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Edge-maximal graphs without θ_7-graphs

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Abstract. Let $\mathcal{G}(n; \theta_{2k+1}, \geq \delta)$ denote the class of non-bipartite θ_{2k+1}-free graphs on n vertices and minimum degree at least δ and let $f(n; \theta_{2k+1}, \geq \delta) = \max\{\mathcal{E}(G) : G \in \mathcal{G}(n; \theta_{2k+1}, \geq \delta)\}$. In this paper we determine an upper bound of $f(n; \theta_7, \geq 25)$ by proving that for large n, $f(n; \theta_7, \geq 25) \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3$. Our result confirm the conjecture made in [1], "Some extermal problems in graph theory", Ph.D thesis, Curtin University of Technology, Australia (2007), in case $k = 3$ and $\delta = 25$.

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§1. Introduction

For our purposes a graph G is finite, undirected and has no loops or multiple edges. We denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. The cardinalities of these sets are denoted by $v(G)$ and $\mathcal{E}(G)$, respectively. The cycle on n vertices is denoted by C_n. Let C be a cycle in a graph G, an edge in $E(G[C])\backslash E(C))$ is called a chord of C. Further, a graph G has a θ_k- graph if G has a cycle C_k with a chord. The circumference of a graph G is denoted by $c(G)$ and defined to be the length of longest cycle. Let G be a graph and $u \in V(G)$. The degree of a vertex u in G, denoted by $d_G(u)$, is the number of edges of G incident to u. The neighbour set of a vertex u of G in a subgraph H of G, denoted by $N_H(u)$, consists of the vertices of H adjacent to u; we write $d_H(u) = |N_H(u)|$. For vertex disjoint subgraphs H_1 and H_2 of G we let

$$E(H_1, H_2) = \{xy \in E(G) : x \in V(H_1), y \in V(H_2)\}$$

and

$$\mathcal{E}(H_1, H_2) = |E(H_1, H_2)|.$$

For a proper subgraph H of G we write $G[V(H)]$ and $G - V(H)$ simply as $G[H]$ and $G - H$ respectively.

In this paper, we consider the Turán-type extremal problem with the θ-graph being the forbidden subgraph. Since a bipartite graph contains no odd θ-graph, we consider non-bipartite graphs. First, we recall some notation and terminology. For a positive integer n and a set of graphs $\mathcal{F}$, let $\mathcal{G}(n;\mathcal{F},\geq \delta)$ denote the class of non-bipartite $\mathcal{F}$-free graphs on n vertices and minimum degree is at least δ, and

$$f(n;\mathcal{F},\geq \delta) = \max\{\mathcal{E}(G) : G \in \mathcal{G}(n;\mathcal{F},\geq \delta)\}.$$

For simplicity, in case $\delta = 1$, we write $\mathcal{G}(n;\mathcal{F},\geq 1) = \mathcal{G}(n;\mathcal{F})$ and $f(n;\mathcal{F},\geq 1) = f(n;\mathcal{F})$.

An important problem in extremal graph theory is that of determining the values of the function $f(n;\mathcal{F})$. Further, characterized the extremal graphs $\mathcal{G}(n;\mathcal{F})$ where $f(n;\mathcal{F})$ is attained. For a given C_r, the edge maximal graphs of $\mathcal{G}(n;C_r)$ have been studied by a number of authors [4, 5, 7]. Bondy [3] proved that a Hamiltonian graph G on n vertices without a cycle of length r has at most $\frac{1}{2}n^2$ edges with equality holding if and only if n is even and r is odd.

Häggkvist et al. [6] proved that $f(n;C_r) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1$ for all r. This result is sharp only for $r = 3$. Jia [8] proved that for $n \geq 9, f(n;C_5) = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3$, and he characterized the extremal graphs as well. In the same work, Jia conjectured that $f(n;C_{2k+1}) = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3$ for $n \geq 4k+2$. Recently, Bataineh [1] confirmed positively the above conjecture for large n. Moreover, Bataineh conjectured that for $k \geq 3$, $f(n;\theta_{2k+1}) = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3$. Most recently, Bataineh et al. [2], proved that for $n \geq 9$, $f(n;\theta_5) = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1$. In this paper, we confirm the above conjecture in case $k = 3$ by proving that $f(n;\theta_7,\geq 25) \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3$. Further, we give a class of graphs to show that $f(n;\theta_7,\geq 1) \geq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3$.

§2. Edge-maximal θ_7-free graphs

We state a number of results which we make use of in our work.

Lemma 2.1 (Woodall) *Let G be a graph on n vertices with no cycles of length greater than k. Then $\mathcal{E}(G) \leq \frac{1}{2}k(n-1) - \frac{1}{2}r(k-r-1)$ where $r = (n-1) - (k-1)\left\lfloor \frac{(n-1)}{(k-1)} \right\rfloor$.*

Theorem 2.1(Bataineh) *Let $G \in \mathcal{G}(n; C_{2k+1})$. For large n,*

$$f(n; C_{2k+1}) = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3.$$

Furthermore, equaility holds only if and only if $G \in \mathcal{G}^*(n)$ where $\mathcal{G}^*(n)$ is the class of graphs obtained by adding a triangle, two vertices of which are new, to the complete bipartite graph $K_{\left\lfloor \frac{(n-2)}{2} \right\rfloor, \left\lceil \frac{(n-2)}{2} \right\rceil}$.

Lemma 2.2 (Bondy) *Let G be a graph on n vertices with $\mathcal{E}(G) > \left\lfloor \frac{n^2}{4} \right\rfloor$, then $c(G) \geq \left\lfloor \frac{n+3}{2} \right\rfloor$ and G contains the cycles of every length l for $3 \leq l \leq c(G)$.*

In this section we give an upper bound of $f(n; \theta_7, \geq 25)$ by proving that for large n, $f(n; \theta_7, \geq 25) \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3$. We begin with some constructions. Let G_1 be a graph on 7 vertices, G_2 be a graph on 8 vertices, G_3 be a graph on 9 vertices and G_4 be a graph on 10 vertices as shown in Figure 1. Observe that each of G_1, G_2, G_3 and G_4 has no θ_7 as a subgraph and $\mathcal{E}(G_1) = 16, \mathcal{E}(G_2) = 18, \mathcal{E}(G_3) = 21$ and $\mathcal{E}(G_4) = 25$.

Lemma 2.3 *Let G be a graph on $n(7 \leq n \leq 10)$ vertices. If G has no θ_7-graph as subgraph, then*

a) If $n = 7$, then $\mathcal{E}(G) \leq 16$ and the bound is best possible.
b) If $n = 8$, then $\mathcal{E}(G) \leq 18$ and the bound is best possible.
c) If $n = 9$, then $\mathcal{E}(G) \leq 21$ and the bound is best possible.
d) If $n = 10$, then $\mathcal{E}(G) \leq 25$ and the bound is best possible.

Proof. a) $\mathbf{n = 7}$: If G is a bipartite graph, then $\mathcal{E}(G) \leq 12$. Assume that $\mathcal{E}(G) \geq 17$. Then by Lemma 2.2 we have $c(G) \geq 5$ and G is pancyclic. So, we have 3 cases to consider according to the value of $c(G)$.

Case1: $c(G) = 7$. Let C be the cycle of length 7 in G. Observe that, if we add any edge to C, then G would have θ_7 subgraph. So, $\mathcal{E}(G) \leq 7$. This is a contradiction.

Case 2: $c(G) = 6$. Let $x_1x_2x_3x_4x_5x_6x_1$ be the cycle of length 6 in G. Define $A = G[x_1, x_2, x_3, x_4, x_5, x_6]$ and let y be the remaining vertex. Then observe that $\mathcal{E}(y, A) \leq 3$ with equality hold only if $N_A(y) = \{x_i, x_{i+2}, x_{i+4}\}$, otherwise $c(G) = 7$. If $|N_A(y)| = 3$, without loss of generality assume that $N_A(y) = \{x_1, x_3, x_5\}$. Observe that x_2x_4, x_2x_6 and $x_4x_6 \notin E(G)$, otherwise $c(G) = 7$. So,

$$\begin{aligned} \mathcal{E}(A) &\leq 15 - 3 \\ &= 12. \end{aligned}$$

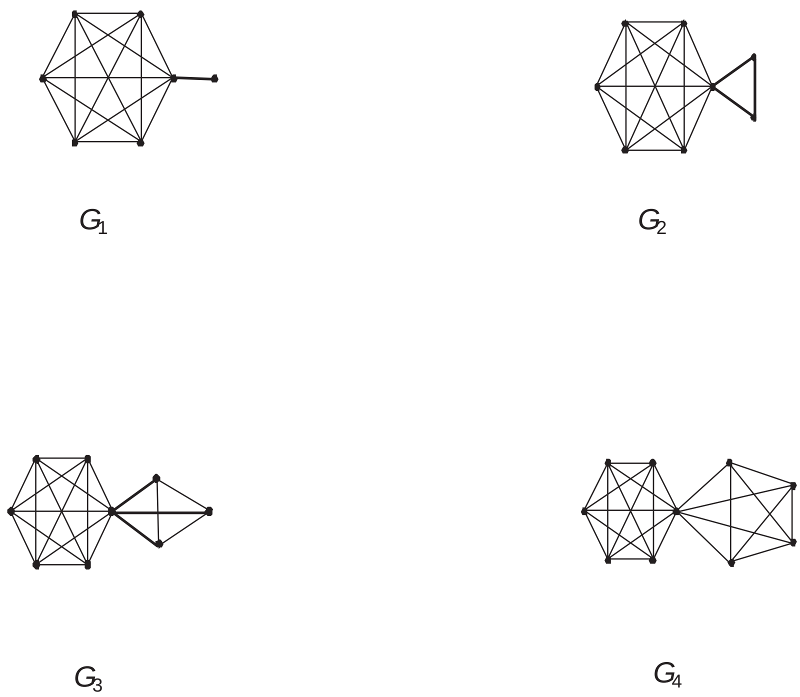

Figure 1:

Thus,

$$\begin{aligned}\mathcal{E}(G) &= \mathcal{E}(A) + \mathcal{E}(y, A)\\ &\leq 12 + 3\\ &= 15.\end{aligned}$$

This is a contradiction. If $|N_A(y)| = 2$, then the neighbors of y must be non-consecutive, otherwise $c(G) = 7$. Also, if $N_A(y) = \{x_i, x_{i+2}\}$, then $x_{i+1}x_{i+5} \notin E(G)$ and $x_{i+1}x_{i+3} \notin E(G)$, otherwise $c(G) = 7$. Furthermore, if $N_A(y) = \{x_i, x_{i+3}\}$, then $x_{i+2}x_{i+5} \notin E(G)$ and $x_{i+1}x_{i+4} \notin E(G)$, otherwise $c(G) = 7$. Thus,

$$\begin{aligned}\mathcal{E}(A) &\leq 15 - 2\\ &= 13.\end{aligned}$$

Consequently, we have,

$$\begin{aligned}\mathcal{E}(G) &= \mathcal{E}(A)+\mathcal{E}(y,A)\\ &= 13+2\\ &= 15.\end{aligned}$$

This is a contradiction. If $|N_A(y)|=1$, then

$$\begin{aligned}\mathcal{E}(G) &= \mathcal{E}(A)+\mathcal{E}(y,A)\\ &\leq 15+1\\ &= 16.\end{aligned}$$

This is a contradiction.
Case 3: $c(G)=5$. Since $c(G)=5$, then by Lemma 2.1, we have

$$\begin{aligned}\mathcal{E}(G) &\leq 9\\ &< 16.\end{aligned}$$

This is a contradiction.

b) $\mathbf{n=8}$: Assume that $\mathcal{E}(G)\geq 19$. Then by Lemma 2.2, we have $c(G)\geq 5$ and G is pancylic. So, we have 3 cases according to the value of $c(G)$.
Case 1: $c(G)\leq 6$. Then by Lemma 2.1, we have

$$\begin{aligned}\mathcal{E}(G) &\leq 15\\ &< 18.\end{aligned}$$

This is a contradiction.
Case 2: $c(G)=7$. Let $x_1x_2x_3x_4x_5x_6x_7x_1$ be the cycle of length 7 in G. Define $A=G[x_1,x_2,x_3,\ x_4,x_5,x_6,x_7]$ and let y be the remaining vertex in G. Observe that if $\mathcal{E}(y,A)\geq 4$, then G would have θ_7 as a subgraph. So $\mathcal{E}(y,A)\leq 3$ with equality hold only if $N_A(y)=\{x_i,x_{i+1},x_{i+2}\}$ or $\{x_i,x_{i+1},x_{i+4}\}$, otherwise θ_7 is produced. Further $\mathcal{E}(G)\leq 7$, thus

$$\begin{aligned}\mathcal{E}(G) &= \mathcal{E}(A)+\mathcal{E}(y,A)\\ &\leq 7+3\\ &= 10.\end{aligned}$$

This is a contradiction.
Case 3: $c(G)=8$. Then G must have a cycle of length 7. From Case 2 we have $\mathcal{E}(G)\leq 10$. This is a contradiction.

c) $\mathbf{n = 9}$: Assume that $\mathcal{E}(G) \geq 22$. Then by Lemma 2.2 we have $c(G) \geq 6$ and G is pancylic. So, we have two cases according to the value of $c(G)$.
Case 1: $7 \leq c(G) \leq 9$. Then G must have a cycle of length 7, say C. Let y_1, y_2 be the remaining vertices in G. Observe that $\mathcal{E}(y_1, C) \leq 3$ and $\mathcal{E}(y_2, C) \leq 3$. Therefore,

$$\begin{aligned}
\mathcal{E}(G) &= \mathcal{E}(C) + \mathcal{E}(y_1, C) + \mathcal{E}(y_2, C) + \mathcal{E}(y_1, y_2) \\
&\leq 7 + 3 + 3 + 1 \\
&= 14.
\end{aligned}$$

This is a contradiction.
Case 2: $c(G) \leq 6$. Then by Lemma 2.1 we have

$$\begin{aligned}
\mathcal{E}(G) &\leq 18 \\
&< 21.
\end{aligned}$$

This is a contradiction.
d) $\mathbf{n = 10}$: Suppose that $\mathcal{E}(G) \geq 26$. Then by Lemma 2.2 we have $c(G) \geq 6$ and G is pancyclic. So, we have two cases according to the value of $c(G)$.
Case 1: $7 \leq c(G) \leq 10$. Then G must have a cycle of length 7. Let $x_1x_2x_3x_4x_5x_6x_7x_1$ be a cycle of length 7 in G and y_1, y_2, y_3 be the remaining vertices in G. Define $A = G[x_1, x_2, x_3, x_4, x_5, x_6, x_7]$ and $B = G[y_1, y_2, y_3]$. Recall that $\mathcal{E}(y_i, A) \leq 3$ for $i = 1, 2, 3$ with equality hold only if $N_A(y_i) = \{x_j, x_{j+1}, x_{j+2}\}$ or $\{x_j, x_{j+1}, x_{j+4}\}$, otherwise G would have θ_7 as a subgraph. Note that $\mathcal{E}(B) \leq 3$. Thus,

$$\begin{aligned}
\mathcal{E}(G) &= \mathcal{E}(B) + \mathcal{E}(B, A) + \mathcal{E}(A) \\
&\leq 3 + 9 + 7 \\
&\leq 19 \\
&< 25.
\end{aligned}$$

This is a contradiction.
Case 2: $c(G) = 6$. Then by Lemma 2.1 we have

$$\begin{aligned}
\mathcal{E}(G) &\leq 23 \\
&< 25.
\end{aligned}$$

This is a contradiction. This completes the proof.

Now we determine the maximum number of edges when θ_7 being the forbidden subgraph.

Theorem 2.2 *For $n \geq 10$, let G be a graph on n vertices. If G has no θ_7 as a subgraph, then*

$$\mathcal{E}(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Proof. Let k be the maximum number of vertex disjoint cycles of length 7 in G. We prove it by induction on the value of k. For $k = 0$, we have by Theorem 2.1, $\mathcal{E}(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$. For $k = 1$. Let $C = x_1x_2x_3x_4x_5x_6x_7x_1$ be a cycle of length 7 in G. Define $R = G - C$. Observe that R has no cycle of length 7. If $|R| \geq 10$, then by induction hypotheses we have

$$\mathcal{E}(R) \leq \left\lfloor \frac{(n-7)^2}{4} \right\rfloor.$$

Now, for any vertex $y \in R$, observe that if $\mathcal{E}(y, C) \geq 4$, then θ_7 is produced. So, $\mathcal{E}(y, C) \leq 3$ for all $y \in R$ with equality hold only if $N_C(y) = \{x_i, x_{i+1}, x_{i+2}\}$ or $N_C(y) = \{x_i, x_{i+1}, x_{i+4}\}$ for $i = 1, 2, \ldots, 7 (\text{mod } 7)$. Otherwise G would have θ_7 subgraph. Thus,

$$\begin{aligned} \mathcal{E}(R, C) &\leq 3\,|R| \\ &= 3(n-7) \\ &= 3n - 21. \end{aligned}$$

So,

$$\begin{aligned} \mathcal{E}(G) &= \mathcal{E}(R) + \mathcal{E}(R, C) + \mathcal{E}(C) \\ &\leq \left\lfloor \frac{(n-7)^2}{4} \right\rfloor + 3n - 21 + 7 \\ &\leq \left\lfloor \frac{n^2 - 14n + 49 + 12n - 56}{4} \right\rfloor \\ &\leq \left\lfloor \frac{n^2 - 2n - 7}{4} \right\rfloor \\ &\leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor - 2 \\ &\leq \left\lfloor \frac{n^2}{4} \right\rfloor. \end{aligned}$$

So we need to consider the case when $|R| \leq 9$. For $|R| = 9$. Then we have

$$\begin{aligned} \mathcal{E}(G) &= \mathcal{E}(R) + \mathcal{E}(R, C) + \mathcal{E}(C) \\ &\leq 21 + 27 + 7 \\ &\leq \left\lfloor \frac{16^2}{4} \right\rfloor. \end{aligned}$$

Similarly, we can do the same arguments for $6 \leq |R| \leq 8$. Now, suppose the result holds when G has less than k vertex-disjoint cycle of length 7.

Let G has k vertex disjoint cycles of length 7 and C be a cycle of length 7 in G. Set $R = G - C$. Note that R has $(k-1)$ vertex disjoint cycles of length 7, thus by induction hypothesis we have

$$\mathcal{E}(R) \leq \left\lfloor \frac{(n-7)^2}{4} \right\rfloor.$$

Also, recall that $\mathcal{E}(y, C) \leq 3$ for all $y \in R$.Thus,

$$\begin{aligned} \mathcal{E}(R, C) &\leq 3\,|R| \\ &= 3(n-7). \end{aligned}$$

So,

$$\begin{aligned} \mathcal{E}(G) &= \mathcal{E}(R) + \mathcal{E}(R, C) + \mathcal{E}(C) \\ &\leq \left\lfloor \frac{(n-7)^2}{4} \right\rfloor + 3(n-7) + 7 \\ &= \left\lfloor \frac{n^2 - 14n + 49}{4} \right\rfloor + 3n - 14 \\ &\leq \left\lfloor \frac{n^2 - 14n + 49 + 12n - 56}{4} \right\rfloor \\ &= \left\lfloor \frac{n^2 - 2n - 7}{4} \right\rfloor \\ &\leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor - 2 \\ &\leq \left\lfloor \frac{n^2}{4} \right\rfloor. \end{aligned}$$

This completes the proof.

We start with the follwoing construction: Let $\mathcal{G}^*(n)$ be the class of graphs obtained by adding a triangle, two vertices of which are new, to the complete bipartite graph $K_{\lfloor \frac{n-2}{2} \rfloor, \lceil \frac{n-2}{2} \rceil}$. Note that if $G \in \mathcal{G}^*(n)$, then G is a free of θ_7. Furthermore, if $G \in \mathcal{G}^*(n)$, then $\mathcal{E}(G) = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3$. Thus, we established that $f(n; \theta_7, \geq 1) \geq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3$. Now, in the following theorem we give an upper bound of $f(n; \theta_7, \geq 25)$.

Theorem 2.3 *For sufficiently large n, let $G \in G(n; \theta_7, \geq 25)$. Then*

$$\mathcal{E}(G) \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3.$$

Proof. Let $G \in \mathcal{G}(n;\theta_7)$. If G has no cycle of length 7 as a subgraph, then by Theorem 2.1 we have that $\mathcal{E}(G) \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3$. So, we need to consider the case when G has a cycle of length 7. Let $x_1x_2x_3x_4x_5x_6x_7x_1$ be a cycle of length 7 in G. Define $A = G[x_1, x_2, x_3, x_4, x_5, x_6, x_7]$. Observe that $\mathcal{E}(A) = 7$. Now, we consider two cases according to whether G has θ_4 as a subgraph or not.

Case 1: G has no θ_4 as a subgraph. Let $x \in G - A$. If $\mathcal{E}(x, A) \geq 4$, then θ_7 is produced. So, $\mathcal{E}(x, A) \leq 3$ with equality hold only if $N_A(x) = \{x_i, x_{i+1}, x_{i+4}\}$ (mod 7), as otherwise G would have θ_7 or θ_4 as a subgraphs. Now, define $B = \{v \in V(G - A) : \mathcal{E}(v, A) = 3\}$.

Claim: $|B| \leq 1$.

Proof of the claim: Suppose that $x, y \in B$ and $x \neq y$, we conceder two cases:

Case I: $xy \in E(G)$. Note that $\mathcal{E}(x, A) = 3$ and $N_A(x) = \{x_i, x_{i+1}, x_{i+4}\}$. So, without loss of generality, we assume that $N_A(x) = \{x_1, x_2, x_5\}$. Then we have the following observations:

1) If y is adjacent to x_1, then the trail $xyx_1x_2xx_1$ would form θ_4 as a subgraph.

2) If y is adjacent to x_2, then the trail $xyx_2x_1xx_2$ would form θ_4 as a subgraph.

3) If y is adjacent to x_4, then the trail $xyx_4x_5x_6x_7x_1xx_5$ would form θ_7 as a subgraph.

4) If y is adjacent to x_5, then the trail $xyx_5x_6x_7x_1x_2xx_1$ would form θ_7 as a subgraph.

5) If y is adjacent to x_6, then the trail $xyx_6x_5x_4x_3x_2xx_5$ would form θ_7 as a subgraph.

From the above observation, we have $\mathcal{E}(y, A) \leq 2$ which contradict that $y \in B$. Thus, $|B| \leq 1$.

Case II: $xy \notin E(G)$. Recall that $N_A(x) = \{x_1, x_2, x_5\}$. We consider two subcases a according to the value of $|N_A(x) \cap N_A(y)|$.

Subcase II.I: $|N_A(x) \cap N_A(y)| = 0$. Since $y \in B$, we have $N_A(y)$ of the form $\{x_i, x_{i+1}, x_{i+4}\}$. This only happen when $N_A(y) = \{x_3, x_4, x_7\}$ or $N_A(y) = \{x_3, x_6, x_7\}$. If $N_A(y) = \{x_3, x_4, x_7\}$, then the trail $xx_5x_4yx_7x_1x_2xx_1$ would form θ_7 as a subgraph. If $N_A(y) = \{x_3, x_6, x_7\}$, then the trail $xx_5x_6yx_7x_1x_2xx_1$ would form θ_7 as a subgraph. Thus, we have $|N_A(x) \cap N_A(y)| > 0$.

Subcase II.II: $|N_A(x) \cap N_A(y)| \geq 1$. Suppose y is adjacent to x_1. Then we have the following observation:

1) If y is adjacent to x_2, then trail $x_1xx_2yx_1x_2$ would form θ_4 as a subgraph.

2) If y is adjacent to x_3, then trail $xx_5x_4x_3yx_1x_2xx_1$ would form θ_7 as asubgraph.

3) If y is adjacent to x_5, then trail $xx_2x_3x_4x_5yx_1xx_5$ would form θ_7 as a

subgraph.

4) If y is adjacent to x_7, then trail $xx_5x_6x_7yx_1x_2xx_1$ would form θ_7 as a subgraph.

From the above observations y can be adjacent only to x_4 and x_6, but the trial $yx_6x_5x_4x_3x_2x_1yx_4$ forms θ_7 as a subgraph. Thus, $\mathcal{E}(y, A) \leq 2$ this is a contradiction. Suppose that $x_2 \in N_A(y) \cap N_A(x)$. Then we have the following observation:

1) $x_1 \notin N_A(y)$ from the previous observations.

2) If y is adjacent to x_3, then the trail $xx_1x_2yx_3x_4x_5xx_2$ would form θ_7 as a subgraph.

3) If y is adjacent to x_5, then the trail $xx_2yx_5x_6x_7x_1xx_5$ would form θ_7 as a subgraph.

4) If y is adjacent to x_7, then the trail $xx_5x_6x_7yx_2x_1xx_2$ would form θ_7 as a subgraph.

Thus, y can be adjacent to at most x_4 and x_6, but the trail $yx_4x_5x_6x_7x_1x_2yx_6$ forms θ_7 as a subgraph. Thus, we have $\mathcal{E}(y, A) \leq 2$. Suppose that $x_5 \in N_A(x) \cap N_A(y)$. Then, we have the following observations:

1) $x_1, x_2 \notin N_A(y)$ form the previous observations.

2) If y is adjacent to x_4, then the trail $xx_1x_2x_3x_4yx_5xx_2$ would form θ_7 as a subgraph.

3) If y is adjacent to x_6, then the trail $xx_5yx_6x_7x_1x_2xx_1$ would form θ_7 as a subgraph.

Thus, y can be adjacent to at most x_3 and x_7, but the trail $yx_3x_2xx_5x_6x_7yx_5$ forms θ_7 as a subgraph. Thus, $\mathcal{E}(y, A) \leq 2$. This is a contradiction. Proof of the claim is complete. Hence,

$$\begin{aligned} \mathcal{E}(G - A, A) &\leq 3\,|B| + 2(|G - A| - |B|) \\ &= 3 + 2(n - 8) \\ &= 2n - 13. \end{aligned}$$

Thus, using Theorem 2.2, we have

$$\begin{aligned}
\mathcal{E}(G) &= \mathcal{E}(G-A)+\mathcal{E}(G-A,A)+\mathcal{E}(A)\\
&\le \left\lfloor \frac{(n-7)^2}{4} \right\rfloor + 2n - 13 + 7\\
&\le \frac{n^2-14n+49+8n-24}{4}\\
&= \frac{n^2-6n+25}{4}\\
&\le \left\lceil \frac{(n-3)^2}{4} \right\rceil + 4\\
&< \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3.
\end{aligned}$$

Case 2: G has θ_4 as a subgraph. Let $x_1x_2x_3x_4$ be θ_4 with x_2x_4 be the chord. Note that the vertices in G have degree more than or equal 25 in G. For $i=1,2,3$, let A_i be a set that consist of 7 neighbors of x_i in G selected so that $A_i \cap A_j = \phi$ for $i \neq j$. Let $T = G[x_1,x_2,x_3,x_4,A_1,A_2,A_3]$ and $H = G-T$. The situation as shown in Figure 2:

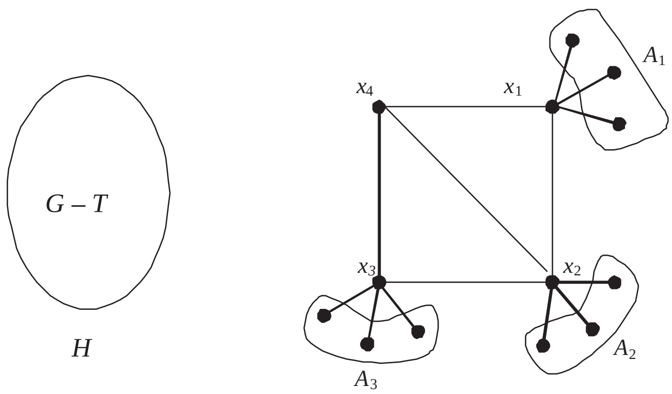

Figure 2:

Let $u \in V(H)$. If u is adjacent to a vertex in one of the sets A_1, A_2 and A_3, then u can not be adjacent to a vertex in the other two sets as otherwise, G would have a θ_7-graph. Thus,

$$\mathcal{E}(\{u\},T) \le 11.$$

Consequently, we have,

$$\mathcal{E}(H,T) \leq 11(n-25).$$

Now,

$$\begin{aligned}
\mathcal{E}(G) &= \mathcal{E}(H) + \mathcal{E}(H,T) + \mathcal{E}(T) \\
&\leq \frac{(n-25)^2}{4} + 11(n-25) + \frac{(25)^2}{4} \\
&\leq \frac{n^2 - 50n + 625 + 44n - 1100 + 625}{4} \\
&\leq \frac{n^2 - 6n + 150}{4} \\
&< \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3.
\end{aligned}$$

This completes the proof.

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