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## Decomposition of symmetric multivariate density function

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**Abstract.** For a  $T$ -variate density function, the present article defines the quasi-symmetry of order  $k$  ( $< T$ ) and the marginal symmetry of order  $k$ , and gives the theorem that the density function is  $T$ -variate permutation symmetric if and only if it is quasi-symmetric and marginal symmetric of order  $k$ . The theorem is illustrated for the multivariate normal density function.

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### §1. Introduction

For analysis of square contingency tables, it is known that the symmetry model holds if and only if both the quasi-symmetry and marginal homogeneity models hold (for example, see Caussinus [3], Tomizawa and Tahata [6]). For multi-way contingency tables, Bhapkar and Darroch [1] defined the complete symmetry, quasi-symmetry and marginal symmetry models, and showed that the complete symmetry model holds if and only if both the quasi-symmetry and marginal symmetry models hold.

By the way, a similar decomposition for bivariate density function (instead of cell probabilities) is given by Tomizawa, Seo and Minaguchi [5]. Let  $X$  and  $Y$  be two continuous random variables with a density function  $f(x, y)$ . The density function  $f(x, y)$  is said to be symmetric if we have

$$f(x, y) = f(y, x) \quad \text{for every } (x, y) \in \mathbf{R}^2;$$

see Tong [7]. Tomizawa, et al. [5] defined quasi-symmetry and marginal homogeneity for the density function, and gave the theorem that the density

function  $f(x, y)$  is symmetric if and only if it is both quasi-symmetric and marginal homogeneous.

Let the support of  $f(x, y)$  denote  $K^2$ , where

$$K^2 = \{(x, y) : f(x, y) > 0\}.$$

We assume that the support of  $f(x, y)$  is an open connected set in  $\mathbf{R}^2$ . Also, let  $\theta(s_1, s_2; t_1, t_2)$  be the odds-ratio for  $X$ -values  $s_1, s_2$  and  $Y$ -values  $t_1, t_2$ ; namely,

$$\theta(s_1, s_2; t_1, t_2) = \frac{f(s_1, t_1)f(s_2, t_2)}{f(s_2, t_1)f(s_1, t_2)}.$$

Then the density function  $f(x, y)$  is said to be quasi-symmetric if we have

$$\theta(s_1, s_2; t_1, t_2) = \theta(t_1, t_2; s_1, s_2)$$

for any  $(s_i, t_j) \in K^2$ . Thus this indicates that the density function is symmetric with respect to the odds-ratio. The density function  $f(x, y)$  is said to be marginal homogeneous if we have

$$f_X(t) = f_Y(t) \quad \text{for every } t \in \mathbf{R},$$

where  $f_X(t)$  and  $f_Y(t)$  are the marginal density functions of  $X$  and  $Y$ , respectively. Now, we are interested in extending the decomposition of the symmetric density function in multivariate case.

In this article, we define the quasi-symmetry and marginal symmetry for multivariate density function, and decompose the symmetry into quasi-symmetry and marginal symmetry. Section 2 provides the decomposition for trivariate density function. Section 3 extends the decomposition to multivariate density function. Section 4 illustrates our decompositions for normal distributions. Section 5 describes some comments.

## §2. Decomposition of trivariate density function

Let  $X_1, X_2$  and  $X_3$  be three continuous random variables with a density function  $f(x_1, x_2, x_3)$ . The density function  $f(x_1, x_2, x_3)$  is said to be permutation symmetric ( $S^3$ ) if for each permutation  $(\pi_1, \pi_2, \pi_3)$  of  $(1, 2, 3)$  and every  $(x_1, x_2, x_3) \in \mathbf{R}^3$ , we have

$$f(x_{\pi_1}, x_{\pi_2}, x_{\pi_3}) = f(x_1, x_2, x_3);$$

see Tong [7], and Fang, Kotz and Ng [4].

Let  $f_{X_1}(x_1)$ ,  $f_{X_2}(x_2)$  and  $f_{X_3}(x_3)$  be the marginal density functions of  $X_1$ ,  $X_2$  and  $X_3$ , respectively. For the density function  $f(x_1, x_2, x_3)$ , we shall define marginal symmetry of order 1 (denoted by  $M_1^3$ ) by

$$M_1^3 : f_{X_1}(t) = f_{X_2}(t) = f_{X_3}(t) \quad \text{for every } t \in \mathbf{R}.$$

Also, we define marginal symmetry of order 2 (denoted by  $M_2^3$ ) by

$$M_2^3 : f_{X_1X_2}(s, t) = f_{X_1X_2}(t, s) = f_{X_1X_3}(s, t) = f_{X_2X_3}(s, t) \quad \text{for every } (s, t) \in \mathbf{R}^2.$$

Thus,  $M_2^3$  indicates that each of marginal distributions of  $(X_1, X_2)$ ,  $(X_1, X_3)$  and  $(X_2, X_3)$  has a same bivariate density function being symmetric. Note that  $M_2^3$  implies  $M_1^3$ .

Let the support of  $f(x_1, x_2, x_3)$  denote  $K^3$ , where

$$K^3 = \{(x_1, x_2, x_3) : f(x_1, x_2, x_3) > 0, a < x_i < b, i = 1, 2, 3, -\infty \leq a < b \leq \infty\}.$$

We assume that the support of  $f(x_1, x_2, x_3)$  is an open connected set in  $\mathbf{R}^3$ . Generally, we can express the density function as

$$(2.1) \quad f(x_1, x_2, x_3) = \mu\alpha_1(x_1)\alpha_2(x_2)\alpha_3(x_3) \times \\ \beta_{12}(x_1, x_2)\beta_{13}(x_1, x_3)\beta_{23}(x_2, x_3)\gamma(x_1, x_2, x_3),$$

where  $(x_1, x_2, x_3) \in K^3$ , and for an arbitrary fixed value  $c \in (a, b)$ ,

$$\alpha_1(c) = 1, \quad \beta_{12}(c, x_2) = \beta_{12}(x_1, c) = 1, \\ \gamma(c, x_2, x_3) = \gamma(x_1, c, x_3) = \gamma(x_1, x_2, c) = 1,$$

with similar properties of  $\alpha_2, \alpha_3, \beta_{13}$  and  $\beta_{23}$ . The terms  $\alpha_i$  correspond to main effects of the variable  $X_i$ ,  $\beta_{ij}$  to interaction effects of  $X_i$  and  $X_j$ , and  $\gamma$  to interaction effect of  $X_1, X_2$  and  $X_3$ . Namely

$$\mu = f(c, c, c), \\ \alpha_1(x_1) = \frac{f(x_1, c, c)}{f(c, c, c)}, \quad \alpha_2(x_2) = \frac{f(c, x_2, c)}{f(c, c, c)}, \quad \alpha_3(x_3) = \frac{f(c, c, x_3)}{f(c, c, c)}, \\ \beta_{12}(x_1, x_2) = \frac{f(x_1, x_2, c)f(c, c, c)}{f(x_1, c, c)f(c, x_2, c)}, \\ \beta_{13}(x_1, x_3) = \frac{f(x_1, c, x_3)f(c, c, c)}{f(x_1, c, c)f(c, c, x_3)}, \\ \beta_{23}(x_2, x_3) = \frac{f(c, x_2, x_3)f(c, c, c)}{f(c, x_2, c)f(c, c, x_3)}, \\ \gamma(x_1, x_2, x_3) = \frac{f(x_1, x_2, x_3)f(x_1, c, c)f(c, x_2, c)f(c, c, x_3)}{f(c, c, c)f(x_1, x_2, c)f(x_1, c, x_3)f(c, x_2, x_3)}.$$

The term  $\alpha_1(x_1)$  indicates the odds of density function with respect to  $X_1$ -values with  $(X_2, X_3) = (c, c)$ . Note that

$$\begin{aligned}\beta_{12}(x_1, x_2) &= \left( \frac{f(x_1, x_2, c)}{f(c, x_2, c)} \right) / \left( \frac{f(x_1, c, c)}{f(c, c, c)} \right) \\ &= \left( \frac{f(x_1, x_2, c)}{f(x_1, c, c)} \right) / \left( \frac{f(c, x_2, c)}{f(c, c, c)} \right),\end{aligned}$$

and

$$\begin{aligned}\gamma(x_1, x_2, x_3) &= \left( \frac{f(x_1, x_2, x_3)f(c, c, x_3)}{f(x_1, c, x_3)f(c, x_2, x_3)} \right) / \left( \frac{f(x_1, x_2, c)f(c, c, c)}{f(x_1, c, c)f(c, x_2, c)} \right) \\ &= \left( \frac{f(x_1, x_2, x_3)f(c, x_2, c)}{f(x_1, x_2, c)f(c, x_2, x_3)} \right) / \left( \frac{f(x_1, c, x_3)f(c, c, c)}{f(x_1, c, c)f(c, c, x_3)} \right) \\ &= \left( \frac{f(x_1, x_2, x_3)f(x_1, c, c)}{f(x_1, x_2, c)f(x_1, c, x_3)} \right) / \left( \frac{f(c, x_2, x_3)f(c, c, c)}{f(c, x_2, c)f(c, c, x_3)} \right).\end{aligned}$$

Thus,  $\beta_{12}(x_1, x_2)$  indicates the odds-ratio of density function with respect to  $(X_1, X_2)$ -values with  $X_3 = c$ . Also  $\gamma(x_1, x_2, x_3)$  indicates the ratio of odds-ratios of density function, i.e., the ratio of odds-ratio with respect to  $(X_1, X_2)$ -values with  $X_3 = x_3$  to that with  $X_3 = c$  (or the ratio of odds-ratio with respect to  $(X_i, X_j)$ -values with  $X_k = x_k$  to that with  $X_k = c$ , where  $(i, j, k) = (1, 3, 2)$  and  $(2, 3, 1)$ ).

The density function is  $S^3$  if and only if it is expressed as the form (2.1) with

$$S^3 : \begin{cases} \alpha_1(x_1) = \alpha_2(x_1) = \alpha_3(x_1), \\ \beta_{12}(x_1, x_2) = \beta_{12}(x_2, x_1) = \beta_{13}(x_1, x_2) = \beta_{23}(x_1, x_2), \\ \gamma(x_{\pi_1}, x_{\pi_2}, x_{\pi_3}) = \gamma(x_1, x_2, x_3). \end{cases}$$

We shall define quasi-symmetry of order 1 (denoted by  $Q_1^3$ ), and order 2 (denoted by  $Q_2^3$ ). We define  $Q_1^3$  by (2.1) with

$$Q_1^3 : \begin{cases} \beta_{12}(x_1, x_2) = \beta_{12}(x_2, x_1) = \beta_{13}(x_1, x_2) = \beta_{23}(x_1, x_2), \\ \gamma(x_{\pi_1}, x_{\pi_2}, x_{\pi_3}) = \gamma(x_1, x_2, x_3). \end{cases}$$

Thus  $Q_1^3$  indicates

$$\begin{aligned}\theta(s_1, s_2; t_1, t_2; u) &= \theta(t_1, t_2; s_1, s_2; u) \\ &= \theta(s_1, s_2; u; t_1, t_2) = \theta(t_1, t_2; u; s_1, s_2) \\ &= \theta(u; s_1, s_2; t_1, t_2) = \theta(u; t_1, t_2; s_1, s_2),\end{aligned}$$

where  $(s_i, t_j, u) \in K^3$  and so on, and

$$\begin{aligned}\theta(s_1, s_2; t_1, t_2; u) &= \frac{f(s_1, t_1, u)f(s_2, t_2, u)}{f(s_2, t_1, u)f(s_1, t_2, u)}, \\ \theta(s_1, s_2; u; t_1, t_2) &= \frac{f(s_1, u, t_1)f(s_2, u, t_2)}{f(s_2, u, t_1)f(s_1, u, t_2)}, \\ \theta(u; s_1, s_2; t_1, t_2) &= \frac{f(u, s_1, t_1)f(u, s_2, t_2)}{f(u, s_2, t_1)f(u, s_1, t_2)}.\end{aligned}$$

because we can see

$$\theta(s_1, s_2; t_1, t_2; u) = \frac{\theta(c, s_1; c, t_1; u)\theta(c, s_2; c, t_2; u)}{\theta(c, s_2; c, t_1; u)\theta(c, s_1; c, t_2; u)},$$

and so on. Therefore  $Q_1^3$  indicates that the density function is symmetric with respect to the odds-ratio.

Also, we define  $Q_2^3$  by (2.1) with

$$Q_2^3 : \gamma(x_{\pi_1}, x_{\pi_2}, x_{\pi_3}) = \gamma(x_1, x_2, x_3).$$

Thus  $Q_2^3$  indicates

$$\begin{aligned}\frac{\theta(s_1, s_2; t_1, t_2; u_1)}{\theta(s_1, s_2; t_1, t_2; u_2)} &= \frac{\theta(t_1, t_2; s_1, s_2; u_1)}{\theta(t_1, t_2; s_1, s_2; u_2)} \\ &= \frac{\theta(s_1, s_2; u_1; t_1, t_2)}{\theta(s_1, s_2; u_2; t_1, t_2)} = \frac{\theta(t_1, t_2; u_1; s_1, s_2)}{\theta(t_1, t_2; u_2; s_1, s_2)} \\ &= \frac{\theta(u_1; s_1, s_2; t_1, t_2)}{\theta(u_2; s_1, s_2; t_1, t_2)} = \frac{\theta(u_1; t_1, t_2; s_1, s_2)}{\theta(u_2; t_1, t_2; s_1, s_2)},\end{aligned}$$

where  $(s_i, t_j, u_k) \in K^3$  and so on; because

$$\frac{\theta(s_1, s_2; t_1, t_2; u_k)}{\theta(s_1, s_2; t_1, t_2; c)} = \frac{\gamma(s_1, t_1, u_k)\gamma(s_2, t_2, u_k)}{\gamma(s_2, t_1, u_k)\gamma(s_1, t_2, u_k)}.$$

Therefore  $Q_2^3$  indicates that the density function is symmetric with respect to the ratio of odds-ratios. We point out that each of  $S^3$ ,  $Q_1^3$  and  $Q_2^3$  does not depend on the value of  $c$  fixed. It is obviously that  $Q_1^3$  implies  $Q_2^3$ . Note that the alternative way of expressing  $Q_1^3$  is

$$Q_1^3 : f(x_1, x_2, x_3) = \theta_1(x_1)\theta_2(x_2)\theta_3(x_3)v(x_1, x_2, x_3),$$

where  $v$  is positive and permutation symmetric function, i.e.,  $v(x_{\pi_1}, x_{\pi_2}, x_{\pi_3}) = v(x_1, x_2, x_3)$ . We obtain the following theorem.

**Theorem 1.** For  $k$  fixed ( $k = 1, 2$ ), the trivariate density function  $f(x_1, x_2, x_3)$  is  $S^3$  if and only if it is both  $Q_k^3$  and  $M_k^3$ .

Referring to Bhapkar and Darroch [1] for discrete probabilities in multi-way contingency tables, we can prove theorem for multivariate density function as follows.

*Proof.* Consider the case of  $k = 1$ . If a density function is  $S^3$ , then it satisfies  $Q_1^3$  and  $M_1^3$ . Assume that it is both  $Q_1^3$  and  $M_1^3$ , and then we shall show that it satisfies  $S^3$ .

Let  $f^*(x_1, x_2, x_3)$  be the density function which satisfies both  $Q_1^3$  and  $M_1^3$ . Since  $f^*(x_1, x_2, x_3)$  satisfies  $Q_1^3$ , we see

$$\log f^*(x_1, x_2, x_3) = \log \theta_1(x_1) + \log \theta_2(x_2) + \log \theta_3(x_3) + \log v(x_1, x_2, x_3),$$

where  $v$  is positive and permutation symmetric function. Let the density  $g(x_1, x_2, x_3)$  be  $c^{-1}v(x_1, x_2, x_3)$  with  $c = \iiint v(x_1, x_2, x_3)dx_1dx_2dx_3$ . Also, since  $f^*(x_1, x_2, x_3)$  satisfies  $M_1^3$ , we see

$$(2.2) \quad f_{X_1}^*(t) = f_{X_2}^*(t) = f_{X_3}^*(t) = \mu(t) \quad \text{for } t \in \mathbf{R},$$

where  $f_{X_1}^*(t)$ ,  $f_{X_2}^*(t)$  and  $f_{X_3}^*(t)$  are the marginal density functions of  $X_1, X_2$  and  $X_3$ , respectively. Consider the arbitrary density function  $f(x_1, x_2, x_3)$  satisfying  $M_1^3$  with

$$(2.3) \quad f_{X_1}(t) = f_{X_2}(t) = f_{X_3}(t) = \mu(t) \quad \text{for } t \in \mathbf{R},$$

where  $f_{X_1}(t)$ ,  $f_{X_2}(t)$  and  $f_{X_3}(t)$  are the marginal density functions of  $X_1, X_2$  and  $X_3$ , respectively. From (2.2) and (2.3), we see

$$(2.4) \quad \iiint \{f(x_1, x_2, x_3) - f^*(x_1, x_2, x_3)\} \times \log \left( \frac{f^*(x_1, x_2, x_3)}{g(x_1, x_2, x_3)} \right) dx_1 dx_2 dx_3 = 0.$$

Using the equation (2.4), we obtain

$$I(f, g) = I(f^*, g) + I(f, f^*),$$

where

$$I(h_1, h_2) = \iiint h_1(x_1, x_2, x_3) \log \left( \frac{h_1(x_1, x_2, x_3)}{h_2(x_1, x_2, x_3)} \right) dx_1 dx_2 dx_3.$$

For  $g$  fixed, we see

$$\min_f I(f, g) = I(f^*, g),$$

and then  $f^*$  uniquely minimizes  $I(f, g)$ .

Let  $f^{**}(x_1, x_2, x_3) = f^*(x_1, x_3, x_2)$ . In a similar way, we also see

$$\iiint \{f(x_1, x_2, x_3) - f^{**}(x_1, x_2, x_3)\} \log \left( \frac{f^{**}(x_1, x_2, x_3)}{g(x_1, x_2, x_3)} \right) dx_1 dx_2 dx_3 = 0,$$

where  $f(x_1, x_2, x_3)$  is  $M_1^3$  with (2.3). Thus, we obtain

$$I(f, g) = I(f^{**}, g) + I(f, f^{**}).$$

For  $g$  fixed, we see

$$\min_f I(f, g) = I(f^{**}, g),$$

and then  $f^{**}$  uniquely minimizes  $I(f, g)$ . Therefore, we see  $f^*(x_1, x_2, x_3) = f^{**}(x_1, x_2, x_3)$ . Thus,  $f^*(x_1, x_2, x_3) = f^*(x_1, x_3, x_2)$ .

Also, in a similar way, we obtain

$$f^*(x_1, x_2, x_3) = f^*(x_2, x_1, x_3) = f^*(x_2, x_3, x_1) = f^*(x_3, x_1, x_2) = f^*(x_3, x_2, x_1).$$

Therefore, we have  $f^*(x_1, x_2, x_3) = f^*(x_{\pi_1}, x_{\pi_2}, x_{\pi_3})$ . Namely  $f^*(x_1, x_2, x_3)$  satisfies  $S^3$ . The case of  $k = 2$  can be proved in a similar way as the case of  $k = 1$ . So the proof is completed.

### §3. Decomposition of multivariate density function

Let  $X_1, \dots, X_T$  be  $T$  continuous random variables with a density function  $f(x_1, \dots, x_T)$ . The density function  $f(x_1, \dots, x_T)$  is said to be permutation symmetric ( $S^T$ ) if for each permutation  $(\pi_1, \dots, \pi_T)$  of  $(1, \dots, T)$  and every  $(x_1, \dots, x_T) \in \mathbf{R}^T$ , we have

$$f(x_{\pi_1}, \dots, x_{\pi_T}) = f(x_1, \dots, x_T);$$

see Tong [7] and Fang et al. [4].

Let the support of  $f(x_1, \dots, x_T)$  denote  $K^T$ , where

$$K^T = \{(x_1, \dots, x_T) : f(x_1, \dots, x_T) > 0, \\ a < x_i < b, i = 1, \dots, T, -\infty \leq a < b \leq \infty\}.$$

We assume that the support of  $f(x_1, \dots, x_T)$  is an open connected set in  $\mathbf{R}^T$ . Generally, we can express the density function as

$$(3.1) \quad f(x_1, \dots, x_T) = \alpha \left[ \prod_{i_1=1}^T \alpha_{i_1}(x_{i_1}) \right] \left[ \prod_{1 \leq i_1 < i_2 \leq T} \alpha_{i_1 i_2}(x_{i_1}, x_{i_2}) \right] \times \dots \\ \times \left[ \prod_{1 \leq i_1 < \dots < i_{T-1} \leq T} \alpha_{i_1 \dots i_{T-1}}(x_{i_1}, \dots, x_{i_{T-1}}) \right] \cdot \alpha_{1 \dots T}(x_1, \dots, x_T),$$

where  $(x_1, \dots, x_T) \in K^T$ , and for an arbitrary fixed value  $c \in (a, b)$ ,

$$\{\alpha_i(c) = \alpha_{i_1 i_2}(c, x_{i_2}) = \dots = \alpha_{1 \dots T}(x_1, \dots, x_{T-1}, c) = 1\}.$$

Then, the density function  $f(x_1, \dots, x_T)$  being  $S^T$  is also expressed as (3.1) with

$$S^T : \alpha_{i_1 \dots i_m}(x_{i_1}, \dots, x_{i_m}) = \alpha_{i_1 \dots i_m}(x_{\pi_{i_1}}, \dots, x_{\pi_{i_m}}) = \alpha_{j_1 \dots j_m}(x_{i_1}, \dots, x_{i_m}) \\ (m = 1, \dots, T; 1 \leq i_1 < \dots < i_m \leq T; 1 \leq j_1 < \dots < j_m \leq T),$$

where  $(\pi_{i_1}, \dots, \pi_{i_m})$  is permutation of  $(i_1, \dots, i_m)$ .

For  $k = 1, \dots, T-1$ , we shall define quasi-symmetry of order  $k$  (denoted by  $Q_k^T$ ) by (3.1) with

$$Q_k^T : \alpha_{i_1 \dots i_m}(x_{i_1}, \dots, x_{i_m}) = \alpha_{i_1 \dots i_m}(x_{\pi_{i_1}}, \dots, x_{\pi_{i_m}}) = \alpha_{j_1 \dots j_m}(x_{i_1}, \dots, x_{i_m}) \\ (m = k+1, \dots, T; 1 \leq i_1 < \dots < i_m \leq T; 1 \leq j_1 < \dots < j_m \leq T).$$

Also, for  $k = 1, \dots, T-1$ , we shall define marginal symmetry of order  $k$  (denoted by  $M_k^T$ ) by

$$M_k^T : f_{X_{i_1} \dots X_{i_k}}(x_{i_1}, \dots, x_{i_k}) = f_{X_{i_1} \dots X_{i_k}}(x_{\pi_{i_1}}, \dots, x_{\pi_{i_k}}) = f_{X_{j_1} \dots X_{j_k}}(x_{i_1}, \dots, x_{i_k}) \\ (1 \leq i_1 < \dots < i_k \leq T; 1 \leq j_1 < \dots < j_k \leq T),$$

where  $f_{X_{i_1} \dots X_{i_k}}$  is the marginal density function of  $(X_{i_1}, \dots, X_{i_k})$ . Then we obtain the following theorem.

**Theorem 2.** *For  $k$  fixed ( $k = 1, \dots, T-1$ ), the multivariate density function  $f(x_1, \dots, x_T)$  is  $S^T$  if and only if it is both  $Q_k^T$  and  $M_k^T$ .*

The proof of Theorem 2 is omitted because it is obtained in a similar way to the proof of Theorem 1.

#### §4. Symmetry of multivariate normal density function

**Example 1.** Consider a  $T$ -dimensional random vector  $\mathbf{X} = (X_1, \dots, X_T)'$  having a normal distribution with mean vector  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_T)'$  and covariance matrix  $\boldsymbol{\Sigma}$ . The density function is

$$(4.1) \quad f(x_1, \dots, x_T) = \frac{1}{(2\pi)^{\frac{T}{2}} |\boldsymbol{\Sigma}|^{\frac{1}{2}}} \exp \left\{ -\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right\}.$$

Denote  $\boldsymbol{\Sigma}^{-1}$  by  $\mathbf{A} = (a_{ij})$  with  $a_{ij} = a_{ji}$ . Then the density function can be expressed as

$$f(x_1, \dots, x_T) = C \exp \left\{ -\frac{1}{2} H \right\},$$



where  $C$  is positive constant and

$$H = \sum_{s=1}^T a_{ss}x_s^2 + \sum_{s \neq t} a_{st}x_sx_t - 2 \sum_{s=1}^T \sum_{t=1}^T a_{st}\mu_sx_t.$$

By setting  $c = 0$  without loss of generality, we see

$$(4.2) \quad \begin{aligned} \alpha_i(x_i) &= \exp \left\{ -\frac{1}{2}(a_{ii}x_i^2 - 2 \sum_{s=1}^T a_{si}\mu_sx_i) \right\} \quad (i = 1, \dots, T), \\ \alpha_{ij}(x_i, x_j) &= \exp(-a_{ij}x_ix_j) \quad (i < j), \end{aligned}$$

and for  $m = 3, \dots, T$ ,

$$\alpha_{i_1 \dots i_m}(x_{i_1}, \dots, x_{i_m}) = 1 \quad (1 \leq i_1 < \dots < i_m \leq T).$$

Therefore the density function (4.1) is  $Q_k^T$  for  $k = 2, \dots, T-1$ . Also from (4.2), the density function (4.1) is  $Q_1^T$  if and only if  $\{a_{ij} (= a_{ji})\}$  are constant (e.g., equals  $w$ ) for all  $i < j$ ; namely,  $\Sigma^{-1}$  has the form

$$(4.3) \quad \Sigma^{-1} = \mathbf{D} + wee',$$

where  $\mathbf{D}$  is the  $T \times T$  diagonal matrix,  $\mathbf{e}$  is the  $T \times 1$  vector of 1 elements, and  $w$  is scalar. Although the detail is omitted, then  $\Sigma$  has the form

$$\Sigma = \mathbf{D}^{-1} + d\mathbf{D}^{-1}ee'\mathbf{D}^{-1},$$

where  $d$  is scalar. Therefore, the density function (4.1) is  $Q_1^T$  if and only if  $\Sigma$  has the form

$$(4.4) \quad \Sigma = \begin{pmatrix} b_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & b_T \end{pmatrix} + d \begin{pmatrix} b_1 \\ \vdots \\ b_T \end{pmatrix} (b_1, \dots, b_T).$$

Let  $V(X_i) = \sigma_i^2$  ( $i = 1, \dots, T$ ) and let  $\rho_{ij}$  be the correlation coefficient of  $X_i$  and  $X_j$  ( $i \neq j$ ) with  $|\rho_{ij}| < 1$ . Assume that

(i)  $\sigma_1^2 = \dots = \sigma_T^2 (= \sigma^2)$  and  $\rho_{ij} = \rho$  ( $i < j$ ).

Then

$$\Sigma = \sigma^2(1 - \rho) \left( \mathbf{E} + \frac{\rho}{1 - \rho} ee' \right),$$

where  $\mathbf{E}$  is the  $T \times T$  identity matrix. This satisfies the form (4.4) of  $\Sigma$ . Therefore the density function (4.1) with condition (i) is  $Q_1^T$ .

Next, assume that

(ii)  $\sigma_1^2 = \dots = \sigma_T^2 (= \sigma^2)$ .

From (4.4), then  $Q_1^T$  holds if and only if

$$\begin{cases} \sigma^2 = b_i + db_i^2 & (i = 1, \dots, T), \\ \sigma^2 \rho_{ij} = db_i b_j & (i < j), \end{cases}$$

hold, namely,  $b_1 = \dots = b_T$  since  $|\rho_{ij}| < 1$ . Therefore the density function (4.1) with condition (ii) is  $Q_1^T$  if and only if  $\rho_{ij} = \rho$  for all  $i < j$  hold.

Also, assume that

(iii)  $\rho_{ij} = \rho$  ( $\neq 0$ ) for all  $i < j$ .

Then we see

$$\Sigma = \begin{pmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_T \end{pmatrix} ((1 - \rho)\mathbf{E} + \rho\mathbf{e}\mathbf{e}') \begin{pmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_T \end{pmatrix}.$$

Although the detail is omitted, we can see

$$\Sigma^{-1} = \frac{1}{1 - \rho} \left( \begin{pmatrix} \sigma_1^{-2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_T^{-2} \end{pmatrix} + \frac{1}{m} \begin{pmatrix} \sigma_1^{-1} \\ \vdots \\ \sigma_T^{-1} \end{pmatrix} (\sigma_1^{-1}, \dots, \sigma_T^{-1}) \right),$$

where  $m = -(1 - \rho)/\rho - T$ . Therefore from (4.3), the density function (4.1) with condition (iii) is  $Q_1^T$  if and only if  $\sigma_1^2 = \dots = \sigma_T^2$  holds.

Assume that

(iv)  $\rho_{ij} = 0$  for all  $i < j$ .

Then the density function (4.1) is  $Q_1^T$  because  $\alpha_{ij}(x_i, x_j) = 1$  in (4.2) with  $a_{ij} = 0$  for  $i < j$ .

We shall consider the relationship between the density function (4.1) and  $M_k^T$  ( $k = 1, \dots, T-1$ ). Obviously, the density function (4.1) is  $M_1^T$  if and only if  $\mu_1 = \dots = \mu_T$  and  $\sigma_1^2 = \dots = \sigma_T^2$  hold. Also, for each  $k$  ( $k = 2, \dots, T-1$ ), it is  $M_k^T$  if and only if  $\mu_1 = \dots = \mu_T$ ,  $\sigma_1^2 = \dots = \sigma_T^2$ , and  $\rho_{ij} = \rho$  for all  $i < j$ . Thus, from Theorem 2 we can see that the density function (4.1) with  $\mu_1 = \dots = \mu_T$  and  $\sigma_1^2 = \dots = \sigma_T^2$  is  $S^T$  if and only if it is  $Q_1^T$ . Also, from Theorem 2, the density function (4.1) is  $S^T$  if and only if  $\mu_1 = \dots = \mu_T$ ,  $\sigma_1^2 = \dots = \sigma_T^2$  and  $\rho_{ij} = \rho$  for all  $i < j$  hold.

**Example 2.** Consider a  $T$ -dimensional random vector  $\mathbf{U} = (U_1, \dots, U_T)'$  having a multinomial distribution with

$$\begin{aligned} P(U_1 = u_1, \dots, U_T = u_T | N) = \\ \frac{N!}{u_1! \cdots u_T! (N - \sum_{i=1}^T u_i)!} \pi_1^{u_1} \cdots \pi_T^{u_T} (1 - \sum_{i=1}^T \pi_i)^{N - \sum_{i=1}^T u_i}, \end{aligned}$$

where  $u_i$  is nonnegative integer with  $0 \leq u_i \leq N$ . Let

$$\boldsymbol{\pi} = (\pi_1, \dots, \pi_T)', \quad \hat{\boldsymbol{\pi}} = (\hat{\pi}_1, \dots, \hat{\pi}_T)',$$

where  $\hat{\pi}_i = u_i/N$ . Also let  $\mathbf{X} = \sqrt{N}(\hat{\boldsymbol{\pi}} - \boldsymbol{\pi})$ . Then it is well-known that  $\mathbf{X}$  has asymptotically (as  $N \rightarrow \infty$ ) a  $T$ -variate normal distribution with mean  $T \times 1$  zero vector  $\mathbf{0} = (0, \dots, 0)'$  and covariance matrix

$$(4.5) \quad \boldsymbol{\Sigma} = \mathbf{D} - \boldsymbol{\pi}\boldsymbol{\pi}',$$

where

$$\mathbf{D} = \begin{pmatrix} \pi_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \pi_T \end{pmatrix};$$

see, e.g., Bishop, Fienberg and Holland [2]. So we shall consider the properties of normal distribution having covariance matrix (4.5). We see that  $\boldsymbol{\Sigma}$  in (4.5) satisfies the form (4.4) obtained in Example 1. Therefore the density function of normal distribution  $N(\mathbf{0}, \mathbf{D} - \boldsymbol{\pi}\boldsymbol{\pi}')$  is always  $Q_1^T$ . Also, it is  $Q_k^T$  ( $k = 2, \dots, T - 1$ ).

The marginal distribution of  $X_i$  in  $\mathbf{X} = (X_1, \dots, X_T)'$  is  $N(0, \pi_i(1 - \pi_i))$  for  $i = 1, \dots, T$ . Therefore the density function of  $N(\mathbf{0}, \mathbf{D} - \boldsymbol{\pi}\boldsymbol{\pi}')$  is  $M_1^T$  if and only if  $\pi_1 = \cdots = \pi_T$  holds.

Also two dimensional marginal distribution of  $(X_i, X_j)$  for  $i < j$  has the mean zero vector and the covariance matrix

$$\begin{pmatrix} \pi_i(1 - \pi_i) & -\pi_i\pi_j \\ -\pi_i\pi_j & \pi_j(1 - \pi_j) \end{pmatrix}.$$

Thus, the density function of  $N(\mathbf{0}, \mathbf{D} - \boldsymbol{\pi}\boldsymbol{\pi}')$  is  $M_2^T$  if and only if  $\pi_1 = \cdots = \pi_T$  holds. In a similar way, it is  $M_k^T$  if and only if  $\pi_1 = \cdots = \pi_T$  holds ( $k = 3, \dots, T - 1$ ).

Therefore we can see from Theorem 2 that the density function of  $N(\mathbf{0}, \mathbf{D} - \boldsymbol{\pi}\boldsymbol{\pi}')$  is  $S^T$  if and only if it is  $M_k^T$  ( $k = 1, \dots, T - 1$ ), because it always satisfies  $Q_k^T$ .

### §5. Comments

When an arbitrary density function  $f(x_1, \dots, x_T)$  is not permutation symmetric, Theorem 2 may be useful for knowing the reason, i.e., for  $k$  fixed, which structure of quasi-symmetry of order  $k$  and marginal symmetry of order  $k$  is lacking.

We point out that for a  $T$ -variate normal distribution, if the variances of  $X_1, \dots, X_T$  are the same and the correlation coefficients of  $X_i$  and  $X_j$  for all

$i < j$  are the same, then the density functions is quasi-symmetric of order 1, i.e.,  $Q_1^T$  (as seen in Example 1); however, the converse always does not hold. Indeed, the normal density function with covariance matrix  $\Sigma = D - \pi\pi'$  (in Example 2) is always  $Q_1^T$  even when the variances of  $X_1, \dots, X_T$  are not the same and the correlation coefficients of  $X_i$  and  $X_j$  are not the same for  $1 \leq i < j \leq T$ .

Finally we note that it is difficult to illustrate the decomposition of symmetry for the elliptical distribution instead of the normal distribution in Example of Section 4 because the  $\{\alpha_i(x_i)\}$  and  $\{\alpha_{ij}(x_i, x_j)\}$  are expressed as the ratio of density functions.

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