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Totally vertex-magic cordial labeling

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Abstract. In this paper, we introduce a new labeling called Totally Vertex-Magic Cordial(TVMC) labeling. A graph $G(p, q)$ is said to be TVMC with a constant C if there is a mapping $f : V(G) \cup E(G) \to \{0, 1\}$ such that

$$\left[f(a) + \sum_{b \in N(a)} f(ab)\right] \equiv C \pmod 2$$

for all vertices $a \in V(G)$ and $|n_f(0) - n_f(1)| \leq 1$, where $N(a)$ is the set of vertices adjacent to the vertex a and $n_f(i)(i = 0, 1)$ is the sum of the number of vertices and edges with label i.

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§1. Introduction

All graphs considered here are finite, simple and undirected. The set of vertices and edges of a graph G will be denoted by $V(G)$ and $E(G)$ respectively, and let $p = |V(G)|$ and $q = |E(G)|$. A labeling of a graph G is a mapping that carries a set of graph elements usually the vertices and/or edges, into a set of numbers, usually integers, called labels. Many kinds of labelings have been studied and an excellent survey of graph labeling can be found in Gallian [3]. For all other terminology and notation we follow Harary [4]. The concept of cordial labeling was introduced by Cahit [1]. A binary vertex labeling $f : V(G) \to \{0, 1\}$ induces an edge labeling $f^* : E(G) \to \{0, 1\}$ defined by $f^*(uv) = |f(u) - f(v)|$. Such a labeling is called cordial if the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$ are satisfied, where $v_f(i)$ and $e_{f^*}(i)(i = 0, 1)$ are the number of vertices and edges with label i respectively. A graph is called cordial if it admits cordial labeling.

Totally Magic Cordial(TMC) labeling was introduced by Cahit in [2] as a modification of total edge-magic labeling. A (p, q) graph G is said to have a totally magic cordial labeling with constant C if there exists a mapping $f : V(G) \cup E(G) \to \{0, 1\}$ such that $f(a) + f(b) + f(ab) \equiv C \pmod 2$ for all edges $ab \in E(G)$ provided the condition $|f(0) - f(1)| \leq 1$, where $f(0) = v_f(0) + e_f(0)$, $f(1) = v_f(1) + e_f(1)$ and $v_f(i)$, $e_f(i)(i = 0, 1)$ are the number of vertices and edges with label i, respectively. It is proved that the graphs $K_{m,n}(m, n > 1)$, trees and K_n for $n = 2, 3, 5$ or 6 have TMC labeling.

J. A. MacDougall et al. introduced the concept of vertex-magic total labeling in [6]. A one-to-one map λ from $V \cup E$ onto the integers $\{1, 2, ..., p + q\}$ is a vertex-magic total labeling if there is a constant k so that for every vertex x, $\lambda(x) + \sum \lambda(xy) = k$, where the sum is over all vertices y adjacent to x. The sum $\lambda(x) + \sum \lambda(xy)$ is called the weight of the vertex x and is denoted by $\mathrm{wt}(x)$. The constant k is called the magic constant for λ. In this paper, we modify the vertex-magic total labeling into a new labeling called totally vertex magic cordial labeling and we examine the totally vertex magic cordiality of some graphs.

§2. Totally vertex-magic cordial labeling

In this section, we define totally vertex-magic cordial labeling and we prove vertex-magic total graph is totally vertex-magic cordial.

Definition 2.1. *A (p, q) graph G is said to have a totally vertex-magic cordial (TVMC) labeling with constant C if there is a mapping $f : V(G) \cup E(G) \to \{0, 1\}$ such that*

$$\left[f(a) + \sum_{b \in N(a)} f(ab) \right] \equiv C \pmod 2$$

for all vertices $a \in V(G)$ provided the condition, $|n_f(0) - n_f(1)| \leq 1$ is held, where $N(a)$ is the set of vertices adjacent to a vertex a and $n_f(i)(i = 0, 1)$ is the sum of the number of vertices and edges with label i.

A graph is called totally vertex-magic cordial if it admits totally vertex-magic cordial labeling .

Theorem 2.2. *If G is a vertex-magic total graph then G is totally vertex-magic cordial.*

Proof. Let f be a vertex-magic total labeling of a graph G with p vertices and q edges and with weight k. Define $g : V(G) \cup E(G) \to \{0, 1\}$ by $g(v) \equiv f(v)$

(mod 2) if $v \in V(G)$ and $g(e) \equiv f(e) \pmod 2$ if $e \in E(G)$. Then, $C = 0$ if k is even and $C = 1$ if k is odd. Since there are exactly $\lceil \frac{p+q}{2} \rceil$ odd integers and $\lfloor \frac{p+q}{2} \rfloor$ even integers in the set $\{1, 2, 3, ..., p + q\}$ we have, $|n_f(0) - n_f(1)| \leq 1$. Hence, g is a totally vertex-magic cordial labeling of G. □

§3. Totally vertex-magic cordial labeling of a complete graph K_n

H. K. Krishnappa et al. [5] proved that $K_n (n \geq 1)$ admits vertex-magic total labeling. In this section, we use another technique to prove $K_n (n \geq 1)$ is totally vertex-magic cordial. Let $V = \{v_i | 1 \leq i \leq n\}$ be the vertex set and $E = \{v_i v_j | i \neq j, 1 \leq i, j \leq n\}$ be the edge set of K_n. We use the following symmetric matrix to label the vertices and the edges of K_n, which is called the label matrix for K_n.

$$\begin{bmatrix} e_{11} & e_{21} & e_{31} & e_{41} & e_{51} & . & . & . & e_{n1} \\ e_{21} & e_{22} & e_{32} & e_{42} & e_{52} & . & . & . & e_{n2} \\ e_{31} & e_{32} & e_{33} & e_{43} & e_{53} & . & . & . & e_{n3} \\ e_{41} & e_{42} & e_{43} & e_{44} & e_{54} & . & . & . & e_{n4} \\ e_{51} & e_{52} & e_{53} & e_{54} & e_{55} & . & . & . & e_{n5} \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ e_{n1} & e_{n2} & e_{n3} & e_{n4} & e_{n5} & . & . & . & e_{nn} \end{bmatrix}$$

The entries in the main diagonal represent the vertex labels, $f(v_i) = e_{ii}$ and the other entries e_{ij}, $i \neq j$ represent the edge labels, $f(v_i v_j) = e_{ij}$. Thus the weight of a vertex v_i is the sum of the elements either in the i^{th} row or in the i^{th} column.

Theorem 3.1. *The complete graph K_n is TVMC for all $n \geq 1$.*

Proof. Let K_n be the complete graph with n vertices. We consider the following three cases:
Case i. $n \equiv 0 \pmod 4$.
We construct the label matrix for K_n as follows:

$$e_{ij} = \begin{cases} 0 & \text{when} \quad i + j \equiv 0, 1 \pmod 4, \\ 1 & \text{when} \quad i + j \equiv 2, 3 \pmod 4. \end{cases}$$

Then for each vertex v_r, $1 \leq r \leq n$, the weight $\mathrm{wt}(v_r)$ is the sum of the elements in the r^{th} row or in the r^{th} column. Hence,

$$\mathrm{wt}(v_r) = \sum_{j=1}^{r} e_{rj} + \sum_{i=r+1}^{n} e_{ir} = \frac{n}{2} \equiv 0 \pmod 2.$$

Also $n_f(0) = n_f(1) = \frac{n^2+n}{4}$. Therefore, $|n_f(0) - n_f(1)| = 0$.
Case ii. $n \equiv 2 \pmod 4$.

We construct the label matrix as follows: when $j \equiv 0, 1 \pmod 4$,

$$e_{ij} = \begin{cases} 1 & \text{if } i \text{ is odd,} \\ 0 & \text{if } i \text{ is even} \end{cases}$$

and when $j \equiv 2, 3 \pmod 4$,

$$e_{ij} = \begin{cases} 0 & \text{if } i \text{ is odd,} \\ 1 & \text{if } i \text{ is even.} \end{cases}$$

Then

$$\mathrm{wt}(v_r) = \sum_{j=1}^{r} e_{rj} + \sum_{i=r+1}^{n} e_{ir} = \frac{n}{2} \equiv 1 \pmod 2.$$

Also $n_f(0) = \frac{n^2+n-2}{4}$ and $n_f(1) = \frac{n^2+n+2}{4}$. Hence, $|n_f(0) - n_f(1)| = 1$.
Case iii. n is odd.

We construct the label matrix as follows: when $i + j \leq n$,

$$e_{ij} = \begin{cases} 1 & \text{if } i \text{ is odd,} \\ 0 & \text{if } i \text{ is even} \end{cases}$$

and when $i + j > n$,

$$e_{ij} = \begin{cases} 1 & \text{if } j \text{ is odd,} \\ 0 & \text{if } j \text{ is even.} \end{cases}$$

We have

$$\begin{aligned} \mathrm{wt}(v_r) &= \sum_{j=1}^{r} e_{rj} + \sum_{i=r+1}^{n-r} e_{ir} + \sum_{i=n-r+1}^{n} e_{ir} \text{ if } 1 \leq r < \frac{n+1}{2}; \\ \mathrm{wt}(v_r) &= \sum_{j=1}^{r-1} e_{rj} + \sum_{i=r}^{n} e_{ir} \text{ if } r = \frac{n+1}{2}; \\ \mathrm{wt}(v_r) &= \sum_{j=1}^{n-r} e_{rj} + \sum_{j=n-r+1}^{r-1} e_{rj} + \sum_{i=r}^{n} e_{ir} \text{ if } \frac{n+1}{2} < r < n; \\ \text{and } \mathrm{wt}(v_r) &= \sum_{j=1}^{n} e_{rj} \text{ if } r = n. \end{aligned}$$

The weights of the vertices for $n = 4k + 1$ and $n = 4k + 3$ are summarized in the following tables:

When $n = 4k + 1$,

	$1 \leq r < \frac{n+1}{2}$	$r = \frac{n+1}{2}$	$\frac{n+1}{2} < r < n$	$r = n$
r is odd	$2k + r$ $\equiv 1 \pmod 2$	$n \times (r \bmod 2)$ $\equiv 1 \pmod 2$	$6k - r + 2$ $\equiv 1 \pmod 2$	$\frac{n+1}{2}$ $\equiv 1 \pmod 2$
r is even	$2k - r + 1$ $\equiv 1 \pmod 2$	-	$r - 2k - 1$ $\equiv 1 \pmod 2$	-

When $n = 4k + 3$,

	$1 \leq r < \frac{n+1}{2}$	$r = \frac{n+1}{2}$	$\frac{n+1}{2} < r < n$	$r = n$
r is odd	$2k + r + 1$ $\equiv 0 \pmod 2$	-	$6k - r + 5$ $\equiv 0 \pmod 2$	$\frac{n+1}{2}$ $\equiv 0 \pmod 2$
r is even	$2k - r + 2$ $\equiv 0 \pmod 2$	$n \times (r \bmod 2)$ $\equiv 0 \pmod 2$	$r - 2k - 2$ $\equiv 0 \pmod 2$	-

Also if $n = 4k + 1$,then $n_f(0) = \frac{n^2+n-2}{4}$, $n_f(1) = \frac{n^2+n+2}{4}$; if $n = 4k + 3$, then $n_f(0) = n_f(1) = \frac{n^2+n}{4}$ and hence, $|n_f(0) - n_f(1)| \leq 1$. Therefore, K_n is TVMC for all $n \geq 1$. $\square$

§4. Totally vertex-magic cordial labeling of a complete bipartite graph $K_{m,n}$

J. A. MacDougall et al. [6] proved that there is a vertex-magic total labeling for a complete bipartite graph $K_{m,m}$ for all $m > 1$. Also they conjectured that there is a vertex-magic total labeling for a complete bipartite graph $K_{m,m+1}$.

In this section, we prove the bipartite graph $K_{m,n}$ admits TVMC labeling whenever $|m - n| \leq 1$. We consider the complete bipartite graph $K_{m,n}$ with the vertex set $\{u_1, u_2, ..., u_m, v_1, v_2, ..., v_n\}$ and the edge set $\{e_{ij} = u_i v_j | 1 \leq i \leq m, 1 \leq j \leq n\}$. We use the following $(m + 1) \times (n + 1)$ matrix to label the vertices and the edges of $K_{m,n}$:

$$\begin{bmatrix} - & | & c_{01} & c_{02} & \cdots & c_{0n} \\ -- & -- & -- & -- & -- & -- \\ c_{10} & | & c_{11} & c_{12} & \cdots & c_{1n} \\ c_{20} & | & c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & | & \vdots & \vdots & & \vdots \\ c_{m0} & | & c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

The entries in the first row $c_{i0}(1 \leq i \leq m)$ represent the labels of the vertices $u_i(1 \leq i \leq m)$, the entries in the first column $c_{0j}(1 \leq j \leq n)$ represent the labels of the vertices $v_j(1 \leq j \leq n)$ and the other entries c_{ij} represent the labels of the edges $u_i v_j(1 \leq i \leq m, 1 \leq j \leq n)$. That is, $f(u_i) = c_{i0}$, $f(v_j) = c_{0j}$ and $f(u_i v_j) = c_{ij}$ for $1 \leq i \leq m$, $1 \leq j \leq n$.

Lemma 4.1. *$K_{m,m+1}$ is TVMC for all $m \geq 1$.*

Proof. Define

$$c_{ij} = \begin{cases} 1 & \text{if} \quad i = 0 \text{ or } j = 0 \quad \text{and} \quad i + j \text{ is odd,} \\ 0 & \text{if} \quad i = 0 \text{ or } j = 0 \quad \text{and} \quad i + j \text{ is even,} \\ 1 & \text{if} \quad i \neq 0, j \neq 0 \quad \text{and} \quad i + j \leq m + 1, \\ 0 & \text{if} \quad i \neq 0, j \neq 0 \quad \text{and} \quad i + j > m + 1. \end{cases}$$

Then $n_f(0) = \frac{m^2+3m}{2}$, $n_f(1) = \frac{m^2+3m+2}{2}$ and hence, $|n_f(0) - n_f(1)| = 1$. The weights of vertices u_i and v_j are summarized in the following table:

	i		j	
	Even	Odd	Even	Odd
m is even	$m + 1 - i$ $\equiv 1 \pmod 2$	$m + 2 - i$ $\equiv 1 \pmod 2$	$m + 1 - j$ $\equiv 1 \pmod 2$	$m + 2 - j$ $\equiv 1 \pmod 2$
m is odd	$m + 1 - i$ $\equiv 0 \pmod 2$	$m + 2 - i$ $\equiv 0 \pmod 2$	$m + 1 - j$ $\equiv 0 \pmod 2$	$m + 2 - j$ $\equiv 0 \pmod 2$

Therefore, $K_{m,m+1}$ is TVMC for all $m \geq 1$. □

Lemma 4.2. *$K_{m,m}$ is TVMC if m is odd.*

Proof. Define

$$c_{ij} = \begin{cases} 1 & \text{if} \quad i + j \quad \text{is odd,} \\ 0 & \text{if} \quad i + j \quad \text{is even.} \end{cases}$$

Then $n_f(0) = \frac{m^2+2m-1}{2}$, $n_f(1) = \frac{m^2+2m+1}{2}$ and hence, $|n_f(0) - n_f(1)| = 1$. The weight of each vertex is

$$\frac{m+1}{2} \equiv \begin{cases} 1 \pmod 2 & \text{if} \quad m \equiv 1 \pmod 4, \\ 0 \pmod 2 & \text{if} \quad m \equiv 3 \pmod 4. \end{cases}$$

Therefore, $K_{m,m}$ is TVMC for odd values of m. □

Lemma 4.3. *$K_{m,m}$ is TVMC if $m \equiv 0 \pmod 4$.*

Proof. Let $m = 4k$. Define $c_{i0} = 0$, $c_{0j} = 0$ and for $i \neq 0$ and $j \neq 0$,

$$c_{ij} = \begin{cases} 1 & \text{if } |i - j| = 0, 1, 2, ..., \frac{m}{4} \text{ and } \frac{3m}{4}, ..., m - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $\mathrm{wt}(v_j) = \mathrm{wt}(u_i) = \frac{m}{2} + 1 = 2k + 1 \equiv 1 \pmod 2$ for all i and j. Also $n_f(0) = n_f(1) = \frac{m^2+2m}{2}$. Thus, $|n_f(0) - n_f(1)| = 0$. Hence, $K_{m,m}$ is TVMC for $m \equiv 0 \pmod 4$. □

Lemma 4.4. *$K_{m,m}$ is TVMC if $m \equiv 2 \pmod 4$.*

Proof. Let $m = 4k+2$. Define $c_{i0} = 0$, $c_{0j} = 1$ and for $i \neq 0$ and $j \neq 0$,

$$c_{ij} = \begin{cases} 1 & \text{if} \quad j \text{ is odd,} \\ 0 & \text{if} \quad j \text{ is even.} \end{cases}$$

Then, $\mathrm{wt}(v_j) = m+1 \equiv 1 \pmod 2$ if j is odd, $\mathrm{wt}(v_j) = 1$ if j is even and $\mathrm{wt}(u_i) = \frac{m}{2} \equiv 1 \pmod 2$. Also $n_f(0) = n_f(1) = \frac{m^2+2m}{2}$ and hence, $|n_f(0) - n_f(1)| = 0$. Thus, $K_{m,m}$ is TVMC for $m \equiv 2 \pmod 4$. □

Lemma 4.5. *$K_{m,n}$ is TVMC if $|m-n| \leq 1$.*

Proof. The proof follows from Lemmas 4.1, 4.2, 4.3 and 4.4. □

§5. Totally vertex-magic cordial(TVMC) labelings of some graphs

J. A. MacDougall et al. [6] proved that not all trees have a vertex-magic total labeling. Also J. A. MacDougall et al. [7] proved that the friendship graph T_n has no vertex-magic total labeling for $n > 3$. In the subsequent theorems we prove all trees are TVMC, the friendship graph T_n for $n \geq 1$ is TVMC and also we examine the totally vertex magic cordiality of flower graph, $P_n + P_2$ and $G + \overline{K}_{2m}$.

Theorem 5.1. *If G is a (p,q) graph with $|p-q| \leq 1$, then G is TVMC with $C = 1$.*

Proof. Assign 0 to all the edges and 1 to all the vertices of G. Then weight of each vertex is 1 and $|n_f(0) - n_f(1)| = |p-q| \leq 1$. Hence, G is TVMC. □

Corollary 5.2. *All cycles($n \geq 3$), trees and unicycle graphs are TVMC with $C = 1$.*

A flower graph Fl_n is constructed from a wheel W_n by attaching a pendant edge at each vertex of the n-cycle and by joining each pendant vertex to the central vertex. We prove that Fl_n admits TVMC labeling.

Theorem 5.3. *The flower graph Fl_n for $n \geq 3$ is TVMC with $C = 0$.*

Proof. Let $V = \{u, u_i, v_i | 1 \leq i \leq n\}$ be the vertex set and $E = \{uu_i, u_iv_i, uv_i | 1 \leq i \leq n\} \cup \{u_ju_{j+1} | 1 \leq j \leq n-1\} \cup \{u_nu_1\}$ be the edge set for $n \geq 3$. Clearly, $|V| = 2n+1$ and $|E| = 4n$. Define $f : V \cup E \to \{0,1\}$ as follows: For $1 \leq i \leq n$, $f(u_i) = 1$, $f(v_i) = 0$, $f(uu_i) = 1$, $f(u_iv_i) = 0$, $f(uv_i) = 0$ and for $1 \leq j \leq n-1$, $f(u_ju_{j+1}) = 1$, $f(u_nu_1) = 1$ and

$$f(u) = \begin{cases} 0 & \text{if} \quad \text{n is even,} \\ 1 & \text{if} \quad \text{n is odd.} \end{cases}$$

We prove that the weight of each vertex is constant modulo 2.

$$\mathrm{wt}(u) = f(u) + \sum_{i=1}^{n} f(uv_i) + \sum_{i=1}^{n} f(uu_i) = \begin{cases} n & \text{if} \quad n \text{ is even}, \\ n+1 & \text{if} \quad n \text{ is odd}. \end{cases}$$

Hence, $\mathrm{wt}(u) \equiv 0 \pmod 2$. Further, for $1 \leq i \leq n$, $\mathrm{wt}(u_i) = 4 \equiv 0 \pmod 2$ and $\mathrm{wt}(v_i) = 0$. Also $|n_f(0) - n_f(1)| \leq 1$. Therefore, Fl_n is TVMC for $n \geq 3$. □

The friendship graph $T_n (n \geq 1)$ consists of n triangles with a common vertex.

Theorem 5.4. *The friendship graph T_n for $n \geq 1$ is TVMC with $C = 0$.*

Proof. Let $V = \{u, u_i, v_i | 1 \leq i \leq n\}$ and $E = \{uu_i, u_iv_i, uv_i | 1 \leq i \leq n\}$ be the vertex set and the edge set, respectively. Define $f : V \cup E \to \{0, 1\}$ as follows: $f(u_i) = 0$, $f(v_i) = 1$ and $f(u) = \begin{cases} 0 & \text{if} \quad n \text{ is even}, \\ 1 & \text{if} \quad n \text{ is odd}. \end{cases}$ For $1 \leq i \leq \lceil \frac{n}{2} \rceil$, $f(uu_i) = 0$, $f(u_iv_i) = 0$, $f(v_iu) = 1$, and for $\lceil \frac{n}{2} \rceil < i \leq n$, $f(uu_i) = 1$, $f(u_iv_i) = 1$ and $f(v_iu) = 0$. It can easily be verified that $\mathrm{wt}(u_i) \equiv \mathrm{wt}(v_i) \equiv \mathrm{wt}(u) \equiv 0 \pmod 2$. Also $n_f(0) = \lceil \frac{5n+1}{2} \rceil$ and $n_f(1) = \lfloor \frac{5n+1}{2} \rfloor$. Hence, $|n_f(0) - n_f(1)| \leq 1$. Therefore, T_n for $n \geq 1$ is TVMC with $C = 0$. □

Let G and H be any two graphs. Let u be any vertex of G and v be any vertex of H. Then $G@H$ is a graph obtained by identifying the vertices u and v.

Theorem 5.5. *If G is TVMC with $C = 1$, then $G@T$ is also TVMC with $C = 1$ for any tree T.*

Proof. Let f be the TVMC labeling of G with $C = 1$. Assign 0 to all the edges and 1 to all the vertices of T. Identify a vertex $u \in V(G)$ with a vertex $v \in V(T)$ and take this new vertex as w. Define a labeling g for $G@T$ as follows:

$$g(a) = \begin{cases} f(a) & \text{if} \quad a \in V(G), \\ 1 & \text{if} \quad a \in V(T) \text{ and } a \neq w, \end{cases}$$

and

$$g(e) = \begin{cases} f(e) & \text{if} \quad e \in E(G), \\ 0 & \text{if} \quad e \in E(T). \end{cases}$$

Then the weight of the identified vertex w is,

$$\begin{aligned}\mathrm{wt}_{G@T}(w) &= g(w) + \sum_{x\in N(w)} g(xw)\\ &= f(u) + \sum_{\substack{x\in N(u)\\ inG}} f(xu) + \sum_{\substack{y\in N(u)\\ inT}} f(yu)\\ &= f(u) + \sum_{\substack{x\in N(u)\\ inG}} f(xu)\\ &= \mathrm{wt}_G(u) \equiv 1 \pmod 2.\end{aligned}$$

For each $a \in V(G@T)$ with $a \neq w$, $\mathrm{wt}_{G@T}(a) = \mathrm{wt}_G(a) \equiv 1 \pmod 2$ if $a \in V(G)$ and $\mathrm{wt}_{G@T}(a) = 1$ if $a \in V(T)$. Also $|n_g(0) - n_g(1)| = |n_f(0) - n_f(1)| \leq 1$. Hence, $G@T$ is also TVMC with $C = 1$. □

The join of two graphs G_1 and G_2 is denoted by $G_1 + G_2$ and it consists of $G_1 \cup G_2$ and all the lines joining $V(G_1)$ with $V(G_2)$.

Theorem 5.6. *$P_n + P_2$ is TVMC for $n \geq 1$.*

Proof. Let $G = P_n + P_2$. We denote the vertices of P_n in G by $u_1, u_2, \ldots, u_n$ and the vertices of P_2 in G by u, v. Then $V(G) = V(P_n) \cup V(P_2)$ and $E(G) = \{uv, u_iu_{i+1} | 1 \leq i \leq n-1\} \cup \{uu_i, vu_i | 1 \leq i \leq n\}$. Clearly $|V(G)| = n+2$ and $|E(G)| = 3n$. Define $f : V(G) \cup E(G) \to \{0, 1\}$ as follows:
Case i. n is odd.
Let $f(u) = f(v) = 0$, $f(u_i) = 0$, $f(uv) = 1$, $f(uu_i) = f(vu_i) = 1$ for $1 \leq i \leq n$ and $f(u_iu_{i+1}) = 0$ for $1 \leq i \leq n-1$. Then

$$\mathrm{wt}(u) = f(u) + f(uv) + \sum_{i=1}^{n} f(uu_i) = 1 + n \equiv 0 \pmod 2,$$

$$\mathrm{wt}(v) = f(v) + f(uv) + \sum_{i=1}^{n} f(vu_i) = 1 + n \equiv 0 \pmod 2$$

and for $1 \leq i \leq n$, $\mathrm{wt}(u_i) = 2 \equiv 0 \pmod 2$. Also $n_f(0) = n_f(1) = 2n+1$. Thus, $|n_f(0) - n_f(1)| = 0$.
Case ii. $n = 2k$ and k is odd.
Let $f(u) = f(v) = 0$, $f(u_i) = 1$, $f(uv) = 1$ for $1 \leq i \leq n$; $f(uu_i) = f(vu_{k+i}) = 1$, $f(uu_{k+i}) = f(vu_i) = 0$ for $1 \leq i \leq k$ and $f(u_iu_{i+1}) = 0$ for $1 \leq i < n$. Hence $\mathrm{wt}(u) = \mathrm{wt}(v) = k+1 \equiv 0 \pmod 2$ and $\mathrm{wt}(u_i) = 2 \equiv 0 \pmod 2$ for $1 \leq i \leq n$. Also $n_f(0) = n_f(1) = 2n+1$. Thus, $|n_f(0) - n_f(1)| = 0$.
Case iii. $n = 2k$ and k is even.

Let $f(u) = f(v) = 0$, $f(u_i) = 1$, $f(uv) = 1$, for $1 \leq i \leq n$; $f(uu_i) = f(vu_i) = 1$, $f(uu_{k+i}) = f(vu_{k+i}) = 0$ for $1 \leq i \leq k$ and $f(u_iu_{i+1}) = 0$ for $1 \leq i < n$. Hence, $\mathrm{wt}(u) = \mathrm{wt}(v) = k+1 \equiv 1 \pmod 2$, $\mathrm{wt}(u_i) = 3 \equiv 1 \pmod 2$ for $1 \leq i \leq k$ and $\mathrm{wt}(u_i) = 1$ for $k+1 \leq i \leq n$. Also $n_f(0) = n_f(1) = 2n+1$. Thus, $|n_f(0) - n_f(1)| = 0$. □

Theorem 5.7. *Let $G(p,q)$ be a TVMC graph with constant $C = 0$ where p is odd. Then $G + \overline{K}_{2m}$ is TVMC with $C = 1$ if m is odd and with $C = 0$ if m is even.*

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_p\}$, $V(\overline{K}_{2m}) = \{v_1, v_2, ..., v_m, ..., v_{2m}\}$ and $E(G + \overline{K}_{2m}) = E(G) \cup \{u_iv_j | 1 \leq i \leq p, 1 \leq j \leq 2m\}$. Let f be the TVMC labeling of G with $C = 0$. Define TVMC labeling g of $G + \overline{K}_{2m}$ as follows: $g(x) = f(x)$ if $x \in V(G) \cup E(G)$, for $1 \leq j \leq p$,

$$g(u_jv_i) = \begin{cases} 0 & \text{if} \quad 1 \leq i \leq m, \\ 1 & \text{if} \quad m < i \leq 2m. \end{cases}$$

When m is odd,

$$g(v_i) = \begin{cases} 1 & \text{if} \quad 1 \leq i \leq m, \\ 0 & \text{if} \quad m < i \leq 2m \end{cases}$$

and when m is even,

$$g(v_i) = \begin{cases} 0 & \text{if} \quad 1 \leq i \leq m, \\ 1 & \text{if} \quad m < i \leq 2m. \end{cases}$$

Now we find the weight of the vertices by considering the following two cases:

Case i. m is odd.

For $v_i \in V(\overline{K}_{2m})$,

$$\mathrm{wt}_{G+\overline{K}_{2m}}(v_i) = g(v_i) + \sum_{j=1}^{p} g(u_jv_i) = 1 \text{ if } 1 \leq i \leq m,$$
$$\mathrm{wt}_{G+\overline{K}_{2m}}(v_i) = p \equiv 1 \pmod 2 \text{ if } m < i \leq 2m$$

and for $u_j \in V(G)$,

$$\begin{aligned} \mathrm{wt}_{G+\overline{K}_{2m}}(u_j) &= \mathrm{wt}_G(u_j) + \sum_{i=1}^{m} g(u_jv_i) + \sum_{i=m+1}^{2m} g(u_jv_i) \\ &= \mathrm{wt}_G(u_j) + m \equiv 1 \pmod 2. \end{aligned}$$

Case ii. m is even.

For $v_i \in V(\overline{K}_{2m})$,

$$\mathrm{wt}_{G+\overline{K}_{2m}}(v_i) = 0 \text{ if } 1 \leq i \leq m,$$
$$\mathrm{wt}_{G+\overline{K}_{2m}}(v_i) = 1 + p \equiv 0 \pmod 2 \quad \text{if } m < i \leq 2m$$

and for $u_j \in V(G)$,

$$\begin{aligned}\mathrm{wt}_{G+\overline{K}_{2m}}(u_j) &= \mathrm{wt}_G(u_j) + \sum_{i=1}^{m} g(u_j v_i) + \sum_{i=m+1}^{2m} g(u_j v_i)\\ &= \mathrm{wt}_G(u_j) + m \equiv 0 \pmod 2.\end{aligned}$$

Also $n_g(0) = n_f(0) + m(p+1)$, $n_g(1) = n_f(1) + m(p+1)$ and hence $|n_g(0) - n_g(1)| = |n_f(0) - n_f(1)| \leq 1$. Therefore, $G + \overline{K}_{2m}$ is TVMC. □

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