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Multivariate normality test using Srivastava's skewness and kurtosis

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Abstract. In this paper, we consider the multivariate normality test based on the sample measures of multivariate skewness and kurtosis defined by Srivastava [11]. Koizumi et al. [4] proposed test statistics M_1 and M_2 using Srivastava's sample skewness and kurtosis, which are asymptotically distributed as χ^2-distribution. We propose a new test statistic M_3 by taking account of the variance of M_2 under the normality. In order to evaluate the accuracy of the proposed test statistic, the numerical results by a Monte Carlo simulation for some selected values of parameters are presented.

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§1. Introduction

In statistical analysis, the test for normality is an important problem. This problem has been considered by many authors. For the univariate case, the test statistic using order statistic derived by Shapiro and Wilk [10] is one of the most famous and essential tests for normality. Another approach for testing normality uses sample skewness and kurtosis separately. D'Agostino [2] derived the test statistic using sample skewness. For the test statistic using sample kurtosis, Anscombe and Glynn [1] proposed the test statistic distributed as standard normal distribution. Jarque and Bera [3] proposed the bivariate test using univariate sample skewness and kurtosis. The improved Jarque-Bera (JB) test statistics have been considered by many authors (see, e.g. Urzúa [12] and Nakagawa et al. [5]).

Mardia [6] and Srivastava [11] gave different definitions of multivariate sample skewness and kurtosis, and discussed some test statistics using these measures for assessing multivariate normality. Mardia and Foster [7] proposed the test statistics using Mardia's sample skewness and kurtosis. Okamoto and Seo [8] derived the improved approximate χ^2 test statistic using multivariate sample skewness of Srivastava [11], which is more accurate than Srivastava's χ^2 test statistic. The test statistics using the multivariate sample kurtosis of Srivastava [11] were discussed by Seo and Ariga [9]. The test statistics M_1 and M_2 using Srivastava's sample skewness and kurtosis that are asymptotically distributed as χ^2-distribution were proposed by Koizumi et al. [4]. However, for a small N, there is difference between the upper percentiles of distributions of their statistics and the χ^2-distribution. Thus, it seems that the multivariate normality test based on M_1 or M_2, though applicable, is not appropriate. Our purpose is to propose a new test statistic M_3 by taking account of the variance of M_2 under the normality. We investigate the accuracies of variances, upper percentiles, type I errors and powers for the multivariate JB test statistics M_1, M_2 and M_3 via a Monte Carlo simulation for selected values of parameters.

§2. Srivastava's measures of multivariate skewness and kurtosis

Let $\boldsymbol{x}$ be a p-dimensional random vector with mean vector $\boldsymbol{\mu}$ and covariance matrix $\Sigma = \Gamma D_\lambda \Gamma'$, where $\Gamma = (\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \ldots, \boldsymbol{\gamma}_p)$ is an orthogonal matrix and $D_\lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p)$. Note that $\lambda_1, \lambda_2, \ldots, \lambda_p$ are the eigenvalues of Σ. Then, Srivastava [11] defined the population measures of multivariate skewness and kurtosis as

$$\beta_{1,p}^2 = \frac{1}{p}\sum_{i=1}^{p}\left\{\frac{\mathrm{E}[(y_i-\theta_i)^3]}{\lambda_i^{\frac{3}{2}}}\right\}^2,$$

$$\beta_{2,p} = \frac{1}{p}\sum_{i=1}^{p}\frac{\mathrm{E}[(y_i-\theta_i)^4]}{\lambda_i^2},$$

respectively, where $y_i = \boldsymbol{\gamma}_i'\boldsymbol{x}$ and $\theta_i = \boldsymbol{\gamma}_i'\boldsymbol{\mu}$ $(i = 1, 2, \ldots, p)$. We note that $\beta_{1,p}^2 = 0$, $\beta_{2,p} = 3$ under a multivariate normal population.

Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$ be samples of size N from a multivariate population. Let $\overline{\boldsymbol{x}}$ and $S = HD_\omega H'$ be the sample mean vector and sample covariance matrix given as

$$\overline{\boldsymbol{x}} = \frac{1}{N}\sum_{j=1}^{N}\boldsymbol{x}_j,$$

$$S = \frac{1}{N}\sum_{j=1}^{N}(\boldsymbol{x}_j-\overline{\boldsymbol{x}})(\boldsymbol{x}_j-\overline{\boldsymbol{x}})',$$

respectively, where $H = (\boldsymbol{h}_1, \boldsymbol{h}_2, \ldots, \boldsymbol{h}_p)$ is an orthogonal matrix and $D_\omega = \text{diag}\,(\omega_1, \omega_2, \ldots, \omega_p)$. We note that

$$\omega_i = \boldsymbol{h}_i' S \boldsymbol{h}_i = \frac{1}{N}\sum_{j=1}^{N}(y_{ij} - \overline{y}_i)^2, \ i = 1, 2, \ldots, p,$$

where $y_{ij} = \boldsymbol{h}_i'\boldsymbol{x}_j$ $(i = 1, 2, \ldots, p,\ j = 1, 2, \ldots, N)$, $\overline{y}_i = N^{-1}\sum_{j=1}^{N} y_{ij}$ $(i = 1, 2, \ldots, p)$. Then, Srivastava [11] defined the sample measures of multivariate skewness and kurtosis as

$$b_{1,p}^2 = \frac{1}{p}\sum_{i=1}^{p}\left\{\frac{1}{\omega_i^{\frac{3}{2}}}\sum_{j=1}^{N}\frac{(y_{ij} - \overline{y}_i)^3}{N}\right\}^2 = \frac{1}{p}\sum_{i=1}^{p}\left\{\frac{m_{3i}}{m_{2i}^{\frac{3}{2}}}\right\}^2,$$

$$b_{2,p} = \frac{1}{p}\sum_{i=1}^{p}\frac{1}{\omega_i^2}\sum_{j=1}^{N}\frac{(y_{ij} - \overline{y}_i)^4}{N} = \frac{1}{p}\sum_{i=1}^{p}\frac{m_{4i}}{m_{2i}^2},$$

respectively, where $m_{\nu i} = N^{-1}\sum_{j=1}^{N}(y_{ij} - \overline{y}_i)^\nu$.

Koizumi et al. [4] proposed two test statistics for multivariate normality:

$$M_1 = Np\left\{\frac{b_{1,p}^2}{6} + \frac{(b_{2,p} - 3)^2}{24}\right\} \xrightarrow{d} \chi_{p+1}^2,$$

$$M_2 = \frac{p b_{1,p}^2}{\mathrm{E}[b_{1,p}^2]} + \frac{(b_{2,p} - \mathrm{E}[b_{2,p}])^2}{\mathrm{Var}[b_{2,p}]} \xrightarrow{d} \chi_{p+1}^2$$

for large N, where the expectation of $b_{1,p}^2$, and the expectation and variance of $b_{2,p}$ are given by

$$\mathrm{E}[b_{1,p}^2] \fallingdotseq \frac{6(N-2)}{(N+1)(N+3)},$$

$$\mathrm{E}[b_{2,p}] \fallingdotseq \frac{3(N-1)}{N+1},$$

$$\mathrm{Var}[b_{2,p}] \fallingdotseq \frac{24N(N-2)(N-3)}{p(N+1)^2(N+3)(N+5)},$$

respectively under the normality.

§3. New multivariate JB test statistic using the variance of M_2

The test statistic M_2 was introduced in Koizumi et al. [4] so that the accuracy of the upper percentile for the approximate test statistic is better than that of the test statistic M_1 for small N. However, for a small N, it seems that there is difference between the the upper percentiles of the distributions of M_2 and the χ^2-distribution. Hence, we propose a new test statistic to be closer to the upper percentile of the χ^2-distribution by using the variance of M_2. The idea of our proposal of M_3 is that $\mathrm{E}[M_3] = p+1$ and $\mathrm{Var}[M_3] = 2(p+1)$.

Theorem 1. *For a large N, the test statistic M_3*

$$M_3 = cM_2 + (1-c)(p+1)$$

is asymptotically distributed as a χ^2_{p+1}-distribution, where

$$c = \left\{ \frac{2(p+1)}{\mathrm{Var}[M_2]} \right\}^{\frac{1}{2}}.$$

In Appendix, $\mathrm{Var}[M_2]$ will be derived under the normality, as follows:

$$\begin{aligned}\mathrm{Var}[M_2] \fallingdotseq\ & \frac{2}{pN(N-2)(N-3)(N+5)(N+7)(N+9)(N+11)(N+13)} \\ & \times\{p(p+1)N^8 + 2(29p^2+110p+135)N^7 \\ & +(859p^2+3055p+702)N^6 + 2(1058p^2-217p-7272)N^5 \\ & -(21665p^2+71105p+38844)N^4 \\ & -2(13471p^2+10792p-96183)N^3 \\ & +3(44759p^2+130587p+134898)N^2 + 90(767p-6222)N \\ & +81000\}\ (N \neq 2,3). \end{aligned} \tag{3.1}$$

§4. Simulation studies

The accuracies of variances, upper percentiles, type I errors and powers of the multivariate JB test statistics M_1, M_2 and M_3 are evaluated via a Monte Carlo simulation study. Simulation parameters are as follows: $p = 3,\ 10,\ 20,\ 30$; $N = 20,\ 50,\ 100,\ 200,\ 400,\ 800$ $(p < N)$; and significance level $\alpha = 0.05$. As a numerical experiment, we carry out 1,000,000 replications.

First, we compare variance (3.1) with simulated variances derived by Monte Carlo simulation. In Table 1, "M_2" denotes values calculated using (3.1). M_2 and simulated values are almost the same for all parameters. Next, we check $\mathrm{Var}[M_3] = 2(p+1)$. In Table 2, "$M_3$" represents variance $\mathrm{Var}[M_3] = 2(p+1)$ and "*Simulation*" is simulated variance of the test statistic M_3 derived by Monte Carlo simulation. It can be seen from Table 2 that M_3 has almost the same variance as the χ^2_{p+1}-distribution for all parameters. Table 3 gives the values of the upper 5 percentiles of M_1, M_2 and M_3. When N is small, they show that difference between M_3 and χ^2_{p+1}-distribution is smaller than that between χ^2_{p+1}-distribution and M_2. Table 4 gives the values of type I errors of M_1, M_2 and M_3. They show that M_3 is closer to 0.05 than others when N is small. We note that if type I error is smaller than 0.05, the test is conservative. M_1 is always conservative and M_2 is not conservative. M_3 is not conservative and the approximate accuracy of M_3 is outstanding except when p is small. Table 5 gives the values of the powers of M_1, M_2 and M_3, where each element of the sample is generated using χ^2_5-distribution. The power of M_2 is the highest. Although Laplace distribution, lognormal distribution and beta distribution are also used as samples, the same tendency was seen.

In conclusion, the simulation results indicate that M_3 has almost the same variance and the upper 5 percentile as the χ^2_{p+1}-distribution even for a small sample size. Although the test statistic M_2 may have the best power, type I error of M_2 has far exceeded 0.05 for a small sample size. In addition, the upper 5 percentile of M_3 is the closest to that of χ^2_{p+1}-distribution for a small sample size. Therefore, the multivariate JB test statistic M_3 proposed in this paper is useful for the multivariate normality test.

§A. Derivation of (3.1)

In this appendix, we calculate variance $\mathrm{Var}[M_2]$ as follows:

$$\mathrm{Var}[M_2] = \mathrm{Var}[T_1] + \mathrm{Var}[T_2] + 2\mathrm{Cov}[T_1, T_2],$$

where

$$T_1 = \frac{pb_{1,p}^2}{\mathrm{E}[b_{1,p}^2]}, \quad T_2 = \frac{(b_{2,p} - \mathrm{E}[b_{2,p}])^2}{\mathrm{Var}[b_{2,p}]}.$$

Now, we derive the moments under the hypothesis that $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ are i.i.d. from $N_p(\boldsymbol{\mu}, \Sigma)$. For large N, we get

$$y_{ij} = \boldsymbol{h}_i' \boldsymbol{x}_j \xrightarrow{d} N(\boldsymbol{\gamma}_i' \boldsymbol{\mu}, \lambda_i)$$

because $\boldsymbol{h}_i \to \boldsymbol{\gamma}_i$ with probability one (see Srivastava [11]). Thus, $y_{i1}, y_{i2}, \ldots, y_{iN}$ are asymptotically independently normally distributed. Further, the following expression was described by Srivastava [11] for large N.

$$\mathrm{E}[m_{\nu i}^k m_{2i}^{-\nu k/2}]\mathrm{E}[m_{2i}^{\nu k/2}] \fallingdotseq \mathrm{E}[m_{\nu i}^k].$$

We checked the above equation numerically. In order to simplify calculation of a moment, dependence of y_{ij} and $\overline{y}_i$ is avoided as follows. Let $\overline{y}_i^{(\alpha)}$ be a mean defined on the subset of $y_{i1}, y_{i2}, \ldots, y_{iN}$, that is,

$$\overline{y}_i^{(\alpha)} = \frac{1}{N-1} \sum_{j=1, j\neq\alpha}^{N} y_{ij}.$$

Then, $y_{i\alpha}$ is asymptotically independent of $\overline{y}_i^{(\alpha)}$. Without a loss of generality, we calculate the moments with $\boldsymbol{\mu} = \mathbf{0}$ and $\lambda_i = 1$, that is, $\Sigma = I$, because $b_{1,p}^2$ and $b_{2,p}$ are hardly influenced by Σ for a large N. In addition, we put

$$\overline{y}_i^{(\alpha)} = \frac{1}{\sqrt{N-1}} Z_i.$$

Since Z_i is distributed as a standard normal distribution for large N, the odd order moments approximately equal zero for calculating moments and

$$\mathrm{E}[Z_i^{2k}] \risingdotseq (2k-1) \cdots 5 \cdot 3 \cdot 1, \ \ k = 1, 2, \ldots .$$

We note that

$$\begin{aligned}
\mathrm{Var}[T_1] &= \frac{p^2}{(\mathrm{E}[b_{1,p}^2])^2} \mathrm{Var}[b_{1,p}^2], \\
\mathrm{Var}[T_2] &= \frac{1}{(\mathrm{Var}[b_{2,p}])^2} \Big[\mathrm{E}[b_{2,p}^4] - 4\mathrm{E}[b_{2,p}]\mathrm{E}[b_{2,p}^3] \\
&\qquad + \mathrm{E}[b_{2,p}^2] \big\{ 8(\mathrm{E}[b_{2,p}])^2 - \mathrm{E}[b_{2,p}^2] \big\} - 4(\mathrm{E}[b_{2,p}])^4 \Big], \\
\mathrm{Cov}[T_1, T_2] &= \frac{p}{\mathrm{E}[b_{1,p}^2]\mathrm{Var}[b_{2,p}]} \big\{ \mathrm{Cov}[b_{1,p}^2, b_{2,p}^2] - 2\mathrm{E}[b_{2,p}]\mathrm{Cov}[b_{1,p}^2, b_{2,p}] \big\},
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{Cov}[b_{1,p}^2, b_{2,p}] &= \mathrm{E}[b_{1,p}^2 b_{2,p}] - \mathrm{E}[b_{1,p}^2]\mathrm{E}[b_{2,p}], \\
\mathrm{Cov}[b_{1,p}^2, b_{2,p}^2] &= \mathrm{E}[b_{1,p}^2 b_{2,p}^2] - \mathrm{E}[b_{1,p}^2]\mathrm{E}[b_{2,p}^2].
\end{aligned}$$

Since Okamoto and Seo [8] derived $\mathrm{Var}[b_{1,p}^2]$, we have only to derive the moments $\mathrm{E}[b_{2,p}^3]$, $\mathrm{E}[b_{2,p}^4]$, $\mathrm{E}[b_{1,p}^2 b_{2,p}]$ and $\mathrm{E}[b_{1,p}^2 b_{2,p}^2]$ under the normality.

Now, we consider the expectation $\mathrm{E}[b_{2,p}^3]$. We have

$$\begin{aligned}
\mathrm{E}[b_{2,p}^3] &= \mathrm{E}\left[\left\{ \frac{1}{p} \sum_{i=1}^{p} \frac{m_{4i}}{m_{2i}^2} \right\}^3 \right] \\
&= \frac{1}{p^3} \left\{ \sum_{i} \mathrm{E}\left[\frac{m_{4i}^3}{m_{2i}^6} \right] + \sum_{i \neq j} \mathrm{E}\left[\frac{m_{4i}^2 m_{4j}}{m_{2i}^4 m_{2j}^2} \right] + \sum_{i \neq j \neq k \neq i} \mathrm{E}\left[\frac{m_{4i} m_{4j} m_{4k}}{m_{2i}^2 m_{2j}^2 m_{2k}^2} \right] \right\},
\end{aligned}$$

where

$$\mathrm{E}[m_{4i}] = \left(1 - \frac{1}{N}\right)^4 \mathrm{E}[C_{i\alpha}^4],$$

$$\mathrm{E}[m_{4i}^2] = \frac{1}{N^2}\left(1 - \frac{1}{N}\right)^8 \Big\{N\mathrm{E}[C_{i\alpha}^8] + N(N-1)\mathrm{E}[C_{i\alpha}^4 C_{i\beta}^4]\Big\},$$

$$\mathrm{E}[m_{4i}^3] = \frac{1}{N^3}\left(1 - \frac{1}{N}\right)^{12} \Big\{N\mathrm{E}[C_{i\alpha}^{12}] + 3N(N-1)\mathrm{E}[C_{i\alpha}^8 C_{i\beta}^4]$$
$$+N(N-1)(N-2)\mathrm{E}[C_{i\alpha}^4 C_{i\beta}^4 C_{i\gamma}^4]\Big\},$$

$$\mathrm{E}[m_{2i}^2] = \frac{1}{N^2}\left(1 - \frac{1}{N}\right)^4 \Big\{N\mathrm{E}[C_{i\alpha}^4] + N(N-1)\mathrm{E}[C_{i\alpha}^2 C_{i\beta}^2]\Big\},$$

$$\mathrm{E}[m_{2i}^4] = \frac{1}{N^4}\left(1 - \frac{1}{N}\right)^8 \Big\{N\mathrm{E}[C_{i\alpha}^8] + 4N(N-1)\mathrm{E}[C_{i\alpha}^6 C_{i\beta}^2]$$
$$+3N(N-1)\mathrm{E}[C_{i\alpha}^4 C_{i\beta}^4] + 6N(N-1)(N-2)\mathrm{E}[C_{i\alpha}^4 C_{i\beta}^2 C_{i\gamma}^2]$$
$$+N(N-1)(N-2)(N-3)\mathrm{E}[C_{i\alpha}^2 C_{i\beta}^2 C_{i\gamma}^2 C_{i\delta}^2]\Big\},$$

$$\mathrm{E}[m_{2i}^6] = \frac{1}{N^6}\left(1 - \frac{1}{N}\right)^{12} \Big\{N\mathrm{E}[C_{i\alpha}^{12}] + 6N(N-1)\mathrm{E}[C_{i\alpha}^{10} C_{i\beta}^2]$$
$$+15N(N-1)\mathrm{E}[C_{i\alpha}^8 C_{i\beta}^4] + 15N(N-1)(N-2)\mathrm{E}[C_{i\alpha}^8 C_{i\beta}^2 C_{i\gamma}^2]$$
$$+10N(N-1)\mathrm{E}[C_{i\alpha}^6 C_{i\beta}^6] + 60N(N-1)(N-2)\mathrm{E}[C_{i\alpha}^6 C_{i\beta}^4 C_{i\gamma}^2]$$
$$+20N(N-1)(N-2)(N-3)\mathrm{E}[C_{i\alpha}^6 C_{i\beta}^2 C_{i\gamma}^2 C_{i\delta}^2]$$
$$+15N(N-1)(N-2)\mathrm{E}[C_{i\alpha}^4 C_{i\beta}^4 C_{i\gamma}^4]$$
$$+45N(N-1)(N-2)(N-3)\mathrm{E}[C_{i\alpha}^4 C_{i\beta}^4 C_{i\gamma}^2 C_{i\delta}^2]$$
$$+15N(N-1)(N-2)(N-3)(N-4)\mathrm{E}[C_{i\alpha}^4 C_{i\beta}^2 C_{i\gamma}^2 C_{i\delta}^2 C_{i\epsilon}^2]$$
$$+N(N-1)(N-2)(N-3)(N-4)(N-5)$$
$$\times\mathrm{E}[C_{i\alpha}^2 C_{i\beta}^2 C_{i\gamma}^2 C_{i\delta}^2 C_{i\epsilon}^2 C_{i\zeta}^2]\Big\}, \qquad (\alpha, \beta, \gamma, \delta, \epsilon, \zeta \text{ are all distinct})$$

and $C_{i\alpha} = y_{i\alpha} - \overline{y}_i^{(\alpha)}$. It is easy to calculate them because $y_{i\alpha}$ and $\overline{y}_i^{(\alpha)}$ are asymptotically independent.

After extensive calculations, we obtain

$$\begin{aligned}
\mathrm{E}[m_{4i}] &\risingdotseq \frac{3(N-1)^2}{N^2},\\
\mathrm{E}[m_{4i}^2] &\risingdotseq \frac{3(N-1)(3N^3+23N^2-63N+45)}{N^4},\\
\mathrm{E}[m_{4i}^3] &\risingdotseq \frac{27(N-1)(N^5+27N^4+226N^3-1098N^2+1725N-945)}{N^6},\\
\mathrm{E}[m_{2i}^2] &\risingdotseq \frac{(N-1)(N+1)}{N^2},\\
\mathrm{E}[m_{2i}^4] &\risingdotseq \frac{(N-1)(N+1)(N+3)(N+5)}{N^4},\\
\mathrm{E}[m_{2i}^6] &\risingdotseq \frac{(N-1)(N+1)(N+3)(N+5)(N+7)(N+9)}{N^6},
\end{aligned}$$

and we can obtain the expectation for $b_{2,p}^3$ as

$$\begin{aligned}
\mathrm{E}[b_{2,p}^3] \risingdotseq 27\{&p^2N^7+p(21p+8)N^6+(137p^2+80p+64)N^5\\
&+(197p^2-176p-640)N^4-(693p^2+1664p-2112)N^3\\
&-(809p^2-4776p+2560)N^2+3(697p^2-1008p+256)N\\
&-945p^2\}\times\frac{1}{p^2(N+1)^3(N+3)(N+5)(N+7)(N+9)}.
\end{aligned}$$

Similarly, we get the expectations $\mathrm{E}[b_{2,p}^4]$, $\mathrm{E}[b_{1,p}^2 b_{2,p}]$ and $\mathrm{E}[b_{1,p}^2 b_{2,p}^2]$ as follows:

$$\begin{aligned}
\mathrm{E}[b_{2,p}^4] \risingdotseq \frac{27}{p^3}\{&3p^3N^{12}+12p^2(13p+4)N^{11}\\
&+2p(1659p^2+984p+416)N^{10}\\
&+12(3069p^3+2424p^2+1504p+960)N^9\\
&+(221565p^3+165312p^2+46016p-85248)N^8\\
&+8(78663p^3-3996p^2-105568p-34368)N^7\\
&+12(6687p^3-276552p^2-155312p+249984)N^6\\
&-8(435261p^3+584736p^2-1781584p+278496)N^5\\
&-3(1164721p^3-7721536p^2-2766528p+7525120)N^4\\
&+12(770105p^3+1570500p^2-7438208p+4324608)N^3\\
&+18(330851p^3-4060968p^2+5142144p-1830912)N^2\\
&-540(28283p^3-72072p^2+36608p-7680)N+6081075p^3\}\\
\times&\frac{1}{(N+1)^4(N+3)^2(N+5)^2(N+7)(N+9)(N+11)(N+13)},
\end{aligned}$$

$$\mathrm{E}[b_{1,p}^2 b_{2,p}] \doteqdot \frac{18(N-2)\{pN^3+(11p+12)N^2+(23p-36)N-35p\}}{p(N+1)^2(N+3)(N+5)(N+7)},$$

$$\begin{aligned}\mathrm{E}[b_{1,p}^2 b_{2,p}^2] \doteqdot{}& \frac{18(N-2)}{p^2(N+1)^3(N+3)^2(N+5)(N+7)(N+9)(N+11)}\\ &\times\{3p^2N^7+p(99p+80)N^6+(1203p^2+1544p+1440)N^5\\ &\quad+(6315p^2+5920p-6048)N^4\\ &\quad+(10737p^2-22160p-12768)N^3\\ &\quad-3(4853p^2+22480p-20704)N^2\\ &\quad-9(3887p^2-10824p+2752)N+31185p^2\}.\end{aligned}$$

Thus, we can obtain

$$\mathrm{Var}[T_1] \doteqdot \frac{6N(N^3+37N^2+11N-313)}{(N-2)(N+5)(N+7)(N+9)} \quad \text{(see [8])},$$

$$\mathrm{Var}[T_2] \doteqdot \frac{2(N+1)^2(N^5+123N^4-67N^3-2667N^2+4842N+5400)}{N(N-2)(N-3)(N+7)(N+9)(N+11)(N+13)},$$

$$\mathrm{Cov}[b_{1,p}^2, b_{2,p}] \doteqdot \frac{216N(N-2)(N-3)}{p(N+1)^2(N+3)(N+5)(N+7)},$$

$$\begin{aligned}\mathrm{Cov}[b_{1,p}^2, b_{2,p}^2] \doteqdot{}& \frac{432N(N-2)(N-3)}{p^2(N+1)^3(N+3)^2(N+5)(N+7)(N+9)(N+11)}\\ &\quad\times\{3pN^4+6(11p+10)N^3+24(17p-3)N^2\\ &\quad\quad+2(207p-374)N-891p+344\},\end{aligned}$$

$$\mathrm{Cov}[T_1, T_2] \doteqdot \frac{12(15N^3-18N^2-187N+86)}{(N-2)(N+7)(N+9)(N+11)},$$

which yield (3.1).

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Table 1: Variance of M_2.

p	N	M_2	*Simulation*
3	20	17.6	18.3
	50	15.6	15.8
	100	12.9	13.1
	200	10.8	10.8
	400	9.5	9.4
	800	8.8	8.8
10	20	34.3	36.8
	50	31.5	31.8
	100	28.0	28.0
	200	25.4	25.4
	400	23.8	23.8
	800	22.9	23.0

p	N	M_2	*Simulation*
20	50	55.8	56.4
	100	50.7	50.7
	200	46.9	46.8
	400	44.6	44.7
	800	43.3	43.4
30	50	80.2	81.6
	100	73.5	73.6
	200	68.5	68.5
	400	65.4	65.5
	800	63.8	63.7

Table 2: Variance of M_3.

p	N	M_3	*Simulation*
3	20	8.0	8.3
	50	8.0	8.1
	100	8.0	8.1
	200	8.0	8.0
	400	8.0	7.9
	800	8.0	8.0
10	20	22.0	23.6
	50	22.0	22.2
	100	22.0	22.0
	200	22.0	22.0
	400	22.0	22.0
	800	22.0	22.0

p	N	M_3	*Simulation*
20	50	42.0	42.5
	100	42.0	42.1
	200	42.0	41.9
	400	42.0	42.0
	800	42.0	42.0
30	50	62.0	63.1
	100	62.0	62.1
	200	62.0	62.1
	400	62.0	62.0
	800	62.0	61.9

Table 3: The upper 5 percentiles of M_1, M_2 and M_3.

p	N	M_1	M_2	M_3	χ^2_{p+1}
3	20	6.8	11.2	8.9	9.5
	50	8.4	10.6	8.7	9.5
	100	9.0	10.2	8.9	9.5
	200	9.3	9.9	9.0	9.5
	400	9.4	9.7	9.2	9.5
	800	9.5	9.6	9.4	9.5
10	20	15.0	22.5	20.2	19.7
	50	17.9	21.4	19.7	19.7
	100	18.9	20.8	19.7	19.7
	200	19.3	20.3	19.7	19.7
	400	19.5	20.0	19.7	19.7
	800	19.6	19.9	19.7	19.7
20	50	29.8	34.9	33.1	32.7
	100	31.3	34.0	32.9	32.7
	200	32.0	33.4	32.8	32.7
	400	32.4	33.1	32.7	32.7
	800	32.5	32.9	32.7	32.7
30	50	41.1	47.6	45.6	45.0
	100	43.1	46.6	45.3	45.0
	200	44.1	45.9	45.2	45.0
	400	44.6	45.5	45.1	45.0
	800	44.8	45.2	45.0	45.0

Table 4: Type I errors of M_1, M_2 and M_3.

p	N	M_1	M_2	M_3
3	20	0.021	0.070	0.042
	50	0.037	0.064	0.040
	100	0.043	0.060	0.041
	200	0.046	0.056	0.043
	400	0.048	0.054	0.045
	800	0.050	0.053	0.048
10	20	0.013	0.079	0.055
	50	0.032	0.070	0.050
	100	0.041	0.064	0.050
	200	0.046	0.058	0.050
	400	0.048	0.055	0.050
	800	0.049	0.053	0.050
20	50	0.027	0.072	0.054
	100	0.037	0.064	0.052
	200	0.043	0.058	0.051
	400	0.047	0.055	0.051
	800	0.048	0.053	0.051
30	50	0.023	0.072	0.055
	100	0.035	0.064	0.053
	200	0.042	0.059	0.052
	400	0.046	0.055	0.051
	800	0.048	0.052	0.050

Table 5: Powers of M_1, M_2 and M_3.

p	N	M_1	M_2	M_3
3	20	0.264	0.414	0.331
	50	0.756	0.805	0.742
	100	0.960	0.967	0.954
	200	0.998	0.998	0.998
	400	1.000	1.000	1.000
	800	1.000	1.000	1.000
10	20	0.154	0.349	0.294
	50	0.586	0.690	0.645
	100	0.903	0.929	0.916
	200	0.995	0.997	0.996
	400	1.000	1.000	1.000
	800	1.000	1.000	1.000

p	N	M_1	M_2	M_3
20	50	0.386	0.532	0.487
	100	0.729	0.800	0.778
	200	0.964	0.975	0.972
	400	1.000	1.000	1.000
	800	1.000	1.000	1.000
30	50	0.278	0.441	0.399
	100	0.563	0.667	0.640
	200	0.879	0.913	0.906
	400	0.995	0.997	0.997
	800	1.000	1.000	1.000

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