\ =\ & b\int_0^t\left(t-\tau\right)^{-\frac{3}{5}}K\left(x\left(t-\tau\right)^{-\frac{2}{5}}\right)\ \tau^{\beta-\frac{4}{5}}\ d\tau\\
&+O\left(\int_0^t\left(t-\tau\right)^{-\frac{3}{5}}\ \tau^{\beta-\frac{4}{5}-\delta}d\tau\right)\\
=\ & bt^{\beta-\frac{2}{5}}\int_0^1\left(1-z\right)^{-\frac{3}{5}}K\left(\chi\left(1-z\right)^{-\frac{2}{5}}\right)z^{\beta-\frac{4}{5}}\ dz+O\left(t^{\beta-\frac{2}{5}-\delta}\right)\\
=\ & bt^{\beta-\frac{2}{5}}\Psi\left(\chi\right)+O\left(t^{\beta-\frac{2}{5}-\delta}\right)
\end{aligned}$$

for all $t>0$ if $0<\delta<\frac{1}{5}$. Thus the second asymptotic formula of the lemma is true. Lemma 4 is proved. □

We now estimate the nonlinearity $\mathcal{N}(u,u_x) = u^\rho u_x^\sigma$ in the norms $\mathbf{Y}_\gamma^{\alpha,\lambda}$.

Lemma 5. *Let $\rho \geq 1$, $\sigma \in \left[1, \frac{3}{2}\right)$. Then the estimate is true*

$$\|\mathcal{N}(u,u_x) - \mathcal{N}(v,v_x)\|_{\mathbf{Y}_\gamma^{\alpha,\lambda}} \leq \|u-v\|_{\mathbf{X}^{\alpha,\beta}} \left(\|u\|_{\mathbf{X}^{\alpha,\beta}} + \|v\|_{\mathbf{X}^{\alpha,\beta}}\right)^{\sigma+\rho-1}$$

with $\gamma = 1 - \frac{2}{5}\sigma$, $\lambda = 1 + \beta(\sigma+\rho) - \frac{2}{5}(\rho + 2\sigma - 1)$.

Proof. We have the estimates $\|\phi\|_{\mathbf{L}^{1,\alpha}} \leq \langle t\rangle^{\beta+\frac{2}{5}\alpha} \|\phi\|_{\mathbf{X}^{\alpha,\beta}}$, $\|\phi\|_{\mathbf{L}^\infty} \leq \langle t\rangle^{\beta-\frac{2}{5}} \|\phi\|_{\mathbf{X}^{\alpha,\beta}}$, $\|\partial_x\phi\|_{\mathbf{L}^\infty} \leq \langle t\rangle^{\beta-\frac{2}{5}} t^{-\frac{2}{5}} \|\phi\|_{\mathbf{X}^{\alpha,\beta}}$. Hence

$$\begin{aligned}
&\|u^\rho u_x^\sigma - v^\rho v_x^\sigma\|_{\mathbf{L}^{1,\alpha}} \\
\leq\ & C\|u-v\|_{\mathbf{L}^{1,\alpha}} \left(\|u\|_{\mathbf{L}^\infty}^{\rho-1} \|u_x\|_{\mathbf{L}^\infty}^{\sigma} + \|v\|_{\mathbf{L}^\infty}^{\rho-1} \|v_x\|_{\mathbf{L}^\infty}^{\sigma}\right) \\
& + C\|u_x - v_x\|_{\mathbf{L}^\infty} \left(\|u\|_{\mathbf{L}^{1,\alpha}} \|u\|_{\mathbf{L}^\infty}^{\rho-1} \|u_x\|_{\mathbf{L}^\infty}^{\sigma-1} + \|v\|_{\mathbf{L}^{1,\alpha}} \|v\|_{\mathbf{L}^\infty}^{\rho-1} \|v_x\|_{\mathbf{L}^\infty}^{\sigma-1}\right) \\
\leq\ & C\langle t\rangle^{\frac{2}{5}\alpha+\beta(\sigma+\rho)-\frac{2}{5}(\rho+\sigma-1)} t^{-\frac{2}{5}\sigma} \|u-v\|_{\mathbf{X}^{\alpha,\beta}} \left(\|u\|_{\mathbf{X}^{\alpha,\beta}} + \|v\|_{\mathbf{X}^{\alpha,\beta}}\right)^{\sigma+\rho-1} \\
\leq\ & C\langle t\rangle^{\lambda-\gamma+\frac{2}{5}\alpha} t^{\gamma-1} \|u-v\|_{\mathbf{X}^{\alpha,\beta}} \left(\|u\|_{\mathbf{X}^{\alpha,\beta}} + \|v\|_{\mathbf{X}^{\alpha,\beta}}\right)^{\sigma+\rho-1}
\end{aligned}$$

with $\gamma = 1 - \frac{2}{5}\sigma$, $\lambda = 1 + \beta(\sigma+\rho) - \frac{2}{5}(\rho + 2\sigma - 1)$ for $\alpha \in [0,1]$. In the same manner

$$\begin{aligned}
&\|u^\rho u_x^\sigma - v^\rho v_x^\sigma\|_{\mathbf{L}^\infty} \\
\leq\ & C\|u-v\|_{\mathbf{L}^\infty} \left(\|u\|_{\mathbf{L}^\infty}^{\rho-1} \|u_x\|_{\mathbf{L}^\infty}^{\sigma} + \|v\|_{\mathbf{L}^\infty}^{\rho-1} \|v_x\|_{\mathbf{L}^\infty}^{\sigma}\right) \\
& + C\|u_x - v_x\|_{\mathbf{L}^\infty} \left(\|u\|_{\mathbf{L}^\infty} \|u\|_{\mathbf{L}^\infty}^{\rho-1} \|u_x\|_{\mathbf{L}^\infty}^{\sigma-1} + \|v\|_{\mathbf{L}^\infty} \|v\|_{\mathbf{L}^\infty}^{\rho-1} \|v_x\|_{\mathbf{L}^\infty}^{\sigma-1}\right) \\
\leq\ & C\langle t\rangle^{-\frac{2}{5}+\beta(\sigma+\rho)-\frac{2}{5}(\rho+\sigma-1)} t^{-\frac{2}{5}\sigma} \|u-v\|_{\mathbf{X}^{\alpha,\beta}} \left(\|u\|_{\mathbf{X}^{\alpha,\beta}} + \|v\|_{\mathbf{X}^{\alpha,\beta}}\right)^{\sigma+\rho-1} \\
\leq\ & C\langle t\rangle^{\lambda-\gamma-\frac{2}{5}} t^{\gamma-1} \|u-v\|_{\mathbf{X}^{\alpha,\beta}} \left(\|u\|_{\mathbf{X}^{\alpha,\beta}} + \|v\|_{\mathbf{X}^{\alpha,\beta}}\right)^{\sigma+\rho-1}.
\end{aligned}$$

Thus the estimate of the lemma is true. Lemma 5 is proved. □

§3. Proof of Theorem 1

Using the Duhamel formula (2.8) we rewrite problem (1.1) in the form of the integral equation

$$u = \mathcal{G}u_0 + \mathcal{I}\left(f - \mathcal{N}(u,u_x)\right) + \mathcal{J}h. \tag{3.1}$$

We apply the contraction mapping principle in the ball

$$\mathbf{X}_\varepsilon^{\alpha,0} = \left\{\phi \in \mathbf{X}^{\alpha,0} : \ \|u\|_{\mathbf{X}^{\alpha,0}} \leq \varepsilon\right\},$$

where $\varepsilon > 0$ is sufficiently small. For $v \in \mathbf{X}_{\varepsilon}^{\alpha,0}$ we define the transformation $\mathcal{M}(v)$ by the formula

$$\mathcal{M}(v) = \mathcal{G}u_0 + \mathcal{I}(f - \mathcal{N}(v, v_x)) + \mathcal{J}h. \tag{3.2}$$

First we prove that $\|\mathcal{M}(v)\|_{\mathbf{X}^{\alpha,0}} \leq \varepsilon$. By the conditions of the theorem using estimates of Lemmas 2 and 5 we find form (3.2)

$$\begin{aligned} &\|\mathcal{M}(v)\|_{\mathbf{X}^{\alpha,0}} \leq \|\mathcal{G}u_0\|_{\mathbf{X}^{\alpha,0}} + \|\mathcal{I}(f - \mathcal{N}(v, v_x))\|_{\mathbf{X}^{\alpha,0}} + \|\mathcal{J}h\|_{\mathbf{X}^{\alpha,0}} \\ \leq\ & C\|u_0\|_{\mathbf{L}^{1,\alpha}} + C\|u_0\|_{\mathbf{L}^{\infty}} + C\|f\|_{\mathbf{Y}_{\gamma}^{\alpha,\nu}} + C\|\mathcal{N}(v, v_x)\|_{\mathbf{Y}_{\gamma}^{\alpha,\lambda}} + C\|h\|_{\mathbf{Z}^0} \\ \leq\ & C\|u_0\|_{\mathbf{L}^{1,\alpha}} + C\|u_0\|_{\mathbf{L}^{\infty}} + C\|f\|_{\mathbf{Y}_{\gamma}^{\alpha,\nu}} + C\|h\|_{\mathbf{Z}^0} + C\|v\|_{\mathbf{X}^{\alpha,0}}^{\sigma+\rho} \leq \varepsilon, \end{aligned}$$

since $\lambda = 1 - \frac{2}{5}(\rho + 2\sigma - 1) < 0$ and $\varepsilon > 0$ is small enough. Therefore $\mathcal{M}$ transforms $\mathbf{X}_{\varepsilon}^{\alpha,0}$ into itself. In the same way we estimate the difference

$$\begin{aligned} &\|\mathcal{M}(w) - \mathcal{M}(v)\|_{\mathbf{X}^{\alpha,0}} \leq \|\mathcal{I}(\mathcal{N}(v, v_x) - \mathcal{N}(w, w_x))\|_{\mathbf{X}^{\alpha,0}} \\ \leq\ & C\|w - v\|_{\mathbf{X}^{\alpha,0}} (\|w\|_{\mathbf{X}^{\alpha,0}} + \|v\|_{\mathbf{X}^{\alpha,0}})^{\sigma+\rho-1} \leq C\varepsilon^{\sigma+\rho-1}\|w - v\|_{\mathbf{X}^{\alpha,0}}, \end{aligned}$$

where $w, v \in \mathbf{X}_{\varepsilon}^{\alpha,0}$. Thus $\mathcal{M}$ is a contraction mapping in $\mathbf{X}_{\varepsilon}^{\alpha,0}$, therefore there exists a unique solution $u \in \mathbf{X}_{\varepsilon}^{\alpha,0}$ to the integral equation (3.1) and the problem (1.1).

We now prove the asymptotics. Since

$$\|\mathcal{N}(u, u_x)\|_{\mathbf{Y}_{\gamma}^{\alpha,\lambda}} \leq C\|u\|_{\mathbf{X}^{\alpha,0}}^{\sigma+\rho} \leq C\varepsilon^{\sigma+\rho},$$

applying estimates of Lemma 3 we find from the integral representation (3.1)

$$\begin{aligned} u(t) &= \mathcal{G}u_0 + \mathcal{I}(f - \mathcal{N}(u, u_x)) + \mathcal{J}h \\ &= (\theta + \vartheta_1 - \vartheta_2)\, t^{-\frac{2}{5}} B\left(xt^{-\frac{2}{5}}\right) + O\left(t^{-\frac{2}{5}-\frac{2}{5}\alpha}\|u_0\|_{\mathbf{L}^{1,\alpha}}\right) \\ &\quad + O\left(t^{-\frac{2}{5}-\delta}\left(\|f\|_{\mathbf{Y}_{\gamma}^{\alpha,\nu}} + \|\mathcal{N}(u, u_x)\|_{\mathbf{Y}_{\gamma}^{\alpha,\lambda}} + \|h\|_{\mathbf{Z}^{\mu}}\right)\right), \end{aligned}$$

where $0 < \delta < \frac{2}{5}\alpha$, $\delta \leq \min(-\lambda, -\nu, -\mu)$, $\theta = \int_0^\infty u_0(y)\, dy$,
$\vartheta_1 = \int_0^\infty \int_0^\infty f(\tau, y)\, dyd\tau$ and

$$\vartheta_2 = \int_0^\infty \int_0^\infty \mathcal{N}(u, u_x)\, dyd\tau.$$

Theorem 1 is proved.

§4. Proof of Theorem 2

We apply the contraction mapping principle in

$$\mathbf{X}_{\varepsilon}^{0,\beta}=\left\{\phi\in\mathbf{X}^{0,\beta}:\ \ \|u\|_{\mathbf{X}^{0,\beta}}\leq\varepsilon\right\},$$

where $\varepsilon>0$ is sufficiently small. For $v\in\mathbf{X}_{\varepsilon}^{0,\beta}$ we define the mapping $\mathcal{M}(v)$ by formula (3.2). First we prove that

$$\|\mathcal{M}(v)\|_{\mathbf{X}^{0,\beta}}\leq\varepsilon.$$

By the conditions of the theorem using estimates of Lemmas 2 and 5 we have from (3.2)

$$\begin{aligned}\|\mathcal{M}(v)\|_{\mathbf{X}^{0,\beta}} &\leq \|\mathcal{G}u_0\|_{\mathbf{X}^{0,\beta}}+\|\mathcal{I}(f-\mathcal{N}(v,v_x))\|_{\mathbf{X}^{0,\beta}}+\|\mathcal{J}h\|_{\mathbf{X}^{0,\beta}}\\ &\leq C\|u_0\|_{\mathbf{L}^1}+C\|u_0\|_{\mathbf{L}^\infty}+C\|f\|_{\mathbf{Y}_{\gamma}^{0,\beta}}\\ &\qquad +C\|\mathcal{N}(v,v_x)\|_{\mathbf{Y}_{\gamma}^{0,\lambda}}+C\|h\|_{\mathbf{Z}^{\beta}}\\ &\leq C\|u_0\|_{\mathbf{L}^1}+C\|u_0\|_{\mathbf{L}^\infty}+C\|f\|_{\mathbf{Y}_{\gamma}^{0,\beta}}+C\|h\|_{\mathbf{Z}^{\beta}}+C\|v\|_{\mathbf{X}^{0,\beta}}^{\sigma+\rho}\\ &\leq \varepsilon.\end{aligned}$$

Here we can choose λ such that $1+\beta(\sigma+\rho)-\frac{2}{5}(\rho+2\sigma-1)<\lambda<\beta$, since $1+\beta(\sigma+\rho)-\frac{2}{5}(\rho+2\sigma-1)<\beta$ if $\rho+\sigma>\frac{5-2\sigma}{2-5\beta}+1$. Therefore $\mathcal{M}$ transforms $\mathbf{X}_{\varepsilon}^{0,\beta}$ into itself. In the same way we estimate the difference

$$\begin{aligned}&\|\mathcal{M}(w)-\mathcal{M}(v)\|_{\mathbf{X}^{0,\beta}}\leq\|\mathcal{I}(t)(\mathcal{N}(v,v_x)-\mathcal{N}(w,w_x))\|_{\mathbf{X}^{0,\beta}}\\ \leq\ & C\|w-v\|_{\mathbf{X}^{0,\beta}}\left(\|w\|_{\mathbf{X}^{0,\beta}}+\|v\|_{\mathbf{X}^{0,\beta}}\right)^{\sigma+\rho-1}\leq C\varepsilon^{\sigma+\rho-1}\|w-v\|_{\mathbf{X}^{0,\beta}}\end{aligned}$$

where $w,v\in\mathbf{X}_{\varepsilon}^{0,\beta}$. Thus $\mathcal{M}$ is a contraction mapping in $\mathbf{X}_{\varepsilon}^{0,\beta}$, therefore there exists a unique solution $u\in\mathbf{X}_{\varepsilon}^{0,\beta}$ to the integral equation (3.1) and the problem (1.1).

We now prove the asymptotics. Since

$$\|\mathcal{N}(u,u_x)\|_{\mathbf{Y}_{\gamma}^{0,\lambda}}\leq C\|u\|_{\mathbf{X}^{0,\beta}}^{\sigma+\rho}\leq C\varepsilon^{\sigma+\rho},$$

using Lemma 2 we obtain the decay estimate

$$\mathcal{I}\mathcal{N}(u,u_x)=O\left(t^{\beta-\frac{2}{5}-\delta}\right)$$

with $\delta=\beta-\lambda>0$. By the first estimate of Lemma 2 we get

$$\mathcal{G}u_0=O\left(t^{-\frac{2}{5}}\|u_0\|_{\mathbf{L}^1}\right).$$

Then by the integral representation (3.1) and the estimates of Lemma 4 we find

$$\begin{aligned} u(t) &= \mathcal{I}f + \mathcal{J}h + \mathcal{G}u_0 - \mathcal{I}\mathcal{N}(u, u_x) \\ &= t^{\beta-\frac{2}{5}} \left(a\Phi(\chi) + b\Psi(\chi)\right) + O\left(t^{\beta-\frac{2}{5}-\delta}\right) \end{aligned}$$

for $t \geq 1$, where $0 < \delta < \min\left(\beta, \frac{1}{5}, \frac{2}{5}\alpha, \beta - \lambda\right)$. Theorem 2 is proved.

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