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## Symmetry properties of lightlike hypersurfaces in indefinite Sasakian manifolds

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**Abstract.** In this paper, we investigate symmetry properties of lightlike hypersurfaces in indefinite Sasakian manifolds, tangent to the structure vector field. Theorems on locally symmetric, semi-symmetric and Ricci semi-symmetric lightlike hypersurfaces are obtained. We show that, under some conditions, these hypersurfaces are totally geodesic. The non-existence conditions of specific lightlike hypersurfaces are given. We prove, under a certain condition, that in lightlike hypersurfaces of an indefinite Sasakian space form, tangent to the structure vector field, the local symmetry and semi-symmetry notions are equivalent. This equivalence is extended to the Ricci semi-symmetry notion when the lightlike hypersurfaces are considered to be  $\eta$ -totally umbilical.

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### §1. Introduction

It is natural to impose condition on semi-Riemannian manifold that its Riemannian curvature tensor  $R$  be parallel, that is, have vanishing covariant differential,  $\nabla R = 0$ , where  $\nabla$  is the Levi-Civita connection on semi-Riemannian manifold and  $R$  is the corresponding curvature tensor. Such a manifold is said to be *locally symmetric*. This class of manifolds contains one of manifolds of constant curvature. A semi-Riemannian manifold is called semi-symmetric, if  $R \cdot R = 0$ , which is the integrability condition of  $\nabla R = 0$ . The semi-symmetric manifolds have been classified, in Riemannian case, by Szabo in [19] and [20]. A semi-Riemannian manifold is called Ricci semi-symmetric, if  $R \cdot Ric = 0$ .

We are interested to answer to the following question: “Are conditions  $\nabla R = 0$  and  $R \cdot R = 0$  equivalent on lightlike hypersurfaces of semi-Riemannian

manifolds?” This equivalence is not true in general. Ryan [17] raised the following question for hypersurfaces of Euclidean spaces in 1972: Are conditions  $R \cdot R = 0$  and  $R \cdot Ric = 0$  equivalent for hypersurfaces of Euclidean spaces? However, there are many results which contributed to the solution of the above question in the affirmative under some conditions (see [5], [6], [16] and references therein). In [1], the authors gave an explicit example of a hypersurface in Euclidean space  $E^{n+1}$  ( $n \geq 4$ ) that is Ricci semi-symmetric but not semi-symmetric (see [7] for another example). This result shows that the conditions  $R \cdot R = 0$  and  $R \cdot Ric = 0$  also are not equivalent for hypersurfaces of Euclidean space in general. In [7] a survey on Ricci semi-symmetric spaces and contributions to the solution of above problem are given. In virtue of results given by Günes, Sahin and Kiliç ([10], Theorem 3.1) and Sahin ([18], Theorem 4.2), we see that the conditions  $\nabla R = 0$  and  $R \cdot R = 0$  are equivalent for lightlike hypersurfaces of semi-Euclidean space under conditions  $Ric(E, X) = 0$  and  $A_N E$  a vector field non-null. In this paper we give an affirmative answer to this question for lightlike hypersurfaces of an indefinite Sasakian space form  $\overline{M}(c)$ , under some conditions (Theorem 8 and Theorem 12).

The general theory of lightlike submanifolds was introduced and presented in [9] by K.L. Duggal and A. Bejancu. The theory of lightlike submanifolds is a new area of differential geometry and it is very different from Riemannian geometry as well as semi-Riemannian geometry.

In the present paper, we study the symmetry properties of lightlike hypersurfaces in indefinite Sasakian manifolds, tangent to the structure vector field, by particularly paying attention to the locally symmetric, semi-symmetric and Ricci semi-symmetric lightlike hypersurfaces. The paper is organized as follows. In section 2, we recall some basic definitions and formulas for indefinite Sasakian manifolds supported by an example and lightlike hypersurfaces of semi-Riemannian manifolds. In section 3, we give the decomposition of almost contact metrics of lightlike hypersurfaces in indefinite Sasakian manifolds which are tangential to the structure vector field. In section 4, we consider a lightlike hypersurface  $M$  of an indefinite Sasakian space form  $\overline{M}(c)$  and study local symmetry conditions on this hypersurface. It is known (cf. [10]) that in locally symmetric semi-Riemannian manifold  $\overline{M}$ , the locally symmetric lightlike hypersurfaces are totally geodesic, under condition that the vector field  $A_N E$  is non-null. Here we show that there are no locally symmetric lightlike hypersurfaces in indefinite Sasakian space form  $\overline{M}(c \neq 1)$ . On the other hand we prove that, in indefinite Sasakian space form  $\overline{M}(c = 1)$ , any lightlike hypersurface is totally geodesic (Theorem 2). We give some theorems on totally contact umbilical,  $\eta$ -totally umbilical lightlike hypersurfaces of an indefinite Sasakian space form. We also prove, in the same section, that local symmetry property of a screen integrable lightlike hypersurface of an indefinite Sasakian space form is related with local symmetry property of leaves of its screen

distribution (Theorem 5). In section 5, we study semi-symmetric lightlike hypersurfaces of indefinite Sasakian space forms. We give a characterization of semi-symmetric lightlike hypersurfaces and we prove, under a certain condition, that in lightlike hypersurfaces of an indefinite Sasakian space form, tangent to the structure vector field, the local symmetry and semi-symmetry notions are equivalent (Theorem 8). In section 6, we give a characterization of Ricci semi-symmetric lightlike hypersurfaces of an indefinite Sasakian space form, tangent to the structure vector field. We show that, under a certain condition, a Ricci semi-symmetric lightlike hypersurfaces of indefinite Sasakian space form  $\bar{M}(c = 1)$  are totally geodesic (Theorem 11). By Theorem 12, we extend the equivalence given in Theorem 8 to Ricci semi-symmetry notion when the lightlike hypersurfaces are considered to be  $\eta$ -totally umbilical.

§2. Preliminaries

Let  $\bar{M}$  be a  $(2n + 1)$ -dimensional manifold endowed with an almost contact structure  $(\bar{\phi}, \xi, \eta)$ , i.e.  $\bar{\phi}$  is a tensor field of type  $(1, 1)$ ,  $\xi$  is a vector field, and  $\eta$  is a 1-form satisfying

$$(2.1) \quad \bar{\phi}^2 = -\mathbb{I} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \bar{\phi} = 0, \quad \bar{\phi}\xi = 0 \text{ and } \text{rank } \bar{\phi} = 2n.$$

Then  $(\bar{\phi}, \xi, \eta, \bar{g})$  is called an almost contact metric structure on  $\bar{M}$  if  $(\bar{\phi}, \xi, \eta)$  is an almost contact structure on  $\bar{M}$  and  $\bar{g}$  is a semi-Riemannian metric on  $\bar{M}$  such that, for any vector field  $\bar{X}, \bar{Y}$  on  $\bar{M}$ ,

$$(2.2) \quad \begin{aligned} \bar{g}(\xi, \xi) &= \varepsilon = \pm 1, \quad \eta(\bar{X}) = \varepsilon \bar{g}(\xi, \bar{X}), \\ \bar{g}(\bar{\phi}\bar{X}, \bar{\phi}\bar{Y}) &= \bar{g}(\bar{X}, \bar{Y}) - \varepsilon \eta(\bar{X})\eta(\bar{Y}). \end{aligned}$$

If, moreover,  $d\eta(\bar{X}, \bar{Y}) = -\bar{g}(\bar{\phi}\bar{X}, \bar{Y})$  and  $(\bar{\nabla}_{\bar{X}}\bar{\phi})\bar{Y} = \bar{g}(\bar{X}, \bar{Y})\xi - \varepsilon\eta(\bar{Y})\bar{X}$ , where  $\bar{\nabla}$  is the Levi-Civita connection for the semi-Riemannian metric  $\bar{g}$ , we call  $\bar{M}$  an *indefinite Sasakian manifold*. From the first equation of (2.2),  $\xi$  is never a lightlike vector field on  $\bar{M}$ .

Sasakian manifolds with indefinite metrics have been first considered by Takahashi [21]. Their importance for physics have been point out by Duggal [8]. We have two classes of indefinite Sasakian manifolds [8]:  $\xi$  is spacelike ( $\varepsilon = 1$  and the index of  $\bar{g}$  is an even number  $\nu = 2r$ ) and  $\xi$  is timelike ( $\varepsilon = -1$  and the index of  $\bar{g}$  is an odd number  $\nu = 2r + 1$ ).

Takahashi [21] shows that it suffices to consider those indefinite almost contact manifolds with space-like  $\xi$ . Hence, from now on, we shall restrict ourselves to the case of  $\xi$  a space-like unit vector (that is  $\bar{g}(\xi, \xi) = 1$ ).

In this case, the equality

$$(\bar{\nabla}_{\bar{X}}\bar{\phi})\bar{Y} = \bar{g}(\bar{X}, \bar{Y})\xi - \eta(\bar{Y})\bar{X}$$

implies  $\bar{\nabla}_{\bar{X}}\xi = -\bar{\phi}(\bar{X})$ ,  $\xi$  is Killing vector field and  $(\bar{\nabla}_{\bar{X}}\eta)\bar{Y} = -\bar{g}(\bar{\phi}\bar{X}, \bar{Y})$  (see [3] for details).

As an example, we have

**Example 1.** Let  $\mathbb{R}^7$  be the 7-dimensional real number space. We consider  $\{x_i\}_{1 \leq i \leq 7}$  as cartesian coordinates on  $\mathbb{R}^7$  and define with respect to the natural field of frames  $\left\{\frac{\partial}{\partial x_i}\right\}$  a tensor field  $\bar{\phi}$  of type (1, 1) by its matrix:

$$(2.3) \quad \begin{aligned} \bar{\phi}\left(\frac{\partial}{\partial x_1}\right) &= \frac{\partial}{\partial x_4}, & \bar{\phi}\left(\frac{\partial}{\partial x_2}\right) &= -\frac{\partial}{\partial x_5}, & \bar{\phi}\left(\frac{\partial}{\partial x_3}\right) &= \frac{\partial}{\partial x_6}, \\ \bar{\phi}\left(\frac{\partial}{\partial x_4}\right) &= -\frac{\partial}{\partial x_1} - x_4\frac{\partial}{\partial x_7}, & \bar{\phi}\left(\frac{\partial}{\partial x_5}\right) &= \frac{\partial}{\partial x_2} + x_5\frac{\partial}{\partial x_7}, \\ \bar{\phi}\left(\frac{\partial}{\partial x_6}\right) &= -\frac{\partial}{\partial x_3} - x_6\frac{\partial}{\partial x_7}, & \bar{\phi}\left(\frac{\partial}{\partial x_7}\right) &= 0. \end{aligned}$$

The differential 1-form  $\eta$  is defined by

$$(2.4) \quad \eta = \frac{1}{2}(dx_7 - x_4dx_1 - x_5dx_2 - x_6dx_3).$$

The vector field  $\xi$  is defined by  $\xi = 2\frac{\partial}{\partial x_7}$ . It is easy to check (2.1) and thus  $(\bar{\phi}, \xi, \eta)$  is an almost contact structure on  $\mathbb{R}^7$ . Finally we define the metric  $\bar{g}$  on  $\mathbb{R}^7$  by

$$(2.5) \quad \begin{aligned} \bar{g} &= \frac{1}{4} \{ (x_4^2 - 1)dx_1^2 + (x_5^2 + 1)dx_2^2 + (x_6^2 + 1)dx_3^2 - dx_4^2 + dx_5^2 + dx_6^2 \\ &\quad + dx_7^2 + x_4x_5dx_1 \otimes dx_2 + x_4x_5dx_2 \otimes dx_1 + x_4x_6dx_1 \otimes dx_3 \\ &\quad + x_4x_6dx_3 \otimes dx_1 + x_5x_6dx_2 \otimes dx_3 + x_5x_6dx_3 \otimes dx_2 - x_4dx_1 \otimes dx_7 \\ &\quad - x_4dx_7 \otimes dx_1 - x_5dx_2 \otimes dx_7 - x_5dx_7 \otimes dx_2 - x_6dx_3 \otimes dx_7 \\ &\quad - x_6dx_7 \otimes dx_3 \}, \end{aligned}$$

with respect to the natural field of frames. It is easy to check that  $\bar{g}$  is a semi-Riemannian metric of index 2 and  $(\bar{\phi}, \xi, \eta, \bar{g})$  given by (2.3)-(2.5) is a Sasakian structure on  $\mathbb{R}^7$ . Therefore,  $(\mathbb{R}^7, \bar{\phi}, \xi, \eta, \bar{g})$  is an indefinite Sasakian space form of constant  $\bar{\phi}$ -sectional curvature  $c = -3$ .

A plane section  $\sigma$  in  $T_p\bar{M}$  is called a  $\bar{\phi}$ -section if it is spanned by  $\bar{X}$  and  $\bar{\phi}\bar{X}$ , where  $\bar{X}$  is a unit tangent vector field orthogonal to  $\xi$ . The sectional curvature of a  $\bar{\phi}$ -section  $\sigma$  is called a  $\bar{\phi}$ -sectional curvature. A Sasakian manifold  $\bar{M}$  with constant  $\bar{\phi}$ -sectional curvature  $c$  is said to be a *Sasakian space form* and is denoted by  $\bar{M}(c)$ . The curvature tensor  $\bar{R}$  of a Sasakian space form  $\bar{M}(c)$  is

given in [15]: for any  $\bar{X}, \bar{Y}, \bar{Z} \in \Gamma(T\bar{M})$ ,

$$\begin{aligned}
 \bar{R}(\bar{X}, \bar{Y})\bar{Z} &= \frac{c+3}{4} (\bar{g}(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{X}, \bar{Z})\bar{Y}) + \frac{c-1}{4} (\eta(\bar{X})\eta(\bar{Z})\bar{Y} \\
 (2.6) \quad &- \eta(\bar{Y})\eta(\bar{Z})\bar{X} + \bar{g}(\bar{X}, \bar{Z})\eta(\bar{Y})\xi - \bar{g}(\bar{Y}, \bar{Z})\eta(\bar{X})\xi \\
 &+ \bar{g}(\bar{\phi}\bar{Y}, \bar{Z})\bar{\phi}\bar{X} - \bar{g}(\bar{\phi}\bar{X}, \bar{Z})\bar{\phi}\bar{Y} - 2\bar{g}(\bar{\phi}\bar{X}, \bar{Y})\bar{\phi}\bar{Z}).
 \end{aligned}$$

Let  $(\bar{M}, \bar{g})$  be a  $(2n + 1)$ -dimensional semi-Riemannian manifold with constant index  $\nu$ ,  $0 < \nu < 2n + 1$  and let  $(M, g)$  be a hypersurface of  $\bar{M}$ , with  $g = \bar{g}|_M$ .  $M$  is said to be a lightlike hypersurface of  $\bar{M}$  if  $g$  is of constant rank  $2n - 1$  and the normal bundle  $TM^\perp$  is a distribution of rank 1 on  $M$ . A complementary bundle of  $TM^\perp$  in  $TM$  is a rank  $2n - 1$  non-degenerate distribution over  $M$ . It is called a screen distribution and is often denoted by  $S(TM)$ . A lightlike hypersurface endowed with a specific screen distribution is denoted by the triple  $(M, g, S(TM))$ . As  $TM^\perp$  lies in the tangent bundle, the following result has an important role in studying the geometry of a lightlike hypersurface [9].

**Theorem 1. (Duggal-Bejancu)** *Let  $(M, g, S(TM))$  be a lightlike hypersurface of  $(\bar{M}, \bar{g})$ . Then there exists a unique vector bundle  $tr(TM)$  of rank 1 over  $M$  such that for any non-zero section  $E$  of  $TM^\perp$  on a coordinate neighborhood  $\mathcal{U} \subset M$ , there exists a unique section  $N$  of  $tr(TM)$  on  $\mathcal{U}$  satisfying*

$$(2.7) \quad \bar{g}(N, E) = 1, \quad \bar{g}(N, N) = \bar{g}(N, W) = 0, \quad \forall W \in \Gamma(S(TM)|_{\mathcal{U}})$$

Throughout the paper, all manifolds are supposed to be paracompact and smooth. We denote by  $\Gamma(F)$  the smooth sections of the vector bundle  $F$ . Also by  $\perp$  and  $\oplus$  we denote the orthogonal and nonorthogonal direct sum of two vector bundles. By Theorem 1 we may write down the following decompositions

$$(2.8) \quad TM = S(TM) \perp TM^\perp,$$

$$(2.9) \quad T\bar{M} = TM \oplus tr(TM) = S(TM) \perp \{TM^\perp \oplus tr(TM)\}.$$

Let  $\bar{\nabla}$  be the Levi-Civita connection on  $(\bar{M}, \bar{g})$ , then by using the decomposition (2.9) and considering a normalizing pair  $\{E, N\}$  as in Theorem 1, we have the Gauss and Weingarten formulae in the form, for any  $X, Y \in \Gamma(TM|_{\mathcal{U}})$ ,

$$(2.10) \quad \bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \quad \text{and} \quad \bar{\nabla}_X N = -A_N X + \nabla_X^\perp N,$$

where  $\nabla_X Y, A_N X \in \Gamma(TM)$ .  $\nabla$  is an induced symmetric linear connection on  $M$ ,  $\nabla^\perp$  is a linear connection on the vector bundle  $tr(TM)$ ,  $h$  is a symmetric bilinear form and  $A_N$  is the shape operator of  $M$ .

Equivalently, consider a normalizing pair  $\{E, N\}$  as in Theorem 1. Then (2.10) takes the form, for any  $X, Y \in \Gamma(TM|_{\mathcal{U}})$ ,

$$(2.11) \quad \bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N \quad \text{and} \quad \bar{\nabla}_X N = -A_N X + \tau(X)N.$$

It is important to mention that the second fundamental form  $B$  is independent of the choice of screen distribution, in fact, from (2.11), we obtain  $X, Y \in \Gamma(TM|_{\mathcal{U}})$ ,

$$(2.12) \quad B(X, Y) = \bar{g}(\bar{\nabla}_X Y, E) \quad \text{and} \quad \tau(X) = \bar{g}(\nabla_X^\perp N, E).$$

Let  $P$  be the projection morphism of  $TM$  on  $S(TM)$  with respect to the orthogonal decomposition (2.8). We have for any  $X, Y \in \Gamma(TM|_{\mathcal{U}})$ ,

$$(2.13) \quad \nabla_X PY = \nabla_X^* PY + C(X, PY)E \quad \text{and} \quad \nabla_X E = -A_E^* X - \tau(X)E,$$

where  $\nabla_X^* PY$  and  $A_E^* X$  belong to  $\Gamma(S(TM))$ .  $C$ ,  $A_E^*$  and  $\nabla^*$  are called the local second fundamental form, the local shape operator and the induced connection on  $S(TM)$ . The induced linear connection  $\nabla$  is not a metric connection and we have

$$(2.14) \quad (\nabla_X g)(Y, Z) = B(X, Y)\theta(Z) + B(X, Z)\theta(Y), \quad \forall X, Y \in \Gamma(TM|_{\mathcal{U}}),$$

where  $\theta$  is a differential 1-form locally defined on  $M$  by  $\theta(X) := \bar{g}(N, X)$ ,  $\forall X \in \Gamma(TM)$ . The local second fundamental form  $B$  of  $M$  satisfies  $B(X, PY) = g(A_E^* X, PY)$ ,  $B(X, E) = 0$ ,  $B(A_E^* X, Y) = B(X, A_E^* Y)$  and  $g(A_E^* X, N) = 0$ . The local second fundamental form  $C$  of  $S(TM)$  satisfies  $C(X, PY) = g(A_N X, PY)$ .

Denote by  $\bar{R}$  and  $R$  the Riemann curvature tensors of  $\bar{M}$  and  $M$ , respectively. From Gauss-Codazzi equations [9], we have the following, for any  $X, Y, Z \in \Gamma(TM|_{\mathcal{U}})$ ,

$$(2.15) \quad \begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z + B(X, Z)A_N Y - B(Y, Z)A_N X \\ &\quad + \{(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \\ &\quad + \tau(X)B(Y, Z) - \tau(Y)B(X, Z)\}N, \end{aligned}$$

$$(2.16) \quad \bar{g}(\bar{R}(X, Y)Z, N) = \bar{g}(R(X, Y)Z, N),$$

$$(2.17) \quad \begin{aligned} \bar{g}(\bar{R}(X, Y)PZ, N) &= (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) \\ &\quad + \tau(Y)C(X, PZ) - \tau(X)C(Y, PZ), \end{aligned}$$

$$(2.18) \quad \bar{g}(\bar{R}(X, Y)E, N) = C(Y, A_E^* X) - C(X, A_E^* Y) - 2d\tau(X, Y).$$

**§3. Lightlike hypersurfaces of indefinite Sasakian manifolds**

Let  $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$  be an indefinite Sasakian manifold and let  $(M, g)$  be a lightlike hypersurface, tangent to the structure vector field  $\xi$  ( $\xi \in TM$ ). If  $E$  is a local section of  $TM^\perp$ , then  $\bar{g}(\bar{\phi}E, E) = 0$ , and  $\bar{\phi}E$  is tangent to  $M$ . Thus  $\bar{\phi}(TM^\perp)$  is a distribution on  $M$  of rank 1 such that  $\bar{\phi}(TM^\perp) \cap TM^\perp = \{0\}$ . This enables us to choose a screen distribution  $S(TM)$  such that it contains  $\bar{\phi}(TM^\perp)$  as a vector subbundle. If we consider a local section  $N$  of  $tr(TM)$ . Since  $\bar{g}(\bar{\phi}N, E) = -\bar{g}(N, \bar{\phi}E) = 0$ , we deduce that  $\bar{\phi}E$  belongs to  $S(TM)$ . On the other hand, since  $\bar{g}(\bar{\phi}N, N) = 0$ , we see that the component of  $\bar{\phi}N$  with respect to  $E$  vanishes. Thus  $\bar{\phi}N \in \Gamma(S(TM))$ . From the last equation of (2.2), we have  $\bar{g}(\bar{\phi}N, \bar{\phi}E) = 1$ . Therefore,  $\bar{\phi}(TM^\perp) \oplus \bar{\phi}(tr(TM))$  (direct sum but not orthogonal) is a non-degenerate vector subbundle of  $S(TM)$  of rank two.

It is known [4] that if  $M$  is tangent to the structure vector field  $\xi$ , then  $\xi$  belongs to  $S(TM)$ . using this and since  $\bar{g}(\bar{\phi}E, \xi) = \bar{g}(\bar{\phi}N, \xi) = 0$ , there exists a non-degenerate distribution  $D_0$  of rank  $2n - 4$  on  $M$  such that

$$(3.1) \quad S(TM) = \{\bar{\phi}(TM^\perp) \oplus \bar{\phi}(tr(TM))\} \perp D_0 \perp \langle \xi \rangle,$$

where  $\langle \xi \rangle = Span\{\xi\}$ . It is easy to check that the distribution  $D_0$  is invariant under  $\bar{\phi}$ , i.e.  $\bar{\phi}(D_0) = D_0$ .

**Example 2.** Let  $M$  be a hypersurface of  $(\mathbb{R}^7, \bar{\phi}, \xi, \eta, \bar{g})$  (indefinite Sasakian manifold defined in Example 1) given by

$$M = \{(x_1, \dots, x_7) \in \mathbb{R}^7 : x_5 = x_4\},$$

where  $(x_1, \dots, x_7)$  is a local coordinate system in  $\mathbb{R}^7$ . Thus, the tangent space  $TM$  is spanned by  $\{U_i\}_{1 \leq i \leq 6}$ , where  $U_1 = \frac{\partial}{\partial x_1}$ ,  $U_2 = \frac{\partial}{\partial x_2}$ ,  $U_3 = \frac{\partial}{\partial x_3}$ ,  $U_4 = \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5}$ ,  $U_5 = \frac{\partial}{\partial x_6}$ ,  $U_6 = \xi$  and the distribution  $TM^\perp$  of rank 1 is spanned by  $E = \frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_5}$ . It follows that  $TM^\perp \subset TM$ . Then  $M$  is a 6-dimensional lightlike hypersurface of  $\mathbb{R}^7$ . Also, the transversal bundle  $tr(TM)$  is spanned by  $N = 2(-\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5})$ . On the other hand, by using the almost contact structure  $(\bar{\phi}, \xi, \eta)$  of  $\mathbb{R}^7$  and also by taking into account the decomposition of screen distribution  $S(TM)$  given in (3.1), the distribution  $D_0$  is spanned by  $\{U_5, \bar{\phi}U_5 = -U_3 - \frac{1}{2}x_6\xi\}$ , and the distributions  $\langle \xi \rangle$ ,  $\bar{\phi}(TM^\perp)$  and  $\bar{\phi}(tr(TM))$  are spanned, respectively, by  $\xi$ ,  $\bar{\phi}E = -U_1 + U_2$  and  $\bar{\phi}N = 2(U_1 + U_2 + x_4\xi)$ . Hence  $M$  is a lightlike hypersurface of  $(\mathbb{R}^7, \bar{\phi}, \xi, \eta, \bar{g})$ .

Moreover, from (2.8) and (2.9) we obtain the decompositions

$$(3.2) \quad TM = \{\bar{\phi}(TM^\perp) \oplus \bar{\phi}(tr(TM))\} \perp D_0 \perp \langle \xi \rangle \perp TM^\perp,$$

$$(3.3) \quad T\bar{M}|_M = \{\bar{\phi}(TM^\perp) \oplus \bar{\phi}(tr(TM))\} \perp D_0 \perp \langle \xi \rangle \perp (TM^\perp \oplus tr(TM)).$$

Now, we consider the distributions on  $M$ ,

$$(3.4) \quad D := TM^\perp \perp \bar{\phi}(TM^\perp) \perp D_0 \quad \text{and} \quad D' := \bar{\phi}(tr(TM)).$$

Then  $D$  is invariant under  $\bar{\phi}$  and

$$(3.5) \quad TM = (D \oplus D') \perp \langle \xi \rangle .$$

Let us consider the local lightlike vector fields  $U := -\bar{\phi}N$ ,  $V := -\bar{\phi}E$ . Then, from (3.5), any  $X \in \Gamma(TM)$  is written as

$$(3.6) \quad X = RX + QX + \eta(X)\xi, \quad QX = u(X)U,$$

where  $R$  and  $Q$  are the projection morphisms of  $TM$  into  $D$  and  $D'$ , respectively, and  $u$  is a differential 1-form locally defined on  $M$  by  $u(X) = g(X, V)$ . Applying  $\bar{\phi}$  to (3.6), using (2.1) and noting that  $\bar{\phi}^2 N = -N$ , we obtain

$$(3.7) \quad \bar{\phi}X = \phi X + u(X)N,$$

where  $\phi$  is a tensor field of type  $(1, 1)$  defined on  $M$  by  $\phi X := \bar{\phi}RX$ , for any  $X \in \Gamma(TM)$ . Again, applying  $\bar{\phi}$  to (3.7) and using (2.1), we also have

$$(3.8) \quad \phi^2 X = -X + \eta(X)\xi + u(X)U, \quad \forall X \in \Gamma(TM)$$

Now applying  $\phi$  to the equation (3.8) and since  $\phi U = 0$ , we obtain  $\phi^3 + \phi = 0$ , which shows that  $\phi$  is an  $f$ -structure [9] of constant rank.

As was proved in Bejancu-Duggal [9] any non-degenerate real hypersurface of an indefinite almost Hermitian manifold  $M$  inherits an almost contact metric structure. However, this is not the case for a lightlike hypersurface of the indefinite Sasakian manifold. More precisely, by using (2.2) and (3.7) we derive that, for any  $X, Y \in \Gamma(TM)$ ,

$$(3.9) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) - u(Y)v(X) - u(X)v(Y),$$

where  $v$  is a 1-form locally defined on  $M$  by  $v(X) = g(X, U)$ ,  $\forall X \in \Gamma(TM)$ .

By direct calculations, we have the following useful identities

$$(3.10) \quad \nabla_X \xi = -\phi X,$$

$$(3.11) \quad B(X, \xi) = -u(X),$$

$$(3.12) \quad C(X, \xi) = -v(X),$$

$$(3.13) \quad B(X, U) = C(X, V),$$

$$(3.14) \quad (\nabla_X u)Y = -B(X, \phi Y) - u(Y)\tau(X),$$

$$(3.15) \quad (\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X - B(X, Y)U + u(Y)A_N X.$$



**§4. Locally symmetric lightlike hypersurfaces in indefinite Sasakian spaces form**

Let  $M$  be a lightlike hypersurface of an indefinite Sasakian space form  $\overline{M}(c)$  with  $\xi \in TM$ . Let us consider the pair  $\{E, N\}$  on  $\mathcal{U} \subset M$  (Theorem 1). By using (2.6), (2.15) and (3.7), and comparing the tangential and transversal parts of the both sides, we have, for any  $X, Y, Z \in \Gamma(TM)$ ,

$$(4.1) \quad R(X, Y)Z = \frac{c+3}{4}\{g(Y, Z)X - g(X, Z)Y\} + \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi + \bar{g}(\bar{\phi}Y, Z)\phi X - \bar{g}(\bar{\phi}X, Z)\phi Y - 2\bar{g}(\bar{\phi}X, Y)\phi Z\} + B(Y, Z)A_N X - B(X, Z)A_N Y,$$

and

$$(4.2) \quad (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = \tau(Y)B(X, Z) - \tau(X)B(Y, Z) + \frac{c-1}{4}\{\bar{g}(\bar{\phi}Y, Z)u(X) - \bar{g}(\bar{\phi}X, Z)u(Y) - 2\bar{g}(\bar{\phi}X, Y)u(Z)\}.$$

A lightlike hypersurface  $(M, g, S(TM))$  of a semi-Riemannian manifold  $(\overline{M}, \bar{g})$  is said to be locally symmetric, if and only if, for any  $X, Y, Z, T, W \in \Gamma(TM)$  and  $N \in \Gamma(tr(TM))$ , the following hold ([10]):

$$(4.3) \quad g((\nabla_W R)(X, Y)Z, PT) = 0 \quad \text{and} \quad \bar{g}((\nabla_W R)(X, Y)Z, N) = 0.$$

That is,

$$(\nabla_W R)(X, Y)Z = 0.$$

Using the lemma 3.2 in [10], for any  $W, X, Y, Z \in \Gamma(TM)$ ,  $T \in \Gamma(S(TM))$  and  $N \in \Gamma(tr(TM))$ , we have,

$$(4.4) \quad \begin{aligned} \bar{g}((\nabla_W \bar{R})(X, Y)Z, T) &= g((\nabla_W R)(X, Y)Z, T) + (\nabla_W B)(X, Z)C(Y, T) \\ &+ B(X, Z)g((\nabla_W A_N)Y, T) - (\nabla_W B)(Y, Z)C(X, T) \\ &- B(Y, Z)g((\nabla_W A_N)X, T) - B(Y, Z)\tau(X)C(W, T) \\ &+ (\nabla_Y B)(X, Z)C(W, T) - (\nabla_X B)(Y, Z)C(W, T) \\ &+ B(X, Z)\tau(Y)C(W, T) - B(W, X)\bar{R}(N, Y, Z, T) \\ &- B(W, Y)\bar{R}(X, N, Z, T) - B(W, Z)\bar{R}(X, Y, N, T) \end{aligned}$$

and

$$(4.5) \quad \begin{aligned} \bar{g}((\nabla_W \bar{R})(X, Y)Z, N) &= g((\nabla_W R)(X, Y)Z, N) \\ &+ B(X, Z)g((\nabla_W (A_N Y)), N) - B(Y, Z)g((\nabla_W (A_N X)), N) \\ &- B(W, X)\bar{R}(N, Y, Z, N) - B(W, Y)\bar{R}(X, N, Z, N). \end{aligned}$$

**Lemma 1.** *Let  $(\overline{M}(c), \overline{g})$  be an indefinite Sasakian space form and  $\overline{R}$  the Riemann curvature tensor of Levi-Civita connection  $\overline{\nabla}$ . Then we have, for any  $W, X, Y, Z \in \Gamma(T\overline{M})$ ,*

$$(4.6) \quad (\overline{\nabla}_W \overline{R})(X, Y)Z = \frac{c-1}{4} \{-\overline{g}(\overline{\phi}W, X)\eta(Z)Y - \overline{g}(\overline{\phi}W, Z)\eta(X)Y \\ + \overline{g}(\overline{\phi}W, Y)\eta(Z)X + \overline{g}(\overline{\phi}W, Z)\eta(Y)X - \overline{g}(X, Z)\overline{g}(\overline{\phi}W, Y)\xi \\ - \overline{g}(X, Z)\eta(Y)\overline{\phi}W + \overline{g}(Y, Z)\overline{g}(\overline{\phi}W, X)\xi + \overline{g}(Y, Z)\eta(X)\overline{\phi}W \\ + \overline{g}(W, Y)\eta(Z)\overline{\phi}X - \overline{g}(W, Z)\eta(Y)\overline{\phi}X + \overline{g}(\overline{\phi}Y, Z)\overline{g}(W, X)\xi \\ - \overline{g}(\overline{\phi}Y, Z)\eta(X)W - \overline{g}(W, X)\eta(Z)\overline{\phi}Y + \overline{g}(W, Z)\eta(X)\overline{\phi}Y \\ - \overline{g}(\overline{\phi}X, Z)\overline{g}(W, Y)\xi + \overline{g}(\overline{\phi}X, Z)\eta(Y)W - 2\overline{g}(W, X)\eta(Y)\overline{\phi}Z \\ + 2\overline{g}(W, Y)\eta(X)\overline{\phi}Z - 2\overline{g}(\overline{\phi}X, Y)\overline{g}(W, Z)\xi + 2\overline{g}(\overline{\phi}X, Y)\eta(Z)W\}.$$

*Proof.* Using the relation (2.6), let decompose the Riemann curvature  $\overline{R}$  on  $\overline{M}$  by  $\overline{R} = \overline{R}_1 + \overline{R}_2$ , where, for any  $X, Y, Z \in \Gamma(TM)$ ,

$$(4.7) \quad \overline{R}_1(X, Y)Z = \frac{c+3}{4} \{\overline{g}(Y, Z)X - \overline{g}(X, Z)Y\},$$

$$(4.8) \quad \overline{R}_2(X, Y)Z = \frac{c-1}{4} \{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \overline{g}(X, Z)\eta(Y)\xi \\ - \overline{g}(Y, Z)\eta(X)\xi + \overline{g}(\overline{\phi}Y, Z)\overline{\phi}X - \overline{g}(\overline{\phi}X, Z)\overline{\phi}Y - 2\overline{g}(\overline{\phi}X, Y)\overline{\phi}Z\}.$$

By covariant derivation of  $\overline{R}$ , we have,

$$(4.9) \quad (\overline{\nabla}_W \overline{R})(X, Y)Z = (\overline{\nabla}_W \overline{R}_2)(X, Y)Z = \overline{\nabla}_W(\overline{R}_2(X, Y)Z) - \overline{R}_2(\overline{\nabla}_W X, Y)Z \\ - \overline{R}_2(X, \overline{\nabla}_W Y)Z - \overline{R}_2(X, Y)\overline{\nabla}_W Z.$$

By direct calculation, using (4.8) and the definition of covariant derivative of differential forms, we obtain the result.  $\square$

It is known that in locally symmetric semi-Riemannian manifold  $\overline{M}$ , the locally symmetric lightlike hypersurfaces are totally geodesic lightlike hypersurfaces if the vector field  $A_N E$  is non-null (see [10]). Also, by using Lemma 1, we infer that all locally symmetric indefinite Sasakian space forms  $\overline{M}(c)$  have  $\overline{\phi}$ -sectional curvature  $c = 1$ . So, in indefinite Sasakian space form  $\overline{M}(c)$  we have the following.

**Theorem 2.** *There are no locally symmetric lightlike hypersurfaces of indefinite Sasakian space forms  $\overline{M}(c)$  ( $c \neq 1$ ), tangent to the structure vector field  $\xi$ . Moreover, if  $M$  is a lightlike hypersurface of an indefinite Sasakian space form  $\overline{M}(c = 1)$  of constant curvature  $c = 1$  with  $\xi \in TM$ , then  $M$  is locally symmetric if and only if it is totally geodesic.*

*Proof.* Let  $M$  be a locally symmetric lightlike hypersurface of an indefinite Sasakian space form  $\overline{M}(c)$ . Suppose  $c \neq 1$ . From (4.6), we have, for any  $W, X, Y, Z \in \Gamma(TM)$ ,

$$\begin{aligned}
 (4.10) \quad & \overline{g}((\overline{\nabla}_W \overline{R})(X, Y)Z, N) \\
 &= \frac{c-1}{4} \{ -\overline{g}(\overline{\phi}W, X)\eta(Z)\theta(Y) - \overline{g}(\overline{\phi}W, Z)\eta(X)\theta(Y) \\
 &\quad + \overline{g}(\overline{\phi}W, Y)\eta(Z)\theta(X) + \overline{g}(\overline{\phi}W, Z)\eta(Y)\theta(X) - \overline{g}(X, Z)\eta(Y)v(W) \\
 &\quad + \overline{g}(Y, Z)\eta(X)v(W) + \overline{g}(W, Y)\eta(Z)v(X) - \overline{g}(W, Z)\eta(Y)v(X) \\
 &\quad - \overline{g}(\overline{\phi}Y, Z)\eta(X)\theta(W) - \overline{g}(W, X)\eta(Z)v(Y) + \overline{g}(W, Z)\eta(X)v(Y) \\
 &\quad + \overline{g}(\overline{\phi}X, Z)\eta(Y)\theta(W) - 2\overline{g}(W, X)\eta(Y)v(Z) + 2\overline{g}(W, Y)\eta(X)v(Z) \\
 &\quad + 2\overline{g}(\overline{\phi}X, Y)\eta(Z)\theta(W) \}.
 \end{aligned}$$

From relation (2.6), we have  $\overline{R}(E, N, E, N) = \frac{c+3}{4}$ . By taking  $X = E$  and  $Z = E$  in (4.5) and (4.10), we obtain for any  $Y, W \in \Gamma(TM)$ ,

$$(4.11) \quad -\frac{c+3}{4}B(W, Y) = \overline{g}((\overline{\nabla}_W \overline{R})(E, Y)E, N) = \frac{c-1}{4}\overline{g}(\overline{\phi}W, E)\eta(Y).$$

That is,

$$(4.12) \quad -\frac{c+3}{4}B(W, Y) = \frac{c-1}{4}u(W)\eta(Y).$$

Since the local second fundamental form  $B$  is symmetric, the relation (4.12) leads to

$$(c-1) \{u(W)\eta(Y) - u(Y)\eta(W)\} = 0.$$

Taking  $Y = \xi$  and  $W = U$ , we have  $c = 1$  which is a contradiction. Hence, the claim hold. On the other hand, let  $M$  is a lightlike hypersurface of an indefinite Sasakian space form  $\overline{M}(c = 1)$  of constant curvature  $c = 1$  with  $\xi \in TM$ . If  $M$  is locally symmetric, we have again the relation (4.12) which implies  $B = 0$ . Conversely if  $(M, g)$  is totally geodesic, using (4.4), (4.5) and (4.6),  $g((\nabla_W R)(X, Y)Z, PT) = 0$  and  $\overline{g}((\nabla_W R)(X, Y)Z, N) = 0$ , that is,  $M$  is locally symmetric. This completes the proof.  $\square$

Theorem 2 generates some lightlike geometric aspects on locally symmetric lightlike hypersurfaces of an indefinite Sasakian space form  $\overline{M}(c = 1)$  by using, for instance, the Duggal-Bejancu Theorem ([9], Theorem 2.2 page 88).

Note that the result of Theorem 2 is similar to the one from Theorem 3.1 in [10] where the ambient manifold was considered to be locally symmetric together with a supplementary condition on the shape operator of its sub-manifold. This is not the case in our considered ambient manifold. In case of Sasakian manifolds, Theorem 2 contains Theorem 3.1 in [10].

Now, we pay attention to a specific example of the non-existence of lightlike locally symmetric hypersurfaces in indefinite Sasakian space forms  $\overline{M}(c)$  ( $c \neq 1$ ), tangent to the structure vector field  $\xi$  (Theorem 2).

A submanifold  $M$  is said to be totally umbilical lightlike hypersurface of a semi-Riemannian manifold  $\overline{M}$  if the local second fundamental form  $B$  of  $M$  satisfies ([9])

$$(4.13) \quad B(X, Y) = \rho g(X, Y), \quad \forall X, Y \in \Gamma(TM),$$

where  $\rho$  is a smooth function on  $\mathcal{U} \subset M$ . The Gauss formula implies that  $\overline{\phi}X = -\overline{\nabla}_X \xi = -\nabla_X \xi - B(X, \xi)N$ . Since  $\overline{\phi}\xi = 0$ , we have  $B(\xi, \xi) = 0$ .

If we assume that  $M$  is totally umbilical lightlike hypersurface of a semi-Riemannian manifold  $\overline{M}$ , then we have  $B(X, Y) = \rho g(X, Y)$ , for any  $X, Y \in \Gamma(TM)$ , which implies that  $0 = B(\xi, \xi) = \rho$ . Hence  $M$  is totally geodesic. Also,  $\overline{\phi}X = \phi X - \rho \eta(X)N = \phi X$ , that is  $M$  is invariant in  $\overline{M}$ . It follows from this that a Sasakian  $\overline{M}(c)$  does not admit any non-totally geodesic, totally umbilical lightlike hypersurface. From this point of view, Bejancu [2] considered the concept of totally contact umbilical semi-invariant submanifolds. The notion of totally contact umbilical submanifolds was first defined by Kon [11]. We follow Bejancu [2] definition of totally contact umbilical submanifolds and state the following definition for totally contact umbilical lightlike hypersurfaces.

A submanifold  $M$  is said to be totally contact umbilical lightlike hypersurface of a semi-Riemannian manifold  $\overline{M}$  if the second fundamental form  $h$  of  $M$  satisfies ([15])

$$(4.14) \quad h(X, Y) = \{g(X, Y) - \eta(X)\eta(Y)\}H + \eta(X)h(Y, \xi) + \eta(Y)h(X, \xi),$$

for any  $X, Y \in \Gamma(TM)$ , where  $H$  is a normal vector field on  $M$  (that is,  $H = \lambda N$ ,  $\lambda$  being a smooth function on  $\mathcal{U} \subset M$ ). The notion of totally contact umbilical submanifolds of Sasakian manifolds corresponds to that of totally umbilical submanifolds of Kählerian manifolds (see [11] for more details). The totally contact umbilical condition (4.14) can be rewritten as,

$$(4.15) \quad h(X, Y) = B(X, Y)N = \{B_1(X, Y) + B_2(X, Y)\}N,$$

where  $B_1(X, Y) = \lambda \{g(X, Y) - \eta(X)\eta(Y)\}$  and  $B_2(X, Y) = -\eta(X)u(Y) - \eta(Y)u(X)$ . If the  $\lambda = 0$  (that is,  $B_1 = 0$ ), then the lightlike hypersurface  $M$  is said to be totally contact geodesic. The notion of totally contact geodesic submanifolds of Sasakian manifolds corresponds to that of totally geodesic submanifolds of Kählerian manifolds.

In [12], Massamba showed that if  $M$  is a totally contact umbilical lightlike hypersurface of an indefinite Sasakian space form  $\overline{M}(c)$  with  $\xi \in TM$ , that

is, the second fundamental form  $h$  of  $M$  satisfies (4.14), then  $c = -3$  and  $\lambda$  satisfies the partial differential equations

$$(4.16) \quad E \cdot \lambda + \lambda\tau(E) - \lambda^2 = 0$$

and

$$(4.17) \quad PX \cdot \lambda + \lambda\tau(PX) = 0, \quad \forall X \in \Gamma(TM).$$

These equations are similar to those of the indefinite Kählerian case (see [9] for details). However, there are non trivial differences arising in the details of the proof [12]. We also note that the partial differential equations (4.16) and the modified (4.17),  $PX \cdot \lambda + \lambda\tau(PX) = 0$  with  $PX \in \Gamma(S(TM) - \langle \xi \rangle)$  (that is, we exclude the partial differential equation in terms of  $\xi$ ) arise when the submanifold  $M$  is a  $D \oplus D'$ -totally umbilical lightlike hypersurface, that is,  $B(X, Y) = \rho g(X, Y)$ , for any  $X, Y \in \Gamma(D \oplus D')$ . Because, in the direction of  $D \oplus D'$ , the function  $\rho$  is nowhere vanishing. In general, such a concept is called proper totally umbilical [9]. The terminology of proper also is going to be used in the case of totally contact umbilical, that is, when, in the relation (4.14), the smooth function  $\lambda$  is nowhere vanishing.

Suppose  $c = -3$ , then the relation (4.1) becomes

$$(4.18) \quad \begin{aligned} R(X, Y)Z &= \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y - g(X, Z)\eta(Y)\xi \\ &+ g(Y, Z)\eta(X)\xi - \bar{g}(\bar{\phi}Y, Z)\phi X + \bar{g}(\bar{\phi}X, Z)\phi Y + 2\bar{g}(\bar{\phi}X, Y)\phi Z \\ &+ B(Y, Z)A_N X - B(X, Z)A_N Y. \end{aligned}$$

From (2.8), the curvature tensor  $R$  is written as

$$(4.19) \quad \begin{aligned} R(X, Y)Z &= R(PX, PY)PZ + \theta(X)R(E, PY)PZ + \theta(Y)R(PX, E)PZ \\ &+ \theta(Z)R(PX, PY)E + \theta(X)\theta(Z)R(E, PY)E \\ &+ \theta(Y)\theta(Z)R(PX, E)E, \end{aligned}$$

where, in particular and using (4.18), the component  $R(E, \cdot)E$  is given by

$$(4.20) \quad R(E, PY)E = 3u(PY)V.$$

Using (4.18), the covariant derivative of  $R$  is given by, for any  $W \in \Gamma(TM)$ ,

$$\begin{aligned}
(4.21) \quad & (\nabla_W R)(E, PY)E = \nabla_W R(E, PY)E - R(\nabla_W E, PY)E \\
& - R(E, \nabla_W PY)E - R(E, PY)\nabla_W E \\
& = 3W.u(PY)V + 3u(PY)\nabla_W V + \bar{g}(\bar{\phi}PY, E)\phi\nabla_W E \\
& - \bar{g}(\bar{\phi}\nabla_W E, E)\phi PY - 2\bar{g}(\bar{\phi}\nabla_W E, PY)\phi E + \bar{g}(\bar{\phi}\nabla_W PY, E)\phi E \\
& - 2\bar{g}(\bar{\phi}E, \nabla_W PY)\phi E - \eta(PY)\eta(\nabla_W E)E + \bar{g}(\bar{\phi}PY, \nabla_W E)\phi E \\
& - \bar{g}(\bar{\phi}E, \nabla_W E)\phi PY - 2\bar{g}(\bar{\phi}E, PY)\phi\nabla_W E - B(PY, \nabla_W E)A_N E \\
& = 3W.u(PY)V + 3u(PY)\nabla_W V + u(PY)\phi\nabla_W E - u(\nabla_W E)\phi PY \\
& + 2\bar{g}(\bar{\phi}\nabla_W E, PY)V - u(\nabla_W PY)V - 2u(\nabla_W PY)V \\
& - \eta(PY)\eta(\nabla_W E)E - \bar{g}(\bar{\phi}PY, \nabla_W E)V + u(\nabla_W E)\phi PY \\
& + 2u(PY)\phi\nabla_W E - B(PY, \nabla_W E)A_N E
\end{aligned}$$

which implies

$$\begin{aligned}
\bar{g}((\nabla_W R)(E, PY)E, N) &= 3u(PY)\bar{g}(\nabla_W V, N) + u(PY)\bar{g}(\phi\nabla_W E, N) \\
&\quad - \eta(PY)\eta(\nabla_W E) + 2u(PY)\bar{g}(\phi\nabla_W E, N) \\
(4.22) \quad &= -\eta(PY)u(W).
\end{aligned}$$

Taking  $PY = \xi$  and  $W = U$  in (4.22), we obtain  $\bar{g}((\nabla_\xi R)(E, \xi)E, N) = -1$ . This means that a totally contact umbilical lightlike hypersurfaces of an indefinite Sasakian space form  $\bar{M}(c)$  with  $\xi \in TM$  cannot be locally symmetric. Therefore, *there are no totally contact umbilical lightlike hypersurfaces of indefinite Sasakian space forms  $\bar{M}(c)$  with  $\xi \in TM$  which are locally symmetric.*

Apart from totally contact umbilical lightlike hypersurfaces, we have

**Example 3.** Let  $M$  be a hypersurface of  $(\mathbb{R}^7, \bar{\phi}, \xi, \eta, \bar{g})$ , of Example 2, given by

$$M = \{(x_1, \dots, x_7) \in \mathbb{R}^7 : x_5 = x_4\},$$

where  $(x_1, \dots, x_7)$  is a local coordinate system in  $\mathbb{R}^7$ . As explained in Example 2,  $M$  is a lightlike hypersurface of  $\mathbb{R}^7$  having a local quasi-orthogonal field of frames  $\{U_1 = \frac{\partial}{\partial x_1}, U_2 = \frac{\partial}{\partial x_2}, U_3 = \frac{\partial}{\partial x_3}, U_4 = E = \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5}, U_5 = \frac{\partial}{\partial x_6}, U_6 = \xi, N = 2(-\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5})\}$  along  $M$ . Then, by straightforward calculations, we

obtain

$$\begin{aligned}
 \bar{\nabla}_{U_1}N &= x_4(U_1 + U_2) + \frac{1}{2}(1 + 2x_4^2)\xi, \\
 \bar{\nabla}_{U_2}N &= x_4(U_1 + U_2) + \frac{1}{2}(2x_4^2 - 1)\xi, \\
 \bar{\nabla}_{U_3}N &= x_6(U_1 + U_2) + x_4x_6\xi, \\
 \bar{\nabla}_{U_6}N &= U, \quad \bar{\nabla}_{U_4}N = \bar{\nabla}_{U_5}N = 0, \\
 \nabla_{U_1}E &= -\frac{1}{2}x_4V - \frac{1}{4}\xi, \quad \nabla_{U_2}E = -\frac{1}{2}x_4V - \frac{1}{4}\xi, \\
 \nabla_{U_3}E &= -\frac{1}{2}x_6V, \quad \nabla_{U_6}E = V, \\
 \nabla_{U_4}E &= \nabla_{U_5}E = 0.
 \end{aligned}$$

Using these equations above, the differential 1-form  $\tau$  vanishes i.e.  $\tau(X) = 0$ , for any  $X \in \Gamma(TM)$ . So, from Gauss and Weingarten formulae we infer

$$\begin{aligned}
 A_NU_1 &= -x_4(U_1 + U_2) - \frac{1}{2}(1 + 2x_4^2)\xi, \\
 A_NU_2 &= -x_4(U_1 + U_2) - \frac{1}{2}(2x_4^2 - 1)\xi, \\
 A_NU_3 &= -x_6(U_1 + U_2) - x_4x_6\xi, \\
 A_NU_6 &= -U, \quad A_NU_4 = A_NU_5 = 0, \\
 A_E^*U_1 &= \frac{1}{2}x_4V + \frac{1}{4}\xi, \quad A_E^*U_2 = \frac{1}{2}x_4V + \frac{1}{4}\xi, \\
 A_E^*U_3 &= \frac{1}{2}x_6V, \quad A_E^*U_4 = A_E^*U_5 = 0, \quad A_E^*U_6 = -V.
 \end{aligned}$$

One of the components of the covariant derivative of the curvature tensor  $R$  of  $M$  is given

$$\begin{aligned}
 (\nabla_{U_1}R)(\xi, E)\xi &= \nabla_{U_1}R(\xi, E)\xi - R(\nabla_{U_1}\xi, E)\xi - R(\xi, \nabla_{U_1}E)\xi \\
 &\quad - R(\xi, E)\nabla_{U_1}\xi \\
 (4.23) \qquad &= \frac{1}{2}x_4V + \frac{1}{4}\xi - \frac{1}{2}x_4V \\
 &= \frac{1}{4}\xi,
 \end{aligned}$$

which implies

$$g((\nabla_{U_1}R)(\xi, E)\xi, \xi) = g\left(\frac{1}{4}\xi, \xi\right) = \frac{1}{4}.$$

This means that  $M$  is a lightlike hypersurface of an indefinite Sasakian space form  $(\mathbb{R}^7, \bar{\phi}, \xi, \eta, \bar{g})$  of constant curvature  $c = -3$  non locally symmetric.

Next, we give an example on the second assertion of Theorem 2. The second fundamental form  $h = B \otimes N$  of  $M$  is said to be parallel if

$$(4.24) \quad (\nabla_X h)(Y, Z) = 0,$$

for any  $X, Y, Z \in \Gamma(TM)$ . That is,

$$(4.25) \quad (\nabla_X B)(Y, Z) = -\tau(X)B(Y, Z).$$

A submanifold of a semi-Riemannian manifold with parallel fundamental form  $h$  is called a parallel submanifold. So, as was proved in [15], there are no parallel lightlike hypersurfaces of indefinite Sasakian space forms  $\overline{M}(c \neq 1)$ , tangent to the structure vector field  $\xi$ .

If  $M$  is parallel, then, by Lemma 3.6 in [15],  $c = 1$  and from (4.1), the curvature tensor  $R$  of  $M$  is given by, for any  $X, Y, Z \in \Gamma(TM)$ ,

$$(4.26) \quad R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + B(Y, Z)A_N X - B(X, Z)A_N Y.$$

Using (4.25), the covariant derivative of  $R$  is given by

$$\begin{aligned} (\nabla_W R)(X, Y)Z &= (\nabla_W g)(Y, Z)X - (\nabla_W g)(X, Z)Y - (\nabla_W B)(X, Z)A_N Y \\ &\quad + (\nabla_W B)(Y, Z)A_N X - B(X, Z)(\nabla_W A_N)Y \\ &\quad + B(Y, Z)(\nabla_W A_N)X \\ (4.27) \quad &= \{B(W, Y)\theta(Z) + B(W, Z)\theta(Y)\}X - \{B(W, X)\theta(Z) \\ &\quad + B(W, Z)\theta(X)\}Y + \tau(W)B(X, Z)A_N Y \\ &\quad - \tau(W)B(Y, Z)A_N X - B(X, Z)(\nabla_W A_N)Y \\ &\quad + B(Y, Z)(\nabla_W A_N)X. \end{aligned}$$

Taking  $X = Z = E$  into (4.27) and since  $B(\cdot, E) = 0$ , we have

$$(4.28) \quad \bar{g}((\nabla_W R)(E, Y)Z, N) = B(W, Y).$$

We have the following result.

**Theorem 3.** *Let  $\overline{M}(c)$  be an indefinite Sasakian space form. Let  $M$  be a lightlike hypersurface of  $\overline{M}(c)$  with  $\xi \in TM$ . If  $M$  is parallel, then  $M$  is locally symmetric if and only if  $M$  is totally geodesic.*

*Proof.* The converse of the Theorem follows from (4.27).  $\square$

Note that the covariant derivative of the second fundamental form  $h$  depends on  $\nabla$ ,  $N$  and  $\tau$  which depend on the choice of the screen vector bundle.



Suppose a screen  $S(TM)$  changes to another screen  $S(TM)'$ . Following are the local transformation equations due to this change (see [9], page 87):

$$\begin{aligned}
 W'_i &= \sum_{j=1}^{2n-1} W_i^j (W_j - \epsilon_j c_j E), \\
 N' &= N - \frac{1}{2} \left\{ \sum_{i=1}^{2n-1} \epsilon_i (c_i)^2 \right\} E + \sum_{i=1}^{2n-1} c_i W_i, \\
 \tau'(X) &= \tau(X) + B(X, N' - N), \\
 (4.29) \quad \nabla'_X Y &= \nabla_X Y + B(X, Y) \left\{ \frac{1}{2} \left( \sum_{i=1}^{2n-1} \epsilon_i (c_i)^2 \right) E - \sum_{i=1}^{2n-1} c_i W_i \right\},
 \end{aligned}$$

where  $\{W_i\}$  and  $\{W'_i\}$  are the local orthonormal bases of  $S(TM)$  and  $S(TM)'$  with respective transversal sections  $N$  and  $N'$  for the same null section  $E$ . Here  $c_i$  and  $W_i^j$  are smooth functions on  $\mathcal{U}$  and  $\{\epsilon_1, \dots, \epsilon_{2n-1}\}$  is the signature of the basis  $\{W_1, \dots, W_{2n-1}\}$ . The covariant derivatives  $\nabla$  of  $h = B \otimes N$  and  $\nabla'$  of  $h' = B \otimes N'$  in the screen distributions  $S(TM)$  and  $S(TM)'$ , respectively, are related as follows: for any  $X, Y, Z \in \Gamma(TM)$ ,

$$(4.30) \quad \bar{g}((\nabla'_X h')(Y, Z), E) = \bar{g}((\nabla_X h)(Y, Z), E) + \mathcal{L}(X, Y, Z),$$

where  $\mathcal{L}$  is given by  $\mathcal{L}(X, Y, Z) = B(X, Y)B(Z, W) + B(X, Z)B(Y, W) + B(Y, Z)B(X, W)$ , with  $W = \sum_{i=1}^{2n-1} c_i W_i$ . It is easy to check that the parallelism of  $h$  is independent of the screen distribution  $S(TM)$  ( $\nabla' h' \equiv \nabla h$ ) if and only if the second fundamental form  $B$  of  $M$  vanishes identically on  $M$ .

As is showed above, a totally contact umbilical lightlike hypersurfaces of an indefinite Sasakian space form  $\bar{M}(c)$  with  $\xi \in TM$  is not locally symmetric but it may contain a distribution in which one of the components of the covariant derivative of curvature tensor  $R$  vanishes. Next we give a characterization of this kind of submanifold containing such a vanishing condition on the curvature tensor  $R$ . For any  $W \in \Gamma(TM)$ ,

$$\begin{aligned}
 (4.31) \quad (\nabla_W R)(U, E)V &= \nabla_W R(U, E)V - R(\nabla_W U, E)V - R(U, \nabla_W E)V \\
 &\quad - R(U, E)\nabla_W V \\
 &= 2 \nabla_W E - W(\lambda)A_N E - \lambda \nabla_W A_N E - 2\tau(W)E + \lambda\tau(W)A_N E \\
 &\quad + u(W)\xi + 2\tau(W)E + \lambda^2 u(W)A_N U + \lambda A_N \nabla_W E + u(A_N W)V \\
 &\quad - 2\phi \nabla_W V - \lambda\tau(W)A_N E \\
 &= 2 \nabla_W E - W(\lambda)A_N E - \lambda \nabla_W A_N E + \lambda\tau(W)A_N E + u(W)\xi \\
 &\quad + \lambda^2 u(W)A_N U + \lambda A_N \nabla_W E + u(A_N W)V - 2\phi \nabla_W V - \lambda\tau(W)A_N E.
 \end{aligned}$$

Consequently

$$\begin{aligned}
 \bar{g}((\nabla_W R)(U, E)V, N) &= 2\bar{g}(\nabla_W E, N) - \lambda\bar{g}(\nabla_W A_N E, N) - 2\bar{g}(\phi\nabla_W V, N) \\
 &= -2\tau(W) + \lambda\bar{g}(A_N E, \bar{\nabla}_W N) + 2\tau(W) \\
 (4.32) \qquad \qquad \qquad &= -\lambda\bar{g}(A_N E, A_N W).
 \end{aligned}$$

**Theorem 4.** *Let  $(M, g, S(TM))$  be a totally contact umbilical lightlike hypersurface of an indefinite Sasakian space form  $(\bar{M}(c), \bar{g})$  with  $\xi \in TM$  such that  $\bar{g}((\nabla_E R)(U, E)V, N) = 0$  and  $A_N E$  is not a null vector field. Then  $M$  is totally contact geodesic.*

*Proof.* The proof follows straightforward from (4.32).  $\square$

From Theorem 4, we obtain

**Corollary 1.** *There are no proper totally contact umbilical lightlike hypersurfaces of indefinite Sasakian space forms  $\bar{M}(c)$  with  $\xi \in TM$  such that  $\bar{g}((\nabla_E R)(U, E)V, N) = 0$  and  $A_N E$  is not a null vector field.*

Let  $M$  be a lightlike hypersurface of an indefinite Sasakian manifolds  $\bar{M}$  with  $\xi \in TM$ . It is easy to check that  $M$  is  $(D \perp \langle \xi \rangle, D')$ -mixed totally geodesic, that is, for any  $X \in \Gamma(D \perp \langle \xi \rangle)$ ,  $B(X, U) = 0$ , if and only if,  $A_N X \in \Gamma(\bar{\phi}(TM^\perp) \perp D_0 \perp \langle \xi \rangle)$ ,  $\forall X \in \Gamma(D \perp \langle \xi \rangle)$  [14]. In particular  $A_N E \in \Gamma(\bar{\phi}(TM^\perp) \perp D_0)$ , since  $g(A_N E, \xi) = 0$ . That is,

$$(4.33) \qquad A_N E = v(A_N E)V + \sum_{i=1}^{2n-4} \frac{C(E, F_i)}{g(F_i, F_i)} F_i,$$

where  $\{F_i\}_{1 \leq i \leq 2n-4}$  is an orthogonal basis of  $D_0$  and  $g(F_i, F_i) \neq 0$ . This means that, in a  $(D \perp \langle \xi \rangle, D')$ -mixed totally geodesic lightlike hypersurface of an indefinite Sasakian manifolds  $\bar{M}$  with  $\xi \in TM$ ,  $A_N E$  is not a null vector. Moreover, if the lightlike vector field  $V$  is parallel with respect to  $\nabla$ ,  $(\nabla_W R)(U, E)V = 0$  which implies that  $\bar{g}((\nabla_E R)(U, E)V, N) = 0$ .

Therefore, there exist vector fields on  $M$  which satisfy the conditions given in the Theorem 4 and the Corollary 1.

A submanifold  $M$  is said to be an  $\eta$ -totally umbilical lightlike hypersurface of a semi-Riemannian manifold  $\bar{M}$  if the second fundamental form  $h$  of  $M$  satisfies ([15]), for any  $X, Y \in \Gamma(TM)$ ,

$$(4.34) \qquad h(X, Y) = \lambda \{g(X, Y) - \eta(X)\eta(Y)\} N.$$

From this definition, we can deduce that the totally contact umbilical lightlike hypersurface  $M$  of  $\bar{M}$  is also  $\eta$ -totally umbilical in the direction of  $D \perp \langle \xi \rangle$ , since the 1-form  $u$  vanishes in that direction.

If  $M$  is an  $\eta$ -totally umbilical lightlike hypersurface of an indefinite Sasakian manifold  $(\bar{M}, \bar{g})$  with  $\xi \in TM$ , we have, for any  $X, Y, Z \in \Gamma(TM)$ ,

$$(4.35) \quad g((\nabla_X h)(Y, Z), E) = (\nabla_X B_1)(Y, Z) + \lambda\tau(X) \{g(Y, Z) - \eta(Y)\eta(Z)\},$$

where  $B_1$  is defined in (4.15). Putting  $Z = \xi$  in (4.35), we obtain

$$(4.36) \quad \begin{aligned} g((\nabla_X h)(Y, \xi), E) &= (\nabla_X B_1)(Y, \xi) + \lambda\tau(X) \{g(Y, \xi) - \eta(Y)\eta(\xi)\} \\ &= \lambda\bar{g}(\bar{\phi}X, Y). \end{aligned}$$

If the second fundamental form  $h$  of the lightlike hypersurface  $M$  is parallel, then, we have  $0 = g((\nabla_X h)(Y, \xi), E) = \lambda\bar{g}(\bar{\phi}X, Y)$  which leads, by taking  $X = E$  and  $Y = U$ , to  $\lambda\bar{g}(\bar{\phi}E, U) = 0$ , that is  $\lambda = 0$ . Hence,  $B(X, Y) = 0$ . This means that an  $\eta$ -totally umbilical parallel lightlike hypersurface  $M$  of an indefinite Sasakian manifold  $\bar{M}$  with  $\xi \in TM$  is totally geodesic which implies that it admits a *metric connection* (see [13] and [15] for details).

Also, in an  $\eta$ -totally umbilical lightlike hypersurface  $M$  of an indefinite Sasakian space form  $\bar{M}(c)$  of constant curvature  $c$  with  $\xi \in TM$ , we have, for any  $Y, Z \in \Gamma(TM)$ ,

$$(4.37) \quad (\nabla_E B)(Y, Z) - (\nabla_Y B)(E, Z) = \frac{3}{4}(c - 1)u(Y)u(Z) - \tau(E)B(Y, Z).$$

By direct calculation, the left hand side gives

$$(4.38) \quad \begin{aligned} (\nabla_E B)(Y, Z) - (\nabla_Y B)(E, Z) &= \{g(Y, Z) - \eta(Y)\eta(Z)\} E \cdot \lambda \\ &+ \lambda \{E \cdot g(Y, Z) - \eta(Y)E \cdot \eta(Z) - \eta(Z)E \cdot \eta(Y)\} \\ &- \lambda \{g(\nabla_E Y, Z) - \eta(\nabla_E Y)\eta(Z)\} - \lambda \{g(Y, \nabla_E Z) - \eta(Y)\eta(\nabla_E Z)\} \\ &+ \lambda \{g(\nabla_Y E, Z) - \eta(\nabla_Y E)\eta(Z)\}. \end{aligned}$$

Putting pieces (4.37) and (4.38) together and taking  $Y = Z = U$ , we obtain,  $\frac{3}{4}(c - 1)u(U)u(U) = 0$ , that is,  $c = 1$ . We have

**Lemma 2.** *There are no  $\eta$ -totally umbilical lightlike hypersurfaces of indefinite Sasakian space forms  $\bar{M}(c \neq 1)$  with  $\xi \in TM$ .*

Also, it has been proved in [13] that, when  $M$  is an  $\eta$ -totally umbilical lightlike hypersurface of an indefinite Sasakian space form  $\bar{M}(c)$  of constant curvature  $c$  with  $\xi \in TM$ , the smooth function  $\lambda$  defined in (4.34) also satisfies the partial differential equations (4.16) and (4.17).

**Proposition 3.** *Let  $(M, g, S(TM))$  be an  $\eta$ -totally umbilical lightlike hypersurface of an indefinite Sasakian space form  $(\bar{M}(c), \bar{g})$  with  $\xi \in TM$ . If the second fundamental form  $h$  of  $M$  is parallel, then  $M$  is locally symmetric.*

*Proof.* The proof follows from a direct calculation and the results above.  $\square$

It is known that lightlike submanifolds whose screen distribution is integrable have interesting properties. For any  $X, Y \in \Gamma(TM)$ ,

$$(4.39) \quad u([X, Y]) = B(X, \phi Y) - B(\phi X, Y).$$

It is easy to check that the distribution  $D \perp \langle \xi \rangle$  is integrable if and only if  $B(X, \phi Y) = B(\phi X, Y), \forall X, Y \in \Gamma(TM)$ .

In the following this property is considered.

**Definition 1.** Let  $(M, g, S(TM))$  be a screen integrable lightlike hypersurface of a semi-Riemannian manifold  $(\bar{M}, \bar{g})$ . A leaf  $M'$  of  $S(TM)$  immersed in  $\bar{M}$  as a non-degenerate submanifold is said to be locally symmetric if the induced curvature  $R^*$  of Levi-Civita connection  $\nabla^*$  satisfies

$$(4.40) \quad (\nabla_W^* R^*)(X, Y)Z = 0, \quad \forall W, X, Y, Z \in \Gamma(TM').$$

In the following theorem, we show that local symmetry property of a screen integrable lightlike hypersurface of an indefinite Sasakian space form is closely related to the local symmetry property of leaves of its screen distribution.

**Lemma 4.** Let  $(M, g, S(TM))$  be a locally symmetric lightlike hypersurface of an indefinite Sasakian space form  $\bar{M}(c)$  with  $\xi \in TM$ . Then, for any  $X, Y, Z, T \in \Gamma(TM)$ , we have,

$$R(E, Y, Z, T) = 0, \quad R(X, E, Z, T) = 0, \quad R(X, Y, E, T) = 0.$$

*Proof.* By Theorem 2, we have  $c = 1$  which implies  $B = 0$  and the proof is completed by using relations (4.1).  $\square$

In the sequel, we need the following Lemma.

**Lemma 5.** Let  $(M, g, S(TM))$  be a lightlike hypersurface of a semi Riemannian manifold  $(\bar{M}, \bar{g})$ . Then, for any  $X, Y \in \Gamma(TM)$  and  $Z \in \Gamma(S(TM))$ ,

$$g((\nabla_X A_E^*)Y, Z) = (\nabla_X B)(Y, Z) \quad \text{and} \quad g((\nabla_X A_N)Y, Z) = (\nabla_X C)(Y, Z).$$

*Proof.* For any  $X, Y \in \Gamma(TM)$  and  $Z \in \Gamma(S(TM))$  we have

$$\begin{aligned} g((\nabla_X A_E^*)Y, Z) &= g(\nabla_X^*(A_E^*Y), Z) - g(A_E^*(\nabla_X Y), Z) \\ &= X.g(A_E^*Y, Z) - g(A_E^*Y, \nabla_X^*Z) - B(\nabla_X Y, Z) \\ &= X.B(Y, Z) - B(Y, \nabla_X Z) - B(\nabla_X Y, Z) \\ &= (\nabla_X B)(Y, Z). \end{aligned}$$

The second relation is obtained by similar calculation.  $\square$

**Theorem 5.** *Let  $(M, g, S(TM))$  be a screen integrable lightlike hypersurface of an indefinite Sasakian space form  $\bar{M}(c)$  with  $\xi \in TM$ . If  $M$  is locally symmetric, then any leaf  $M'$  of  $S(TM)$  immersed in  $\bar{M}$  as a non-degenerate submanifold is locally symmetric.*

*Proof.* Using Gauss and Weingarten equations, we have for any  $X, Y, Z \in \Gamma(TM')$ ,

$$(4.41) \quad \begin{aligned} R(X, Y)Z &= R^*(X, Y)Z + C(X, Z)A_E^*Y - C(Y, Z)A_E^*X \\ &\quad + \{(\nabla_X C)(Y, Z) - (\nabla_Y C)(X, Z) + \tau(Y)C(X, Z) \\ &\quad - \tau(X)C(Y, Z)\}E, \end{aligned}$$

where  $(\nabla_X C)(Y, Z) = X.C(Y, Z) - C(\nabla_X^*Y, Z) - C(Y, \nabla_X^*Z)$ .

By covariant derivative, we have for any  $W, X, Y, Z \in \Gamma(TM')$ ,

$$(4.42) \quad \begin{aligned} (\nabla_W R)(X, Y)Z &= (\nabla_W^* R^*)(X, Y)Z + (\nabla_W C)(X, Z)A^*Y \\ &\quad - (\nabla_W C)(Y, Z)A^*X + C(X, Z)(\nabla_W A^*)Y - C(Y, Z)(\nabla_W A^*)X \\ &\quad - \{(\nabla_X C)(Y, Z) - (\nabla_Y C)(X, Z) + \tau(Y)C(X, Z) \\ &\quad - \tau(X)C(Y, Z)\}A^*W + \{(\nabla_W \nabla_X C)(Y, Z) - (\nabla_W \nabla_Y C)(X, Z) \\ &\quad + C(X, Z)(\nabla_W \tau)Y - C(Y, Z)(\nabla_W \tau)X + \tau(Y)(\nabla_W C)(X, Z) \\ &\quad - \tau(X)(\nabla_W C)(Y, Z) + \tau(W)(\nabla_Y C)(X, Z) - \tau(W)(\nabla_X C)(Y, Z) \\ &\quad + \tau(W)\tau(X)C(Y, Z) - \tau(W)\tau(Y)C(X, Z) + C(X, Z)C(W, A^*Y) \\ &\quad - C(Y, Z)C(W, A^*X) + (\nabla_{\nabla_W^*Y} C)(X, Z) - (\nabla_{\nabla_W^*X} C)(Y, Z) \\ &\quad + C(W, R^*(X, Y)Z)\}E - R(C(W, X)E, Y)Z - R(X, C(W, Y)E)Z \\ &\quad - R(X, Y)C(W, Z)E. \end{aligned}$$

So, for any  $W, X, Y, Z, T \in \Gamma(TM')$ , we have,

$$(4.43) \quad \begin{aligned} g((\nabla_W R)(X, Y)Z, T) &= g((\nabla_W^* R^*)(X, Y)Z, T) + B(Y, T)(\nabla_W C)(X, Z) \\ &\quad - B(X, T)(\nabla_W C)(Y, Z) + C(X, Z)g((\nabla_W A^*)Y, T) \\ &\quad - C(Y, Z)g((\nabla_W A^*)X, T) + B(W, T)(\nabla_Y C)(X, Z) \\ &\quad - B(W, T)(\nabla_X C)(Y, Z) + B(W, T)\tau(X)C(Y, Z) \\ &\quad - B(W, T)\tau(Y)C(X, Z) - C(W, X)R(E, Y, Z, T) \\ &\quad - C(W, Y)R(X, E, Z, T) - C(W, Z)R(X, Y, E, T). \end{aligned}$$

By virtue of Lemma 5, we have

$$g((\nabla_W A^*)Y, T) = (\nabla_W B)(Y, T).$$

If  $M$  is locally symmetric, then, using Theorem 2,  $c = 1$  and  $B = 0$ . By Lemma 4,  $g((\nabla_W^* R^*)(X, Y)Z, T) = 0$ , that is  $M'$  is locally symmetric in  $\bar{M}$ .  $\square$

### §5. Semi-symmetric lightlike hypersurfaces in indefinite Sasakian spaces form

In this section, we deal with semi-symmetric lightlike hypersurfaces in indefinite Sasakian spaces form, tangent to the structure vector field  $\xi$ . First of all, a lightlike hypersurface  $M$  of a semi-Riemannian manifold  $\overline{M}$  is said to be semi-symmetric if the following condition is satisfied ([18])

$$(5.1) \quad (R(W_1, W_2) \cdot R)(X, Y, Z, T) = 0, \quad \forall W_1, W_2, X, Y, Z, T \in \Gamma(TM),$$

where  $R$  is the induced Riemann curvature on  $M$ . This is equivalent to

$$-R(R(W_1, W_2)X, Y, Z, T) - \dots - R(X, Y, Z, R(W_1, W_2)T) = 0.$$

In general the condition (5.1) is not equivalent to  $(R(W_1, W_2) \cdot R)(X, Y)Z = 0$  like in the non-degenerate case. Indeed, by direct calculation we have, for any  $W_1, W_2, X, Y, Z, T \in \Gamma(TM)$ ,

$$(5.2) \quad \begin{aligned} (R(W_1, W_2) \cdot R)(X, Y, Z, T) &= g((R(W_1, W_2) \cdot R)(X, Y)Z, T) \\ &+ (R(W_1, W_2) \cdot g)(R(X, Y)Z, T). \end{aligned}$$

In the sequel, we need the following proposition.

**Proposition 6.** *Let  $M$  be a lightlike hypersurface of an indefinite Sasakian space form  $\overline{M}(c)$  with  $\xi \in TM$ . Then, for any  $W_1, W_2, Y, T \in \Gamma(TM)$  and  $E \in \Gamma(TM^\perp)$ , we have*

$$(5.3) \quad \begin{aligned} (\overline{R}(W_1, W_2) \cdot \overline{R})(E, Y, E, T) &= (R(W_1, W_2) \cdot R)(E, Y, E, T) \\ &- B(W_1, Y)R(E, A_N W_2, E, T) + B(W_2, Y)R(E, A_N W_1, E, T) \\ &+ B(Y, R(W_1, W_2)E)g(A_N E, T) - B(W_1, T)R(E, Y, E, A_N W_2) \\ &+ B(W_2, T)R(E, Y, E, A_N W_1) - \{(\nabla_{W_1} B)(W_2, T) - (\nabla_{W_2} B)(W_1, T) \\ &+ \tau(W_1)B(W_2, T) - \tau(W_2)B(W_1, T)\} \overline{R}(E, Y, E, N) \\ &- \{(\nabla_{W_1} B)(W_2, Y) - (\nabla_{W_2} B)(W_1, Y) + \tau(W_1)B(W_2, Y) \\ &- \tau(W_2)B(W_1, Y)\} \overline{R}(E, N, E, T) - \theta(T) \{(\nabla_E B)(Y, R(W_1, W_2)E) \\ &- (\nabla_Y B)(E, R(W_1, W_2)E) + \tau(E)B(Y, R(W_1, W_2)E)\}. \end{aligned}$$

*Proof.* The proof follows from direct calculation by using  $(\nabla_X B)(Y, E) = (\nabla_Y B)(X, E)$ .  $\square$

Next, we investigate the effect of semi-symmetry condition on geometry of lightlike hypersurfaces in an indefinite Sasakian space form.

A submanifold  $M$  of a semi-Riemannian manifold is said to be  $(\overline{\phi}(TM^\perp), D \oplus D')$ -mixed totally geodesic if its second fundamental form  $h$  satisfies  $h(X, Y) = 0$  (equivalently  $B(X, Y) = 0$ ), for any  $X \in \Gamma(\overline{\phi}(TM^\perp))$  and  $Y \in \Gamma(D \oplus D')$ .

**Theorem 6.** *Let  $M$  be a semi-symmetric lightlike hypersurface of indefinite Sasakian space form  $\overline{M}(c)$  with  $\xi \in TM$ . Then, at least one of the following holds:*

- (i)  $c = 1$ ,
- (ii)  $A_N E = 0$ ,
- (iii)  $M$  is  $(\overline{\phi}(TM^\perp), D \oplus D')$ -mixed totally geodesic.

*Proof.* Let  $M$  be a semi-symmetric lightlike hypersurface of an indefinite Sasakian space form  $\overline{M}(c)$  of constant curvature  $c$  with  $\xi \in \Gamma(TM)$ . From (4.1) we have  $R(E, X)E = \frac{3(c-1)}{4}u(X)\overline{\phi}E$  and since, by using (2.6),  $\overline{R}(E, N, E, X) = 0$  and  $\overline{R}(E, X, E, N) = 0$ , by taking  $W_1 = E$  and  $W_2 = U$  into (5.3), we obtain, for any  $Y, T \in \Gamma(TM)$ ,

$$\begin{aligned}
 (\overline{R}(E, U) \cdot \overline{R})(E, Y, E, T) &= B(Y, R(E, U)E)g(A_N E, T) \\
 (5.4) \qquad \qquad \qquad &= -\frac{3(c-1)}{4}B(Y, V)g(A_N E, T).
 \end{aligned}$$

By direct calculation, the left-hand side is given by  $(\overline{R}(E, U) \cdot \overline{R})(E, Y, E, T) = 0$ . This equation implies, using (5.4),  $\frac{3(c-1)}{4}B(Y, V)g(A_N E, T) = 0$ , for any  $Y, T \in \Gamma(TM)$ , which completes the proof.  $\square$

**Theorem 7.** *Let  $M$  be a lightlike hypersurface of an indefinite Sasakian space form  $\overline{M}(c = 1)$ , with  $\xi \in TM$  and  $A_N E$  is a not null vector field. Then  $M$  is semi-symmetric if and only if it is totally geodesic.*

*Proof.* Suppose that  $c = 1$  and the vector field  $A_N E$  non-null on  $M$ . Then the curvature tensor  $R$  satisfies (4.26) and we have, for any  $X, Y, Z, T \in \Gamma(TM)$ ,

$$\begin{aligned}
 (5.5) \quad (R(E, X) \cdot R)(E, Y, Z, T) &= -B(X, Y)B(A_N E, Z)g(A_N E, T) \\
 &\quad - B(Y, A_N E)B(X, Z)g(A_N E, T) - B(Y, Z)B(X, T)g(A_N E, A_N E).
 \end{aligned}$$

If  $M$  is semi-symmetric, the left-hand side of (5.5) vanishes and we have,

$$\begin{aligned}
 0 &= B(X, Y)B(A_N E, Z)g(A_N E, T) + B(Y, A_N E)B(X, Z)g(A_N E, T) \\
 &\quad + B(Y, Z)B(X, T)g(A_N E, A_N E).
 \end{aligned}$$

which leads, by taking  $T = \xi$  and  $X = U$ , to  $0 = B(Y, Z)g(A_N E, A_N E)$ , that is  $B(Y, Z) = 0$ , for any  $Y, Z \in \Gamma(TM)$ . Conversely, suppose that  $B(X, Y) = 0$ . Then, using the relation (4.26),  $R(W_1, W_2) \cdot R = 0$ , that is,  $M$  is semi-symmetric.  $\square$

In virtue of Theorem 2 and Theorem 7, we have the following result.

**Theorem 8.** *Let  $M$  be a lightlike hypersurface of an indefinite Sasakian space form  $\overline{M}(c = 1)$  with  $\xi \in TM$  and  $A_N E$  is a not null vector field. Then  $M$  is locally symmetric if and only if  $M$  is semi-symmetric.*

**Theorem 9.** *Let  $M$  be a semi-symmetric lightlike hypersurface of an indefinite Sasakian space form  $\overline{M}(c)$  with  $\xi \in TM$ . If  $M$  is totally contact umbilical, then  $M$  is totally contact geodesic or  $A_N E = 0$ .*

*Proof.* Let  $M$  be a totally contact umbilical lightlike hypersurface of an indefinite Sasakian space forms  $\overline{M}(c)$  with  $\xi \in TM$ . Then, as was mentioned above,  $c = -3$ . If  $M$  is semi-symmetric. Using relations (4.14) and (5.4), we obtain

$$(5.6) \quad -\frac{3}{4}(c-1)\lambda u(Y)g(A_N E, PT) = 0, \quad \forall Y, T \in \Gamma(TM),$$

which leads, by taking  $Y = \overline{\phi}N$ , to  $\lambda = 0$  or  $A_N E = 0$ . This completes the proof.  $\square$

**Corollary 2.** *There are no proper totally contact umbilical lightlike hypersurfaces of indefinite Sasakian space forms  $\overline{M}(c)$  with  $\xi \in TM$  and  $A_N E \neq 0$  which are semi-symmetric.*

From Theorem 9, we deduce the following result.

**Proposition 7.** *Let  $M$  be a semi-symmetric lightlike hypersurface of an indefinite Sasakian space form  $\overline{M}(c)$  with  $\xi \in TM$  such that  $A_N E \neq 0$ . If  $M$  is  $\eta$ -totally contact umbilical, then  $M$  is totally geodesic.*

## §6. Ricci semi-symmetric lightlike hypersurfaces in indefinite Sasakian spaces form

In this section, we study Ricci semi-symmetric lightlike hypersurfaces of an indefinite Sasakian spaces form, tangent to the structure vector field  $\xi$ . We prove that Ricci semi-symmetric lightlike hypersurfaces are totally geodesic under some condition.

A lightlike submanifold  $M$  of a semi-Riemannian manifold  $\overline{M}$  is said to be Ricci semi-symmetric if the following condition is satisfied ([7])

$$(6.1) \quad (R(W_1, W_2) \cdot Ric)(X, Y) = 0, \quad \forall W_1, W_2, X, Y \in \Gamma(TM),$$

where  $R$  and  $Ric$  are induced Riemannian curvature and Ricci tensor on  $M$ , respectively. The latter condition is equivalent to

$$-Ric((R(W_1, W_2)X, Y) - Ric(X, (R(W_1, W_2)Y) = 0.$$



By Proposition 5 in [14], we have, for any  $X, Y \in \Gamma(TM)$ ,

$$(6.2) \quad Ric(X, Y) = ag(X, Y) - b\eta(X)\eta(Y) + B(X, Y)trA_N - B(A_NX, Y),$$

where  $a = \frac{(2n+1)(c+3)-8}{4}$  and  $b = \frac{(2n+1)(c-1)}{4}$  and trace  $tr$  is written with respect to  $g$  restricted to  $S(TM)$ .

In the following theorem we give result which shows the effect of Ricci semi-symmetric condition on the geometry of lightlike hypersurfaces of an indefinite Sasakian space form.

**Theorem 10.** *Let  $M$  be a Ricci semi-symmetric lightlike hypersurface of an indefinite Sasakian space form  $\bar{M}(c)$  with  $\xi \in TM$ . Then either  $c = 1$  or  $Ric(E, V) = 0$ . Moreover, if  $c = 1$ , then either  $M$  is totally geodesic or  $Ric(E, A_N E) = 0$ .*

*Proof.* Let  $M$  be a lightlike hypersurface of an indefinite Sasakian space form  $\bar{M}(c)$  with  $\xi \in \Gamma(TM)$ . We have, for any  $X, Y$ ,

$$(6.3) \quad (R(E, X) \cdot Ric)(E, Y) = \frac{c-1}{4} \{3au(X)u(Y) + 3u(X)B(V, Y)trA_N - 3u(X)B(A_NV, Y) - \bar{g}(\bar{\phi}X, Y)B(A_NE, V) + u(Y)B(A_NE, \phi X) + 2u(X)B(A_NE, \phi Y)\} + B(X, Y)B(A_NE, A_NE).$$

If  $M$  is Ricci semi-symmetric, then, by taking  $Y = E$  into (6.3), we obtain

$$\frac{3}{4}(c-1)u(X)B(\phi E, A_NE) = 0$$

which implies, for  $X = \bar{\phi}N$ ,  $\frac{3}{4}(c-1)Ric(E, \phi E) = 0$ , since  $B(\phi E, A_NE) = -Ric(E, \phi E)$ . On the other hand, suppose that  $c = 1$ . Using (6.3) and  $B(A_NE, A_NE) = -Ric(E, A_NE)$ , we have  $B(X, Y)Ric(E, A_NE) = 0$  which completes the proof.  $\square$

From Theorem 10, we have the following result.

**Theorem 11.** *Let  $M$  be a lightlike hypersurface of an indefinite Sasakian space form  $\bar{M}(c = 1)$  with  $\xi \in TM$  and  $Ric(E, A_NE) \neq 0$ . Then  $M$  is Ricci semi-symmetric if and only if  $M$  is totally geodesic.*

*Proof.* The converse follows from (4.26), (6.1) and (6.2).  $\square$

Let  $M$  be a lightlike hypersurface of an indefinite Sasakian space form  $\bar{M}(c)$  with  $\xi \in TM$ . If  $M$  is  $\eta$ -totally umbilical, then, by Lemma 2,  $c = 1$  and using the relation (6.3), we have

$$(6.4) \quad \begin{aligned} (R(E, X) \cdot Ric)(E, Y) &= B(X, Y)B(A_NE, A_NE) \\ &= \lambda^2 \{g(X, Y) - \eta(X)\eta(Y)\} g(A_NE, A_NE) \end{aligned}$$

which leads, by taking  $X = V$  and  $Y = U$ , to

$$(6.5) \quad (R(E, V) \cdot Ric)(E, U) = \lambda^2 g(A_N E, A_N E)$$

and we have the following result.

**Proposition 8.** *Let  $M$  be an  $\eta$ -totally umbilical lightlike hypersurface of an indefinite Sasakian space form  $\bar{M}(c)$  with  $\xi \in TM$  and  $A_N E$  is a not null vector field. Then  $M$  is Ricci semi-symmetric if and only if  $M$  is totally geodesic.*

By Theorem 8 and Proposition 8, we have

**Theorem 12.** *In  $\eta$ -totally umbilical lightlike hypersurfaces of indefinite Sasakian space forms  $\bar{M}(c)$ , tangent to the structure vector field  $\xi$  such that  $A_N E$  is a not null vector field, the conditions (4.3), (5.1) and (6.1) are equivalent.*

It is well known that the second fundamental form and the shape operator of a non-degenerate hypersurface (in general, submanifold) are related by means of the metric tensor field. Contrary to this, we see from (2.10)- (2.13) that in the case of lightlike hypersurfaces, there are interrelations between these geometric objects and those of its screen distributions. So, the geometry of lightlike hypersurfaces depends on the vector bundles  $S(TM)$ ,  $S(TM^\perp)$  and  $N(TM)$ . However, it is important to investigate the relationship between some geometrical objects induced, studied above, with the change of the screen distributions. In this case, it is known that the local second fundamental form of  $M$  on  $\mathcal{U}$  is independent of the choice of the above vector bundles. This means that all results of this paper which depend only on  $B$  are stable with respect to any change of those vector bundles.

Let  $P$  and  $P'$  be projections of  $TM$  on  $S(TM)$  and  $S(TM)'$ , respectively with respect to the orthogonal decomposition of  $TM$ . So, any vector field  $X$  on  $M$  can be written as

$$X = PX + \theta(X)E = P'X + \theta(X)E + \omega(X)E,$$

where  $\omega$  is the dual 1-form of  $W = \sum_{i=1}^{2n-1} c_i W_i$ , characteristic vector field of the screen change, with respect to the induced metric  $g$  of  $M$  defined as  $\omega(\cdot) = g(\cdot, W)$ . Then, using (4.29) we have  $P'X = PX - \omega(X)E$  and  $C'(X, P'Y) = C'(X, PY)$ , for any  $X, Y \in \Gamma(TM)$ . The relationship between the second fundamental forms  $C$  and  $C'$  of the screen distribution  $S(TM)$  and  $S(TM)'$ , respectively, is given by (using (4.29))

$$(6.6) \quad C'(X, PY) = C(X, PY) - \frac{1}{2}\omega(\nabla_X PY + B(X, Y)W).$$

All results above depending only on the the local second fundamental form  $C$  (making equations non unique) are independent of the screen distribution  $S(TM)$  if and only if  $\omega(\nabla_X PY + B(X, Y)W) = 0, \forall X, Y \in \Gamma(TM)$ .

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