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Topology of strongly polar weighted homogeneous links

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Abstract. We consider a canonical S^1 action on S^3 which is defined by $(\rho, (z_1, z_2)) \mapsto (z_1\rho^p, z_2\rho^q)$ for $\rho \in S^1$ and $(z_1, z_2) \in S^3 \subset \mathbb{C}^2$. We consider a link consisting of finite orbits of this action, where some of the orbits are reversely oriented. Such a link appears as a link of a certain type of mixed polynomials. We study the space of such links and show smooth degeneration relations.

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§1. Introduction

Complex hypersurface singularities have been studied by many authors since Milnor proposed so called Milnor fibration theorem ([6]). However for the complement of real algebraic links of real codimension two, the existence of the fibration structure on the complement is not always the case. Second author proposed to study this type of links from complex singularity point of view in [8]. Under a certain *strongly non-degenerate condition on the Newton boundary*, he proved the existence of the fibration. The class of links which come from mixed polynomials contains many interesting links which never comes from complex analytic links.

We consider a mixed polynomial $f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{\nu,\mu} c_{\nu,\mu}\mathbf{z}^\nu\bar{\mathbf{z}}^\mu$ where $\mathbf{z} = (z_1, \dots, z_n)$, $\bar{\mathbf{z}} = (\bar{z}_1, \dots, \bar{z}_n)$, $\mathbf{z}^\nu = z_1^{\nu_1} \cdots z_n^{\nu_n}$ for $\nu = (\nu_1, \dots, \nu_n)$ (respectively $\bar{\mathbf{z}}^\mu = \bar{z}_1^{\mu_1} \cdots \bar{z}_n^{\mu_n}$ for $\mu = (\mu_1, \dots, \mu_n)$).

Definition 1. We say $f(\mathbf{z}, \bar{\mathbf{z}})$ is *a mixed weighted homogeneous polynomial of radial weight type* $(q_1, \dots, q_n; d_r)$ *and of polar weight type* $(p_1, \dots, p_n; d_p)$ if

$$\sum_{j=1}^n q_j(\nu_j + \mu_j) = d_r, \quad \sum_{j=1}^n p_j(\nu_j - \mu_j) = d_p, \quad \text{if } c_{\nu,\mu} \neq 0.$$

Let f be a mixed weighted homogeneous polynomial. Using a polar coordinate (r,η) of $\mathbb{C}^*$ where $r>0$ and $\eta\in S^1$ with $S^1=\{\eta\in\mathbb{C}\,|\,|\eta|=1\}$, we define *a polar* $\mathbb{C}^*$*-action* on $\mathbb{C}^n$ by

$$(r,\eta)\circ\mathbf{z}=(r^{q_1}\eta^{p_1}z_1,\dots,r^{q_n}\eta^{p_n}z_n),\quad (r,\eta)\in\mathbb{R}^+\times S^1$$
$$(r,\eta)\circ\bar{\mathbf{z}}=\overline{(r,\eta)\circ\mathbf{z}}=(r^{q_1}\eta^{-p_1}\bar{z}_1,\dots,r^{q_n}\eta^{-p_n}\bar{z}_n).$$

More precisely, it is a $\mathbb{R}_+\times S^1$-action. Then f satisfies the functional equality

$$f((r,\eta)\circ(\mathbf{z},\bar{\mathbf{z}}))=r^{d_r}\eta^{d_p}f(\mathbf{z},\bar{\mathbf{z}}). \tag{1.1}$$

This notion was introduced by Ruas-Seade-Verjovsky [12] and Cisneros-Molina [3].

A mixed polynomial $f(\mathbf{z},\bar{\mathbf{z}})$ is called *strongly polar weighted homogeneous* if the polar weight and the radial weight coincide, i.e., $p_j=q_j$, $1\le j\le n$.

In this case, the $\mathbb{C}^*$ action is simply defined by

$$\zeta\circ\mathbf{z}=(z_1\zeta^{p_1},\dots,z_n\zeta^{p_n}),\quad \zeta\in\mathbb{C}^*.$$

In this paper, we study the geometry of the links defined by strongly polar weighted homogeneous mixed polynomials.

§2. Cobordism of links

First of all we have to point out that the topology of mixed links is very particular and we recall some classical results and definitions in the case of knots and algebraic links.

Let K be a closed $(2k-1)$-dimensional manifold embedded in the $(2k+1)$-dimensional sphere S^{2k+1}. We suppose that K is $(k-2)$-connected if $k\ge 2$. When K is orientable, we further assume that it is oriented. Then we call K or its (oriented) isotopy class an $2k-1$-*knot*.

First, recall that a manifold with boundary Y embedded in a manifold X with boundary is said to be *properly embedded* if $\partial Y=\partial X\cap Y$ and Y is transverse to ∂X, then we define

Definition 2. Two $(2k-1)$-knots K_0 and K_1 in S^{2k+1} are said to be *cobordant* if there exists a properly embedded $(2k)$-dimensional manifold X of $S^{2k+1}\times[0,1]$ such that

(1) X is diffeomorphic to $K_0\times[0,1]$, and

(2) $\partial X=(K_0\times\{0\})\cup(K_1\times\{1\})$.

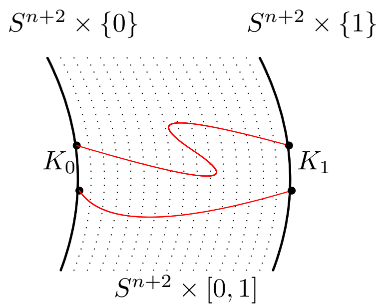

Figure 1: A cobordism between K_0 and K_1

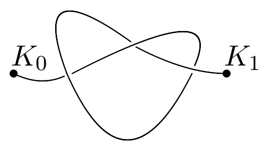

Figure 2: A cobordism which is not an isotopy

The manifold X is called a *cobordism* between K_0 and K_1. When the knots are oriented, we say that K_0 and K_1 are *oriented cobordant* (or simply *cobordant*) if there exists an oriented cobordism X between them such that

$$\partial X = (-K_0 \times \{0\}) \cup (K_1 \times \{1\}),$$

where $-K_0$ is obtained from K_0 by reversing the orientation.

It is clear that isotopic knots are always cobordant. However, the converse is not true in general (see Fig. 2).

For a classification of high dimensional knots up to cobordism we refer to [2].

Let us study one example of dimensional one links. We denote by T^+ and T^- respectively the one dimensional right and the left trefoil knots (which are both mixed links). We know that T^+ and T^- are cobordant, see [11] p. 219 ; but let us give here the idea of the proof.

Precisely, we denote by S^3_+ (resp. S^3_-) the upper (resp. lower) hemisphere of the unit 3-sphere $\partial D^4 = S^3 \hookrightarrow \mathbf{R}^4$. Set $\mathcal{E}$ be the equatorial hyperplane of D^4, and let $\pi : \mathbf{R}^4 \to \mathcal{E}$ the orthogonal projection onto $\mathcal{E}$.

One can suppose that T^+ and T^-, which is the mirror image of T^+, are

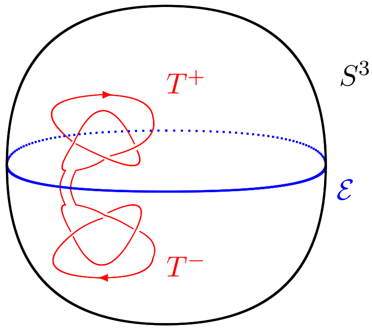

Figure 3: The connected sum of the trefoil knot and its inverse in S^3

embedded in S^3_+ and S^3_- respectively such that

$$T^- = -\big(\pi(T^+) \times [0,1]\big) \cap S^3_-.$$

(In the last formula, the sign is necessary to have the right orientation.)

Then we construct the connected sum $\mathcal{O} = T^+ \# T^-$ of T^+ and T^- in S^3; we illustrate this construction in Fig. 3.

Set $\tilde{T}^+$ (resp. $\tilde{T}^-$) the intersection $\tilde{T}^+ = \mathcal{O} \cap S^3_+$ (resp. $\tilde{T} = \mathcal{O} \cap S^3_-$). One can assume that the connected sum $\mathcal{O}$ is made in order to have

$$\tilde{T}^- = -\big(\pi(\tilde{T}^+) \times [0,1]\big) \cap S^3_-.$$

Now, if we denote

$$\mathcal{D} = \big(\pi(\tilde{T}^+) \times [0,1]\big) \cap D^4,$$

then $\mathcal{D}$ is homeomorphic to a 2-disk since $\pi(\tilde{T}^+)$ is a 1-disk. Moreover $\partial\mathcal{D} = \mathcal{O} = T^+ \# T^-$. Since the knots T^+ and T^- are homeomorphic to a sphere, then to prove that they are cobordant it is sufficient to prove that their connected sum bounds a disk [5]. But $\mathcal{O}$ bounds a 2-disk embedded in D^4 then $\mathcal{O}$ is null cobordant, and, T^+ and T^- are cobordant.

In [4] D. T. Lê proved that the Alexander polynomial determines the topological type of the link of an isolated singularity of a complex analytic curve and moreover he proved that cobordant links are isotopic since the product of their Alexander polynomials is a square.

In the case of mixed links things are different. For example the two trefoil knots T^+ and T^- are cobordant but not isotopic mixed links. Recall that they are not isotopic since they have distinct Jones polynomials.

Remark 3. Moreover, since the trivial knot $\mathcal{O}$ is a mixed link, then the connected sum of mixed one dimensional links can be a mixed link contrary to the classical case as proved by N. A'Campo [1].

§3. Strongly polar weighted homogeneous links

Hereafter we consider strongly polar weighted homogeneous polynomial $f(\mathbf{z}, \bar{\mathbf{z}})$ of two variables i.e., $n = 2$ with weight vector $P = {}^t(p, q)$. Here we assume that $\gcd(p, q) = 1$. We assume that f is convenient and non-degenerate so that the link $L = f^{-1}(0) \cap S^3$ is smooth. Let $\mathcal{M}(P; d_p)$ be the space of strongly polar weighted homogeneous mixed polynomials of the polar degree d_p, which is non-degenerate convenient and let $\mathcal{L}(P; d_p)$ be the associated oriented links. Hereafter we denote simply $\mathcal{M}, \mathcal{L}$ for $\mathcal{M}(P; d_p)$ and $\mathcal{L}(P; d_p)$ respectively. We have a canonical mapping $\pi : \mathcal{M} \to \mathcal{L}$ defined by $\pi(f)$ the link defined by $f^{-1}(0) \cap S^3$. A difficulty in the mixed polynomial situation is that for a fixed link, there exist an infinitely many mixed polynomials which define the link.

Let d_r, d_p be the radial and polar degrees respectively. As f is assumed to be convenient, f contains monomials $z_1^{a_1} \bar{z}_1^{b_1}$ and $z_2^{a_2} \bar{z}_2^{b_2}$ such that

$$p(a_1 + b_1) = q(a_2 + b_2) = d_r, \quad p(a_1 - b_1) = q(a_2 - b_2) = d_p.$$

Therefore

$$\frac{p}{q} = \frac{a_2}{a_1} = \frac{b_2}{b_1}$$

and we see that $p|a_2, b_2$ and $q|a_1, b_1$ and thus $pq|d_r, d_p$. As our link is S^1 invariant, its component is a finite union of orbits of the action. Recall that the associated S^1-action is defined by

$$S^1 \times S^3 \to S^3,\ (\rho, (z_1, z_2)) \mapsto (z_1\rho^p, z_2\rho^q),\ \rho \in S^1$$

Let $P = (p, q)$ be the primitive weight vector of f. P is fixed throughout this paper. Note that L is stable under the action, by the Euler equiality

$$f(\rho \circ \mathbf{z}) = \rho^{d_p} f(\mathbf{z}).$$

Two orbit $z_1 = 0$ and $z_2 = 0$ are singular but by the covenience assumption, our link has only regular orbits.

3.1. Coordinates of the orbits

Take a regular orbit L. We can take a point $X = (\beta_1, \beta_2) \in L \subset S^3 \subset \mathbb{C}^2$ such that β_1 is a positive number. β_1 and $|\beta_2|$ are unique by L but β_2 is not unique. The umbiguity is the action of $\mathbb{Z}/p\mathbb{Z}$. Thus $|\beta_2| = \sqrt{1 - \beta_1^2}$ and the argument of β_2 is unique mudulo $2\pi/p$. Thus the space of the regular orbits is isomorphic to the punctured disk $\Delta^* := \{\xi = r\rho \in \mathbb{C} \,|\, 0 < r < 1,\ \rho \in S^1\}$, by the correspondence $\beta_2 \mapsto \beta_2^p \in \Delta^*$. For $u = r^p\, e^{i\theta} \in \Delta_p^*$, we associate the regular orbit

$$K(u) := \{(\rho^p \sqrt{1 - r^2}, \rho^q r e^{i\theta/p}) \,|\, \rho \in S^1\}, \quad u = re^{i\theta} \in \Delta^*. \tag{3.1}$$

Consider a strongly polar weighted homogeneous polynomial for arbitrary non-negative integer k:

$$(3.2)\quad \begin{cases} \ell_{u,k}(\mathbf{z}) & := z_1^{q+kq}\bar{z}_1^{kq} - \alpha_{u,k} z_2^{p+kp}\bar{z}_2^{kp} = z_1^q \|z_1^q\|^{2k} - \alpha_{u,k} z_2^p \|z_2^p\|^{2k} \\ \bar{\ell}_{u,k}(\mathbf{z}) & := \bar{z}_1^q \|z_1^q\|^{2k} - \overline{\alpha}_{u,k} \bar{z}_2^p \|z_2^p\|^{2k} \quad \text{where} \end{cases}$$

$$(3.3)\qquad \alpha_{u,k} = \tfrac{(1-r^2)^{q(1/2+k)}}{r^{p(1+2k)}e^{i\theta}}.$$

Note that the polar degrees of $\alpha_{u,k}$ are pq but the radial degrees are different and they are given as

$$\operatorname{rdeg} \ell_{u,k} = (2k+1)pq.$$

Observation 1. *The polynomials $\ell_{u,k}$ define $K(u)$ and $\bar{\ell}_{u,k}$ defines $K(u)$ with reversed orientation for any $k = 0, 1, \dots$*

Hereafter we simply use the notation:

$$\ell_u(\mathbf{z}) := \ell_{u,0}(\mathbf{z}) = z_1^q - \frac{(1-r^2)^{q/2}}{r^p e^{i\theta}} z_2^p, \; u = r^p\, e^{i\theta}.$$

Let $\mathcal{L}(P; dpq, r)$ be the subspace of $L(P; dpq)$ which has $d + 2r$ components where r components are negatively oriented. First we prepare the next lemma:

Lemma 2. *The moduli space $\mathcal{L}(P; dpq, r)$ is connected and therefore any two links of this moduli has the same topology.*

Proof. Note that $\mathcal{L}(P; dpq, r)$ are parametrized by

$$M_{d,r} := (\Delta^*)^{d+2r} \setminus \Xi$$

where $\Xi = \{\mathbf{u} = (u_1, \dots, u_{d+2r}) \in \Delta^{*(d+2r)} \,|\, u_i = u_j \,(\exists i, j,\, i \neq j)\}$. Thus it is easy to see that M_{d+2r} is connected. $\mathbf{u}$ corresponds to the link $\cup_{i=1}^{d+2r} K(u_i)$ where $K(u_j)$ are reversely oriented for $j = d + r + 1, \dots, d + 2r$. □

3.2. Typical degeneration

We consider an important degeneration of links $L(t)$, $t \in \mathbb{C}$ which is defined by the family of strongly polar weighted homogeneous polynomials:

$$f(\mathbf{z}, \bar{\mathbf{z}}, t) = -2z_2^{2p}\bar{z}_2^p + z_1^{2q}\bar{z}_1^q + t z_2^{2p}\bar{z}_1^q.$$

Using Proposition 1 ([7]), we see that the degeneration locus is given as the following real semi-algebraic variety

$$\Sigma := \{t \in \mathbb{C} \,|\, t = \frac{2s-1}{s^2}, \exists s \in S^1\}$$

Figure 1 shows the graph of Σ. Let Ω be the bounded region surrounded by Σ. By Example 59 in [8], we can see the following.

Proposition 3. *For any $t \in \Omega$, $L(t)$ has one link component, while for $t \in \mathbb{C} \setminus \bar{\Omega}$(= the outside of Σ), $L(t)$ has three components.*

Proof. Let us consider the weighted projective space $\mathbb{P}^1(P) := \mathbb{C}^2 \setminus \{O\}/\mathbb{C}^*$ by the above $\mathbb{C}^*$-action. For $U := \mathbb{P}^1(P) \cap \{z_1 z_2 \neq 0\}$, it is easy to see that $u := z_2^p/z_1^q$ is a coordinate function. Our link corresponds to the solutions (=zero points) of

$$-2u^2\bar{u} + tu^2 + 1 = 0$$

and there exists one solution (respectively 3 solutions) for each $t \in \Omega$ (resp. $t \notin \bar{\Omega}$). See Example 59, [8] or [9]. □

We consider the point $-3 \in \Sigma$ which is a smoot point of Σ. There are two components for $L(-3)$ ($u = 1/2$ and $u = -1$) and the component passing through $(1, e^{i\pi/p})$ is a doubled component. Here we are considering the link on the sphere of radius $\sqrt{2}$, $S_{\sqrt{2}}$ for simplicity. Let us consider the variety:

$$\mathcal{W} = \{(z_1, z_2, t) \in S_{\sqrt{2}} \times \mathbb{R} \,|\, -3 - \varepsilon \leq t \leq -3 + \varepsilon,\ f(\mathbf{z}, \bar{\mathbf{z}}, t) = 0\}, \quad \varepsilon \ll 1.$$

The following is the key assertion.

Lemma 4. *$\mathcal{W}$ is a smooth manifold with boundary $L(-3-\varepsilon) \cup -L(-3+\varepsilon)$.*

Proof. Let $f(\mathbf{z}, \bar{\mathbf{z}}, t) = g(\mathbf{z}, \bar{\mathbf{z}}, t) + i\, h(\mathbf{z}, \bar{\mathbf{z}}, t)$. We assert that $\mathcal{W}$ is a complete intersection variety. For this purpose, we show that three 1-forms $dg, dh, d\rho$ are independent on $L(-3)$, where $\rho(\mathbf{z}) = \|\mathbf{z}\|^2$. As the polynomial f is strongly polar weighted homogeneous, it is enough to check the assertion on a point $\tilde{\mathbf{z}}_0 = (1, \alpha, -3) \in W$ where $\alpha = e^{i\pi/p}$. For the calculation's simplicity, we use the base $\{dz_1, d\bar{z}_1, dz_2, d\bar{z}_2, dt\}$ of the complexified cotangent space. Using the equalities $g = (f + \bar{f})/2$, $h = (f - \bar{f})/(2i)$, we get

$$\begin{pmatrix} dg(\tilde{\mathbf{z}}_0) \\ dh(\tilde{\mathbf{z}}_0) \\ d\rho(\tilde{\mathbf{z}}_0) \end{pmatrix} = A \begin{pmatrix} dz_1 \\ d\bar{z}_1 \\ dz_2 \\ d\bar{z}_2 \\ dt \end{pmatrix} \qquad \text{where}$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ -2\,iq & 2\,iq & 2\,ip\bar{\alpha} & -2\,ip\alpha & 0 \\ 1 & 1 & \bar{\alpha} & \alpha & 0 \end{bmatrix}$$

Thus it is easy to see that $\operatorname{rank} \mathrm{A} = 3$. □

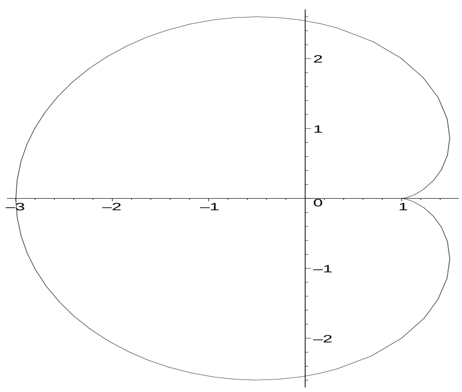

Figure 4: Σ

3.3. Milnor fibrations

Take $\mathbf{u} = (u_1, \dots, u_{d+2r}) \in M_{d,r}$ and consider the corresponding link $L(\mathbf{u}) = \cup_{j=1}^{d+2r} K(u_j)$ with $d+2r$ components and the last r components are negatively oriented. Let $f(\mathbf{z})$ be a strongly polar weighted homogeneous polynomial which defines $L(\mathbf{u})$ with $\operatorname{pdeg} f = dpq$ and $\operatorname{rdeg} f = (d+2s)pq$ with $s \geq r$. For example, we can take

$$g(\mathbf{z}) = \ell_{u_1,s-r}(\mathbf{z}) \prod_{j=2}^{d+r} \ell_{u_j}(\mathbf{z}) \prod_{j=d+r+1}^{d+2r} \bar{\ell}_{u_j}(\mathbf{z}).$$

Let F be the Milnor fiber of f: $F = \{\mathbf{z} \in S^3 \mid f(\mathbf{z}) > 0\}$. As we assume that $L(\mathbf{u})$ has no singular orbit, $f(\mathbf{z})$ is a convenient mixed polynomial. Thus it contains monomials $z_1^{(d+s)q} \bar{z}_1^{qs}$ and $z_2^{(d+s)p} \bar{z}_2^{ps}$. The monodromy $h : F \to F$ is defined by $h(\mathbf{z}) = e^{2\pi i/dpq} \circ \mathbf{z}$ and it is the restriction of S^1-action to $\mathbb{Z}_{dpq} \subset S^1$. Thus we have a commutative diagram:

$$\begin{array}{ccc} F & \hookrightarrow & S^3 \setminus L(\mathbf{u}) \\ & \searrow \xi & \downarrow \pi \\ & & \mathbb{P}^1(P) \setminus W \end{array}$$

where W is $d+2r$ points corresponding to the components of $L(\mathbf{u})$ and π, ξ are canonical quotient mapping by S^1 and $\mathbb{Z}_{dpq}$ respectively. As F is a $\mathbb{Z}_{dpq}$ cyclic covering over $\mathbb{P}^1 \setminus W$, with two singular points $(0,1)$ and $(1,0)$. Over these two points, the corresponding fibers are q, p points respectively. Thus we have

Proposition 5. *(cf. Theorem 65,[8]) The Euler charactersitic of F is given as*

$$\chi(F) = -(d+2r)dpq + p + q$$

Note that $\chi(F)$ depends on the number of components $d+2r$ but it does not depend on the radial degree $(d+2s)pq$. Thus we see that, under fixed polar and radial degrees, there are $s+1$ different topologies among their Milnor fibrations. The components types can be $d+2r$, $r=0,\dots,s$.

§4. Main result

Consider a smooth family of strongly polar weighted homogeneous links $L(t) \in \mathcal{L}(P;dpq), 0 \le t \le 1$ with weight $P = {}^t(p,q)$ such that

(1) the variety $W = \{(\mathbf{z},t) \in S^3 \times [0,1] \,|\, \mathbf{z} \in L(t)\}$ is a smooth variety of codimension two.

(2) There exists t_0 such that $0 < t_0 < 1$ and

$$\begin{cases} L(t) \in \mathcal{L}(P;dpq,r-1) & t < t_0 \\ L(t) \in \mathcal{L}(P;dpq,r) & t > t_0 \end{cases}$$

The link $L(t_0)$ is singular. One component is the limit of two components with opposite orientations. We call such a family *a smooth elimination of a pair of links.*

Theorem 6. *For any link $L \in \mathcal{L}(P;dpq,r)$ with $r>0$, there exists a smooth elimination family $L(t)$ of a pair of links with $L(0)=L$ and $L(1) \in \mathcal{L}(P; dpq, r-1)$.*

Proof. First represent L by an explicit mixed polynomial described in §3.3. Choose two positive components and one negative component. By the connectivity of the moduli space, we may assume that these three components are descrived by $L(-3)$ in the explicit family $L(t)$ in §3.2. So we can write $L = L' \cup L(-3)$. We apply the degenration process using $L(t)$. For this purpose, we may assume that other components in L' does not take any components of $L(t)$ for $-3 \le t \le 0$. Then it is easy to see that $L = L' \cup L(-3)$ degenerate into $L' \cup L(0)$ in which two components has disappeared. □

Corollary 7. *For any link $L \in \mathcal{L}(P;dpq,r)$ with $r>0$, r pairs of links with opposite orientations can be eliminated successively to a link $L' \in \mathcal{L}(P;dpq,0)$ of positive link. L' is isomorphic to a holomorphic torus link defined by*

$$z_1^{qd} - z_2^{pd} = 0.$$

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