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On supermagic coverings of fans and ladders

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Abstract. A simple graph G admits an *H-covering* if every edge in $E(G)$ belongs to a subgraph of G isomorphic to H. The graph G is said to be *H-magic* if there exists a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \cdots, |V(G) \cup E(G)|\}$ such that for every subgraph H' of G isomorphic to H, $\sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)$ is constant. Additionally, G is said to be *H-supermagic* if $f(V(G)) = \{1, 2, 3, \cdots, |V(G)|\}$. In this paper, we study H-supermagic labelings of two classes of connected graph namely fans and ladders.

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§1. Introduction

We consider finite, undirected and simple graphs. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively. Let H be a graph. An *edge-covering* of G is a family of subgraphs $H_1, \ldots, H_k$ such that each edge of $E(G)$ belongs to at least one of the subgraphs H_i, $1 \leq i \leq k$. If every H_i is isomorphic to a given graph H then we say that G admits an H-covering. Suppose G admits an H-covering. A total labeling $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, |V(G) \cup E(G)|\}$ is said an *H-magic labeling* of G if for every subgraph H' of G isomorphic to H, $\sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)$ is constant. An H-magic labeling f is said an *H-supermagic labeling* if $f(V(G)) = \{1, 2, 3, \ldots, |V(G)|\}$. A graph that admits H-(super)magic labeling is called *H-(super)magic*. The sum of all vertex labels and all edge labels on H (under a labeling f) is denoted by $\sum f(H)$. In Figure 1, we show C_4-magic and C_4-supermagic labelings of L_4.

The H-supermagic labeling was first introduced by Gutiérrez and Lladó [5] in 2005. They considered star-supermagic and path-supermagic labelings of some graphs. In [8], Lladó and Moragas gave C_n-supermagic labelings of

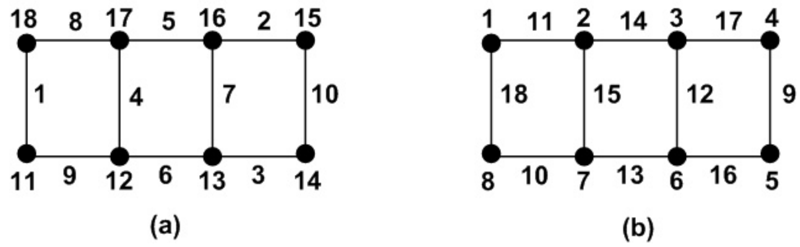

Figure 1: (a). a C_4-magic labeling of L_4 (b). a C_4-supermagic labeling of L_4

wheels, windmills, prisms and theta graphs. Cycles-supermagic labeling of chain graphs kC_n-snake, triangle ladders TL_n, grids $P_m \times P_n$, for $n = 2, 3, 4, 5$, and books B_n can be found in [13]. Maryati et al. [9] proved that some classes of trees such as subdivision of stars, shrubs, and banana tree graphs are P_h-supermagic for some h and prove that certain shackles and amalgamations of a connected graph H are H-supermagic [10].

For $H \cong P_2$, an H-supermagic graph is also called a *super edge-magic graph*. The notion of a super edge-magic graph was introduced by Enomoto at al. [2] as a particular type of edge-magic graph given by Kotzig and Rosa [6]. There are many graphs that have been proved to be (super) edge-magic graphs, see for instance [3, 11, 12, 14, 15]. For further information about (super) edge-magic graphs, see [4]. The H-magic labeling is related to a face-magic labeling of a plane graph introduced by Lih [7]. A total labeling f of a plane graph is said to be *face-magic* if for every positive integer s, all s-sided faces have the same weight. The weight of a face under a labeling f is the sum of labels carried by the edges and vertices surrounding it. Lih [7] allows different weights for different s. If a plane graph G contains only n-sided faces, then face-magic labeling of G is also C_n-magic labeling. Other results about this labeling can be found in [1].

In this paper, we study C_m and F_m-supermagic labelings of fans F_n, and C_m and L_m-supermagic labelings of ladders L_n for all possible values of m and n.

§2. Supermagic coverings of fans

In this section we consider C_m and F_m-supermagic labelings of the fans F_n. We define the fans $F_n \cong P_n + \{c\}$ as a graph with

$$V(F_n) = \{c, x_i | i = 1, 2, 3, \ldots, n\}$$

and

$$E(F_n) = \{cx_i | i = 1, 2, 3, \ldots, n\} \cup \{x_i x_{i+1} | i = 1, 2, 3, \ldots, n-1\}.$$

In [7], Lih proved that F_n is C_3-supermagic for every n except $n \equiv 2 \pmod 4$. Furthermore, Ngurah et al. [13] proved that F_n is C_3-supermagic for any $n \geq 2$. In the following theorem, we show that F_n is C_m-supermagic for any integer $4 \leq m \leq \lfloor \frac{n+4}{2} \rfloor$.

Theorem 1. *Let $n \geq 4$ be a positive integer. Then the fan F_n is C_m-supermagic for any integer $4 \leq m \leq \lfloor \frac{n+4}{2} \rfloor$.*

Proof. First, label every vertex in the following way.

- Label the vertex c with 1.

Case 1: $n \equiv 0 \pmod{m-1}$

- Label x_1, x_m, x_{2m-1}, x_{3m-2}, x_{4m-3}, …, x_{n-m+2} with $2, 3, 4, 5, 6, \ldots$, $\frac{n}{m-1} + 1$, respectively.
- For $1 \leq k \leq m-2$, label x_{1+k}, x_{m+k}, x_{2m+k-1}, x_{3m+k-2}, x_{4m+k-3}, …, $x_{n-m+k+2}$ with $k(\frac{n}{m-1}) + 2$, $k(\frac{n}{m-1}) + 3$, $k(\frac{n}{m-1}) + 4$, $k(\frac{n}{m-1}) + 5$, $k(\frac{n}{m-1}) + 6$, …, $(k+1)(\frac{n}{m-1}) + 1$, respectively.

Case 2: $n \equiv 1 \pmod{m-1}$

- Label x_1, x_m, x_{2m-1}, x_{3m-2}, x_{4m-3}, …, x_{n-m+1}, x_n with $2, 3, 4, 5, 6$, …, $\frac{n-1}{m-1} + 1$, $\frac{n-1}{m-1} + 2$, respectively.
- For $1 \leq k \leq m-2$, label x_{1+k}, x_{m+k}, x_{2m+k-1}, x_{3m+k-2}, x_{4m+k-3}, …, $x_{n-m+k+1}$ with $k(\frac{n-1}{m-1}) + 3$, $k(\frac{n-1}{m-1}) + 4$, $k(\frac{n-1}{m-1}) + 5$, $k(\frac{n-1}{m-1}) + 6$, $k(\frac{n-1}{m-1}) + 7$, …, $(k+1)(\frac{n-1}{m-1}) + 2$, respectively.

Case 3: $n \equiv t \pmod{m-1}$, where $t = 2, 3, 4, \ldots, m-2$

- Label x_1, x_m, x_{2m-1}, x_{3m-2}, x_{4m-3}, …, $x_{n-m-t+2}$, x_{n-t+1} with $2, 3, 4, 5, 6$, …, $\frac{n-t}{m-1} + 1$, $\frac{n-t}{m-1} + 2$, respectively.
- Label x_{n-t+2}, x_{n-t+3}, x_{n-t+4}, …, x_n with $2(\frac{n-t}{m-1}) + 3$, $3(\frac{n-t}{m-1}) + 4$, $4(\frac{n-t}{m-1}) + 5$, …, $t(\frac{n-t}{m-1}) + t + 1$, respectively.
- For $1 \leq k \leq t$, label x_{1+k}, x_{m+k}, x_{2m+k-1}, x_{3m+k-2}, x_{4m+k-3}, …, $x_{n-m-t+k+2}$ with $\gamma_1^k + 2$, $\gamma_1^k + 3$, $\gamma_1^k + 4$, $\gamma_1^k + 5$, $\gamma_1^k + 6$, …, $\gamma_1^k + (\frac{n-t}{m-1} + 1)$, respectively, where $\gamma_1^k = k(\frac{n-t}{m-1} + 1)$.

- For $t+1 \leq k \leq m-2$, label x_{1+k}, x_{m+k}, x_{2m+k-1}, x_{3m+k-2}, x_{4m+k-3}, $\ldots$, $x_{n-m-t+k+2}$ with γ_2^k+2, γ_2^k+3, γ_2^k+4, γ_2^k+5, γ_2^k+6, $\ldots$, $\gamma_2^k+(\frac{n-t}{m-1})+1$, respectively, where $\gamma_2^k = k(\frac{n-t}{m-1}) + t$.

Next, label every edge as follows.

- For $1 \leq i \leq n$, label cx_i with $3n+1-i$.

For labeling the remaining edges, let $e_i = x_i x_{i+1}$, $1 \leq i \leq n-1$, and let $q = n-1$.

Case 1: $q \equiv 0 \pmod{m-2}$

- Label e_1, e_{m-1}, e_{2m-3}, e_{3m-5}, e_{4m-7}, $\ldots$, e_{q-m+3} with $n+2$, $n+3$, $n+4$, $n+5$, $n+6$, $\ldots$, $n+\frac{q}{m-2}+1$, respectively.

- For $1 \leq k \leq m-3$, label e_{1+k}, e_{m-1+k}, e_{2m-3+k}, e_{3m-5+k}, e_{4m-7+k}, $\ldots$, $e_{q-m+3+k}$ with γ_3^k+2, γ_3^k+3, γ_3^k+4, γ_3^k+5, γ_3^k+6, $\ldots$, $\gamma_3^k+(\frac{q}{m-2})+1$, respectively, where $\gamma_3^k = k(\frac{q}{m-2}) + n$.

Case 2: $q \equiv 1 \pmod{m-2}$

- Label e_1, e_{m-1}, e_{2m-3}, e_{3m-5}, e_{4m-7}, $\ldots$, e_q with $n+2$, $n+3$, $n+4$, $n+5$, $n+6$, $\ldots$, $n+\frac{q-1}{m-2}+2$, respectively.

- For $1 \leq k \leq m-3$, label e_{1+k}, e_{m-1+k}, e_{2m-3+k}, e_{3m-5+k}, e_{4m-7+k}, $\ldots$, $e_{q-m+2+k}$ with γ_4^k+3, γ_4^k+4, γ_4^k+5, γ_4^k+6, γ_4^k+7, $\ldots$, $\gamma_4^k+(\frac{q-1}{m-2})+2$, respectively, where $\gamma_4^k = k(\frac{q-1}{m-2}) + n$.

Case 3: $q \equiv t \pmod{m-2}$, where $t = 2, 3, 4, \ldots, m-3$

- Label e_1, e_{m-1}, e_{2m-3}, e_{3m-5}, e_{4m-7}, $\ldots$, e_{q-t+1} with $n+2$, $n+3$, $n+4$, $n+5$, $n+6$, $\ldots$, $n+\frac{q-t}{m-2}+2$, respectively.

- Label e_{q-t+2}, e_{q-t+2}, e_{q-t+3}, $\ldots$, e_q with $n+2(\frac{q-t}{m-2})+3$, $n+3(\frac{q-t}{m-2})+4$, $n+4(\frac{q-t}{m-2})+5$, $\ldots$, $n+t(\frac{q-t}{m-2})+t+1$, respectively.

- For $1 \leq k \leq t$, label e_{1+k}, e_{m-1+k}, e_{2m-3+k}, e_{3m-5+k}, e_{4m-7+k}, $\ldots$, $e_{q-t-m+3+k}$ with γ_5^k+2, γ_5^k+3, γ_5^k+4, γ_5^k+5, γ_5^k+6, $\ldots$, $\gamma_5^k+\frac{q-t}{m-2}+1$, respectively,$\gamma_5^k = k(\frac{q-t}{m-2}+1) + n$.

- For $t+1 \leq k \leq m-3$, label e_{1+k}, e_{m-1+k}, e_{2m-3+k}, e_{3m-5+k}, e_{4m-7+k}, $\ldots$, $e_{q-t-m+3+k}$ with γ_6^k+2, γ_6^k+3, γ_6^k+4, γ_6^k+5, γ_6^k+6, $\ldots$, $\gamma_6^k+\frac{q-t}{m-2}+1$, respectively, $\gamma_6^k = k(\frac{q-t}{m-2}) + n + t$.

Let us denote the total labeling defined above by h. It can be checked that $h(V(F_n)) = \{1, 2, 3, \ldots, n+1\}$; for $1 \leq i \leq n-m+1$, $h(x_i) = h(x_{i+m-1}) - 1$, $h(x_i x_{i+1}) = h(x_{i+m-2} x_{i+m-1}) - 1$, and $h(cx_i) + h(cx_{i+m-2}) = h(cx_{i+1}) + h(cx_{i+m-1}) + 2$.

For $1 \leq i \leq n-m+2$, let $C_m^{(i)}$ be the subcycle of F_n with $V(C_m^{(i)}) = \{c, x_j | i \leq j \leq i+m-2\}$ and $E(C_m^{(i)}) = \{cx_i, cx_{i+m-2}\} \cup \{x_j x_{j+1} | i \leq j \leq i+m-3\}$. It is easy to verify that for $1 \leq i \leq n-m+1$, $\sum h(C_m^{(i)}) = \sum h(C_m^{(i+1)})$. Thus, for $1 \leq i \leq n-m+2$, $\sum h(C_m^{(i)})$ is constant. Hence, F_n is C_m-supermagic for any integer $4 \leq m \leq \lfloor \frac{n+4}{2} \rfloor$. □

Next, we consider fan-supermagic labelings of fan. Notice that F_n is $C_3 \cong F_2$-supermagic [13] and F_n is trivially F_n-supermagic. In the following theorem, we show that F_n is F_m-supermagic for all remaining possible values of m.

Theorem 2. *Let $n \geq 4$ be a positive integer. The fan F_n is F_m-supermagic for every integer $3 \leq m \leq n-1$.*

Proof. Define a total labeling of F_n as follows.

- For $1 \leq i \leq n-1$, label $x_i x_{i+1}$ with $n+1+i$.
- For $1 \leq i \leq n$, label cx_i with $3n+1-i$.
- Label the vertex c with 1.

For the remaining vertices, we consider three following cases.

Case 1: $n \equiv 0 \pmod{m}$

- Label x_1, x_{m+1}, x_{2m+1}, x_{3m+1}, $\ldots$, x_{n-m+1} with $2, 3, 4, 5, \ldots, \frac{n}{m}+1$, respectively.
- For $1 \leq k \leq m-1$, label x_{1+k}, x_{m+1+k}, x_{2m+1+k}, x_{3m+1+k}, $\ldots$, $x_{n-m+1+k}$ with $k(\frac{n}{m})+2, k(\frac{n}{m})+3, k(\frac{n}{m})+4, k(\frac{n}{m})+5, \ldots, (k+1)(\frac{n}{m})+1$, respectively.

Case 2: $n \equiv 1 \pmod{m}$

- Label x_1, x_{m+1}, x_{2m+1}, x_{3m+1}, $\ldots$, x_{n-m}, x_n with $2, 3, 4, 5, \ldots, \frac{n-1}{m}+1$, $\frac{n-1}{m}+2$, respectively.
- For $1 \leq k \leq m-1$, label x_{1+k}, x_{m+1+k}, x_{2m+1+k}, x_{3m+1+k}, $\ldots$, x_{n-m+k} with $k(\frac{n-1}{m})+3, k(\frac{n-1}{m})+4, k(\frac{n-1}{m})+5, k(\frac{n-1}{m})+6, \ldots, (k+1)(\frac{n-1}{m})+2$, respectively.

Case 3: $n \equiv t \pmod{m}$, where $t = 2, 3, 4, \ldots, m-1$

- Label $x_1,\ x_{m+1},\ x_{2m+1},\ x_{3m+1},\ \dots,\ x_{n-t+1}$ with $2, 3, 4, 5, \dots, \frac{n-t}{m} + 2$, respectively.
- Label $x_{n-t+2},\ x_{n-t+3},\ x_{n-t+4},\ \dots,\ x_n$ with $2(\frac{n-t}{m}) + 3,\ 3(\frac{n-t}{m}) + 4,\ 4(\frac{n-t}{m}) + 5, \dots, t(\frac{n-t}{m}) + t + 1$, respectively.
- For $1 \le k \le t$, label $x_{1+k},\ x_{m+1+k},\ x_{2m+1+k},\ x_{3m+1+k},\ \dots,\ x_{n-m-t+1+k}$ with $k(\frac{n-t}{m}+1)+2, k(\frac{n-t}{m}+1)+3, k(\frac{n-t}{m}+1)+4, k(\frac{n-t}{m}+1)+5, \dots, (k+1)(\frac{n-t}{m}+1)$, respectively.
- For $t + 1 \le k \le m - 1$, label $x_{1+k},\ x_{m+1+k},\ x_{2m+1+k},\ x_{3m+1+k},\ \dots,\ x_{n-m-t+1+k}$ with $k(\frac{n-t}{m}) + t + 2, k(\frac{n-t}{m}) + t + 3, k(\frac{n-t}{m}) + t + 4, k(\frac{n-t}{m}) + t + 5, \dots, (k + 1)(\frac{n-t}{m}) + t + 1$, respectively.

Denote the total labeling defined above by f. It can be checked that $f(V(F_n)) = \{1, 2, 3, \dots, n + 1\}$; for $1 \le i \le n - m + 1$,

$$\sum_{j=i}^{i+m-1} f(x_j) = -1 + \sum_{j=i+1}^{i+m} f(x_j),$$

$$\sum_{j=i}^{i+m-2} f(x_j x_{j+1}) = 1 - m + \sum_{j=i+1}^{i+m-1} f(x_j x_{j+1}),$$

and

$$\sum_{j=i}^{i+m-1} f(cx_j) = m + \sum_{j=i+1}^{i+m} f(cx_j).$$

For $1 \le i \le n-m+1$, let $F_m^{(i)}$ be the subfan of F_n with $V(F_m^{(i)}) = \{c, x_j | i \le j \le i + m - 1\}$ and $E(F_m^{(i)}) = \{x_j x_{j+1} | i \le j \le i + m - 2\} \cup \{cx_j | i \le j \le i + m - 1\}$. It is a routine procedure to verify that for $1 \le i \le n - m$, $\sum f(F_m^{(i)}) = \sum f(F_m^{(i+1)})$. So, f is an F_m-supermagic labeling of F_n. Hence, F_n is F_m-supermagic. □

In Figure 2, we show a C_4-supermagic labeling of F_8 and an F_4-supermagic labeling of F_{10} as defined in the proof of Theorems 1 and 2, respectively.

§3. Supermagic coverings of ladders

Let $L_n \cong P_n \times P_2$ denote the ladder of order $2n$ and size $3n - 2$. Clearly L_n admits a cycle covering of some even order. As a direct consequence of Lladó and Moragas's result (see Theorem 7 [8]), L_n is C_4-supermagic for odd n. Later, Ngurah et al. [13] solved for the remaining cases. In the next theorem, we show that L_n is also C_{2m}-supermagic for the remaining possible values of m.

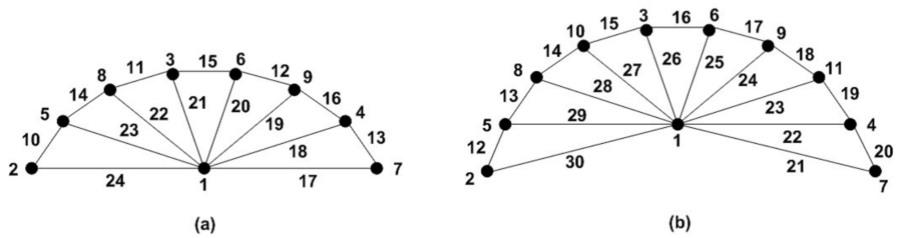

Figure 2: (a). a C_4-supermagic labeling of F_8 (b). an F_4-supermagic labeling of F_{10}

Theorem 3. *Let $n \geq 4$ be a positive integer. Then the ladder L_n is C_{2m}-supermagic for every integer $3 \leq m \leq \lfloor \frac{n}{2} \rfloor + 1$.*

Proof. First, let L_n be a graph with

$$V(L_n) = \{x_i, y_i | 1 \leq i \leq n\}$$

and

$$E(L_n) = \{x_i x_{i+1}, y_i y_{i+1} | 1 \leq i \leq n-1\} \cup \{x_i y_i | 1 \leq i \leq n\}.$$

Next, label every edge in the following way.

- For $1 \leq i \leq n-1$, label $x_i x_{i+1}$ with $2n + i$.
- For $1 \leq i \leq n-1$, label $y_i y_{i+1}$ with $4n - 1 - i$.
- For $1 \leq i \leq n$, label $x_i y_i$ with $5n - 1 - i$.

Label every vertex in the following way.

Case 1: $n \equiv 0 \pmod m$

- Label $x_1, x_{m+1}, x_{2m+1}, x_{3m+1}, \ldots, x_{n-m+1}$ with $1, 2, 3, 4, \ldots, \frac{n}{m}$, respectively.
- For $1 \leq k \leq m-1$, label $x_{1+k}, x_{m+1+k}, x_{2m+1+k}, x_{3m+1+k}, \ldots, x_{n-m+1+k}$ with $k(\frac{n}{m})+1, k(\frac{n}{m})+2, k(\frac{n}{m})+3, k(\frac{n}{m})+4, \ldots, (k+1)(\frac{n}{m})$, respectively.

Case 2: $n \equiv 1 \pmod m$

- Label $x_1, x_{m+1}, x_{2m+1}, x_{3m+1}, \ldots, x_{n-m}, x_n$ with $1, 2, 3, 4, \ldots, (\frac{n-1}{m})$, $(\frac{n-1}{m}) + 1$, respectively.
- For $1 \leq k \leq m-1$, label $x_{1+k}, x_{m+1+k}, x_{2m+1+k}, x_{3m+1+k}, \ldots, x_{n-m+k}$ with $k(\frac{n-1}{m})+2, k(\frac{n-1}{m})+3, k(\frac{n-1}{m})+4, k(\frac{n-1}{m})+5, \ldots, (k+1)(\frac{n-1}{m})+1$, respectively.

Case 3: $n \equiv t \pmod{m}$, where $t = 2, 3, 4, \ldots, m-1$

- Label $x_1, x_{m+1}, x_{2m+1}, x_{3m+1}, \ldots, x_{n-t+1}$ with $1, 2, 3, 4, \ldots, (\frac{n-t}{m})+1$, respectively.
- Label $x_{n-t+2}, x_{n-t+3}, \ldots, x_n$ with $2(\frac{n-t}{m}+1), 3(\frac{n-t}{m}+1), \ldots, t(\frac{n-t}{m}+1)$, respectively.
- For $1 \leq k \leq t$, label x_{1+k}, x_{m+1+k}, x_{2m+1+k}, x_{3m+1+k}, $\ldots$, $x_{n-m-t+1+k}$ with $k(\frac{n-t}{m}+1)+1$, $k(\frac{n-t}{m}+1)+2$, $k(\frac{n-t}{m}+1)+3$, $k(\frac{n-t}{m}+1)+4$, $\ldots$, $k(\frac{n-t}{m}+1)+\frac{n-t}{m}$, respectively.
- For $t+1 \leq k \leq m-1$, label x_{1+k}, x_{m+1+k}, x_{2m+1+k}, x_{3m+1+k}, $\ldots$, $x_{n-m-t+1+k}$ with $k(\frac{n-t}{m})+t+1$, $k(\frac{n-t}{m})+t+2$, $k(\frac{n-t}{m})+t+3$, $k(\frac{n-t}{m})+t+4$, $\ldots$, $(k+1)(\frac{n-t}{m})+t$, respectively.

Finally, for $1 \leq i \leq n$, label y_i with $n +$ (the label of x_i).

Let us denote the total labeling defined above by f. It can be checked that $f(V(L_n)) = \{1, 2, 3, \ldots, 2n\}$; for $1 \leq i \leq n-m$,

$$\begin{aligned} f(x_i) + f(y_i) &= f(x_{m+i}) + f(y_{m+i}) - 2, \\ f(x_iy_i) + f(x_{m+i-1}y_{m+i-1}) &= f(x_{i+1}y_{i+1}) + f(x_{m+i}y_{m+i}) + 2; \end{aligned}$$

for $1 \leq i \leq n-2$,

$$f(x_ix_{i+1}) + f(y_iy_{i+1}) = f(x_{i+1}x_{i+2}) + f(y_{i+1}y_{i+2}).$$

For $1 \leq i \leq n-m+1$, let $C_{2m}^{(i)}$, be the subcycle of L_n with

$$V(C_{2m}^{(i)}) = \{x_j, y_j | i \leq j \leq i+m-1\}$$

and

$$E(C_{2m}^{(i)}) = \{x_jx_{j+1}, y_jy_{j+1} | i \leq j \leq i+m-2\} \cup \{x_iy_i, x_{i+m-1}y_{i+m-1}\}.$$

It is easy to verify that $V(C_{2m}^{(i)}) \cap V(C_{2m}^{(i+1)}) = \{x_j, y_j | i+1 \leq j \leq i+m-1\}$ and $E(C_{2m}^{(i)}) \cap E(C_{2m}^{(i+1)}) = \{x_jx_{j+1}, y_jy_{j+1} | i+1 \leq j \leq i+m-2\}$.

By using these facts, for $1 \leq i \leq n-m$, we obtain

$$\begin{aligned} \sum f(C_{2m}^{(i)}) &= \textstyle\sum_{j=i}^{i+m-1}[f(x_j) + f(y_j)] + \sum_{j=i}^{i+m-2}[f(x_jx_{j+1}) + f(y_jy_{j+1})] + \\ &\quad f(x_iy_i) + f(x_{i+m-1}y_{i+m-1}) \\ &= \textstyle\sum_{j=i+1}^{i+m}[f(x_j) + f(y_j)] + \sum_{j=i+1}^{i+m-1}[f(x_jx_{j+1}) + f(y_jy_{j+1})] + \\ &\quad f(x_{i+1}y_{i+1}) + f(x_{i+m}y_{i+m}) \\ &= \textstyle\sum f(C_{2m}^{(i+1)}). \end{aligned}$$

So, for $1 \leq i \leq n-m+1$, $\sum f(C_{2m}^{(i)})$ is constant. Hence, f is a C_{2m}-supermagic labeling of L_n. $\square$

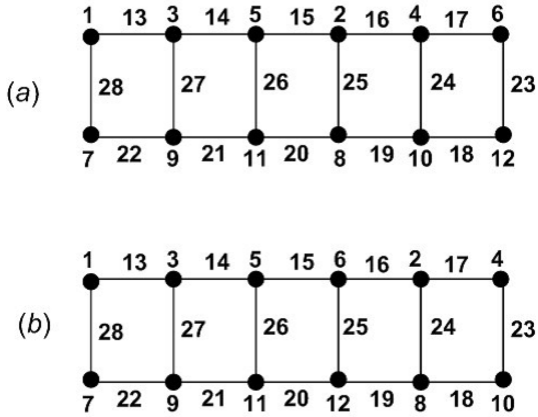

Figure 3: (a). a C_6-supermagic labeling of L_6, (b). a C_8-supermagic labeling of L_6

In Figure 3 we show a C_6-supermagic labeling and a C_8-supermagic labeling of L_6 as defined in the proof of Theorem 3.

Next, we consider a L_m-supermagic labeling of L_n. Notice that, L_n is $L_2 \cong C_4$-supermagic and L_n is trivially L_n-supermagic. So, in the following theorem, we consider a L_m-supermagic labeling of L_n for any integer $3 \leq m \leq n-1$.

Theorem 4. *Let $n \geq 4$ be a positive integer. Then the ladder L_n is L_m-supermagic for every integer $3 \leq m \leq n-1$.*

Proof. For proving this theorem, we define the ladder L_n as a graph with $V(L_n) = \{x_i, y_i | 1 \leq i \leq n\}$ and $E(L_n) = \{x_i x_{i+1}, y_i y_{i+1} | 1 \leq i \leq n-1\} \cup \{x_i y_{n+1-i} | 1 \leq i \leq n\}$.

Define a total labeling of L_n in the following way.

- For $1 \leq i \leq n$, label x_i as in the proof of Theorem 3.
- For $1 \leq i \leq n$, label y_i with $n +$ (the label of x_{n+1-i}).
- For $1 \leq i \leq n-1$, label $x_i x_{i+1}$ with $2n + i$.
- For $1 \leq i \leq n$, label $x_i y_{n+1-i}$ with $5n - 1 - i$.

For labeling $y_i y_{i+1}$, we consider two following cases. First, let $q = n - 2$.
Case 1: $q \equiv 0 \pmod{m-1}$

- Label $y_1 y_2$, $y_m y_{m+1}$, $y_{2m-1} y_{2m}$, $y_{3m-2} y_{3m-1}$, $y_{4m-3} y_{4m-2}$, $\ldots$, $y_{n-m} y_{n-m+1}$, $y_{n-1} y_n$ with $3n$, $3n+1$, $3n+2$, $3n+3$, $3n+4$, $3n+5$, $\ldots$, $3n + (\frac{q}{m-1} - 1)$, $3n + (\frac{q}{m-1})$, respectively.

- For $1 \le k \le m-2$, label $y_{1+k}y_{2+k}$, $y_{m+k}y_{m+k+1}$, $y_{2m+k-1}y_{2m+k}$, $y_{3m+k-2}y_{3m+k-1}$, $y_{4m+k-3}y_{4m+k-2}$, $\cdots$, $y_{n-m+k}y_{n-m+k+1}$ with $3n+k(\frac{q}{m-1})+1$, $3n+k(\frac{q}{m-1})+2$, $3n+k(\frac{q}{m-1})+3$, $3n+k(\frac{q}{m-1})+4$, $3n+k(\frac{q}{m-1})+5$, $\ldots$, $3n+(k+1)(\frac{q}{m-1})$, respectively.

Case 2: $q \equiv t \pmod{m-1}$, where $t = 1, 2, 3, \ldots, m-2$

- Label y_1y_2, y_my_{m+1}, $y_{2m-1}y_{2m}$, $y_{3m-2}y_{3m-1}$, $y_{4m-3}y_{4m-2}$, $\cdots$, $y_{q-t-m+2}y_{q-t-m+3}$, $y_{q-t+1}y_{q-t+2}$ with $3n$, $3n+1$, $3n+2$, $3n+3$, $3n+4$, $3n+5$, $\ldots$, $3n+(\frac{q-t}{m-1}-1)$, $3n+(\frac{q-t}{m-1})$, respectively.

- Label $y_\alpha y_{\alpha+1}$, $y_{\alpha+1}y_{\alpha+2}$, $y_{\alpha+2}y_{\alpha+3}$, $\cdots$, $y_{\alpha+t-1}y_{\alpha+t}$ with $3n+2(\frac{q-t}{m-1})+1$, $3n+3(\frac{q-t}{m-1})+2$, $3n+4(\frac{q-t}{m-1})+3$, $\ldots$, $3n+(t+1)(\frac{q-t}{m-1})+t$, respectively, where $\alpha = q-t+2$.

- For $1 \le k \le t+1$, label $y_{k+1}y_{k+2}$, $y_{m+k}y_{m+k+1}$, $y_{2m+k-1}y_{2m+k}$, $y_{3m+k-2}y_{3m+k-1}$, $y_{4m+k-3}y_{4m+k-2}$, $\cdots$,$y_{q-t-m+k+2}y_{q-t-m+k+3}$, with β_1^k, β_1^k+1, β_1^k+2, β_1^k+3, β_1^k+4, $\ldots$, $\beta_1^k+(\frac{q-t}{m-1}-1)$, respectively, where $\beta_1^k = k(\frac{q-t}{m-1}+1)+3n$.

- For $t+2 \le k \le m-2$, label $y_{k+1}y_{k+2}$, $y_{m+k}y_{m+k+1}$, $y_{2m+k-1}y_{2m+k}$, $y_{3m+k-2}y_{3m+k-1}$, $y_{4m+k-3}y_{4m+k-2}$, $\cdots$,$y_{q-t-m+k+2}y_{q-t-m+k+3}$, with β_2^k, β_2^k+1, β_2^k+2, β_2^k+3, β_2^k+4, $\ldots$, $\beta_2^k+(\frac{q-t}{m-1}-1)$, respectively, where $\beta_2^k = k(\frac{q-t}{m-1})+3n+t+1$.

Let us denote the labeling defined above by g. For $1 \le i \le n-m+1$, it can be checked that

$$\sum_{j=i}^{i+m-1}[g(x_j)+g(y_{n+1-j})] = -2+\sum_{j=i+1}^{i+m}[g(x_j)+g(y_{n+1-j})],$$

$$\sum_{j=i}^{i+m-2} g(x_jx_{j+1}) = 1-m+\sum_{j=i+1}^{i+m-1} g(x_jx_{j+1}),$$

$$\sum_{j=i}^{i+m-2} g(y_{n+1-j}y_{n-j}) = 1+\sum_{j=i+1}^{i+m-1} g(y_{n+1-j}y_{n-j}),$$

and

$$\sum_{j=i}^{i+m-1} g(x_jy_{n+1-j}) = m+\sum_{j=i+1}^{i+m} g(x_jy_{n+1-j}).$$

For $1 \le i \le n-m+1$, let $L_m^{(i)}$ be the subladder of L_n with $V(L_m^{(i)}) = \{x_j, y_{n+1-j} | i \le j \le m+i-1\}$ and $E(L_m^{(i)}) = \{x_jx_{j+1}, y_{n+1-j}y_{n-j} | i \le j \le m+i-2\} \cup \{x_jy_{n+1-j} | i \le j \le m+i-1\}$.

In a similar way as in the proof of Theorem 3, for $1 \leq i \leq n-m$, it is easy to verify that $\sum g(L_m^{(i)}) = \sum g(L_m^{(i+1)})$.

So, $\sum g(L_m^{(i)})$ is constant for all possible values of i. Hence, L_n is L_m-supermagic for every integer $3 \leq m \leq n-1$. □

An example of the labeling obtained in the above proof is showed in Figure 4.

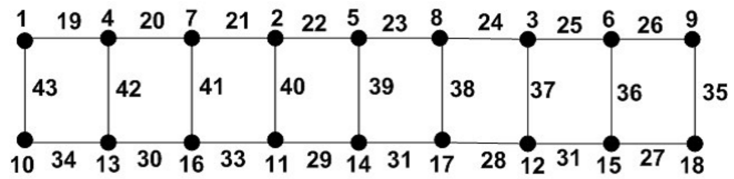

Figure 4: an L_3-supermagic labeling of L_9

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