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Mean Labeling of Some Graphs

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Abstract. Let G be a (p, q) graph and $f : V(G) \to \{0, 1, 2, 3, \ldots, q\}$ be an injection. For each edge $e = uv$, let $f^*(e) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. Then f is called a mean labeling if $\{f^*(e) : e \in E(G)\} = \{1, 2, 3, \ldots, q\}$. A graph that admits a mean labeling is called a *mean graph*. In this paper, we prove $T\hat{o}C_n, T\tilde{o}C_n, T@P_n, T©2P_n$, where T is a T_p-tree, are mean graphs.

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Key words and phrases. Mean labeling, mean graph, T_p-tree.

§1. Introduction

By a graph, we mean a finite simple and undirected one. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. The disjoint union of m copies of the graph G is denoted by mG. The union of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. A vertex of degree one is called a pendant vertex. The corona $G_1 \odot G_2$ of the graphs G_1 and G_2 is obtained by taking one copy of G_1 (with p vertices) and p copies of G_2 and joining the i^{th} vertex of G_1 to every vertex of the i^{th} copy of G_2.

Let T be a tree and u_0 and v_0 be two adjacent vertices in $V(T)$. Let there be two pendant vertices u and v in T such that the length of u_0-u path is equal to the length of v_0-v path. If the edge u_0v_0 is deleted from T and u, v are joined by an edge uv, then such a transformation of T is called an elementary parallel transformation (or an EPT) and the edge u_0v_0 is called a transformable edge. If by a sequence of EPT's T can be reduced to a path, then T is called a T_p-tree (transformed tree) and any such sequence regarded as a composition of mappings (EPT's) denoted by P, is called a parallel transformation of T. The path, the image of T under P is denoted as $P(T)$.

Let T be a T_p-tree with m vertices. Let $T\hat{o}C_n$ be a graph obtained from T and m copies of C_n by identifying a vertex of i^{th} copy of C_n with i^{th} vertex of T. Let $T\tilde{o}C_n$ be a graph obtained from T and m copies of C_n by joining a vertex of i^{th} copy of C_n with i^{th} vertex of T by an edge. Let $T@P_n$ be the graph obtained from T and m copies of P_n by identifying one pendant vertex of i^{th} copy of P_n with i^{th} vertex of T, where P_n is a path of length $n-1$. Let $T©2P_n$ be the graph obtained from T by identifying the pendant vertices of two vertex disjoint paths of equal lengths $n-1$ at each vertex of the T_p-tree T. Terms and notations not defined here are used in the sense of Harary [1].

A graph $G=(p,q)$ with p vertices and q edges is called a mean graph if there is an injective function f that maps $V(G)$ to $\{0,1,2,3,\ldots,q\}$ such that for each edge uv is labeled with $\frac{f(u)+f(v)}{2}$ if $f(u)+f(v)$ is even and $\frac{f(u)+f(v)+1}{2}$ if $f(u)+f(v)$ is odd. Then the resulting edge labels are distinct.

The mean labeling of $P_6 \odot K_1$ is given in Figure 1.

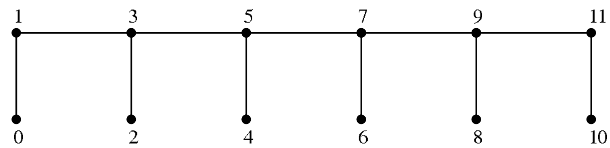

Figure 1

The concept of mean labeling is introduced by S. Somasundaram and R. Ponraj [7] in 2003. They have proved in [4, 5, 8, 9] and [10] the meanness of many standard graphs like $P_n, C_n, K_n (n \le 3)$, the ladder, the triangular snake $K_{2,n}, K_2+mK_1, K_n+2K_2, C_m \cup P_n, P_m \times P_n, P_m \times C_n, C_m \odot K_1, P_m \odot K_1$, the friendship graph, triangular snakes, quadrilateral snakes, K_n if and only if $n \le 3, K_{1,n}$ if and only if $n \le 3$, bistars $B_{m,n} (m \ge n)$ if and only if $m < n+2$, the subdivision graph of the star $K_{1,n}$ if and only if $n < 4$, the friendship graph $C_3^{(t)}$ if and only if $t < 2$. Also they ahve investigated the mean graphs or order less than or equal to 5. In addition, they have proved that the graphs $K_n(n>3), K_{1,n}(n>3), B_{m,n}(m>n+2), S(K_{1,n})(n>4), C_3^{(t)}(t>2)$, the wheel W_n are not mean graphs. M.A. Seoud and M.A. Salim [6] investigated the mean labeling of graphs of order 6. In [2, 3], some constructions of mean graphs and mean labeling of some standard families of graphs are given.

§2. Mean Graphs

Theorem 2.1. *Let T be a T_p-tree on m vertices. Then the graph $T\hat{o}C_n$ is a mean graph.*

Proof. Let T be a T_p-tree with m vertices. By the definition of a T_p-tree, there exists a parallel transformation P of T such that for the path $P(T)$ we have

(i) $V(P(T)) = V(T)$ and (ii) $E(P(T)) = (E(T)\backslash E_d) \cup E_P$, where E_d is the set of edges deleted from T and E_P is the set of edges newly added through the sequence $P = (P_1, P_2, \ldots, P_k)$ of the EPTs P used to arrive at the path $P(T)$. Clearly E_d and E_P have the same number of edges.

Now denote the vertices of $P(T)$ successively as $v_1, v_2, v_3, \ldots, v_m$ starting from one pendant vertex of $P(T)$ right up to other. Let $u_1^i, u_2^i, \ldots, u_n^i$ be the vertices of the i^{th} copy of C_n with $u_1^i = v_i$ for $1 \leq i \leq m$. Then $V(T\hat{o}C_n) = \{u_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$. Let

$$n = \begin{cases} 2k+1 & \text{if } n \text{ is odd} \\ 2k & \text{if } n \text{ is even.} \end{cases}$$

Define $f : V(T\hat{o}C_n) \to \{0, 1, 2, 3, \ldots, q = (n+1)m - 1\}$ as follows:
Case (i). n is odd.

$$\begin{aligned}
f(u_i^{2j-1}) &= 2(n+1)(j-1) + 2(i-1) && for\ 1 \leq j \leq \left\lceil \frac{m}{2} \right\rceil, 1 \leq i \leq k+1, \\
f(u_{k+2}^{2j-1}) &= 2(n+1)(j-1) + n && for\ 1 \leq j \leq \left\lceil \frac{m}{2} \right\rceil, \\
f(u_{k+2+i}^{2j-1}) &= 2(n+1)(j-1) + n - 2i && for\ 1 \leq j \leq \left\lceil \frac{m}{2} \right\rceil, 1 \leq i \leq k-1, \\
f(u_1^{2j}) &= 2(n+1)j - 1 && for\ 1 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor, \\
f(u_i^{2j}) &= 2(n+1)j - 2(i-1) && for\ 1 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor, 2 \leq i \leq k+2, \\
f(u_{k+2+i}^{2j}) &= (n+1)(2j-1) + 2i + 1 && for\ 1 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor, 1 \leq i \leq k-1.
\end{aligned}$$

Let $v_i v_j$ be a transformed edge in T for some indices i and $j, 1 \leq i \leq j \leq m$ and let P_1 be the EPT that deletes the edge $v_i v_j$ and adds the edge $v_{i+t} v_{j-t}$ where t is the distance of v_i from v_{i+t} and also the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent EPTs. Since $v_{i+t} v_{j-t}$ is an edge in the path $P(T)$, $i + t + 1 = j - t$ which implies $j = i + 2t + 1$. The induced label of the edge $v_i v_j$ is given by,

$$(2.1) \qquad f^*(v_i v_j) = f^*(v_i v_{i+2t+1}) = \left\lceil \frac{f(v_i) + f(v_{i+2t+1})}{2} \right\rceil = (n+1)(i+t)$$

and

$$(2.2) \quad f^*(v_{i+t} v_{j-t}) = f^*(v_{i+t} v_{i+t+1}) = \left\lceil \frac{f(v_{i+t} + f(v_{i+t+1})}{2} \right\rceil = (n+1)(i+t)$$

Therefore from (2.1) and (2.2), $f^*(v_i v_j) = f^*(v_{i+t} v_{j-t})$.

Let $e_i^j = u_i^j u_{i+1}^j, e_n^j = u_n^j u_1^j$ for $1 \leq i \leq n-1, 1 \leq j \leq m$.

For each vertex label f, the induced edge label f^* is defined as follows:

$$\begin{aligned}
f^*(v_iv_{i+1}) &= (n+1)i && for\ 1 \le i \le m-1,\\
f^*(e_i^{2j-1}) &= 2(n+1)(j-1)+2i-1 && for\ 1 \le j \le \left\lceil \frac{m}{2} \right\rceil, 1 \le i \le k+1\\
f^*(e_{k+2}^{2j-1}) &= 2(n+1)(j-1)+n-1 && for\ 1 \le j \le \left\lceil \frac{m}{2} \right\rceil,\\
f^*(e_{k+2+i}^{2j-1}) &= 2(n+1)(j-1)+n-1+2i && for\ 1 \le j \le \left\lceil \frac{m}{2} \right\rceil, 1 \le i \le k-1,\\
f^*(e_i^{2j}) &= 2(n+1)j-2i+1 && for\ 1 \le j \le \left\lfloor \frac{m}{2} \right\rfloor, 1 \le i \le k+1,\\
f^*(e_{k+2}^{2j}) &= 2(n+1)j-n+1 && for\ 1 \le j \le \left\lfloor \frac{m}{2} \right\rfloor,\\
f^*(e_{k+2+i}^{2j}) &= 2(n+1)j-n+1+2i && for\ 1 \le j \le \left\lfloor \frac{m}{2} \right\rfloor, 1 \le i \le k-1.
\end{aligned}$$

It can be verified that f is an mean labeling of $T\hat{o}C_n$.
Case (ii). n is even.

$$\begin{aligned}
f(u_i^{2j-1}) &= 2(n+1)(j-1)+2(i-1) && for\ 1 \le j \le \left\lceil \frac{m}{2} \right\rceil, 1 \le i \le k+1,\\
f(u_{k+2}^{2j-1}) &= 2(n+1)(j-1)+n-1 && for\ 1 \le j \le \left\lceil \frac{m}{2} \right\rceil,\\
f(u_{k+2+i}^{2j-1}) &= 2(n+1)(j-1)+n-1-2i && for\ 1 \le j \le \left\lceil \frac{m}{2} \right\rceil, 1 \le i \le k-2,\\
f(u_i^{2j}) &= 2(n+1)j-2i+1 && for\ 1 \le j \le \left\lfloor \frac{m}{2} \right\rfloor, 1 \le i \le k+1\\
f(u_{k+2}^{2j}) &= 2(n+1)j-n+2 && for\ 1 \le j \le \left\lfloor \frac{m}{2} \right\rfloor,\\
f(u_{k+2+i}^{2j}) &= 2(n+1)j-n+2+2i && for\ 1 \le j \le \left\lfloor \frac{m}{2} \right\rfloor, 1 \le i \le k-2.
\end{aligned}$$

Let v_iv_j be a transformed edge in T for some indices i and $j, 1 \le i \le j \le m$ and let P_1 be the EPT that deletes the edge v_iv_j and adds the edge $v_{i+t}v_{j-t}$ where t is the distance of v_i from v_{i+t} and also the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent EPTs. Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, $i+t+1 = j-t$ which implies $j = i+2t+1$. The induced label of the edge v_iv_j is given by,

$$f^*(v_iv_j) = f^*(v_iv_{i+2t+1}) = \left\lceil \frac{f(v_i)+f(v_{i+2t+1})}{2} \right\rceil = (n+1)(i+t) \tag{2.3}$$

and

$$f^*(v_{i+t}v_{j-t}) = f^*(v_{i+t}v_{i+t+1}) = \left\lceil \frac{f(v_{i+t}+f(v_{i+t+1})}{2} \right\rceil = (n+1)(i+t) \tag{2.4}$$

Therefore from (2.3) and (2.4), $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$.

Let $e_i^j = u_i^j u_{i+1}^j, e_n^j = u_n^j u_1^j$ for $1 \le i \le n-1, 1 \le j \le m$.
For each vertex label f, the induced edge label f^* is defined as follows:

$$
\begin{aligned}
f^*(v_i v_{i+1}) &= (n+1)i && for\ 1 \le i \le m-1,\\
f^*(e_i^{2j-1}) &= 2(n+1)(j-1)+2i-1 && for\ 1 \le j \le \left\lceil \frac{m}{2} \right\rceil, 1 \le i \le k\\
f^*(e_{k+i}^{2j-1}) &= 2(n+1)(j-1)+n-2(i-1) && for\ 1 \le j \le \left\lceil \frac{m}{2} \right\rceil, 1 \le i \le k,\\
f^*(e_i^{2j}) &= 2(n+1)j-2i && for\ 1 \le j \le \left\lfloor \frac{m}{2} \right\rfloor, 1 \le i \le k,\\
f^*(e_{k+i}^{2j}) &= 2(n+1)j-n+2i-1 && for\ 1 \le j \le \left\lfloor \frac{m}{2} \right\rfloor, 1 \le i \le k.
\end{aligned}
$$

It can be verified that f is an mean labeling of $T\hat{o}C_n$. Hence $T\hat{o}C_n$ is a mean graph. □

The example for the mean labeling of $T\hat{o}C_6$, where T is a T_p-tree with 11 vertices, is given in Figure 2.

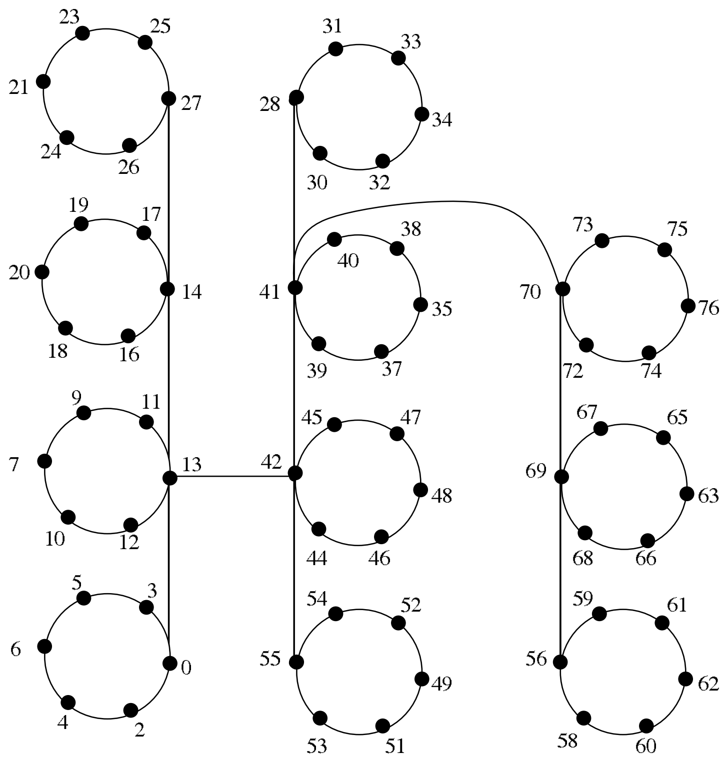

Figure 2

Corollary 2.2. *Let T be a T_p-tree on m vertices. Then the graph $T \odot K_2$ is a mean graph.*

Proof. It follows from Theorem 2.1, by taking $n = 3$. □

Theorem 2.3. *Let T be a T_p-tree on m vertices. Then the graph $T\tilde{o}C_n$ is a mean graph.*

Proof. Let T be a T_p-tree with m vertices. By the definition of a T_p-tree there exists a parallel transformation P of T such that for the path $P(T)$ we have (i) $V(P(T)) = V(T)$ and (ii) $E(P(T)) = (E(T)\ E_d) \cup E_P$, where E_d is the set of edges deleted from T and E_P is the set of edges newly added through the sequence $P = (P_1, P_2, \ldots, P_k)$ of the EPTs P used to arrive at the path $P(T)$. Clearly E_d and E_P have the same number of edges.

Now denote the vertices of $P(T)$ successively as $v_1, v_2, v_3, \ldots, v_m$ starting from one pendant vertex of $P(T)$ right up to other. Let $u_1^i, u_2^i, \ldots, u_n^i$ be the vertices of the i^{th} copy of C_n for $1 \leq i \leq m$. Then $V(T\tilde{o}C_n) = \{v_j, u_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(T\tilde{o}C_n) = E(T) \cup E(C_n) \cup \{v_j u_1^j : 1 \leq j \leq m\}$. Let

$$n = \begin{cases} 2k+1 & \text{if } n \text{ is odd} \\ 2k & \text{if } n \text{ is even.} \end{cases}$$

Define $f : V(T\tilde{o}C_n) \to \{0, 1, 2, 3, \ldots, q = (n+2)m - 1\}$ as follows:

$$\begin{aligned} f(v_{2i-1}) &= 2(n+2)(i-1) && for\ 1 \leq i \leq \left\lceil \frac{m}{2} \right\rceil, \\ f(v_{2i}) &= 2(n+2)i - 1 && for\ 1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor. \end{aligned}$$

Case (i). n is odd.

$$\begin{aligned} f(u_i^{2j-1}) &= 2(n+2)(j-1) + (2i-1) && for\ 1 \leq j \leq \left\lceil \frac{m}{2} \right\rceil, 1 \leq i \leq k+1, \\ f(u_{k+2}^{2j-1}) &= 2(n+2)(j-1) + n + 1 && for\ 1 \leq j \leq \left\lceil \frac{m}{2} \right\rceil, \\ f(u_{k+2+i}^{2j-1}) &= 2(n+2)(j-1) + n + 1 - 2i && for\ 1 \leq j \leq \left\lceil \frac{m}{2} \right\rceil, 1 \leq i \leq k-1, \\ f(u_i^{2j}) &= 2(n+2)j - 2(i-1) - 2 && for\ 1 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor, 1 \leq i \leq k, \\ f(u_{k+1}^{2j}) &= 2(n+2)j - n - 2 && for\ 1 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor, \\ f(u_{k+1+i}^{2j}) &= 2(n+2)j - n + 2(i-1) && for\ 1 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor, 1 \leq i \leq k. \end{aligned}$$

Let $v_i v_j$ be a transformed edge in T for some indices i and $j, 1 \leq i \leq j \leq m$ and let P_1 be the EPT that deletes the edge $v_i v_j$ and adds the edge $v_{i+t} v_{j-t}$ where t is the distance of v_i from v_{i+t} and also the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent

EPTs. Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, $i+t+1=j-t$ which implies $j=i+2t+1$. The induced label of the edge v_iv_j is given by,

$$f^*(v_iv_j)=f^*(v_iv_{i+2t+1})=\left\lceil\frac{f(v_i)+f(v_{i+2t+1})}{2}\right\rceil=(n+2)(i+t) \tag{2.5}$$

and

$$f^*(v_{i+t}v_{j-t})=f^*(v_{i+t}v_{i+t+1})=\left\lceil\frac{f(v_{i+t}+f(v_{i+t+1})}{2}\right\rceil=(n+2)(i+t) \tag{2.6}$$

Therefore from (2.5) and (2.6), $f^*(v_iv_j)=f^*(v_{i+t}v_{j-t})$.

Let $e_i^j=u_i^ju_{i+1}^j, e_n^j=u_n^ju_1^j$ for $1\le i\le n-1, 1\le j\le m$.

For each vertex label f, the induced edge label f^* is defined as follows:

$$\begin{aligned}
f^*(v_iv_{i+1})&=(n+2)i && for\ 1\le i\le m-1,\\
f^*(v_{2j-1}u_1^{2j-1})&=2(n+2)(j-1)+1 && for\ 1\le j\le\left\lceil\frac{m}{2}\right\rceil,\\
f^*(v_{2j}u_1^{2j})&=2(n+2)j-1 && for\ 1\le j\le\left\lceil\frac{m}{2}\right\rceil,\\
f^*(e_i^{2j-1})&=2(n+2)(j-1)+2i && for\ 1\le j\le\left\lceil\frac{m}{2}\right\rceil,\\
& && 1\le i\le k+1\\
f^*(e_{k+1+i}^{2j-1})&=2(n+2)(j-1)+n-2(i-1) && for\ 1\le j\le\left\lceil\frac{m}{2}\right\rceil,\\
& && 1\le i\le k,\\
f^*(e_i^{2j})&=2(n+2)j-2i-1 && for\ 1\le j\le\left\lfloor\frac{m}{2}\right\rfloor,\\
& && 1\le i\le k,\\
f^*(e_{k+i}^{2j})&=2(n+2)j-n-1+2i && for\ 1\le j\le\left\lfloor\frac{m}{2}\right\rfloor,\\
& && 1\le i\le k+1.
\end{aligned}$$

It can be verified that f is an mean labeling of $T\tilde{o}C_n$.

Case (ii). n is even.

$$\begin{aligned}
f(u_i^{2j-1})&=2(n+2)(j-1)+(2i-1) && for\ 1\le j\le\left\lceil\frac{m}{2}\right\rceil, 1\le i\le k+1,\\
f(u_{k+2}^{2j-1})&=2(n+2)(j-1)+n && for\ 1\le j\le\left\lceil\frac{m}{2}\right\rceil,\\
f(u_{k+2+i}^{2j-1})&=2(n+2)(j-1)+n-2i && for\ 1\le j\le\left\lceil\frac{m}{2}\right\rceil, 1\le i\le k-2,\\
f(u_i^{2j})&=2(n+2)j-2(i-1)-2 && for\ 1\le j\le\left\lfloor\frac{m}{2}\right\rfloor, 1\le i\le k+1,\\
f(u_{k+2}^{2j})&=2(n+2)j-n+1 && for\ 1\le j\le\left\lfloor\frac{m}{2}\right\rfloor,\\
f(u_{k+2+i}^{2j})&=2(n+2)j-n+1+2i && for\ 1\le j\le\left\lfloor\frac{m}{2}\right\rfloor, 1\le i\le k-2.
\end{aligned}$$

Let v_iv_j be a transformed edge in T for some indices i and $j, 1 \leq i \leq j \leq m$ and let P_1 be the EPT that deletes the edge v_iv_j and adds the edge $v_{i+t}v_{j-t}$ where t is the distance of v_i from v_{i+t} and also the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent EPTs. Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, $i+t+1 = j-t$ which implies $j = i+2t+1$. The induced label of the edge v_iv_j is given by,

$$f^*(v_iv_j) = f^*(v_iv_{i+2t+1}) = \left\lceil \frac{f(v_i) + f(v_{i+2t+1})}{2} \right\rceil = (n+2)(i+t) \tag{2.7}$$

and

$$f^*(v_{i+t}v_{j-t}) = f^*(v_{i+t}v_{i+t+1}) = \left\lceil \frac{f(v_{i+t} + f(v_{i+t+1})}{2} \right\rceil = (n+2)(i+t) \tag{2.8}$$

Therefore from (2.7) and (2.8), $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$.

Let $e_i^j = u_i^j u_{i+1}^j, e_n^j = u_n^j u_1^j$ for $1 \leq i \leq n-1, 1 \leq j \leq m$.

For each vertex label f, the induced edge label f^* is defined as follows:

$$\begin{aligned}
f^*(v_iv_{i+1}) &= (n+2)i && for\ 1 \leq i \leq m-1,\\
f^*(v_{2j-1}u_1^{2j-1}) &= 2(n+2)(j-1)+1 && for\ 1 \leq j \leq \left\lceil \frac{m}{2} \right\rceil,\\
f^*(v_{2j}u_1^{2j}) &= 2(n+2)j-1 && for\ 1 \leq j \leq \left\lceil \frac{m}{2} \right\rceil,\\
f^*(e_i^{2j-1}) &= 2(n+2)(j-1)+2i && for\ 1 \leq j \leq \left\lceil \frac{m}{2} \right\rceil, 1 \leq i \leq k,\\
f^*(e_{k+i}^{2j-1}) &= 2(n+2)(j-1)+n+3-2i && for\ 1 \leq j \leq \left\lceil \frac{m}{2} \right\rceil, 1 \leq i \leq k,\\
f^*(e_i^{2j}) &= 2(n+2)j-2i-1 && for\ 1 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor, 1 \leq i \leq k,\\
f^*(e_{k+i}^{2j}) &= 2(n+2)j-n+2(i-1) && for\ 1 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor, 1 \leq i \leq k.
\end{aligned}$$

It can be verified that f is a mean labeling of $T\tilde{o}C_n$. Hence $T\tilde{o}C_n$ is a mean graph. □

The example for the mean labeling of $T\tilde{o}C_5$, where T is a T_p-tree with 9 vertices, is given in Figure 3.

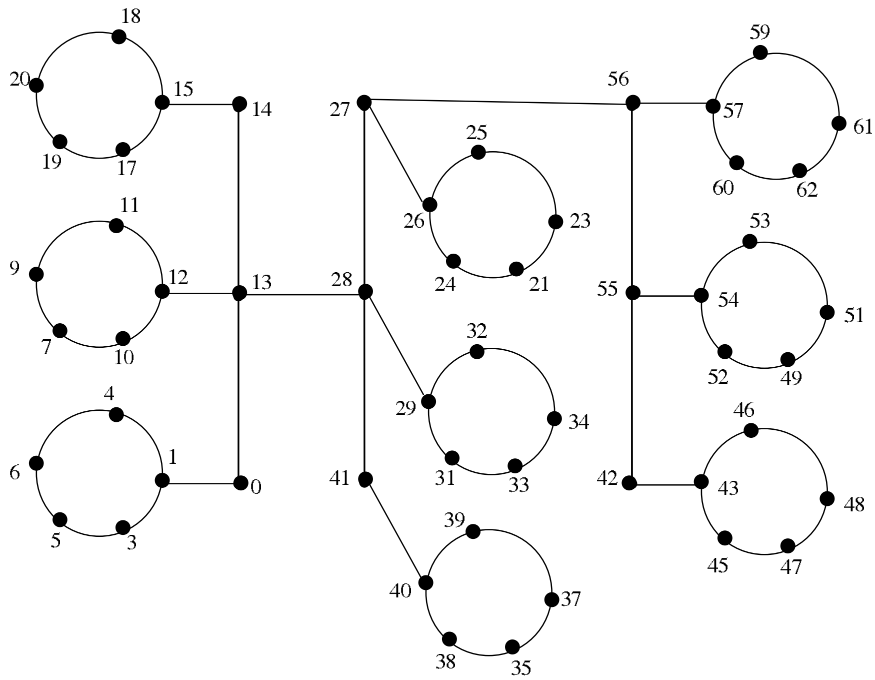

Figure 3

Theorem 2.4. *Let T be a T_p-tree on m vertices. Then the graph $T@P_n$ is a mean graph.*

Proof. Let T be a T_p-tree with m vertices. By the definition of a T_p-tree there exists a parallel transformation P of T such that for the path $P(T)$ we have (i) $V(P(T)) = V(T)$ and (ii) $E(P(T)) = (E(T)\ E_d) \cup E_P$, where E_d is the set of edges deleted from T and E_P is the set of edges newly added through the sequence $P = (P_1, P_2, \ldots, P_k)$ of the EPTs P used to arrive at the path $P(T)$. Clearly E_d and E_P have the same number of edges.

Now denote the vertices of $P(T)$ successively as $v_1, v_2, v_3, \ldots, v_m$ starting from one pendant vertex of $P(T)$ right up to other. Let $u_1^j, u_2^j, u_3^j, \ldots, u_n^j (1 \leq j \leq m)$ be the vertices of j^{th} copy of P_n. Then $V(T@P_n) = \{u_i^j : 1 \leq i \leq n, 1 \leq j \leq m$ with $u_n^j = v_j\}$.

Define $f : V(T@P_n) \to \{0, 1, 2, 3, \ldots, q = mn - 1\}$ as follows:

$$f(u_i^{2j-1}) = 2(j-1)n + i - 1 \ for\ 1 \leq i \leq n, 1 \leq j \leq \left\lceil \frac{m}{2} \right\rceil,$$
$$f(u_i^{2j}) = (2j-1)n + n - i \ for\ 1 \leq i \leq n, 1 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor.$$

Let $v_i v_j$ be a transformed edge in T for some indices i and $j, 1 \leq i \leq j \leq m$ and let P_1 be the EPT that deletes the edge $v_i v_j$ and adds the edge $v_{i+t} v_{j-t}$ where t is the distance of v_i from v_{i+t} and also the distance of v_j from v_{j-t}. Let

P be a parallel transformation of T that contains P_1 as one of the constituent EPTs. Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, $i+t+1=j-t$ which implies $j=i+2t+1$. The induced label of the edge v_iv_j is given by,

$$f^*(v_iv_j) = f^*(v_iv_{i+2t+1}) = \left\lceil \frac{f(v_i)+f(v_{i+2t+1})}{2} \right\rceil = n(i+t) \tag{2.9}$$

and

$$f^*(v_{i+t}v_{j-t}) = f^*(v_{i+t}v_{i+t+1}) = \left\lceil \frac{f(v_{i+t}+f(v_{i+t+1})}{2} \right\rceil = n(i+t). \tag{2.10}$$

Therefore from (2.9) and (2.10), $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$.

Let $e_i^j = u_i^j u_{i+1}^j$ for $1 \le i \le n-1, 1 \le j \le m$.

For each vertex label f, the induced edge label f^* is defined as follows:

$$\begin{aligned} f^*(v_iv_{i+1}) &= ni && for\ 1 \le i \le m-1, \\ f^*(e_i^{2j-1}) &= 2n(j-1)+i && for\ 1 \le i \le n-1, 1 \le j \le \left\lceil \frac{m}{2} \right\rceil, \\ f^*(e_i^{2j}) &= n(2j-1)+n-i && for\ 1 \le i \le n-1, 1 \le j \le \left\lfloor \frac{m}{2} \right\rfloor. \end{aligned}$$

It can be verified that f is an mean labeling of $T@P_n$. Hence $T@P_n$ is a mean graph. □

The example for the mean labeling of $T@P_4$, where T is a T_p-tree with 12 vertices, is given in Figure 4.

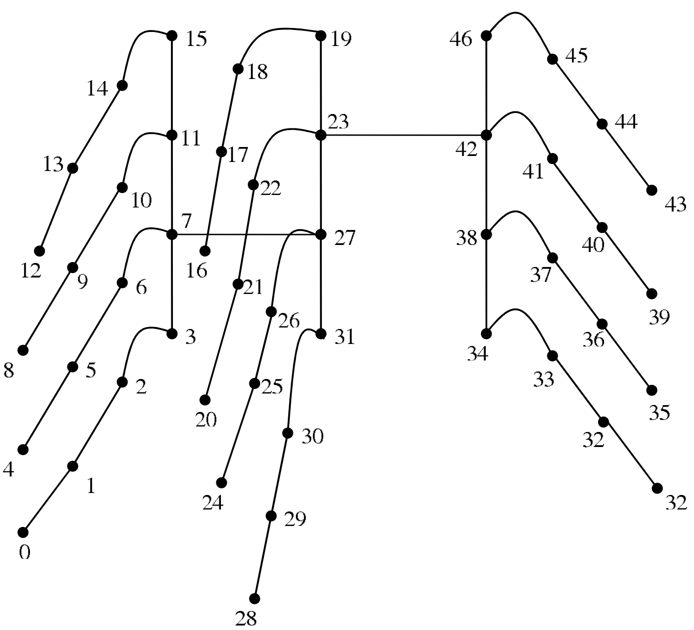

Figure 4

Theorem 2.5. *Let T be a T_p-tree on m vertices. Then the graph $T \copyright 2P_n$ is a mean graph.*

Proof. Let T be a T_p-tree with m vertices. By the definition of a T_p-tree there exists a parallel transformation P of T such that for the path $P(T)$ we have (i) $V(P(T)) = V(T)$ and (ii) $E(P(T)) = (E(T) \backslash E_d) \cup E_P$, where E_d is the set of edges deleted from T and E_P is the set of edges newly added through the sequence $P = (P_1, P_2, \ldots, P_k)$ of the EPTs P used to arrive at the path $P(T)$. Clearly E_d and E_P have the same number of edges.

Now denote the vertices of $P(T)$ successively as $v_1, v_2, v_3, \ldots, v_m$ starting from one pendant vertex of $P(T)$ right up to other. Let $u^j_{1,1}, u^j_{1,2}, u^j_{1,3}, \ldots, u^j_{1,n}$ and $u^j_{2,1}, u^j_{2,2}, u^j_{2,3}, \ldots, u^j_{2,n} (1 \leq j \leq m)$ be the vertices of the two vertex disjoint paths joined with j^{th} vertex of T such that $v_j = u^j_{1,n} = u^j_{2,n}$. Then $V(T \copyright P_n) = \{v_j, u^j_{1,i}, u^j_{2,i} : 1 \leq i \leq n, 1 \leq j \leq m \text{ with } v_j = u^j_{1,n} = u^j_{2,n}\}$.

Define $f : V(T \copyright 2P_n) \to \{0, 1, 2, 3, \ldots, q = m(2n-1) - 1\}$ as follows:

$$\begin{aligned} f(u^j_{1,i}) &= (2n-1)(j-1) + i - 1 && \textit{for } 1 \leq i \leq n, 1 \leq j \leq m, \\ f(u^j_{2,n+1-i}) &= (2n-1)(j-1) + n + i - 2 && \textit{for } 2 \leq i \leq n, 1 \leq j \leq m. \end{aligned}$$

Let $v_i v_j$ be a transformed edge in T for some indices i and $j, 1 \leq i \leq j \leq m$ and let P_1 be the EPT that deletes the edge $v_i v_j$ and adds the edge $v_{i+t} v_{j-t}$ where t is the distance of v_i from v_{i+t} and also the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent EPTs. Since $v_{i+t} v_{j-t}$ is an edge in the path $P(T)$, $i + t + 1 = j - t$ which implies $j = i + 2t + 1$. The induced label of the edge $v_i v_j$ is given by,

$$f^*(v_i v_j) = f^*(v_i v_{i+2t+1}) = \left\lceil \frac{f(v_i) + f(v_{i+2t+1})}{2} \right\rceil = (2n-1)(i+t) \tag{2.11}$$

and

$$f^*(v_{i+t} v_{j-t}) = f^*(v_{i+t} v_{i+t+1}) = \left\lceil \frac{f(v_{i+t} + f(v_{i+t+1})}{2} \right\rceil = (2n-1)(i+t). \tag{2.12}$$

Therefore from (2.11) and (2.12), $f^*(v_i v_j) = f^*(v_{i+t} v_{j-t})$.

Let $e^j_{1,i} = u^j_{1,i} u^j_{1,i+1}$ for $1 \leq i \leq n-1, 1 \leq j \leq m$, $e^j_{2i} = u^j_{2,i} u^j_{2,i+1}$ for $1 \leq i \leq n-1, 1 \leq j \leq m$ and $e_j = v_j v_{j+1}$ for $1 \leq j \leq m-1$.

For each vertex label f, the induced edge label f^* is defined as follows:

$$\begin{aligned} f^*(v_i v_{i+1}) &= (2n-1)i && \textit{for } 1 \leq i \leq m-1, \\ f^*(e^j_{1,i}) &= (2n-1)(j-1) + i && \textit{for } 1 \leq i \leq n-1, 1 \leq j \leq m, \\ f^*(e^j_{2,n+1-i}) &= (2n-1)(j-1) + n + i - 2 && \textit{for } 2 \leq i \leq n, 1 \leq j \leq m. \end{aligned}$$

It can be verified that f is a mean labeling of $T©2P_n$. Hence $T©2P_n$ is a mean graph. □

The example for the mean labeling of $T©2P_3$, where T is a T_p-tree with 11 vertices, is given in Figure 5.

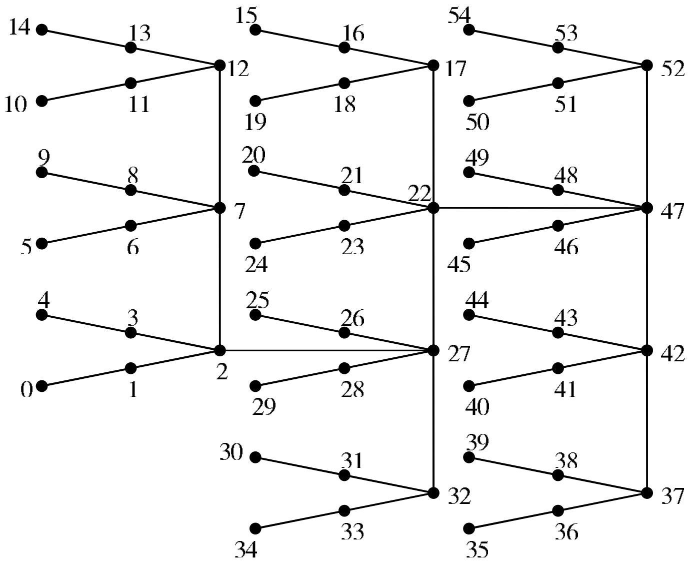

Figure 5

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