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Hochschild cohomology of a class of weakly symmetric algebras with radical cube zero

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Abstract. In this paper we provide an explicit minimal projective bimodule resolution for some weakly symmetric algebras with radical cube zero. Then by using this resolution we compute the dimension of its Hochschild cohomology groups.

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§1. Introduction

Throughout this paper, let k be a field and Γ the quiver with m vertices and $2m$ arrows as follows:

$$\begin{tikzcd} 0 \arrow[loop left, "a_0"] \arrow[r, shift left, "a_1"] & 1 \arrow[l, shift left, "\overline{a}_1"] \arrow[r, shift left, "a_2"] & \cdots \arrow[l, shift left, "\overline{a}_2"] \arrow[r, shift left, "a_{m-1}"] & m-1 \arrow[l, shift left, "\overline{a}_{m-1}"] \arrow[loop right, "a_m"] \end{tikzcd}$$

for an integer $m \geq 1$. Let e_i be the trivial path corresponding to the vertex i. All paths are written from left to right.

Now, to define the algebra A. we treat the following three cases separately.

(1) In the case $m \geq 3$, Γ is the quiver above and then we define the algebra A by $k\Gamma/I$ where I is the ideal generated by the following uniform elements:

$$a_1\overline{a}_1 - a_0^2, \quad a_m^2 - \overline{a}_{m-1}a_{m-1}, \quad \overline{a}_1 a_0, \quad a_m\overline{a}_{m-1},$$
$$a_i\overline{a}_i - \overline{a}_{i-1}a_{i-1}, \quad a_j a_{j+1}, \quad \overline{a}_{l+1}\overline{a}_l,$$

for $2 \leq i \leq m-1$, $0 \leq j \leq m-1$ and $1 \leq l \leq m-2$. In this case, the following elements form a k-basis of A:

$$e_i, \; a_j, \; \overline{a}_l, \; a_r\overline{a}_r, \; a_m^2 \quad \text{for } 0 \leq i \leq m-1; 0 \leq j \leq m; 1 \leq l, r \leq m-1.$$

(2) In the case $m = 2$, Γ is the quiver

$$\begin{tikzcd} 0 \arrow[loop left, "a_0"] \arrow[r, shift left, "a_1"] & 1 \arrow[l, shift left, "\overline{a}_1"] \arrow[loop right, "a_2"] \end{tikzcd}$$

Then we define the algebra A by $k\Gamma/I$ where I is the ideal generated by the following uniform elements:

$$a_1\overline{a}_1 - a_0^2,\ a_2^2 - \overline{a}_1 a_1,\ \overline{a}_1 a_0,\ a_0 a_1, a_1 a_2 \text{ and } a_2\overline{a}_1.$$

In this case, the following elements form a k-basis of A:

$$e_i,\ a_j,\ \overline{a}_1,\ a_1\overline{a}_1,\ a_2^2 \quad \text{for } 0 \leq i \leq 1, 0 \leq j \leq 2.$$

(3) In the case $m = 1$, Γ is the quiver

$$\begin{tikzcd} 0 \arrow[loop left, "a_0"] \arrow[loop right, "a_1"] \end{tikzcd}$$

Then we define the algebra A by $k\Gamma/I$ where I is the ideal generated by the following uniform elements:

$$a_1^2 - a_0^2,\ a_0 a_1 \text{ and } a_1 a_0.$$

In this case, the following elements form a k-basis of A:

$$e_0,\ a_0,\ a_1,\ a_1^2$$

So we have $\dim_k A = 4m$ for $m \geq 1$. It is known that A is a Koszul weakly symmetric algebra with radical cube zero and it belongs to the class of weakly symmetric tame algebras of type $\widetilde{\mathbb{Z}}_{m-1}$ introduced in [1]. Moreover we see that A is a special biserial algebra of [9].

In [7], Snashall and Solberg defined the support varieties for finitely generated modules over a finite dimensional algebra by using the Hochschild cohomology ring modulo nilpotence. Furthermore, in [2], Erdmann, Holloway, Snashall, Solberg and Taillefer introduced some reasonable "finiteness conditions," denoted by (Fg), for any finite dimensional algebra, and they showed that if a finite dimensional algebra satisfies (Fg), then the support varieties have a lot of analogous properties of support varieties for finite group algebras.

Recently, in [8], Snashall and Taillefer described the Hochschild cohomology rings for algebras in a class of certain special biserial weakly symmetric algebras (which does not contain A). They gave explicit generators and relations of the Hochschild cohomology rings modulo nilpotence for algebras in this class, and then used the Hochschild cohomology ring to show that some of these algebras satisfy (Fg).

In [3], Erdman and Solberg gave necessary and sufficient conditions for any Koszul algebra to satisfy (Fg). Consequently, they showed that A satisfies (Fg). So the Hochschild cohomology ring of A is finitely generated as an algebra. On the other hand, in the case where $m = 2$ and $\operatorname{char} k \neq 2$, A is precisely the principal block of the tame Hecke algebra $H_q(S_5)$ for $q = -1$. In this case, a k-basis of the Hochschild cohomology groups of A was described by Schroll and Snashall in [6]. They proved independently that A satisfies (Fg), and gave some properties of the support varieties for modules over A.

In this paper, we provide an explicit minimal projective bimodule resolution of A for $m \geq 3$ and $m = 1$; see [3] for the case $m = 2$, and then compute the dimension of the Hochschild cohomology groups of A and give a k-basis of the Hochschild cohomology groups in a way similar to that in [8]; see also [3] and [6].

The contents of this paper are organized as follows. In Section 2, with the same notation as that in [8], we determine sets $\mathcal{G}^n$ $(n \geq 0)$, introduced in [5], for the right A-module $A/\mathfrak{r}$, where $\mathfrak{r}$ denotes the radical of A. Then, using $\mathcal{G}^n$, we construct a minimal projective resolution $(P_\bullet, \partial_\bullet)$ of A as an A-A-bimodule (Theorem 2.3). In Section 3, we first determine the dimension of the Hochschild cohomology groups for $m \geq 3$ (Theorem 3.5), and then we give an explicit k-basis of the Hochschild cohomology groups (Propositions 3.7, 3.8 and 3.9). In Section 4, using the same arguments as in Sections 2 and 3, we determine the dimension of the Hochschild cohomology groups in the case $m = 1$ (Theorem 4.4).

Throughout this paper, for any arrow a in Γ, we denote the origin of a by $o(a)$ and the terminus by $t(a)$. We write $\otimes_k$ as $\otimes$ for simplicity, and we denote the enveloping algebra $A^{\mathrm{op}} \otimes A$ by A^e.

§2. A projective bimodule resolution for A

In this section, we give an explicit minimal projective bimodule resolution

$$(P_\bullet, \partial_\bullet): \quad \cdots \xrightarrow{\partial_4} P_3 \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} A \to 0$$

of $A = k\Gamma/I$ for $m \geq 3$ by using the argument in [4].

Let $B = kQ/I'$ be any finite-dimensional algebra with a finite quiver Q and an admissible ideal I' in kQ. Denote the radical of B by J. In [5], Green, Solberg and Zacharia introduced subsets $\mathcal{G}^n$ $(n \geq 0)$ of kQ, and used the subsets to give a minimal projective resolution of the right B-module B/J. First we briefly recall the construction of $\mathcal{G}^n$. Let $\mathcal{G}^0$ the set of all vertices of Q, $\mathcal{G}^1$ the set of all arrows of Q and $\mathcal{G}^2$ a minimal set of generators of I. In [5], the authors proved that for each $n \geq 3$ there is a subset $\mathcal{G}^n$ of kQ satisfying the following two conditions:

(a) Each of the elements x of $\mathcal{G}^n$ is a uniform element satisfying

$$x = \sum_{y \in \mathcal{G}^{n-1}} y r_y = \sum_{z \in \mathcal{G}^{n-2}} z s_z \quad \text{for unique } r_y, s_z \in kQ.$$

(b) There is a minimal projective B-resolution of B/J

$$(R_\bullet, \delta_\bullet) : \quad \cdots \xrightarrow{\delta_4} R_3 \xrightarrow{\delta_3} R_2 \xrightarrow{\delta_2} R_1 \xrightarrow{\delta_1} R_0 \xrightarrow{\delta_0} B/J \to 0,$$

satisfying the following conditions:

(i) For each $j \geq 0$, $R_j = \bigoplus_{x \in \mathcal{G}^j} t(x)B$.

(ii) For each $j \geq 1$, the differential $\delta_j : R_j \to R_{j-1}$ is defined by

$$t(x)\lambda \longmapsto \sum_{y \in \mathcal{G}^{j-1}} r_y t(x)\lambda \quad \text{for } x \in \mathcal{G}^j \text{ and } \lambda \in B,$$

where r_y are elements in the expression (a).

In [4], Green, Hartman, Marcos and Solberg used the set $\mathcal{G}^n$ to give a minimal projective bimodule resolution for any finite dimensional Koszul algebra. This set also appears in the papers [3], [6] and [8] in constructing minimal projective bimodule resolutions. In this section, following this approach, we construct the set $\mathcal{G}^n$, and then find a minimal projective bimodule resolution of A for $m \geq 3$.

2.1. A construction of the sets $\mathcal{G}^n$

Now we fix an integer $m \geq 3$. In order to give sets $\mathcal{G}^n$ $(n \geq 0)$ for $A/\mathfrak{r}$ where $\mathfrak{r}$ denotes the radical of A, we first define morphisms of quivers $\phi^i = (\phi^i_0, \phi^i_1) : \Delta \to \Gamma$ for $i = 0, 1, \ldots, m-1$. Let Δ be the following locally finite quiver with vertices (x, y) and arrows $b^{(x,y)} : (x, y) \to (x+1, y)$ and

$c^{(x,y)} : (x,y) \to (x, y+1)$ for integers $x, y \geq 0$.

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\Big\uparrow{\scriptstyle c^{(0,4)}} & & \Big\uparrow{\scriptstyle c^{(1,4)}} & & \Big\uparrow{\scriptstyle c^{(2,4)}} & & \Big\uparrow{\scriptstyle c^{(3,4)}} & & \\
(0,4) & \xrightarrow{b^{(0,4)}} & (1,4) & \xrightarrow{b^{(1,4)}} & (2,4) & \xrightarrow{b^{(2,4)}} & (3,4) & \xrightarrow{b^{(3,4)}} & \cdots \\
\Big\uparrow{\scriptstyle c^{(0,3)}} & & \Big\uparrow{\scriptstyle c^{(1,3)}} & & \Big\uparrow{\scriptstyle c^{(2,3)}} & & \Big\uparrow{\scriptstyle c^{(3,3)}} & & \\
(0,3) & \xrightarrow{b^{(0,3)}} & (1,3) & \xrightarrow{b^{(1,3)}} & (2,3) & \xrightarrow{b^{(2,3)}} & (3,3) & \xrightarrow{b^{(3,3)}} & \cdots \\
\Big\uparrow{\scriptstyle c^{(0,2)}} & & \Big\uparrow{\scriptstyle c^{(1,2)}} & & \Big\uparrow{\scriptstyle c^{(2,2)}} & & \Big\uparrow{\scriptstyle c^{(3,2)}} & & \\
(0,2) & \xrightarrow{b^{(0,2)}} & (1,2) & \xrightarrow{b^{(1,2)}} & (2,2) & \xrightarrow{b^{(2,2)}} & (3,2) & \xrightarrow{b^{(3,2)}} & \cdots \\
\Big\uparrow{\scriptstyle c^{(0,1)}} & & \Big\uparrow{\scriptstyle c^{(1,1)}} & & \Big\uparrow{\scriptstyle c^{(2,1)}} & & \Big\uparrow{\scriptstyle c^{(3,1)}} & & \\
(0,1) & \xrightarrow{b^{(0,1)}} & (1,1) & \xrightarrow{b^{(1,1)}} & (2,1) & \xrightarrow{b^{(2,1)}} & (3,1) & \xrightarrow{b^{(3,1)}} & \cdots \\
\Big\uparrow{\scriptstyle c^{(0,0)}} & & \Big\uparrow{\scriptstyle c^{(1,0)}} & & \Big\uparrow{\scriptstyle c^{(2,0)}} & & \Big\uparrow{\scriptstyle c^{(3,0)}} & & \\
(0,0) & \xrightarrow{b^{(0,0)}} & (1,0) & \xrightarrow{b^{(1,0)}} & (2,0) & \xrightarrow{b^{(2,0)}} & (3,0) & \xrightarrow{b^{(3,0)}} & \cdots
\end{array}
$$

For any integer z, let $Q(z)$ be the quotient and $\overline{z}$ the remainder when we divide z by m. Then we have $0 \leq \overline{z} \leq m-1$. We denote the sets of vertices of Δ and Γ by Δ_0 and Γ_0, respectively. Also, we denote the sets of arrows of Δ and Γ by Δ_1 and Γ_1, respectively. For each $i = 0, 1, \ldots, m-1$, we define the maps $\phi_0^i : \Delta_0 \to \Gamma_0$ and $\phi_1^i : \Delta_1 \to \Gamma_1$ by

(1) For $(x,y) \in \Delta_0$

$$\phi_0^i(x,y) := \begin{cases} e_{\overline{x-y+i}} & \text{if } Q(x-y+i) \in 2\mathbb{Z}, \\ e_{m-1-\overline{x-y+i}} & \text{if } Q(x-y+i) \notin 2\mathbb{Z}. \end{cases}$$

(2) For $b^{(x,y)}, c^{(x,y)} \in \Delta_1$

$$\phi_1^i(b^{(x,y)}) := \begin{cases} a_{\overline{x-y+i}+1} & \text{if } Q(x-y+i) \in 2\mathbb{Z}, \\ \overline{a}_{m-1-\overline{x-y+i}} & \text{if } Q(x-y+i) \notin 2\mathbb{Z}, \end{cases}$$

$$\phi_1^i(c^{(x,y)}) := \begin{cases} \overline{a}_{\overline{x-y+i}} & \text{if } Q(x-y+i) \in 2\mathbb{Z}, \\ a_{m-\overline{x-y+i}} & \text{if } Q(x-y+i) \notin 2\mathbb{Z}. \end{cases}$$

where we put $\overline{a}_0 := a_0$ for our convenience.

Then, for all $i = 0, 1, \ldots, m-1$ and arrows $b^{(x,y)}$ and $c^{(x,y)}$ in Δ, we have

$$\begin{aligned} o(\phi_1^i(b^{(x,y)})) &= o(\phi_1^i(c^{(x,y)})) = \phi_0^i(x,y), \\ t(\phi_1^i(b^{(x,y)})) &= \phi_0^i(x+1,y), \\ t(\phi_1^i(c^{(x,y)})) &= \phi_0^i(x,y+1). \end{aligned}$$

Thus ϕ_1^i is a morphism of quivers. Note that ϕ_1^i naturally induces the map between the set of paths of Δ and that of Γ as follows:

$$\phi_1^i(p_1 \cdots p_r) = \phi_1^i(p_1) \cdots \phi_1^i(p_r),$$

for a path $p_1 \cdots p_r (r \geq 1)$ of Δ where p_j is an arrow for $1 \leq j \leq r$.

Now, we can define the sets $\mathcal{G}^n$ $(n \geq 0)$ for $A/\mathfrak{r}$ in the way similar to that in [8]. Let $g^0_{0,0,i} = e_i$ for $i = 0, 1, \ldots, m-1$. For integers $n \geq 1$, $x, y \geq 0$ with $x + y = n$ and $i = 0, 1, \ldots, m-1$, we define the element $g^n_{x,y,i}$ in $k\Gamma$ by

$$g^n_{x,y,i} := \sum_p (-1)^{s_p} \phi_1^i(p),$$

where

- p ranges over all paths in Δ starting at $(0,0)$ and ending with (x,y); and
- s_p is an integer determined as follows: If we write $p = p_1 p_2 \ldots p_n$ with p_j arrows in Δ for $1 \leq j \leq n$, then $s_p = \sum_{p_j = c^{(x',y')}} j$ where x' and y' are positive integers with $x' + y' = j - 1$.

For each $n \geq 0$, we put

$$\mathcal{G}^n := \{g^n_{x,n-x,i} \mid 0 \leq x \leq n \text{ and } 0 \leq i \leq m-1\}.$$

Then, for $n = 0, 1, 2$, $\mathcal{G}^n$ can be described as follows:

$$\begin{aligned} &\mathcal{G}^0 = \{e_0, e_1, \ldots, e_{m-1}\}, \\ &\mathcal{G}^1 = \{a_1, \ldots, , a_m, -a_0 - \overline{a}_1, -\overline{a}_2, \ldots, -\overline{a}_{m-1}\}, \\ &\mathcal{G}^2 = \\ &\{-\phi_1^i(c^{(0,0)}c^{(0,1)}), \phi_1^i(b^{(0,0)}c^{(1,0)}) - \phi_1^i(c^{(0,0)}b^{(0,1)}), \phi_1^i(b^{(0,0)}b^{(1,0)}) \mid 0 \leq i \leq m-1\} \\ &= \{-a_0 a_1, -\overline{a}_1 a_0, -\overline{a}_i \overline{a}_{i-1}, a_1 \overline{a}_1 - a_0^2, a_{j+1}\overline{a}_{j+1} - \overline{a}_j a_j, a_m^2 - \overline{a}_{m-1} a_{m-1}, \\ &\quad a_{l+1} a_{l+2}, a_m \overline{a}_{m-1} \mid 2 \leq i \leq m-1, 1 \leq j \leq m-2 \text{ and } 0 \leq l \leq m-2\}. \end{aligned}$$

And it is easily seen that $\mathcal{G}^n$ satisfies the conditions (a) and (b) for $m \geq 3$ in the beginning of this section.

As in the beginning of this section, $\mathcal{G}^n$ gives the minimal projective resolution $(R_\bullet, \delta_\bullet)$ of $A/\mathfrak{r}$ defined by (b).

Remark 2.1. The following sequence is a minimal projective resolution of $A/\mathfrak{r}$.

$$(R_\bullet, \delta_\bullet): \quad \cdots \xrightarrow{\delta_4} R_3 \xrightarrow{\delta_3} R_2 \xrightarrow{\delta_2} R_1 \xrightarrow{\delta_1} R_0 \xrightarrow{\delta_0} A/\mathfrak{r} \to 0,$$

where $R_n = \coprod_{0 \le x \le n} t(g^n_{x,n-x,i})A$ for $n \ge 0$, δ_0 is the natural epimorphism and for $n \ge 1$, $0 \le i \le m-1$ and $0 \le x \le n$,

$$\delta_n(t(g^n_{x,n-x,i})) = \begin{cases} (-1)^n t(g^{n-1}_{0,n-1,i})\phi_1^i(c^{(0,n-1)}) & \text{if } x = 0, \\ t(g^{n-1}_{x-1,n-x,i})\phi_1^i(b^{(x-1,n-x)}) + (-1)^n t(g^{n-1}_{x,n-1-x,i})\phi_1^i(c^{(x,n-1-x)}) & \\ & \text{if } 1 \le x \le n-1, \\ t(g^{n-1}_{n-1,0,i})\phi_1^i(b^{(n-1,0)}) & \text{if } x = n. \end{cases}$$

2.2. A minimal projective bimodule resolution of A

To construct a minimal projective bimodule resolution of A, we use the following lemma. The proof of this lemma is straightforward.

Lemma 2.2. *For any positive integers n, and any integers i and x with $0 \le i \le m-1$ and $0 \le x \le n$, we have the following:*

(1) *In the case $i = 0$,*

$$g^n_{x,n-x,0} = \begin{cases} \begin{cases} (-1)^n g^{n-1}_{0,n-1,0}\phi_1^0(c^{(0,n-1)}) \\ \quad = \begin{cases} (-1)^n \phi_1^0(c^{(0,0)}) g^{n-1}_{n-1,0,0} & \textit{if } n \equiv 0,1 \pmod 4, \\ -(-1)^n \phi_1^0(c^{(0,0)}) g^{n-1}_{n-1,0,0} & \textit{if } n \equiv 2,3 \pmod 4, \end{cases} \end{cases} \\ \quad \textit{if } x = 0, \\ \begin{cases} g^{n-1}_{x-1,n-x,0}\phi_1^0(b^{(x-1,n-x)}) + (-1)^n g^{n-1}_{x,n-1-x,0}\phi_1^0(c^{(x,n-1-x)}) \\ \quad = (-1)^{n-x}\phi_1^0(b^{(0,0)}) g^{n-1}_{x-1,n-x,1} \\ \qquad + \begin{cases} (-1)^{n-x}\phi_1^0(c^{(0,0)}) g^{n-1}_{n-1-x,x,0} & \textit{if } n \equiv 0,1 \pmod 4, \\ -(-1)^{n-x}\phi_1^0(c^{(0,0)}) g^{n-1}_{n-1-x,x,0} & \textit{if } n \equiv 2,3 \pmod 4, \end{cases} \end{cases} \\ \quad \textit{if } 1 \le x \le n-1, \\ g^{n-1}_{n-1,0,0}\phi_1^0(b^{(n-1,0)}) = \phi_1^0(b^{(0,0)}) g^{n-1}_{n-1,0,1} \quad \textit{if } x = n. \end{cases}$$

(2) *In the case* $1 \le i \le m-2$,

$$g^n_{x,n-1-x,i} = \begin{cases} (-1)^n g^{n-1}_{0,n-1,i}\phi_1^i(c^{(0,n-1)}) = (-1)^n\phi_1^i(c^{(0,0)})g^{n-1}_{0,n-1,i-1} & \textit{if } x = 0, \\ g^{n-1}_{x-1,n-x,i}\phi_1^i(b^{(x-1,n-x)}) + (-1)^n g^{n-1}_{x,n-1-x,i}\phi_1^i(c^{(x,n-1-x)}) \\ \quad = (-1)^{n-x}\phi_1^i(b^{(0,0)})g^{n-1}_{x-1,n-x,i+1} + (-1)^{n-x}\phi_1^i(c^{(0,0)})g^{n-1}_{x,n-1-x,i-1} \\ \qquad \textit{if } 1 \le x \le n-1, \\ g^{n-1}_{n-1,0,i}\phi_1^i(b^{(n-1,0)}) = \phi_1^i(b^{(0,0)})g^{n-1}_{n-1,0,i+1} & \textit{if } x = n. \end{cases}$$

(3) *In the case* $i = m-1$,

$$g^n_{x,n-x,m-1} = \begin{cases} (-1)^n g^{n-1}_{0,n-1,m-1}\phi_1^{m-1}(c^{(0,n-1)}) = (-1)^n\phi_1^{m-1}(c^{(0,0)})g^{n-1}_{0,n-1,m-2} \\ \quad \textit{if } x = 0, \\ \begin{cases} g^{n-1}_{x-1,n-x,m-1}\phi_1^{m-1}(b^{(x-1,n-x)}) \\ \quad +(-1)^n g^{n-1}_{x,n-1-x,m-1}\phi_1^{m-1}(c^{(x,n-1-x)}) \\ \qquad = \begin{cases} (-1)^{n-x}\phi_1^{m-1}(b^{(0,0)})g^{n-1}_{n-x,x-1,m-1} & \textit{if } n \equiv 0,1 (\operatorname{mod} 4), \\ -(-1)^{n-x}\phi_1^{m-1}(b^{(0,0)})g^{n-1}_{n-x,x-1,m-1} & \textit{if } n \equiv 2,3 (\operatorname{mod} 4), \end{cases} \\ \qquad\quad +(-1)^{n-x}\phi_1^{m-1}(c^{(0,0)})g^{n-1}_{x,n-1-x,m-2}, \end{cases} \\ \quad \textit{if } 1 \le x \le n-1, \\ \begin{cases} g^{n-1}_{n-1,0,m-1}\phi_1^{m-1}(b^{(n-1,0)}) \\ \quad = \begin{cases} \phi_1^{m-1}(b^{(0,0)})g^{n-1}_{0,n-1,m-1} & \textit{if } n \equiv 0,1 (\operatorname{mod} 4), \\ -\phi_1^{m-1}(b^{(0,0)})g^{n-1}_{0,n-1,m-1} & \textit{if } n \equiv 2,3 (\operatorname{mod} 4), \end{cases} \end{cases} \\ \quad \textit{if } x = n. \end{cases}$$

Now, for any integer $n \ge 0$, we define a left A^e-module

$$P_n := \coprod_{g \in \mathcal{G}^n} Ao(g) \otimes t(g)A.$$

Using the argument of [4], by Lemma 2.2, for $n \ge 1$, we define the A^e-homomorphism $\partial_n \colon P_n \to P_{n-1}$ as follows:

(1) In the case where $i=0$,

$$\partial_n(o(g^n_{x,n-x,0})\otimes t(g^n_{x,n-x,0}))=$$
$$\begin{cases} (-1)^n o(g^{n-1}_{0,n-1,0})\otimes \phi^0_1(c^{(0,n-1)}) \\ \quad + \begin{cases} \phi^0_1(c^{(0,0)})\otimes t(g^{n-1}_{n-1,0,0}) & \text{if } n\equiv 0,1 (\operatorname{mod} 4), \\ -\phi^0_1(c^{(0,0)})\otimes t(g^{n-1}_{n-1,0,0}) & \text{if } n\equiv 2,3 (\operatorname{mod} 4), \end{cases} & \text{if } x=0, \\ o(g^{n-1}_{x-1,n-x,0})\otimes \phi^0_1(b^{(x-1,n-x)}) + (-1)^n o(g^{n-1}_{x,n-1-x,0})\otimes \phi^0_1(c^{(x,n-1-x)}) \\ \quad +(-1)^x \phi^0_1(b^{(0,0)})\otimes t(g^{n-1}_{x-1,n-x,1}) \\ \qquad + \begin{cases} (-1)^x\phi^0_1(c^{(0,0)})\otimes t(g^{n-1}_{n-1-x,x,0}) & \text{if } n\equiv 0,1 (\operatorname{mod} 4), \\ -(-1)^x\phi^0_1(c^{(0,0)})\otimes t(g^{n-1}_{n-1-x,x,0}) & \text{if } n\equiv 2,3 (\operatorname{mod} 4), \end{cases} \\ \qquad\quad \text{if } 1\le x\le n-1, \\ o(g^{n-1}_{n-1,0,0})\otimes \phi^0_1(b^{(n-1,0)}) + (-1)^n\phi^0_1(b^{(0,0)})\otimes t(g^{n-1}_{n-1,0,1}) \quad \text{if } x=n. \end{cases}$$

(2) In the case where $1\le i\le m-2$,

$$\partial_n(o(g^n_{x,n-x,i})\otimes t(g^n_{x,n-x,i}))=$$
$$\begin{cases} (-1)^n o(g^{n-1}_{0,n-1,i})\otimes \phi^i_1(c^{(0,n-1)}) + \phi^i_1(c^{(0,0)})\otimes t(g^{n-1}_{0,n-1,i-1}) \quad \text{if } x=0, \\ o(g^{n-1}_{x-1,n-x,i})\otimes \phi^i_1(b^{(x-1,n-x)}) + (-1)^n o(g^{n-1}_{x,n-1-x,i})\otimes \phi^i_1(c^{(x,n-1-x)}) \\ \quad +(-1)^x\phi^i_1(b^{(0,0)})\otimes t(g^{n-1}_{x-1,n-x,i+1}) \\ \qquad +(-1)^x\phi^i_1(c^{(0,0)})\otimes t(g^{n-1}_{x,n-1-x,i-1}) \quad \text{if } 1\le x\le n-1, \\ o(g^{n-1}_{n-1,0,i})\otimes \phi^i_1(b^{(n-1,0)}) + (-1)^n\phi^i_1(b^{(0,0)})\otimes t(g^{n-1}_{n-1,0,i+1}) \quad \text{if } x=n. \end{cases}$$

(3) In the case where $i=m-1$,

$$\partial_n(o(g^n_{x,n-x,m-1})\otimes t(g^n_{x,n-x,m-1}))=$$
$$\begin{cases} (-1)^n o(g^{n-1}_{0,n-1,m-1})\otimes \phi^{m-1}_1(c^{(0,n-1)}) + \phi^{m-1}_1(c^{(0,0)})\otimes t(g^{n-1}_{0,n-1,m-2}) \\ \quad \text{if } x=0, \\ o(g^{n-1}_{x-1,n-x,m-1})\otimes \phi^{m-1}_1(b^{(x-1,n-x)}) \\ \quad +(-1)^n o(g^{n-1}_{x,n-1-x,m-1})\otimes \phi^{m-1}_1(c^{(x,n-1-x)}) \\ \qquad + \begin{cases} (-1)^x\phi^{m-1}_1(b^{(0,0)})\otimes t(g^{n-1}_{n-x,x-1,m-1}) & \text{if } n\equiv 0,1 (\operatorname{mod} 4), \\ -(-1)^x\phi^{m-1}_1(b^{(0,0)})\otimes t(g^{n-1}_{n-x,x-1,m-1}) & \text{if } n\equiv 2,3 (\operatorname{mod} 4), \end{cases} \\ \qquad\quad +(-1)^x\phi^{m-1}_1(c^{(0,0)})\otimes t(g^{n-1}_{x,n-1-x,m-2}), \\ \qquad\qquad \text{if } 1\le x\le n-1, \\ o(g^{n-1}_{n-1,0,m-1})\otimes \phi^{m-1}_1(b^{(n-1,0)}) \\ \quad + \begin{cases} (-1)^n\phi^{m-1}_1(b^{(0,0)})\otimes t(g^{n-1}_{0,n-1,m-1}) & \text{if } n\equiv 0,1 (\operatorname{mod} 4), \\ -(-1)^n\phi^{m-1}_1(b^{(0,0)})\otimes t(g^{n-1}_{0,n-1,m-1}) & \text{if } n\equiv 2,3 (\operatorname{mod} 4), \end{cases} \\ \qquad \text{if } x=n. \end{cases}$$

Then we have a sequence

$$(P_\bullet, \partial_\bullet) : \cdots \to P_n \xrightarrow{\partial_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\pi} A \to 0,$$

where π is the multiplication map. The following result is now immediate from [4].

Theorem 2.3. *The sequence $(P_\bullet, \partial_\bullet)$ is a projective bimodule resolution of the left A^e-module A.*

§3. The dimension of the Hochschild cohomology groups for $m \geq 3$

In this section, we determine the dimension of the Hochschild cohomology groups of A by using the minimal projective A^e-resolution given in Theorem 2.3. Throughout this section, we keep the notation from sections 1 and 2.

By setting $P_n^* := \mathrm{Hom}_{A^e}(P_n, A)$ and $\partial_n^* = \mathrm{Hom}_{A^e}(\partial_n, A)$ for $n \geq 0$, we get the following complex.

$$(P_\bullet^*, \partial_\bullet^*) : \quad 0 \to P_0^* \xrightarrow{\partial_1^*} P_1^* \xrightarrow{\partial_2^*} \cdots \xrightarrow{\partial_{n-1}^*} P_{n-1}^* \xrightarrow{\partial_n^*} P_n^* \xrightarrow{\partial_{n+1}^*} \cdots .$$

Then, for $n \geq 0$, the n-th Hochschild cohomology group $\mathrm{HH}^n(A)$ of A is given by $\mathrm{HH}^n(A) := \mathrm{Ext}^n_{A^e}(A, A) = \mathrm{Ker}\, \partial_{n+1}^* / \mathrm{Im}\, \partial_n^*$.

In the rest of the paper, for an integer $n \geq 0$, we set $p := Q(n)$ and $t := \overline{n}$, that is, p and t are unique integers such that $n = pm + t$ with $p \geq 0$ and $0 \leq t \leq m - 1$.

3.1. The dimension of the Hochschild cohomology groups of A

For an element f in P_n^*, $f(o(g^n_{x,n-x,i}) \otimes t(g^n_{x,n-x,i}))$ is in $o(g^n_{x,n-x,i})At(g^n_{x,n-x,i})$. By definition of $\mathcal{G}^n$ and ϕ_0^i, we have

$$o(g^n_{x,n-x,i})At(g^n_{x,n-x,i}) = \begin{cases} e_iAe_{\overline{2x-n+i}} & \text{if } Q(2x-n+i) \in 2\mathbb{Z}, \\ e_iAe_{m-1-\overline{2x-n+i}} & \text{if } Q(2x-n+i) \notin 2\mathbb{Z}. \end{cases}$$

Moreover, it follows from the definition of A that, for vertices i, j, e_iAe_j has the following basis elements.

$$\begin{cases} e_0, a_0, a_1\overline{a}_1 & \text{if } i = j = 0, \\ e_{m-1}, a_m, a_m^2 & \text{if } i = j = m-1, \\ e_i, a_{i+1}\overline{a}_{i+1} & \text{if } i = j \neq 0, m-1, \\ a_{i+1} & \text{if } j - i = 1, \\ \overline{a}_i & \text{if } j - i = -1, \end{cases}$$

and, in the other cases, e_iAe_j is zero. So we have

$$\dim_k e_iAe_j = \begin{cases} 3 & \text{if } i = j = 0, m-1, \\ 2 & \text{if } i = j \neq 0, m-1, \\ 1 & \text{if } i - j = \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $f(o(g^n_{x,n-x,i}) \otimes t(g^n_{x,n-x,i}))$ can be written as a linear combination of these basis elements in $o(g^n_{x,n-x,i})At(g^n_{x,n-x,i})$. We need to find the conditions on the coefficients in this linear combination when f is in $\operatorname{Ker} \partial^*_{pm+t+1}$. To do this, we consider the four cases (i) p and t are even, (ii) p is even, t is odd, (iii) p is odd, t is even, (iv) p and t are odd.

Lemma 3.1. *We have the image of $o(g^n_{x,n-x,i}) \otimes t(g^n_{x,n-x,i})$ by $f \in P^*_n$ as follows:*

(1) *In the case where p and t are even, we have*

$$f(o(g^n_{x,n-x,i}) \otimes t(g^n_{x,n-x,i})) =$$
$$\begin{cases} \begin{cases} \sigma_{0,\alpha}e_0 + \tau_{0,\alpha}a_0 + \lambda_{0,\alpha}a_1\overline{a}_1 & \textit{if } i = 0, \\ \sigma_{i,\alpha}e_i + \lambda_{i,\alpha}a_{i+1}\overline{a}_{i+1} & \textit{if } 1 \leq i \leq m-2, \\ \sigma_{m-1,\alpha}e_{m-1} + \tau_{m-1,\alpha}a_m + \lambda_{m-1,\alpha}a_m^2 & \textit{if } i = m-1, \end{cases} \\ \quad \textit{if } x = (p-\alpha)m + t/2 \textit{ for } 0 \leq \alpha \leq p, \\ \begin{cases} \mu_{i,0}a_{i+1} & \textit{if } m-1-t/2 \leq i \leq m-2, \beta = 0, \\ \mu_{i,\beta}a_{i+1} & \textit{if } 0 \leq i \leq m-2, 1 \leq \beta \leq p, \\ \mu_{i,p+1}a_{i+1} & \textit{if } 0 \leq i \leq t/2-1, \beta = p+1, \end{cases} \\ \quad \textit{if } x = (p-\beta+1)m + t/2 - i - 1, \\ \begin{cases} \iota_{i,0}\overline{a}_i & \textit{if } m - t/2 \leq i \leq m-1, \gamma = 0, \\ \iota_{i,\gamma}\overline{a}_i & \textit{if } 1 \leq i \leq m-1, 1 \leq \gamma \leq p, \\ \iota_{i,p+1}\overline{a}_i & \textit{if } 1 \leq i \leq t/2, \gamma = p+1, \end{cases} \\ \quad \textit{if } x = (p-\gamma+1)m + t/2 - i, \end{cases}$$

where all the coefficients $\sigma_{i,\alpha}$, $\tau_{i,\alpha}$, $\lambda_{i,\alpha}$, $\mu_{i,\beta}$, and $\iota_{i,\gamma}$ are in k.

(2) *In the case where p is even and t is odd, we have*

$$f(o(g^n_{x,n-x,i}) \otimes t(g^n_{x,n-x,i})) =$$
$$\begin{cases} \begin{cases} \sigma_{0,\alpha}e_0 + \tau_{0,\alpha}a_0 + \lambda_{0,\alpha}a_1\overline{a}_1 & \text{if } i = 0, \\ \sigma_{i,\alpha}e_i + \lambda_{i,\alpha}a_{i+1}\overline{a}_{i+1} & \text{if } 1 \le i \le m-2, \\ \sigma_{m-1,\alpha}e_{m-1} + \tau_{m-1,\alpha}a_m + \lambda_{m-1,\alpha}a_m^2 & \text{if } i = m-1, \end{cases} \\ \quad \text{if } x = (p-\alpha+1)m + (t-1)/2 - i \text{ for } 1 \le \alpha \le p, \\ \begin{cases} \sigma_{i,0}e_i + \lambda_{i,0}a_{i+1}\overline{a}_{i+1} & \text{if } m-(t+1)/2 \le i \le m-2, \\ \sigma_{m-1,0}e_{m-1} + \tau_{m-1,0}a_m + \lambda_{m-1,0}a_m^2 & \text{if } i = m-1, \end{cases} \\ \quad \text{if } x = (p+1)m + (t-1)/2 - i, \\ \begin{cases} \sigma_{0,p+1}e_0 + \tau_{0,p+1}a_0 + \lambda_{0,p+1}a_1\overline{a}_1 & \text{if } i = 0, \\ \sigma_{i,p+1}e_i + \lambda_{i,p+1}a_{i+1}\overline{a}_{i+1} & \text{if } 1 \le i \le (t-1)/2, \end{cases} \\ \quad \text{if } x = (t-1)/2 - i, \\ \mu_{i,\beta}a_{i+1} \quad \text{if } 0 \le i \le m-2, 0 \le \beta \le p, x = (p-\beta)m + (t+1)/2, \\ \iota_{i,\gamma}\overline{a}_i \quad \text{if } 1 \le i \le m-1, 0 \le \gamma \le p, x = (p-\gamma)m + (t-1)/2, \end{cases}$$

where all the coefficients $\sigma_{i,\alpha}$, $\tau_{i,\alpha}$, $\lambda_{i,\alpha}$, $\mu_{i,\beta}$, and $\iota_{i,\gamma}$ are in k.

(3) *In the case where p is odd and m and t are even, and m, p and t are odd, we have*

$$f(o(g^n_{x,n-x,i}) \otimes t(g^n_{x,n-x,i})) =$$
$$\begin{cases} \begin{cases} \sigma_{0,\alpha}e_0 + \tau_{0,\alpha}a_0 + \lambda_{0,\alpha}a_1\overline{a}_1 & \text{if } i = 0, \\ \sigma_{i,\alpha}e_i + \lambda_{i,\alpha}a_{i+1}\overline{a}_{i+1} & \text{if } 1 \le i \le m-2, \\ \sigma_{m-1,\alpha}e_{m-1} + \tau_{m-1,\alpha}a_m + \lambda_{m-1,\alpha}a_m^2 & \text{if } i = m-1, \end{cases} \\ \quad \text{if } x = (p-\alpha)m + (m+t)/2 \text{ for } 1 \le \alpha \le p, \\ \begin{cases} \mu_{i,0}a_{i+1} & \text{if } (m-t)/2 - 1 \le i \le m-2, \beta = 0, \\ \mu_{i,\beta}a_{i+1} & \text{if } 0 \le i \le m-2, 1 \le \beta \le p-1, \\ \mu_{i,p}a_{i+1} & \text{if } 0 \le i \le (m+t)/2 - 1, \beta = p, \end{cases} \\ \quad \text{if } x = (p-\beta)m + (m+t)/2 - i - 1, \\ \begin{cases} \iota_{i,0}\overline{a}_i & \text{if } (m-t)/2 \le i \le m-1, \gamma = 0, \\ \iota_{i,\gamma}\overline{a}_i & \text{if } 1 \le i \le m-1, 1 \le \gamma \le p-1, \\ \iota_{i,p}\overline{a}_i & \text{if } 1 \le i \le (m+t)/2, \gamma = p, \end{cases} \\ \quad \text{if } x = (p-\gamma)m + (m+t)/2 - i, \end{cases}$$

where all the coefficients $\sigma_{i,\alpha}$, $\tau_{i,\alpha}$, $\lambda_{i,\alpha}$, $\mu_{i,\beta}$, and $\iota_{i,\gamma}$ are in k.

(4) *In the case where m is even and p and t are odd, and m and p are odd*

and t is even, we have

$$
f(o(g^n_{x,n-x,i}) \otimes t(g^n_{x,n-x,i})) =
\begin{cases}
\begin{cases}
\sigma_{0,\alpha}e_0 + \tau_{0,\alpha}a_0 + \lambda_{0,\alpha}a_1\overline{a}_1 & \text{if } i = 0,\\
\sigma_{i,\alpha}e_i + \lambda_{i,\alpha}a_{i+1}\overline{a}_{i+1} & \text{if } 1 \le i \le m-2,\\
\sigma_{m-1,\alpha}e_{m-1} + \tau_{m-1,\alpha}a_m + \lambda_{m-1,\alpha}a_m^2 & \text{if } i = m-1,
\end{cases}\\
\quad \text{if } x = (p-\alpha)m + (m+t-1)/2 - i \text{ for } 1 \le \alpha \le p-1,\\
\begin{cases}
\sigma_{0,0}e_0 + \tau_{0,0}a_0 + \lambda_{0,0}a_1\overline{a}_1 & \text{if } i = 0, t = m-1,\\
\sigma_{i,0}e_i + \lambda_{i,0}a_{i+1}\overline{a}_{i+1} & \text{if } (m-1-t)/2 \le i \le m-2,\\
\sigma_{m-1,0}e_{m-1} + \tau_{m-1,0}a_m + \lambda_{m-1,0}a_m^2 & \text{if } i = m-1,
\end{cases}\\
\quad \text{if } x = pm + (m+t-1)/2 - i,\\
\begin{cases}
\sigma_{0,p}e_0 + \tau_{0,p}a_0 + \lambda_{0,p}a_1\overline{a}_1 & \text{if } i = 0,\\
\sigma_{i,p}e_i + \lambda_{i,p}a_{i+1}\overline{a}_{i+1} & \text{if } 1 \le i \le (m-1+t)/2,\\
\sigma_{m-1,p}e_{m-1} + \tau_{m-1,p}a_m + \lambda_{m-1,p}a_m^2 & \text{if } i = m-1, t = m-1,
\end{cases}\\
\quad \text{if } x = (m+t-1)/2 - i,\\
\begin{cases}
\mu_{i,\beta}a_{i+1} & \text{if } 0 \le i \le m-2, 1 \le \beta \le p,\\
\mu_{i,p+1}a_{i+1} & \text{if } 0 \le i \le m-2, \beta = p+1, t = m-1,
\end{cases}\\
\quad \text{if } x = (p-\beta+1)m - (m-t-1)/2,\\
\begin{cases}
\iota_{i,0}\overline{a}_i & \text{if } 1 \le i \le m-1, \gamma = 0, t = m-1,\\
\iota_{i,\gamma}\overline{a}_i & \text{if } 1 \le i \le m-1, 1 \le \gamma \le p,
\end{cases}\\
\quad \text{if } x = (p-\gamma+1)m - (m-t+1)/2,
\end{cases}
$$

where all the coefficients $\sigma_{i,\alpha}$, $\tau_{i,\alpha}$, $\lambda_{i,\alpha}$, $\mu_{i,\beta}$, and $\iota_{i,\gamma}$ are in k.

By Lemma 3.1, we obtain immediately the following corollary.

Corollary 3.2. *We have the dimension of $P_n^* = \operatorname{Hom}_{A^e}(P_n, A)$ for $m \ge 3$ as follows*:

$$
\dim_k P_n^* = \begin{cases}
4mp + 2m + 2t + 2 & \text{if } p \text{ is even},\\
4mp + 2t + 2 & \text{if } p \text{ is odd and } t \ne m-1,\\
4m(p+1) & \text{if } p \text{ is odd and } t = m-1.
\end{cases}
$$

Next, using Theorem 2.3 and Lemma 3.1, we determine the $\operatorname{Im} \partial^*_{n+1}$. We suppose that m is even. In the case where m is odd, we have the similar results.

Lemma 3.3. *With the notation of Lemma 3.1, if m is even, then we have $\partial^*_{n+1}(f)(o(g^{n+1}_{x,n+1-x,i}) \otimes t(g^{n+1}_{x,n+1-x,i}))$ in the following cases.*

(1) *In the case where p and t are even, we have* $\partial^*_{n+1}(f)(o(g^{n+1}_{x,n+1-x,i}) \otimes t(g^{n+1}_{x,n+1-x,i}))$ *as follows*:

(a) *If* $x = (p-\alpha)m + t/2$ *for* $0 \le \alpha \le p-1$,

$$\begin{cases} (-\sigma_{0,\alpha} + \sigma_{0,p-\alpha})a_0 + (\mu_{0,\alpha+1} \\ \quad -\tau_{0,\alpha} + (-1)^{t/2}\iota_{1,\alpha+1} + \tau_{0,p-\alpha})a_1\overline{a}_1 & \text{if } i = 0, \\ (-\sigma_{i,\alpha} + (-1)^{t/2}\sigma_{i-1,\alpha})\overline{a}_i & \text{if } 1 \le i \le m-1. \end{cases}$$

(b) *If* $x = t/2$,

$$\begin{cases} (-\sigma_{0,p} + \sigma_{0,0})a_0 + (-\tau_{0,p} + \tau_{0,0})a_1\overline{a}_1 & \text{if } i = 0, t = 0, \\ (-\sigma_{0,p} + \sigma_{0,0})a_0 + (\mu_{0,p+1} \\ \quad -\tau_{0,p} + (-1)^{t/2}\iota_{1,p+1} + \tau_{0,0})a_1\overline{a}_1 & \text{if } i = 0, t \ge 2, \\ (-\sigma_{i,p} + (-1)^{t/2}\sigma_{i-1,p})\overline{a}_i & \text{if } 1 \le i \le m-1. \end{cases}$$

(c) *If* $x = (p-\alpha)m + t/2 + 1$ *for* $1 \le \alpha \le p$,

$$\begin{cases} (\sigma_{i,\alpha} - (-1)^{t/2}\sigma_{i+1,\alpha})a_{i+1} & \text{for } 0 \le i \le m-2, \\ (\sigma_{m-1,\alpha} - \sigma_{m-1,p-\alpha})a_m + (\tau_{m-1,\alpha} \\ \quad -\iota_{m-1,\alpha} - \tau_{m-1,p-\alpha} - (-1)^{t/2}\mu_{m-2,\alpha})a_m^2 & \text{if } i = m-1. \end{cases}$$

(d) *If* $x = pm + t/2 + 1$,

$$\begin{cases} (\sigma_{i,0} - (-1)^{t/2}\sigma_{i+1,0})a_{i+1} & \text{if } 0 \le i \le m-2, \\ (\sigma_{m-1,0} - \sigma_{m-1,p})a_m + (\tau_{m-1,0} - \tau_{m-1,p})a_m^2 & \text{if } i = m-1, t = 0, \\ (\sigma_{m-1,0} - \sigma_{m-1,p})a_m + (\tau_{m-1,0} \\ \quad -\iota_{m-1,0} - \tau_{m-1,p} - (-1)^{t/2}\mu_{m-2,0})a_m^2 & \text{if } i = m-1, t \ge 2. \end{cases}$$

(e) *If* $x = (p-\beta+1)m + t/2 - i$ *for* $0 \le \beta \le p+1$,

$$\begin{cases} (\mu_{i,0} - \iota_{i,0} + (-1)^{t/2+i}\iota_{i+1,0} + (-1)^{t/2+i}\mu_{i-1,0})a_{i+1}\overline{a}_{i+1} \\ \quad \text{if } m - t/2 \le i \le m-2, \beta = 0, \\ (\mu_{i,\beta} - \iota_{i,\beta} + (-1)^{t/2+i}\iota_{i+1,\beta} + (-1)^{t/2+i}\mu_{i-1,\beta})a_{i+1}\overline{a}_{i+1} \\ \quad \text{if } 1 \le i \le m-2, 1 \le \beta \le p, \\ (\mu_{i,p+1} - \iota_{i,p+1} + (-1)^{t/2+i}\iota_{i+1,p+1} + (-1)^{t/2+i}\mu_{i-1,p+1})a_{i+1}\overline{a}_{i+1} \\ \quad \text{if } 1 \le i \le t/2 - 1, \beta = p+1. \end{cases}$$

(f) *If* $x = n+1$ *and* $i = m-1-t/2$, $(\mu_{i,0} - \iota_{i+1,0})a_{i+1}\overline{a}_{i+1}$.

(g) *If* $x = 0$ *and* $i = t/2$, $(-\iota_{i,p+1} + \mu_{i-1,p+1})a_{i+1}\overline{a}_{i+1}$.

(2) *In the case where p is even and t is odd, we have $\partial^*_{n+1}(f)(o(g^{n+1}_{x,n+1-x,i})\otimes t(g^{n+1}_{x,n+1-x,i}))$ as follows*:

(a) *If $x=(p-\alpha+1)m+(t+1)/2-i$ for $1\le\alpha\le p$,*

$$\begin{cases}(\sigma_{0,\alpha}+\sigma_{0,p-\alpha+2})a_0+(\mu_{0,\alpha-1} \\ \quad +\tau_{0,\alpha}+(-1)^{(t+1)/2}\iota_{1,\alpha-1}+\tau_{0,p-\alpha+2})a_1\overline{a}_1 & \text{if } i=0, \\ (\sigma_{i,\alpha}+(-1)^{(t+1)/2+i}\sigma_{i-1,\alpha})\overline{a}_i & \text{if } 1\le i\le m-1.\end{cases}$$

(b) *If $x=(t+1)/2-i$,*

$$\begin{cases}(\sigma_{0,p+1}+\sigma_{0,1})a_0+(\mu_{0,p}+\tau_{0,p+1} \\ \quad +(-1)^{(t+1)/2}\iota_{1,p}+\tau_{0,1})a_1\overline{a}_1 & \text{if } i=0, \\ (\sigma_{i,p+1}+(-1)^{(t+1)/2+i}\sigma_{i-1,p+1})\overline{a}_i & \text{if } 1\le i\le (t-1)/2, \\ \sigma_{i-1,p+1}\overline{a}_i & \text{if } i=(t+1)/2.\end{cases}$$

(c) *If $x=(p+1)m+(t+1)/2-i$,*

$$\begin{cases}\sigma_{i,0}\overline{a}_i & \text{if } i=m-(t+1)/2, \\ (\sigma_{i,0}+(-1)^{(t+1)/2-i}\sigma_{i-1,0})\overline{a}_i & \text{if } m-(t-1)/2\le i\le m-2, \\ \sigma_{m-1,0}\overline{a}_{m-1} & \text{if } i=m-1 \text{ and } t=1, \\ (\sigma_{m-1,0}-(-1)^{(t+1)/2}\sigma_{m-2,0})\overline{a}_{m-1} & \text{if } i=m-1 \text{ and } t\ge 3.\end{cases}$$

(d) *If $x=(p-\alpha+1)m+(t-1)/2-i$ for $1\le\alpha\le p$,*

$$\begin{cases}(\sigma_{i,\alpha}+(-1)^{(t-1)/2+i}\sigma_{i+1,\alpha})a_{i+1} & \text{if } 0\le i\le m-2, \\ (\sigma_{m-1,\alpha}+\sigma_{m-1,p-\alpha})a_m+(\tau_{m-1,\alpha} \\ \quad +\iota_{m-1,\alpha}+\tau_{m-1,p-\alpha}+(-1)^{(t+1)/2}\mu_{m-2,\alpha})a_m^2 & \text{if } i=m-1.\end{cases}$$

(e) *If $x=(p+1)m+(t-1)/2-i$,*

$$\begin{cases}\sigma_{i+1,0}a_{i+1} & \text{if } i=m-(t+3)/2, \\ (\sigma_{i,0}+(-1)^{(t-1)/2+i}\sigma_{i+1,0})a_{i+1} & \text{if } m-(t+1)/2\le i\le m-2, \\ (\sigma_{m-1,0}+\sigma_{m-1,p})a_m \\ \quad +(\tau_{m-1,0}+\iota_{m-1,0}+\tau_{m-1,p} \\ \qquad +(-1)^{(t+1)/2}\mu_{m-2,0})a_m^2 & \text{if } i=m-1.\end{cases}$$

(f) *If $x=(t-1)/2-i$,*

$$\begin{cases}\sigma_{0,p+1}a_1 & \text{if } i=0 \text{ and } t=1, \\ (\sigma_{0,p+1}+(-1)^{(t-1)/2}\sigma_{1,p+1}a_1 & \text{if } i=0 \text{ and } t\ge 3, \\ (\sigma_{i,p+1}+(-1)^{(t-1)/2+i}\sigma_{i+1,p+1})a_{i+1} & \text{if } 1\le i\le (t-3)/2, \\ \sigma_{i,p+1}a_{i+1} & \text{if } i=(t-1)/2.\end{cases}$$

(g) *If* $x = (p-\beta)m + (t+1)/2$ *for* $0 \le \beta \le p$,

$$(\mu_{i,\beta} + \iota_{i,\beta} + (-1)^{(t+1)/2}\iota_{i+1,\beta} + (-1)^{(t+1)/2}\mu_{i-1,\beta})a_{i+1}\overline{a}_{i+1}.$$

for $1 \le i \le m-2$,

In the case where p is odd, we have the similar results to (1) *and* (2).

In this way, we have the dimension of the Kernel of ∂^*_{n+1}.

Proposition 3.4. (1) *In the case where* m *is even and* $\operatorname{char} k \neq 2$, *we have*

$$\dim_k \operatorname{Ker} \partial^*_{pm+t+1} = \begin{cases} (2p+1)m + p/2 + t + 3 & \textit{if } p \textit{ and } t \textit{ are even,} \\ (2p+1)m + p/2 + t + 1 & \textit{if } p \textit{ is even and } t \textit{ is odd,} \\ 2pm + (p-1)/2 + t + 3 & \textit{if } p \textit{ is odd and } t \textit{ is even,} \\ 2pm + (p-1)/2 + t + 1 & \textit{if } p \textit{ is odd and } t \textit{ is odd}(\neq m-1), \\ (2p+1)m + (p+1)/2 & \textit{if } p \textit{ is odd and } t = m-1. \end{cases}$$

(2) *In the case where* m *is even and* $\operatorname{char} k = 2$, *we have*

$$\dim_k \operatorname{Ker} \partial^*_{pm+t+1} = \begin{cases} (2p+1)m + p/2 + t + 3 & \textit{if } p \textit{ and } t \textit{ are even,} \\ (2p+1)m + p/2 + t + 2 & \textit{if } p \textit{ is even and } t \textit{ is odd,} \\ 2pm + (p-1)/2 + t + 3 & \textit{if } p \textit{ is odd and } t \textit{ is even,} \\ 2pm + (p-1)/2 + t + 2 & \textit{if } p \textit{ is odd and } t \textit{ is odd}(\neq m-1), \\ (2p+1)m + (p+1)/2 + 1 & \textit{if } p \textit{ is odd and } t = m-1. \end{cases}$$

(3) *In the cases where* m *is odd and* $\operatorname{char} k \neq 2$, *we have*

$$\dim_k \operatorname{Ker} \partial^*_{pm+t+1} = \begin{cases} (2p+1)m + p/2 + t + 3 & \textit{if } p \textit{ and } t \textit{ are even,} \\ (2p+1)m + p/2 + t + 1 & \textit{if } p \textit{ is even and } t \textit{ is odd,} \\ 2pm + (p-1)/2 + t + 3 & \textit{if } p \textit{ and } t \textit{ are odd,} \\ 2pm + (p-1)/2 + t + 1 & \textit{if } p \textit{ is odd and } t \textit{ is even}(\neq m-1), \\ (2p+1)m + (p-1)/2 + 1 & \textit{if } p \textit{ is odd and } t = m-1. \end{cases}$$

(4) *In the case where* m *is odd and* $\operatorname{char} k = 2$, *we have*

$$\dim_k \operatorname{Ker} \partial^*_{pm+t+1} = \begin{cases} (2p+1)m + p/2 + t + 3 & \text{if } p \text{ and } t \text{ are even,} \\ (2p+1)m + p/2 + t + 2 & \text{if } p \text{ is even and } t \text{ is odd,} \\ 2pm + (p-1)/2 + t + 3 & \text{if } p \text{ and } t \text{ are odd,} \\ 2pm + (p-1)/2 + t + 2 & \text{if } p \text{ is odd and } t \text{ is even}(\neq m-1), \\ (2p+1)m + (p-1)/2 + 2 & \text{if } p \text{ is odd and } t = m-1. \end{cases}$$

Proof. We consider the case where m, p and t are even and $\operatorname{char} k \neq 2$. In the other cases, we prove this theorem by the same method. If m, p and t are even, $\operatorname{char} k \neq 2$ and $f \in \operatorname{Ker} \partial^*_{pm+t+1}$, then we have the following conditions.

$\sigma_{0,\alpha} = \sigma_{0,p-\alpha}$ for $0 \leq \alpha \leq p/2$,

$\sigma_{0,\alpha} = \sigma_{2l,\alpha} = (-1)^{t/2}\sigma_{2l+1,\alpha}$ for $0 \leq \alpha \leq p, 0 \leq l \leq m/2 - 1$,

$\iota_{1,\beta} = (-1)^{t/2}\tau_{0,\beta-1} - (-1)^{t/2}\tau_{0,p-\beta+1} - (-1)^{t/2}\mu_{0,\beta}$ for $1 \leq \beta \leq p$,

$\iota_{i+1,\beta} = (-1)^{t/2+i}\iota_{i,\beta} - (-1)^{t/2+i}\mu_{i,\beta} - \mu_{i-1,\beta}$ for $1 \leq i \leq m-2, 1 \leq \beta \leq p$,

$\iota_{m-1,\beta} = \tau_{m-1,\beta} - \tau_{m-1,p-\beta} - (-1)^{t/2}\mu_{m-2,\beta}$ for $1 \leq \beta \leq p$,

If $t = 0$, $\begin{cases} \tau_{0,0} = \tau_{0,p}, \\ \tau_{m-1,0} = \tau_{m-1,p}, \end{cases}$

If $t \geq 2$, $\begin{cases} \iota_{1,p+1} = (-1)^{t/2}\tau_{0,p} - (-1)^{t/2}\tau_{0,0} - (-1)^{t/2}\mu_{0,p+1}, \\ \iota_{t/2,p+1} = \mu_{t/2-1,p+1}, \\ \iota_{m-t/2,0} = \mu_{m-1-t/2,0}, \\ \iota_{m-1,0} = \tau_{m-1,0} - \tau_{0,p} - (-1)^{t/2}\mu_{m-2,0}, \end{cases}$

If $t \geq 4$,

$$\begin{cases} \iota_{i+1,p+1} = (-1)^{t/2+i}\iota_{0,p+1} - (-1)^{t/2+i}\mu_{i,p+1} - \mu_{i-1,p+1} & \text{for } 1 \leq i \leq t/2 - 1, \\ \iota_{i+1,0} = (-1)^{t/2+i}\iota_{i,0} - (-1)^{t/2+i}\mu_{i,0} - \mu_{i-1,0} & \text{for } m - t/2 \leq i \leq m-2. \end{cases}$$

Therefore, $\sigma_{i,\alpha}$, $\sigma_{0,p-\alpha'}$, $\tau_{0,p-\alpha'}$, $\tau_{m-1,p-\alpha'}$, $\iota_{i,\beta}$, $\iota_{i',0}$ and $\iota_{i'',p+1}$ are determined by $\sigma_{0,\alpha'}$, $\sigma_{0,t/2}$, $\tau_{0,\alpha'}$, $\tau_{0,t/2}$, $\tau_{m-1,\alpha'}$, $\tau_{m-1,t/2}$, $\mu_{i,\beta}$, $\mu_{i''-1,p+1}$, $\mu_{i'-1,0}$, for $1 \leq i \leq m-1$, $m-t/2 \leq i' \leq m-1$, $1 \leq i'' \leq t/2$, $0 \leq \alpha \leq p$, $0 \leq \alpha' \leq p/2-1$ and $1 \leq \beta \leq p$. Finally $\lambda_{i,\alpha}$ are arbitrary for $0 \leq i \leq m-1$ and $0 \leq \alpha \leq p$. Therefore, in this case the dimension of $\operatorname{Ker} \partial^*_{pm+t+1}$ is $(2p+1)m + p/2 + t + 3$. □

Finally, we give the dimension of the n-th Hochschild cohomology group $\mathrm{HH}^n(A) = \operatorname{Ker} \partial^*_{n+1}/\operatorname{Im} \partial^*_n$ of A for $n \geq 0$. Using Theorem 3.4 and the equation

$$\begin{aligned}\dim_k \mathrm{HH}^n(A) &= \dim_k \operatorname{Ker}\partial_{n+1}^* - \dim_k \operatorname{Im}\partial_n^* \\ &= \dim_k \operatorname{Ker}\partial_{n+1}^* - \dim_k P_{n-1}^* + \dim_k \operatorname{Ker}\partial_n^*,\end{aligned}$$

we obtain the dimension of $\mathrm{HH}^n(A)$ of A for $n \geq 0$ as follows.

Theorem 3.5. *In the case $m \geq 3$, we have $\dim_k \mathrm{HH}^0(A) = m + 3$ and, for $pm + t \geq 1$,*

$$\dim_k \mathrm{HH}^{pm+t}(A) = p + \begin{cases} 3 & \textit{if } p \textit{ is even and } \operatorname{char} k \neq 2, \\ 2 & \textit{if } p \textit{ is odd, } t \neq m-1 \textit{ and } \operatorname{char} k \neq 2, \\ 3 & \textit{if } p \textit{ is odd, } t = m-1 \textit{ and } \operatorname{char} k \neq 2, \\ 4 & \textit{if } p \textit{ is even and } \operatorname{char} k = 2, \\ 3 & \textit{if } p \textit{ is odd, } t \neq m-1 \textit{ and } \operatorname{char} k = 2, \\ 4 & \textit{if } p \textit{ is odd, } t = m-1 \textit{ and } \operatorname{char} k = 2. \end{cases}$$

Remark 3.6. In the case $m = 2$, by Theorem 3.5, we have the dimension of the Hochschild cohomology groups of A given in [6].

3.2. A basis of the Hochschild cohomology groups of A

Using Lemmas 3.3 and Theorem 3.5, we obtain a k-basis of $\mathrm{HH}^n(A)$ for $n \geq 0$.

Proposition 3.7. *Suppose that $m \geq 3$. Then the following elements form a k-basis of the center $Z(A) = \mathrm{HH}^0(A) = \operatorname{Ker}\partial_1^*$ of A.*

$$\sum_{i=0}^{m-1} e_i,\ a_0,\ a_m,\ a_j\overline{a}_j \quad \textit{for } 1 \leq j \leq m.$$

Proposition 3.8. *Suppose $m \geq 3$ and m is even. For each $n = pm + t \geq 1$, the following elements form a k-basis of $\mathrm{HH}^{pm+t}(A)$.*

(1) *In the case where p and t are even, we have a k-basis of $\mathrm{HH}^{pm+t}(A)$ as follows*:

(a) *If $x_1 = (p-\alpha)m + t/2$, $x_2 = \alpha m + t/2$,*

$$\chi_{n,\alpha} : \begin{cases} e_i \otimes \phi_0^i(x_1, n-x_1) \mapsto \begin{cases} e_i & \textit{if } i \textit{ is even}, \\ (-1)^{t/2} e_i & \textit{if } i \textit{ is odd}, \end{cases} \\ e_i \otimes \phi_0^i(x_2, n-x_2) \mapsto \begin{cases} e_i & \textit{if } i \textit{ is even}, \\ (-1)^{t/2} e_i & \textit{if } i \textit{ is odd}, \end{cases} \end{cases}$$

for $0 \leq i \leq m-1$, $0 \leq \alpha \leq p/2$.

(b) *If* $x = pm/2 + t/2$, $\pi_{n,1} : e_0 \otimes \phi_0^0(x, n-x) \mapsto a_0$.

(c) *If* $x = pm/2 + t/2$, $\pi_{n,2} : e_{m-1} \otimes \phi_0^{m-1}(x, n-x) \mapsto a_m$.

(d) *If* $x = (p-\alpha)m + t/2$,
$F_{n,\alpha} : e_0 \otimes \phi_0^0(x, n-x) \mapsto a_1\overline{a}_1 \quad$ *for* $0 \leq \alpha \leq p/2 - 1$.

(e) *If* $x = pm/2 + t/2$, $\operatorname{char} k = 2$, $F_{n,p/2} : e_0 \otimes \phi_0^0(x, n-x) \mapsto a_1\overline{a}_1$.

(2) *In the case where* p *is even and* t *is odd, we have a* k*-basis of* $\mathrm{HH}^{pm+t}(A)$ *as follows*:

(a) *If* $x_1 = (p-\alpha)m + (t-1)/2$ *and* $x_2 = \alpha m + (t-1)/2$,

$$\mu_{n,\alpha} : \begin{cases} e_i \otimes \phi_0^i(x_1, n-x_1) \mapsto \begin{cases} \overline{a}_i & \text{if } i \text{ is even,} \\ (-1)^{(t-1)/2}\overline{a}_i & \text{if } i \text{ is odd,} \end{cases} \\ e_i \otimes \phi_0^i(x_2, n-x_2) \mapsto \begin{cases} \overline{a}_i & \text{if } i \text{ is even}(\neq 0), \\ (-1)^{(t-1)/2}\overline{a}_i & \text{if } i \text{ is odd,} \end{cases} \\ e_{m-1} \otimes \phi_0^{m-1}(x_2+1, n-x_2-1) \mapsto (-1)^{(t+1)/2}a_m, \end{cases}$$

for $0 \leq i \leq m-1$, $0 \leq \alpha \leq p/2 - 1$.

(b) *If* $x = pm/2 + (t-1)/2$ *and* $\operatorname{char} k \neq 2$,

$$\mu_{n,p/2} : \begin{cases} e_i \otimes \phi_0^i(x, n-x) \mapsto \begin{cases} \overline{a}_i & \text{if } i \text{ is even,} \\ (-1)^{(t-1)/2}\overline{a}_i & \text{if } i \text{ is odd,} \end{cases} \\ e_i \otimes \phi_0^i(x+1, n-1-x) \mapsto \begin{cases} -a_{i+1} & \text{if } i \text{ is even,} \\ (-1)^{(t+1)/2}a_{i+1} & \text{if } i \text{ is odd,} \end{cases} \end{cases}$$

for $0 \leq i \leq m-1$.

(c) *If* $x = pm/2 + (t-1)/2$ *and* $\operatorname{char} k = 2$, $\mu_{n,p/2} : e_0 \otimes \phi_0^0(x, n-x) \mapsto a_0$.

(d) *If* $x = pm/2 + (t+1)/2$ *and* $\operatorname{char} k = 2$,
$\mu'_{n,p/2} : e_{m-1} \otimes \phi_0^{m-1}(x, n-x) \mapsto a_m$.

(e) *If* $x = (p-\alpha)m + (t-1)/2$,

$$\nu_{n,\alpha} : \begin{cases} e_0 \otimes \phi_0^0(x+1, n-1-x) \mapsto a_1, \\ e_1 \otimes \phi_0^1(x, n-x) \mapsto (-1)^{(t-1)/2}\overline{a}_1, \end{cases}$$

for $0 \leq \alpha \leq p/2 - 1$.

(f) *If* $x = pm/2 + (t-1)/2$, $E_{n,1} : e_0 \otimes \phi_0^0(x, n-x) \mapsto a_1\overline{a}_1$.

(g) *If* $x = pm/2 + (t+1)/2$, $E_{n,2} : e_{m-1} \otimes \phi_0^{m-1}(x, n-x) \mapsto a_m^2$.

(3) *In the case where* p *is odd and* t *is even, we have a* k*-basis of* $\mathrm{HH}^{pm+t}(A)$ *as follows*:

(a) *If* $x_1 = (p-\alpha-1)m + (m+t)/2$ *and* $x_2 = \alpha m + (m+t)/2$,

$$\chi_{n,\alpha}:\begin{cases} e_i\otimes\phi_0^i(x_1,n-x_1)\mapsto\begin{cases} e_i & \text{if } i \text{ is even},\\ (-1)^{(m+t)/2}e_i & \text{if } i \text{ is odd},\end{cases}\\ e_i\otimes\phi_0^i(x_2,n-x_2)\mapsto\begin{cases} e_i & \text{if } i \text{ is even},\\ (-1)^{(m+t)/2}e_i & \text{if } i \text{ is odd},\end{cases}\end{cases}$$

for $0\le i\le m-1$, $0\le\alpha\le(p-1)/2$.

(b) *If* $x=(p-1)m/2+(m+t)/2$, $\pi_{n,1}: e_0\otimes\phi_0^0(x,n-x)\mapsto a_0$.

(c) *If* $x=(p-1)m/2+(m+t)/2$, $\pi_{n,2}: e_{m-1}\otimes\phi_0^{m-1}(x,n-x)\mapsto a_m$.

(d) *If* $x=(p-\alpha-1)m+(m+t)/2$,
$F_{n,\alpha}: e_0\otimes\phi_0^0(x,n-x)\mapsto a_1\overline{a}_1$ *for* $0\le\alpha\le(p-1)/2-1$.

(e) *If* $x=(p-1)m/2+(m+t)/2$ *and* $\operatorname{char}k=2$,
$F_{n,(p-1)/2}: e_0\otimes\phi_0^0(x,n-x)\mapsto a_1\overline{a}_1$.

(4) *In the case where* p *and* t *are odd, we have a* k*-basis of* $\mathrm{HH}^{pm+t}(A)$ *as follows*:

(a) *If* $x_1=(p-\alpha-1)m+(m+t-1)/2$ *and* $x_2=\alpha m+(m+t-1)/2$,

$$\mu_{n,\alpha}:\begin{cases} e_i\otimes\phi_0^i(x_1,n-x_1)\mapsto\begin{cases} \overline{a}_i & \text{if } i \text{ is even},\\ (-1)^{(m+t-1)/2}\overline{a}_i & \text{if } i \text{ is odd},\end{cases}\\ e_i\otimes\phi_0^i(x_2,n-x_2)\mapsto\begin{cases} \overline{a}_i & \text{if } i \text{ is even}(\neq 0),\\ (-1)^{(m+t-1)/2}\overline{a}_i & \text{if } i \text{ is odd},\end{cases}\\ e_{m-1}\otimes\phi_0^{m-1}(x_2+1,n-x_2-1)\mapsto(-1)^{(m+t+1)/2}a_m,\end{cases}$$

for $0\le i\le m-1$, $0\le\alpha\le(p-1)/2-1$.

(b) *If* $x=(p-1)m/2+(m+t-1)/2$ *and* $\operatorname{char}k\neq 2$,

$$\mu_{n,(p-1)/2}:\begin{cases} e_i\otimes\phi_0^i(x,n-x)\mapsto\begin{cases} \overline{a}_i & \text{if } i \text{ is even},\\ (-1)^{(m+t-1)/2}\overline{a}_i & \text{if } i \text{ is odd},\end{cases}\\ e_i\otimes\phi_0^i(x+1,n-1-x)\mapsto\\ \qquad\begin{cases} -a_{i+1} & \text{if } i \text{ is even},\\ (-1)^{(m+t+1)/2}a_{i+1} & \text{if } i \text{ is odd},\end{cases}\end{cases}$$

for $0\le i\le m-1$.

(c) *If* $x=(p-1)m/2+(m+t-1)/2$ *and* $\operatorname{char}k=2$,
$\mu_{n,(p-1)/2}: e_0\otimes\phi_0^0(x,n-x)\mapsto a_0$.

(d) *If* $x=(p-1)m/2+(m+t+1)/2$ *and* $\operatorname{char}k=2$,
$\mu'_{n,(p-1)/2}: e_{m-1}\otimes\phi_0^{m-1}(x,n-x)\mapsto a_m$.

(e) *If* $x_1=pm+m-1$, $x_2=0$, $t=m-1$,

$$\psi_n:\begin{cases} e_i\otimes\phi_0^i(pm+m-1,0)\mapsto(-1)^i\overline{a}_i,\\ e_i\otimes\phi_0^i(0,pm+m-1)\mapsto(-1)^{i+1}a_{i+1},\end{cases}$$

for $0\le i\le m-1$.

(f) *If* $x = (p-\alpha-1)m + (m+t-1)/2$,
$$\nu_{n,\alpha} : \begin{cases} e_0 \otimes \phi_0^0(x+1, n-1-x) \mapsto a_1, \\ e_1 \otimes \phi_0^1(x, n-x) \mapsto (-1)^{(m+t-1)/2}\overline{a}_1, \end{cases}$$
for $0 \le \alpha \le (p-1)/2 - 1$.

(g) *If* $x = (p-1)m/2 + (m+t-1)/2$, $E_{n,1} : e_0 \otimes \phi_0^0(x, n-x) \mapsto a_1\overline{a}_1$.

(h) *If* $x = (p-1)m/2+(m+t+1)/2$, $E_{n,2} : e_{m-1} \otimes \phi_0^{m-1}(x, n-x) \mapsto a_m^2$.

Proposition 3.9. *Suppose that* $m \ge 3$ *and* m *is odd. For each* $n = pm+t \ge 1$, *the following elements form a* k*-basis of* $\mathrm{HH}^{pm+t}(A)$.

(1) *In the case where* p *and* t *are even, we have a* k*-basis of* $\mathrm{HH}^{pm+t}(A)$ *as follows*:

(a) *If* $x_1 = (p-\alpha)m + t/2$ *and* $x_2 = \alpha m + t/2$,
$$\chi_{n,\alpha} : \begin{cases} e_i \otimes \phi_0^i(x_1, n-x_1) \mapsto \begin{cases} e_i & \text{if } i \text{ is even}, \\ (-1)^{\alpha+t/2}e_i & \text{if } i \text{ is odd}, \end{cases} \\ e_i \otimes \phi_0^i(x_2, n-x_2) \mapsto \begin{cases} (-1)^{\alpha+p/2}e_i & \text{if } i \text{ is even}, \\ (-1)^{(p+t)/2}e_i & \text{if } i \text{ is odd}, \end{cases} \end{cases}$$
for $0 \le i \le m-1$, $0 \le \alpha \le p/2$.

(b) *If* $x = pm/2 + t/2$, $\pi_{n,1} : e_0 \otimes \phi_0^0(x, n-x) \mapsto a_0$.

(c) *If* $x = pm/2 + t/2$, $\pi_{n,2} : e_{m-1} \otimes \phi_0^{m-1}(x, n-x) \mapsto a_m$.

(d) *If* $x = (p-\alpha)m + t/2$,
$F_{n,\alpha} : e_0 \otimes \phi_0^0(x, n-x) \mapsto a_1\overline{a}_1$ *for* $0 \le \alpha \le p/2 - 1$.

(e) *If* $x = pm/2 + t/2$, $\operatorname{char} k = 2$, $F_{n,p/2} : e_0 \otimes \phi_0^0(x, n-x) \mapsto a_1\overline{a}_1$.

(2) *In the case where* p *is even and* t *is odd, we have a* k*-basis of* $\mathrm{HH}^{pm+t}(A)$ *as follows*:

(a) *If* $x_1 = (p-\alpha)m + (t-1)/2$ *and* $x_2 = \alpha m + (t-1)/2$,
$$\mu_{n,\alpha} : \begin{cases} e_i \otimes \phi_0^i(x_1, n-x_1) \mapsto \begin{cases} \overline{a}_i & \text{if } i \text{ is even}, \\ (-1)^{\alpha+(t-1)/2}\overline{a}_i & \text{if } i \text{ is odd}, \end{cases} \\ e_i \otimes \phi_0^i(x_2, n-x_2) \mapsto \\ \quad \begin{cases} (-1)^{\alpha+p/2}\overline{a}_i & \text{if } i \text{ is even}(\neq 0), \\ (-1)^{(p+t-1)/2}\overline{a}_i & \text{if } i \text{ is odd}, \end{cases} \\ e_{m-1} \otimes \phi_0^{m-1}(x_2+1, n-x_2-1) \mapsto (-1)^{\alpha+p/2+1}a_m, \end{cases}$$
for $0 \le i \le m-1$, $0 \le \alpha \le p/2 - 1$.

(b) *If* $x = pm/2 + (t-1)/2$ *and* $\operatorname{char} k \neq 2$,

$$\mu_{n,p/2}:\begin{cases} e_i\otimes\phi_0^i(x,n-x)\mapsto\begin{cases}\overline{a}_i & \textit{if } i \textit{ is even},\\ (-1)^{(p+t-1)/2}\overline{a}_i & \textit{if } i \textit{ is odd},\end{cases}\\ e_i\otimes\phi_0^i(x+1,n-1-x)\mapsto\\ \quad\begin{cases}-a_{i+1} & \textit{if } i \textit{ is even},\\ (-1)^{(p+t+1)/2}a_{i+1} & \textit{if } i \textit{ is odd},\end{cases}\end{cases}$$
for $0\le i\le m-1$.

(c) *If* $x=pm/2+(t-1)/2$ *and* $\operatorname{char}k=2$, $\mu_{n,p/2}: e_0\otimes\phi_0^0(x,n-x)\mapsto a_0$.

(d) *If* $x=pm/2+(t+1)/2$ *and* $\operatorname{char}k=2$,
$\mu'_{n,p/2}: e_{m-1}\otimes\phi_0^{m-1}(x,n-x)\mapsto a_m$.

(e) *If* $x=(p-\alpha)m+(t-1)/2$,
$$\nu_{n,\alpha}:\begin{cases} e_0\otimes\phi_0^0(x+1,n-x+1)\mapsto a_1,\\ e_1\otimes\phi_0^1(x,n-x)\mapsto(-1)^{\alpha+(t-1)/2}\overline{a}_1,\end{cases}$$
for $0\le\alpha\le p/2-1$.

(f) *If* $x=pm/2+(t-1)/2$, $E_{n,1}: e_0\otimes\phi_0^0(x,n-x)\mapsto a_1\overline{a}_1$.

(g) *If* $x=pm/2+(t+1)/2$, $E_{n,2}: e_{m-1}\otimes\phi_0^{m-1}(x,n-x)\mapsto a_m^2$.

(3) *In the case where* p *and* t *are odd, we have a* k*-basis of* $\mathrm{HH}^{pm+t}(A)$ *as follows*:

(a) *If* $x_1=(p-\alpha-1)m+(m+t)/2$ *and* $x_2=\alpha m+(m+t)/2$,
$$\chi_{n,\alpha}:\begin{cases} e_i\otimes\phi_0^i(x_1,n-x_1)\mapsto\begin{cases}e_i & \textit{if } i \textit{ is even},\\ (-1)^{\alpha+(m+t)/2}e_i & \textit{if } i \textit{ is odd},\end{cases}\\ e_i\otimes\phi_0^i(x_2,n-x_2)\mapsto\begin{cases}(-1)^{\alpha+(p-1)/2}e_i & \textit{if } i \textit{ is even},\\ (-1)^{(m+p+t-1)/2}e_i & \textit{if } i \textit{ is odd},\end{cases}\end{cases}$$
for $0\le i\le m-1$, $0\le\alpha\le(p-1)/2$.

(b) *If* $x=(p-1)m/2+(m+t)/2$, $\pi_{n,1}: e_0\otimes\phi_0^0(x,n-x)\mapsto a_0$.

(c) *If* $x=(p-1)m/2+(m+t)/2$, $\pi_{n,2}: e_{m-1}\otimes\phi_0^{m-1}(x,n-x)\mapsto a_m$.

(d) *If* $x=(p-\alpha-1)m+(m+t)/2$,
$F_{n,\alpha}: e_0\otimes\phi_0^0(x,n-x)\mapsto a_1\overline{a}_1$ *for* $0\le\alpha\le(p-1)/2-1$.

(e) *If* $x=(p-1)m/2+(m+t)/2$ *and* $\operatorname{char}k=2$,
$F_{n,(p-1)/2}: e_0\otimes\phi_0^0(x,n-x)\mapsto a_1\overline{a}_1$.

(4) *In the case where* p *is odd and* t *is even, we have a* k*-basis of* $\mathrm{HH}^{pm+t}(A)$ *as follows*:

(a) *If* $x_1=(p-\alpha-1)m+(m+t-1)/2$ *and* $x_2=\alpha m+(m+t-1)/2$,

$$\mu_{n,\alpha}:\begin{cases} e_i\otimes\phi_0^i(x_1,n-x_1)\mapsto\begin{cases}\overline{a}_i & \textit{if } i \textit{ is even,}\\ (-1)^{\alpha+(m+t-1)/2}\overline{a}_i & \textit{if } i \textit{ is odd,}\end{cases}\\ e_i\otimes\phi_0^i(x_2,n-x_2)\mapsto\begin{cases}(-1)^{\alpha+(p-1)/2}\overline{a}_i & \textit{if } i \textit{ is even}(\neq 0),\\ (-1)^{(m+p+t)/2+1}\overline{a}_i & \textit{if } i \textit{ is odd,}\end{cases}\\ e_{m-1}\otimes\phi_0^{m-1}(x_2+1,n-x_2-1)\mapsto(-1)^{\alpha+(p+1)/2}a_m,\end{cases}$$

for $0\le i\le m-1$, $0\le\alpha\le(p-1)/2-1$.

(b) *If* $x=(p-1)m/2+(m+t-1)/2$ *and* $\operatorname{char}k\neq 2$,

$$\mu_{n,(p-1)/2}:\begin{cases} e_i\otimes\phi_0^i(x,n-x)\mapsto\begin{cases}\overline{a}_i & \textit{if } i \textit{ is even,}\\ (-1)^{(m+p+t)/2+1}\overline{a}_i & \textit{if } i \textit{ is odd,}\end{cases}\\ e_i\otimes\phi_0^i(x+1,n-1-x)\mapsto\\ \quad\begin{cases}-a_{i+1} & \textit{if } i \textit{ is even,}\\ (-1)^{(m+p+t)/2}a_{i+1} & \textit{if } i \textit{ is odd,}\end{cases}\end{cases}$$

for $0\le i\le m-1$.

(c) *If* $x=(p-1)m/2+(m+t-1)/2$ *and* $\operatorname{char}k=2$,
$\mu_{n,(p-1)/2}:e_0\otimes\phi_0^0(x,n-x)\mapsto a_0$.

(d) *If* $x=(p-1)m/2+(m+t+1)/2$ *and* $\operatorname{char}k=2$,
$\mu'_{n,(p-1)/2}:e_{m-1}\otimes\phi_0^{m-1}(x,n-x)\mapsto a_m$.

(e) *If* $x_1=pm+m-1$ *and* $x_2=0$, $t=m-1$,

$$\psi_n:\begin{cases} e_i\otimes\phi_0^i(pm+m-1,0)\mapsto(-1)^i\overline{a}_i,\\ e_i\otimes\phi_0^i(0,pm+m-1)\mapsto(-1)^{(p+1)/2+i}a_{i+1},\end{cases}$$

for $0\le i\le m-1$.

(f) *If* $x=(p-\alpha-1)m+(m+t-1)/2$,

$$\nu_{n,\alpha}:\begin{cases} e_0\otimes\phi_0^0(x+1,n-1-x)\mapsto a_1\\ e_1\otimes\phi_0^1(x,n-x)\mapsto(-1)^{\alpha+(m+t+1)/2}\overline{a}_1,\end{cases}$$

for $0\le\alpha\le(p-1)/2-1$.

(g) *If* $x=(p-1)m/2+(m+t-1)/2$, $E_{n,1}:e_0\otimes\phi_0^0(x,n-x)\mapsto a_1\overline{a}_1$.

(h) *If* $x=(p-1)m/2+(m+t+1)/2$, $E_{n,2}:e_{m-1}\otimes\phi_0^{m-1}(x,n-x)\mapsto a_m^2$.

§4. The Hochschild cohomology groups of A for $m=1$

Using the same argument as Sections 2 and 3, we have a k-basis of the Hochschild cohomology groups of A for $m=1$. As in Section 2, we define a left A^e-module

$$P_n:=\coprod_{g\in\mathcal{G}^n}Ao(g)\otimes t(g)A.$$

Then we have the following resolution.

Theorem 4.1. *If $m = 1$, we have a projective bimodule resolution of A:*

$$(P_\bullet, \partial_\bullet) : \cdots \to P_n \xrightarrow{\partial_n} P_{n-1} \to \cdots \to P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\pi} A \to 0,$$

where π is the multiplication map, and for $n \geq 1$, ∂_n: $P_n \to P_{n-1}$ are defined as follows:

(1) *If $x = 0$, then*

$$\begin{aligned}&\partial_n(o(g^n_{0,n,0}) \otimes t(g^n_{0,n,0}))\\ =&(-1)^n e_0 \otimes \phi^0_1(c^{(0,n-1)}) + \begin{cases} a_0 \otimes t(g^{n-1}_{n-1,0,0}) & \text{if } n \equiv 0, 1 (\operatorname{mod} 4), \\ -a_0 \otimes t(g^{n-1}_{n-1,0,0}) & \text{if } n \equiv 2, 3 (\operatorname{mod} 4). \end{cases}\end{aligned}$$

(2) *If $1 \leq x \leq n-1$, then*

$$\begin{aligned}&\partial_n(o(g^n_{x,n-x,0}) \otimes t(g^n_{x,n-x,0}))\\ =&e_0 \otimes \phi^0_1(b^{(x-1,n-x)}) + (-1)^n e_0 \otimes \phi^0_1(c^{(x,n-1-x)})\\ &+ \begin{cases} (-1)^x a_1 \otimes t(g^{n-1}_{n-x,x-1,0}) + (-1)^x a_0 \otimes t(g^{n-1}_{n-1-x,x,0}) \\ \hfill \text{if } n \equiv 0, 1 (\operatorname{mod} 4), \\ -(-1)^x a_1 \otimes t(g^{n-1}_{n-x,x-1,0}) - (-1)^x a_0 \otimes t(g^{n-1}_{n-1-x,x,0}) \\ \hfill \text{if } n \equiv 2, 3 (\operatorname{mod} 4). \end{cases}\end{aligned}$$

(3) *If $x = n$, then*

$$\begin{aligned}&\partial_n(o(g^n_{n,0,0}) \otimes t(g^n_{n,0,0}))\\ =&e_0 \otimes \phi^0_1(b^{(n-1,0)}) + \begin{cases} (-1)^n a_1 \otimes t(g^{n-1}_{0,n-1,0}) & \text{if } n \equiv 0, 1 (\operatorname{mod} 4), \\ -(-1)^n a_1 \otimes t(g^{n-1}_{0,n-1,0}) & \text{if } n \equiv 2, 3 (\operatorname{mod} 4). \end{cases}\end{aligned}$$

We set $P^*_n := \operatorname{Hom}_{A^e}(P_n, A)$ and $\partial^*_n := \operatorname{Hom}_{A^e}(\partial_n, A)$. Then we have the complex

$$(P^*_\bullet, \partial^*_\bullet) : 0 \to P^*_0 \xrightarrow{\partial^*_1} P^*_1 \to \cdots \to P^*_{n-1} \xrightarrow{\partial^*_n} P^*_n \to \cdots .$$

Lemma 4.2. *For an element f in P^*_n, we have*

$$f(o(g^n_{x,n-x,0}) \otimes t(g^n_{x,n-x,0})) = \sigma_{0,x} e_0 + \tau_{0,x} a_0 + \mu_{0,x} a_1 + \lambda_{0,x} a_1^2 \text{ for } 0 \leq x \leq n.$$

So we have $\dim_k P^*_n = 4(n+1)$.

By Theorem 4.1 and Lemma 4.2, we can determine the image of ∂^*_{n+1}.

Lemma 4.3. *With the notation of Lemma* 4.2*, we have* $\partial^*_{n+1}(f)(o(g^n_{x,n+1-x,0})\otimes t(g^n_{x,n+1-x,0}))$ *in the following cases.*

(1) *In the case of* $x = 0$*, we have* $\partial^*_{n+1}(f)(o(g^n_{x,n+1-x,0}) \otimes t(g^n_{x,n+1-x,0}))$ *as follows*:

$$\begin{cases} (-\sigma_{0,0} + (-1)^{n/2}\sigma_{0,n})a_0 + (-\tau_{0,0} + (-1)^{n/2}\tau_{0,n})a_1^2 & \text{if } n \in 2\mathbb{Z}, \\ (-1)^{n/2}\sigma_{0,n}a_0 + \sigma_{0,0}a_1 + (\mu_{0,0} + (-1)^{(n+1)/2}\tau_{0,n})a_1^2 & \text{if } n \notin 2\mathbb{Z}. \end{cases}$$

(2) *In the case of* $1 \le x \le n$*, we have* $\partial^*_{n+1}(f)(o(g^n_{x,n+1-x,0}) \otimes t(g^n_{x,n+1-x,0}))$ *as follows*:

$$\begin{cases} (-\sigma_{0,x} + (-1)^{x+n/2}\sigma_{0,n-x})a_0 \\ \quad +(\sigma_{0,x-1} + (-1)^{x+n/2}\sigma_{0,n+1-x})a_1 \\ \quad\quad +(\mu_{0,x-1} - \tau_{0,x} + (-1)^{x+n/2}\mu_{0,n+1-x} \\ \quad\quad\quad +(-1)^{x+n/2}\tau_{0,n-x})a_1^2 & \text{if } n \in 2\mathbb{Z}, \\ (\sigma_{0,x-1} + (-1)^{x+(n+1)/2}\sigma_{0,n-x})a_0 \\ \quad +(\sigma_{0,x} + (-1)^{x+(n+1)/2}\sigma_{0,n+1-x})a_1 \\ \quad\quad +(\tau_{0,x-1} + \mu_{0,x} + (-1)^{x+(n+1)/2}\mu_{0,n+1-x} \\ \quad\quad\quad +(-1)^{x+(n+1)/2}\tau_{0,n-x})a_1^2 & \text{if } n \notin 2\mathbb{Z}. \end{cases}$$

(3) *In the case of* $x = n + 1$*, we have* $\partial^*_{n+1}(f)(o(g^n_{x,n+1-x,0}) \otimes t(g^n_{x,n+1-x,0}))$ *as follows*:

$$\begin{cases} (\sigma_{0,n} + (-1)^{n/2}\sigma_{0,0})a_1 + (\mu_{0,n} - (-1)^{n/2}\mu_{0,0})a_1^2 & \text{if } n \in 2\mathbb{Z}, \\ \sigma_{0,n}a_0 + (-1)^{(n+1)/2}\sigma_{0,0}a_1 + (\tau_{0,n} + (-1)^{(n+1)/2}\mu_{0,0})a_1^2 & \text{if } n \notin 2\mathbb{Z}. \end{cases}$$

By Lemma 4.3, we obtain the following results.

Theorem 4.4. *By Lemma* 4.3,*, we have*

$$\dim_k \mathrm{HH}^n(A) = \begin{cases} 4 & \text{if } n = 0, \\ n+3 & \text{if } n \ge 1 \text{ and } \operatorname{char} k \ne 2, \\ n+4 & \text{if } n \ge 1 \text{ and } \operatorname{char} k = 2. \end{cases}$$

For $m = 1$, the algebra A is commutative. So the 0-th Hochschild cohomology group $\mathrm{HH}^0(A)$ of A coincides with A. And we give a k-basis of the n-th Hochschild cohomology groupc $\mathrm{HH}^n(A)$ of A for $n \ge 1$.

Proposition 4.5. *For each* $n \ge 1$*, the following elements form a* k*-basis of* $\mathrm{HH}^n(A)$.

(1) *If n is even,*

(a) *For $0 \leq x \leq n/2-1$,*

$$\chi_{n,x} : \begin{cases} e_0 \otimes \phi_0^0(x, n-x) \mapsto e_0, \\ e_0 \otimes \phi_0^0(n-x, x) \mapsto (-1)^{x+n/2+1} e_0, \end{cases}$$

(b) $\chi_{n,n/2} : e_0 \otimes \phi_0^0(n/2, n/2) \mapsto e_0$.

(c) $\pi_{n,1} : e_0 \otimes \phi_0^0(n/2, n/2) \mapsto a_0$.

(d) $\pi_{n,2} : e_0 \otimes \phi_0^0(n/2, n/2) \mapsto a_1$.

(e) *For $0 \leq x \leq n/2-1$, $F_{n,x} : e_0 \otimes \phi_0^0(x, n-x) \mapsto a_1^2$.*

(f) *If $\operatorname{char} k = 2$, $F_{n,n/2} : e_0 \otimes \phi_0^0(n/2, n/2) \mapsto a_1^2$.*

(2) *If n is odd,*

(a) *For $0 \leq x \leq (n-3)/2$,*

$$\mu_{n,x} : \begin{cases} e_0 \otimes \phi_0^0(x, n-x) \mapsto a_0, \\ e_0 \otimes \phi_0^0(n-1-x, x+1) \mapsto (-1)^{x+(n-1)/2+1} a_0, \end{cases}$$

(b) *If $\operatorname{char} k \neq 2$, $\mu_{n,(n-1)/2} : \begin{cases} e_0 \otimes \phi_0^0((n-1)/2, (n+1)/2) \mapsto a_0, \\ e_0 \otimes \phi_0^0((n+1)/2, (n-1)/2) \mapsto -a_1. \end{cases}$*

(c) *If $\operatorname{char} k = 2$, $\mu_{n,(n-1)/2} : e_0 \otimes \phi_0^0((n-1)/2, (n+1)/2) \mapsto a_0$.*

(d) *For $0 \leq x \leq (n-1)/2$,*

$$\nu_{n,x} : \begin{cases} e_0 \otimes \phi_0^0(x, n-x) \mapsto a_1, \\ e_0 \otimes \phi_0^0(n-x, x) \mapsto (-1)^{x+(n-1)/2} a_0, \end{cases}$$

(e) *If $\operatorname{char} k = 2$, $\nu_{n,(n+1)/2} : e_0 \otimes \phi_0^0((n+1)/2, (n-1)/2) \mapsto a_1$.*

(f) $E_{n,0} : e_0 \otimes \phi_0^0(0, n) \mapsto a_1^2$.

(g) $E_{n,1} : e_0 \otimes \phi_0^0(1, n-1) \mapsto a_1^2$.

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