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# Symmetry and asymmetry models and decompositions of models for contingency tables

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**Abstract.** For analyzing square contingency tables, Bowker [14] proposed the symmetry model. Caussinus [16] proposed the quasi-symmetry model and gave a decomposition of model such that the symmetry model holds if and only if both the quasi-symmetry and the marginal homogeneity models hold. Bhapkar and Darroch [13] gave the similar theorem for multi-way contingency tables. For square tables and for multi-way tables, the present paper (1) reviews various models of symmetry and asymmetry, (2) reviews the decompositions of models, (3) gives some figures which indicate the relationships among various models, and (4) gives a new decomposition of symmetry model.

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## §1. Introduction

For the analysis of two-way contingency tables, we are usually interested in whether or not the independence between the row and column classifications holds. However, for the analysis of square contingency tables with the same row and column classifications, we are interested in whether or not the row classification is symmetric with the column classification, instead of the independence, and how the row classification is symmetric or asymmetric with the column classification, because in square contingency tables there is a strong association between two classifications and there is not statistical independence between them.

Consider an $r \times r$ square (i.e., two-way) contingency table with the same row and column classifications. Let $X_1$ and $X_2$ denote the row and column variables, respectively. Let $p_{ij}$ denote the probability that an observation will fall in the $i$th row and $j$th column of the table $(i = 1, \ldots, r; j = 1, \ldots, r)$. Note that $\{p_{ij}\}$ are unknown. We are interested in various models which indicate the structure of $\{p_{ij}\}$. As one of models of various kinds of symmetry, Bowker [14] considered the symmetry model, which indicates the structure of symmetry for cell probabilities $\{p_{ij}\}$. Stuart [42] gave the marginal homogeneity model for the marginal probabilities of $X_1$ and $X_2$. Caussinus [16] considered the quasi-symmetry model for $\{p_{ij}\}$. Also many models, which describe the structures of various asymmetry, are proposed; for instance, McCullagh's [35] conditional symmetry model, Goodman's [18] diagonals-parameter symmetry model, Agresti's [1] linear diagonals-parameter symmetry model, Agresti's [5, p.429] ordinal quasi-symmetry model, Tomizawa's [62] extended quasi-symmetry model and extended marginal homogeneity model, Tomizawa's [74] cumulative diagonals-parameter symmetry model, and Tahata and Tomizawa's [50] generalized marginal homogeneity model, etc.

Caussinus [16] gave the decomposition of the symmetry model such that the symmetry model holds if and only if both the quasi-symmetry and the marginal homogeneity models hold. Tomizawa [62] gave the decomposition of the conditional symmetry model into the extended quasi-symmetry, the extended marginal homogeneity, and the other models. The decompositions of some symmetry and asymmetry models are given (see Section 3).

Next consider the multi-way contingency tables. For these tables, the symmetry, the quasi-symmetry and the marginal symmetry models are also considered. For example, see Bishop, Fienberg and Holland [15, pp.299-309], Bhapkar and Darroch [13], Agresti [5, p.440], and Tomizawa and Tahata [89]. For multi-way contingency tables, some asymmetry models are proposed; for example, see Yamamoto, Iwashita and Tomizawa [93], Tahata, Yamamoto and Tomizawa [60, 61], and Tahata and Tomizawa [54] (see Section 6). In these articles, the decompositions of the symmetry and asymmetry models in the multi-way tables are given (see Section 7).

The purpose of the present paper is (1) to review various models of symmetry and asymmetry for square contingency tables (Section 2), (2) to review the decompositions of models for square tables (Section 3), (3) to give the figures which indicate the relationships among various models for square tables (Section 4), (4) to give a new decomposition of symmetry model (Section 5), (5) to review models of symmetry for multi-way contingency tables (Section 6), (6) to review the decompositions of models for multi-way tables (Section 7), and (7) to give the figure which indicates the relationships among various quasi-symmetry models for multi-way tables (Section 8).

## §2. Models for square contingency tables

This section reviews various models of symmetry and asymmetry. Consider an $r \times r$ square contingency tables with the same row and column classifications.

### 2.1. Symmetry models

The symmetry (S) model, which was given by Bowker [14], is defined by

$$p_{ij} = \psi_{ij} \quad (i = 1, \ldots, r; j = 1, \ldots, r),$$

where $\psi_{ij} = \psi_{ji}$. This model indicates that the probability that an observation will fall in row category $i$ and column category $j$ is equal to the probability that the observation falls in row category $j$ and column category $i$. Namely, this describes a structure of symmetry of the cell probabilities $\{p_{ij}\}$ with respect to the main diagonal of the table. For the S model see also Bishop et al. [15, p.282], Caussinus [16], McCullagh [34], Goodman [18, 20], Bhapkar [12], van der Heijden, Falguerolles and Leeuw [90], van der Heijden and Mooijaart [91], Agresti and Natarajan [7], Agresti [5, p.424], Andersen [9, p.320], Tomizawa and Tahata [89], and Tomizawa [79], etc.

When we express $\{p_{ij}\}$ as the log-linear model,

$$\log p_{ij} = \lambda + \lambda_1(i) + \lambda_2(j) + \lambda_{12}(ij) \quad (i = 1, \ldots, r; j = 1, \ldots, r), \tag{2.1}$$

the S model can be expressed as equation (2.1) with $\{\lambda_1(i) = \lambda_2(i)\}$ and $\{\lambda_{12}(ij) = \lambda_{12}(ji)\}$; see Bishop et al. [15, p.282].

Caussinus [16] considered the quasi-symmetry (QS) model defined by

$$p_{ij} = \alpha_i \beta_j \psi_{ij} \quad (i = 1, \ldots, r; j = 1, \ldots, r),$$

where $\psi_{ij} = \psi_{ji}$. A special case of this model obtained by putting $\{\alpha_i = \beta_i\}$ is the S model. By putting $\{\gamma_j = \beta_j/\alpha_j\}$ and $\{\phi_{ij} = \alpha_i \alpha_j \psi_{ij}\}$, the QS model may be expressed as

$$p_{ij} = \gamma_j \phi_{ij} \quad (i = 1, \ldots, r; j = 1, \ldots, r),$$

where $\phi_{ij} = \phi_{ji}$. Thus this indicates

$$\frac{p_{ij}}{p_{ji}} = \frac{\gamma_j}{\gamma_i} \quad (i \neq j).$$

Note that we may set $\gamma_1 = 1$ without loss of generality. The QS model can be expressed as equation (2.1) with $\{\lambda_{12}(ij) = \lambda_{12}(ji)\}$. Denote the odds ratio for rows $i$ and $j$ ($> i$), and columns $s$ and $t$ ($> s$) by $\theta_{ij;st}$, where $\theta_{ij;st} = (p_{is}p_{jt})/(p_{js}p_{it})$. The QS model is expressed as

$$\theta_{ij;st} = \theta_{st;ij} \quad (i < j; s < t).$$

Thus the QS model has characterization in terms of symmetry of odds ratios. The QS model also may be expressed as

$$p_{ij}p_{jk}p_{ki} = p_{ji}p_{kj}p_{ik} \quad (1 \le i < j < k \le r).$$

For the QS model, see also, e.g., Agresti [1, 4], Agresti and Lang [6], Bishop et al. [15, p.286], Goodman [18], Bhapkar [12], Bhapkar and Darroch [13], Becker [10], McCullagh [36], Haberman [21, p.490], Plackett [39, p.78], and Tomizawa and Tahata [89], etc.

The marginal homogeneity (MH) model is defined by

$$p_{i\cdot} = p_{\cdot i} \quad (i = 1, \dots, r),$$

where

$$p_{i\cdot} = \sum_{t=1}^{r} p_{it}, \quad p_{\cdot i} = \sum_{s=1}^{r} p_{si}.$$

See, e.g., Stuart [42], Bhapkar [11], Bishop et al. [15, p.294], and Agresti [2]. The MH model indicates that the row marginal distribution is identical to the column marginal distribution. Note that the S model implies the MH model.

Kateri and Papaioannou [28] introduced the generalized quasi-symmetry model (denoted by QS[f]). The QS[f] model is defined by

$$p_{ij} = p_{ij}^{S} F^{-1}(\alpha_i + \gamma_{ij}) \quad (i = 1, \dots, r; j = 1, \dots, r),$$

where $\gamma_{ij} = \gamma_{ji}$, $2p_{ij}^{S} = p_{ij} + p_{ji}$, $F(u) = f'(u)$, and $f$ is a twice-differentiable and strictly convex function on $(0, +\infty)$ with $f(1) = 0$, $f(0) = \lim_{\mu \to 0} f(\mu)$, $0 \cdot f(0/0) = 0$, $0 \cdot f(\mu/0) = \mu f_{\infty}$ with $f_{\infty} = \lim_{\mu \to \infty}[f(\mu)/\mu]$. When $f(u) = u \log u$ ($u > 0$) (i.e., $F^{-1}(x) = e^{x-1}$), the QS[f] model can be expressed as

$$p_{ij} = p_{ij}^{S} \frac{2a_i}{a_i + a_j}$$

where $a_i = \exp(\alpha_i - 1)$. This is identical to the QS model. Note that Kateri and Agresti [27] introduced the simple QS[f] model with $\{\alpha_i\}$ replaced by $\{\alpha u_i\}$ using the known scores $u_1 \le u_2 \le \cdots \le u_r$ (with $u_1 < u_r$).

Goodman [19] and Agresti [3] considered various association models. Especially for analyzing square contingency tables with the same row and column classifications, it may be useful to use the quasi-association models which are defined only off the main diagonal cells. Goodman [19] gave the quasi-uniform association (QU) model defined by

$$p_{ij} = \begin{cases} \alpha_i \beta_j \theta^{ij} & (i \neq j), \\ \psi_{ii} & (i = j). \end{cases}$$

A special case of the QU model obtained by putting $\theta = 1$ is the quasi-independence (quasi null association) model. The QU model is a special case of QS model. Using the known scores $u_1 < u_2 < \cdots < u_r$, Agresti [3] introduced the quasi linear-by-linear association (QLL) model defined by

$$p_{ij} = \begin{cases} \alpha_i \beta_j \theta^{u_i u_j} & (i \neq j), \\ \psi_{ii} & (i = j). \end{cases}$$

This is also a special case of QS model.

Goodman [20] introduced the symmetry plus quasi-independence (SQI) model defined by

$$p_{ij} = \begin{cases} \alpha_i \alpha_j & (i \neq j), \\ \psi_{ii} & (i = j). \end{cases}$$

This model is a special case of the S model obtained by putting $\{\psi_{ij} = \alpha_i \alpha_j\}$, $i \neq j$. Goodman [20] also introduced various generalized independence models and generalized symmetry plus independence models: for example, the triangle non-symmetry plus independence (T) model is defined by

$$p_{ij} = \begin{cases} \alpha_i \alpha_j \tau_1 & (i < j), \\ \alpha_i \alpha_j \tau_2 & (i > j), \\ \psi_{ii} & (i = j). \end{cases}$$

Note that the T model is a special case of the conditional symmetry model in Section 2.2, and the SQI model is a special case of the T model.

Yamamoto and Tomizawa [101] proposed the symmetry plus quasi-uniform association (SQU) model defined by

$$p_{ij} = \begin{cases} \alpha_i \alpha_j \theta^{ij} & (i \neq j), \\ \psi_{ii} & (i = j). \end{cases}$$

This model is an extension of the SQI model. The SQU model is a special case of the S model, and a special case of the QU model.

## 2.2. Asymmetry models for cell probabilities

This section describes some models which indicate the structure of asymmetry although each model in Section 2.1 indicates the structure of symmetry.

The conditional symmetry (CS) model, which was given by McCullagh [35], is defined by

$$p_{ij} = \begin{cases} \delta\psi_{ij} & (i < j), \\ \psi_{ij} & (i \geq j), \end{cases}$$

where $\psi_{ij} = \psi_{ji}$. A special case of this model obtained by putting $\delta = 1$ is the S model. Note that the CS model is equivalent to Read's [40] proportional symmetry model and to a log-linear model by Bishop et al. [15, pp.285-286]. The CS model may be expressed as

$$P(X_1 = i, X_2 = j | X_1 < X_2) = P(X_1 = j, X_2 = i | X_1 > X_2) \quad (i < j);$$

see McCullagh [35].

Goodman [18] proposed the diagonals-parameter symmetry (DPS) model defined by

$$p_{ij} = \begin{cases} \delta_{j-i}\psi_{ij} & (i < j), \\ \psi_{ij} & (i \geq j), \end{cases}$$

where $\psi_{ij} = \psi_{ji}$. A special case of this model obtained by putting $\{\delta_{j-i} = \delta\}$ is the CS model.

Tomizawa [73] proposed the diagonal uniform association symmetry (DUS) model defined by

$$p_{ij} = \begin{cases} \delta_{j-i}\phi_{j-i}^{i-1}\psi_{ij} & (i < j), \\ \psi_{ij} & (i \geq j), \end{cases}$$

where $\psi_{ij} = \psi_{ji}$ (also see Tomizawa and Miyamoto [81]). A special case of this model obtained by putting $\phi_1 = \cdots = \phi_{r-2} = 1$ is the DPS model.

The linear diagonals-parameter symmetry (LDPS) model, which was given by Agresti [1], is defined by

$$p_{ij} = \begin{cases} \delta^{j-i}\psi_{ij} & (i < j), \\ \psi_{ij} & (i \geq j), \end{cases}$$

where $\psi_{ij} = \psi_{ji}$. A special case of this model obtained by putting $\delta = 1$ is the S model. Also the LDPS model is a special case of QS model, which may be expressed as equation (2.1) with $\{\lambda_1(i) = i\lambda_1\}$ and $\{\lambda_2(j) = j\lambda_2\}$. When we assign known scores $u_1 < \cdots < u_r$ to the categories, the LDPS model with $\delta^{j-i}$ replaced by $\delta^{u_j - u_i}$ is the ordinal quasi-symmetry (OQS) model (Agresti [5, p.429]).

Tomizawa [68] proposed the two-ratios-parameter symmetry (2RPS) model, defined by

$$p_{ij} = \begin{cases} \gamma\delta^{j-i}\psi_{ij} & (i < j), \\ \psi_{ij} & (i \geq j), \end{cases}$$

where $\psi_{ij} = \psi_{ji}$ (also see Tahata and Tomizawa [52]). Special cases of this model obtained by putting $\delta = 1$ and $\gamma = 1$ are the CS and LDPS models, respectively.

Tomizawa [71] proposed the polynomial diagonals-parameter symmetry (PDPS) model defined by

$$p_{ij} = \begin{cases} \left(\prod_{k=0}^{r-2} \theta_k^{(j-i)^k}\right) \psi_{ij} & (i < j), \\ \psi_{ij} & (i \geq j), \end{cases}$$

where $\psi_{ij} = \psi_{ji}$. Special cases of this model obtained by putting $\theta_0 = \theta_1 = \cdots = \theta_{r-2} = 1$, $\theta_1 = \cdots = \theta_{r-2} = 1$, $\theta_0 = \theta_2 = \cdots = \theta_{r-2} = 1$, and $\theta_2 = \cdots = \theta_{r-2} = 1$ are the S, CS, LDPS and 2RPS models, respectively. Note that the PDPS model is another expression of the DPS model.

Tahata and Tomizawa [54] considered the generalized linear asymmetry model (denoted by $LS_m$) for a fixed $m$ ($m = 1, \ldots, r-1$), as follows:

$$p_{ij} = \begin{cases} w_{ij}^{(m)} \psi_{ij} & (i < j), \\ \psi_{ij} & (i \geq j), \end{cases}$$

where $\psi_{ij} = \psi_{ji}$ and

$$w_{ij}^{(m)} = \prod_{t=1}^{m} \theta_t^{j^t - i^t}.$$

When $m = 1$ (i.e., $w_{ij}^{(1)} = \theta_1^{j-i}$), this model is the LDPS model. When $m = 2$ (i.e., $w_{ij}^{(2)} = \theta_1^{j-i}\theta_2^{j^2-i^2}$), this model is Tomizawa's [72] extended LDPS (denoted by ELDPS) model.

Tomizawa [62, 63, 66] proposed the extended quasi-symmetry (EQS) model defined by

$$p_{ij} = \alpha_i \beta_j \psi_{ij} \quad (i = 1, \ldots, r; j = 1, \ldots, r),$$

where $\psi_{ij} = \gamma \psi_{ji}$ $(i < j)$; see also Tomizawa and Tahata [89]. A special case of this model obtained by putting $\gamma = 1$ is the QS model. The EQS model also may be expressed as

$$p_{ij} p_{jk} p_{ki} = \gamma p_{ji} p_{kj} p_{ik} \quad (1 \leq i < j < k \leq r).$$

Tomizawa [62, 63, 66] also considered the extended marginal homogeneity (EMH) model defined by

$$p_{i\cdot}^{(\delta)} = p_{\cdot i}^{(\delta)} \quad (i = 1, \ldots, r),$$

where $\delta$ is unspecified and

$$p_{i\cdot}^{(\delta)} = \delta p_{i\cdot}^{-} + p_{ii} + p_{i\cdot}^{+}, \quad p_{\cdot i}^{(\delta)} = p_{\cdot i}^{+} + p_{ii} + \delta p_{\cdot i}^{-},$$

$$p_{i\cdot}^{-} = \sum_{k=1}^{i-1} p_{ik}, \quad p_{i\cdot}^{+} = \sum_{k=i+1}^{r} p_{ik}, \quad p_{\cdot i}^{+} = \sum_{k=1}^{i-1} p_{ki}, \quad p_{\cdot i}^{-} = \sum_{k=i+1}^{r} p_{ki}.$$

A special case of this model obtained by putting $\delta = 1$ is the MH model. The EMH model indicates that the row marginal totals summed by multiplying the probabilities $p_{ij}$ for the lower left triangle cells below main diagonal in the table by the weight $\delta$ $(> 0)$ are equal to the column marginal totals summed by the same way.

Yamamoto, Shinoda and Tomizawa [99] proposed the weighted marginal homogeneity model I (WMH-I) using the scores $u_1 < \cdots < u_r$, as follows:

$$p_{i\cdot}^{-}(\delta) + p_{ii} + p_{i\cdot}^{+} = p_{\cdot i}^{+} + p_{ii} + p_{\cdot i}^{-}(\delta) \quad (i = 1, \ldots, r),$$

where $\delta$ is unspecified and

$$p_{i\cdot}^{-}(\delta) = \sum_{k=1}^{i-1} \delta^{u_i - u_k} p_{ik}, \quad p_{\cdot i}^{-}(\delta) = \sum_{k=i+1}^{r} \delta^{u_k - u_i} p_{ki}.$$

This indicates that the row marginal totals summed by multiplying the probabilities $p_{ij}$ for cell with a distance $i-j$ $(> 0)$ below main diagonal in the table by the weight $\delta^{u_i - u_j}$ $(> 0)$ are equal to the column marginal totals summed by the same way. A special case of this model obtained by putting $\delta = 1$ is the MH model. Yamamoto et al. [99] also proposed the WMH-II model, by using the weight $\delta^{u_j - u_i}$ for cells with a distance $j - i$ $(> 0)$ above main diagonal in the table; although the details are omitted. Especially, when the scores $\{u_i\}$ are the equal-interval scores $\{u_0 + id\}$, the WMH-t (t=I, II) model is identical to Tomizawa's [69] diagonals weighted marginal homogeneity model (DWM-t (t=I, II)).

### 2.3. Asymmetry model for cumulative probabilities

Let

$$G_{ij} = \sum_{s=1}^{i} \sum_{t=j}^{r} p_{st} = P(X_1 \le i, X_2 \ge j) \quad (i < j),$$

and

$$G_{ij} = \sum_{s=i}^{r} \sum_{t=1}^{j} p_{st} = P(X_1 \ge i, X_2 \le j) \quad (i > j).$$

As Tomizawa [74] and Tomizawa and Tahata [89] pointed out, the multiplicative forms of the S and CS models for $\{p_{ij}\}$ can also be expressed similarly as multiplicative forms for $\{G_{ij}\}$, $i \ne j$. Namely, the S model can be expressed as

$$G_{ij} = \Psi_{ij} \quad (i \ne j), \quad p_{ii} = \Psi_{ii},$$

where $\Psi_{ij} = \Psi_{ji}$. The CS model can be expressed as

$$G_{ij} = \begin{cases} \delta\Psi_{ij} & (i < j), \\ \Psi_{ij} & (i > j), \end{cases} \qquad p_{ii} = \Psi_{ii},$$

where $\Psi_{ij} = \Psi_{ji}$. However, the DPS model cannot be expressed as a similar multiplicative form for $\{G_{ij}\}$, $i \neq j$. So, we are also interested in the structure of $\{G_{ij}\}$ instead of $\{p_{ij}\}$. Tomizawa [74] proposed the cumulative diagonals-parameter symmetry (CDPS) model defined by

$$G_{ij} = \begin{cases} \Delta_{j-i}\Psi_{ij} & (i < j), \\ \Psi_{ij} & (i > j), \end{cases} \qquad p_{ii} = \Psi_{ii},$$

where $\Psi_{ij} = \Psi_{ji}$. This model indicates that the cumulative probability that an observation will fall in row category $i$ or below and column category $j$ ($> i$) or above, is $\Delta_{j-i}$ times higher than the cumulative probability that the observation falls in column category $i$ or below and row category $j$ or above. Special cases of the CDPS model obtained by putting $\{\Delta_{j-i} = 1\}$ and $\{\Delta_{j-i} = \Delta\}$ are the S and CS models, respectively.

Tomizawa and Miyamoto [81] proposed the cumulative diagonal uniform association symmetry (CDUS) model defined by

$$G_{ij} = \begin{cases} \Delta_{j-i}\Phi_{j-i}^{i-1}\Psi_{ij} & (i < j), \\ \Psi_{ij} & (i > j), \end{cases} \qquad p_{ii} = \Psi_{ii},$$

where $\Psi_{ij} = \Psi_{ji}$. A special case of this model obtained by putting $\Phi_1 = \cdots = \Phi_{r-2} = 1$ is the CDPS model.

Miyamoto, Ohtsuka and Tomizawa [37] proposed the cumulative linear diagonals-parameter symmetry (CLDPS) model defined by

$$G_{ij} = \begin{cases} \Theta^{j-i}\Psi_{ij} & (i < j), \\ \Psi_{ij} & (i > j), \end{cases} \qquad p_{ii} = \Psi_{ii},$$

where $\Psi_{ij} = \Psi_{ji}$. A special case of this model obtained by putting $\Theta = 1$ is the S model. Miyamoto et al. [37] also proposed the cumulative quasi-symmetry (CQS) model defined by

$$G_{ij} = \alpha_i\beta_j\Psi_{ij} \quad (i \neq j), \quad p_{ii} = \Psi_{ii},$$

where $\Psi_{ij} = \Psi_{ji}$. This model is different from the QS model. The CLDPS model is a special case of CQS model.

Tomizawa, Miyamoto, Yamamoto and Sugiyama [87] proposed the cumulative two-ratios-parameter symmetry (C2RPS) model defined by

$$G_{ij} = \begin{cases} \Gamma\Theta^{j-i}\Psi_{ij} & (i < j), \\ \Psi_{ij} & (i > j), \end{cases} \qquad p_{ii} = \Psi_{ii},$$

where $\Psi_{ij} = \Psi_{ji}$. A special case of this model obtained by putting $\Gamma = 1$ is the CLDPS model.

Tomizawa, Miyamoto and Yamamoto [86] proposed the cumulative polynomial diagonals-parameter symmetry (CPDPS) model defined by

$$G_{ij} = \begin{cases} \left(\prod_{k=0}^{r-2} \Theta_k^{(j-i)^k}\right) \Psi_{ij} & (i < j), \\ \Psi_{ij} & (i > j), \end{cases} \qquad p_{ii} = \Psi_{ii},$$

where $\Psi_{ij} = \Psi_{ji}$. Special cases of this model obtained by putting $\Theta_0 = \Theta_1 = \cdots = \Theta_{r-2} = 1$, $\Theta_1 = \cdots = \Theta_{r-2} = 1$, $\Theta_0 = \Theta_2 = \cdots = \Theta_{r-2} = 1$, and $\Theta_2 = \cdots = \Theta_{r-2} = 1$ are the S, CS, CLDPS and C2RPS models, respectively. Note that the CPDPS model is another expression of the CDPS model.

The cumulative extended quasi-symmetry (CEQS) model, which was given by Tomizawa et al. [87], is defined by

$$G_{ij} = \alpha_i \beta_j \Psi_{ij} \quad (i \neq j), \quad p_{ii} = \Psi_{ii},$$

where $\Psi_{ij} = \gamma \Psi_{ji}$ $(i < j)$. A special case of this model obtained by putting $\gamma = 1$ is the CQS model. The C2RPS model is a special case of CEQS model.

Yamamoto, Tahata and Tomizawa [107] considered a generalization of the C2RPS model as follows: for a fixed $m$ $(m = 1, \ldots, r-1)$,

$$G_{ij} = \begin{cases} \Gamma \Omega_{ij}^{(m)} \Psi_{ij} & (i < j), \\ \Psi_{ij} & (i > j), \end{cases} \qquad p_{ii} = \Psi_{ii},$$

where $\Psi_{ij} = \Psi_{ji}$ and

$$\Omega_{ij}^{(m)} = \prod_{t=1}^{m} \Theta_t^{j^t - i^t}.$$

Yamamoto et al. [107] denoted this model by C2RPS($m$). When $m = 1$ (i.e., $\Omega_{ij}^{(1)} = \Theta_1^{j-i}$), this is the C2RPS model. Yamamoto et al. [107] also denoted the C2RPS($m$) with $\Gamma = 1$ by CLDPS($m$). When $m = 1$, the CLDPS(1) model is the CLDPS model. Note that the C2RPS($m$) (CLDPS($m$)) model is a special case of CEQS (CQS) model. We point out that when $m = r - 1$, the C2RPS($r-1$) model is equivalent to the CEQS model, and also the CLDPS($r-1$) model is equivalent to the CQS model. Note that Yamamoto and Tomizawa [102] and Yamamoto, Ohama and Tomizawa [98] introduced the generalized LDPS model and the other generalized CLDPS model although the details are omitted.

McCullagh [35] considered the palindromic symmetry (PS) model defined by

$$G_{ij} = \begin{cases} \Delta \alpha_i \Psi_{ij} & (i < j), \\ \alpha_{i-1} \Psi_{ij} & (i > j), \end{cases} \qquad p_{ii} = \Psi_{ii},$$

where $\Psi_{ij} = \Psi_{ji}$, and $\alpha_1 = 1$ without loss of generality. Special cases of this model by setting $\Delta = 1$ and $\alpha_1 = \cdots = \alpha_{r-1}$ and by setting $\alpha_1 = \cdots = \alpha_{r-1}$ are the S and CS models, respectively (also see Tomizawa [70]). The PS model with $\Delta$ replaced by $\Delta_i$ is McCullagh's [35] generalized palindromic symmetry (GPS) model.

Saigusa, Tahata and Tomizawa [41] considered the extension of PS model (called the $m$-additional parameters palindromic symmetry (PS($m$)) model). For a given $m$ ($m = 1, \ldots, r-1$), the PS($m$) model is given by

$$G_{ij} = \begin{cases} \Delta_i^{(m)} \alpha_i \Psi_{ij} & (i < j), \\ \alpha_{i-1} \Psi_{ij} & (i > j), \end{cases} \qquad p_{ii} = \Psi_{ii},$$

where $\Psi_{ij} = \Psi_{ji}$ and

$$\Delta_i^{(m)} = \prod_{k=0}^{m-1} \Delta_k^{i^k}.$$

Especially, when $m = 1$, the PS(1) model is the PS model, and when $m = r-1$, the PS($r-1$) model is identical to the GPS model.

Iki, Oda and Tomizawa [24] proposed the modified palindromic symmetry (MPS) model defined by

$$G_{ij} = \begin{cases} \beta_i \Psi_{ij} & (i < j; j \neq i+1), \\ \Gamma \beta_i \Psi_{ij} & (j = i+1), \\ \beta_{i-1} \Psi_{ij} & (i > j), \end{cases} \qquad p_{ii} = \Psi_{ii},$$

where $\Psi_{ij} = \Psi_{ji}$. The MPS model is different from the PS model. A special case of MPS model obtained by putting $\Gamma = 1$ and $\{\beta_i = 1\}$ is the S model.

We shall consider the models which indicate the structure of asymmetry for row and column marginal distributions. The MH model may be expressed as

$$G_{i,i+1} = G_{i+1,i} \quad (i = 1, \ldots, r-1).$$

The EMH model may be expressed as

$$G_{i,i+1} = \delta G_{i+1,i} \quad (i = 1, \ldots, r-1).$$

Tomizawa [77] proposed the generalized marginal homogeneity (GMH) model as follows:

$$G_{i,i+1} = \delta \gamma^{i-1} G_{i+1,i} \quad (i = 1, \ldots, r-1).$$

Special cases of this model obtained by putting $\gamma = 1$ and $\gamma = \delta = 1$ are the EMH and MH models, respectively.

Tahata and Tomizawa [50] proposed the $m$-additional parameters marginal homogeneity (MH($m$)) model for a fixed $m$ ($m = 1, \ldots, r-1$), as follows:

$$G_{i,i+1} = \Delta_i^{(m)} G_{i+1,i} \quad (i = 1, \ldots, r-1),$$

where

$$\Delta_i^{(m)} = \prod_{k=0}^{m-1} \psi_k^{i^k}.$$

When $m = 1$ (i.e., $\Delta_i^{(1)} = \psi_0$), this is the EMH model. When $m = 2$ (i.e., $\Delta_i^{(2)} = \psi_0\psi_1^i$), this is the GMH model. Note that when $m = r - 1$, this is the saturated model.

Denote the marginal cumulative logit of $X_t$ $(t = 1, 2)$ by $L_i^{(t)}$ $(i = 1, \ldots, r-1)$. Thus

$$L_i^{(t)} = \text{logit}(F_i^{(t)}) = \log\left(\frac{F_i^{(t)}}{1 - F_i^{(t)}}\right),$$

where

$$F_i^{(t)} = P(X_t \leq i).$$

Agresti [5, p.442] considered the marginal cumulative logistic (L) model as follows:

$$L_i^{(1)} = L_i^{(2)} + \Delta \quad (i = 1, \ldots, r-1).$$

A special case of this model obtained by putting $\Delta = 0$ is the MH model.

Miyamoto, Niibe and Tomizawa [38] proposed the conditional marginal cumulative logistic (CL) model which is the L model with $\{L_i^{(t)}\}$ replaced by $\{L_i^{c(t)}\}$, where for $t = 1, 2; i = 1, \ldots, r-1$,

$$L_i^{c(t)} = \text{logit}(F_i^{c(t)}),$$

$$F_i^{c(t)} = P(X_t \leq i | (X_1, X_2) \neq (s, s), s = 1, \ldots, r).$$

Kurakami, Tahata and Tomizawa [29, 30] proposed the $m$th generalized marginal cumulative logistic models (denoted by L($m$) and CL($m$)) for $m = 1, \ldots, r-1$. The L($m$) model is defined by

$$L_i^{(1)} = L_i^{(2)} + \Delta_i^{(m)} \quad (i = 1, \ldots, r-1),$$

where

$$\Delta_i^{(m)} = \sum_{k=0}^{m-1} i^k \delta_k.$$

This model indicates that the difference between two marginal cumulative logits is the $(m-1)$th order polynomial function of cut-point $i$ of categories $(i = 1, \ldots, r-1)$. A special case of this model obtained by putting $\{\delta_k = 0\}$ is the MH model. When $m = 1$ (i.e., $\Delta_i^{(1)} = \delta_0$), this model is the L model. The CL($m$) model is the extension of the CL model although the detail is omitted.

### 2.4. Point-symmetry and double symmetry models

Wall and Lienert [92] considered the point-symmetry (P) model defined by

$$p_{ij} = \psi_{ij} \quad (i = 1, \ldots, r; j = 1, \ldots, r),$$

where $\psi_{ij} = \psi_{i^*j^*}$, $i^* = r + 1 - i$ and $j^* = r + 1 - j$. This model indicates the structure of point-symmetry of cell probabilities with respect to the center cell (when $r$ is odd) or center point (when $r$ is even) in the table. Also see Tomizawa [64, 67].

Tomizawa [64] proposed the quasi point-symmetry (QP) model defined by

$$p_{ij} = \alpha_i \beta_j \psi_{ij} \quad (i = 1, \ldots, r; j = 1, \ldots, r),$$

where $\psi_{ij} = \psi_{i^*j^*}$ (also see Tahata and Tomizawa [51]). A special case of this model obtained by putting $\{\alpha_i = \alpha_{i^*}\}$ and $\{\beta_j = \beta_{j^*}\}$ is the P model. Tomizawa [64] also considered the marginal point-symmetry (MP) model defined by

$$p_{i\cdot} = p_{i^*\cdot} \quad \text{and} \quad p_{\cdot i} = p_{\cdot i^*} \quad (i = 1, \ldots, r).$$

Tomizawa [65] proposed the double symmetry (DS) model, which has the structure of both S and P, and also proposed the quasi double symmetry (QDS) and the marginal double symmetry (MDS) models, although the details are omitted (also see Yamamoto, Takahashi and Tomizawa [105]).

Tahata and Tomizawa [53] proposed the double linear diagonals-parameter symmetry (D-LDPS) model defined by

$$p_{ij} = \alpha^i \beta^j \psi_{ij} \quad (i = 1, \ldots, r; j = 1, \ldots, r),$$

where $\psi_{ij} = \psi_{ji} = \psi_{i^*j^*} = \psi_{j^*i^*}$. Note that the D-LDPS model implies the LDPS model, and the D-LDPS model implies the QDS model.

### 2.5. The other symmetry models

Agresti [1] described the relationship between the LDPS model and the joint bivariate normal distribution. The LDPS model, the D-LDPS model, and Tahata, Yamamoto and Tomizawa's [56] model may be appropriate for a square ordinal table if it is reasonable to assume an underlying bivariate normal distribution with equal marginal variances.

Similarly, the ELDPS model may be appropriate for a square ordinal table if it is reasonable to assume an underlying bivariate normal distribution with any marginal variances (also see Yamamoto et al. [93]; Tahata, Yamamoto and Tomizawa [61]).

Iki, Ishihara and Tomizawa [23] proposed a model which may be useful if it is reasonable to assume an underlying bivariate $t$-distribution with equal marginal variances.

Yamamoto and Murakami [96] proposed a model which may be useful if it is reasonable to assume an underlying bivariate skew normal distribution, although the detail is omitted.

For square contingency tables with ordered categories, there may be some cases that one wants to analyze them by considering collapsed tables with some adjacent categories combined in the original table. For some models of symmetry for collapsed tables, see, e.g., Tahata, Takazawa and Tomizawa [55], Yamamoto, Tahata and Tomizawa [106], and Yamamoto, Murakami and Tomizawa [97].

## §3. Decompositions of models for square tables

### 3.1. Decomposition

This section reviews the decompositions of models. Caussinus [16] gave the decomposition of the symmetry model as follows.

**Theorem 1.** *The S model holds if and only if both the QS and MH models hold.*

For this decomposition, also see Bishop et al. [15, p.287], Agresti [5, p.429], and Tomizawa and Tahata [89]. We see from Theorem 1 that assuming that the QS model holds true, the hypothesis that the S model holds is equivalent to the hypothesis that the MH model holds.

Tomizawa [62] and Tomizawa and Tahata [89] introduced the balance (BA) model which indicates that the parameter $\gamma$ in the EQS model is equal to the parameter $\delta$ in the EMH model when both models hold, e.g., as follows:

$$\frac{\sum_{i=1}^{r-1} G_{i,i+1}}{\sum_{i=1}^{r-1} G_{i+1,i}} = \frac{\sum_{i<j<k} p_{ij}p_{jk}p_{ki}}{\sum_{i<j<k} p_{ji}p_{kj}p_{ik}}.$$

Tomizawa [62] and Tomizawa and Tahata [89] gave the following theorem.

**Theorem 2.** *The CS model holds if and only if all the EQS, EMH and BA models hold.*

Consider the marginal mean equality (ME) model which indicates $E(X_1) = E(X_2)$, where $E(X_1) = \sum_{i=1}^{r} ip_{i\cdot}$ and $E(X_2) = \sum_{j=1}^{r} jp_{\cdot j}$. The ME model can be expressed as

$$\sum_{i=1}^{r-1} G_{i,i+1} = \sum_{i=1}^{r-1} G_{i+1,i}.$$

Yamamoto et al. [93] and Tahata, Yamamoto and Tomizawa [60] gave the following theorem.

**Theorem 3.** *The S model holds if and only if both the LDPS and ME models hold.*

Tomizawa [69] gave the following theorem.

**Theorem 4.** *For t=I and II, the LDPS model holds if and only if both the QS and DWM-t models hold.*

Consider the global symmetry (GS) model defined by $P(X_1 < X_2) = P(X_1 > X_2)$, i.e., $\sum_{i<j} p_{ij} = \sum_{i>j} p_{ij}$. Read [40] gave the following theorem.

**Theorem 5.** *The S model holds if and only if both the CS and GS models hold.*

Tahata and Tomizawa [52] gave the following theorem.

**Theorem 6.** *The S model holds if and only if all the 2RPS, GS and ME models hold.*

For a fixed $k$ ($k = 1, \dots, r-1$), consider the marginal $k$th moment equality ($\mathrm{MME}_k$) model defined by

$$E(X_1^l) = E(X_2^l) \quad (l = 1, \dots, k).$$

When $k = 1$, this is the ME model. Tahata and Tomizawa [54] gave the following theorem.

**Theorem 7.** *For a fixed $k$ ($k = 1, \dots, r-1$), the S model holds if and only if both the $LS_k$ and $MME_k$ models hold.*

Note that when $k = 1$, Theorem 7 is identical to Theorem 3.

Tomizawa [62, 66, 70] introduced three kinds of modified marginal homogeneity models (denoted by MM-t (t=1, 2, 3)). The MM-1 model is defined by

$$p_{i\cdot}^{+} = \phi p_{\cdot i}^{-} \quad (i = 1, \dots, r-1).$$

The MM-2 model is defined by

$$p_{\cdot i}^{+} = \psi p_{i\cdot}^{-} \quad (i = 2, 3, \dots, r).$$

The MM-3 model is defined by

$$p_{i\cdot}^{+} = \xi p_{\cdot i}^{-} \quad \text{and} \quad p_{\cdot i+1}^{+} = \xi p_{i+1\cdot}^{-} \quad (i = 1, \dots, r-1).$$

Denote the MM-1 model with $\phi = 1$ and the MM-2 model with $\psi = 1$ by $\mathrm{MM}_0$-1 and $\mathrm{MM}_0$-2, respectively. Tomizawa [62, 66] gave the following theorem.

**Theorem 8.** *For t=1 and 2, the S model holds if and only if both the QS and $MM_0$-t models hold.*

Tomizawa [70] gave the following Theorems 9, 10 and 11.

**Theorem 9.** *For t=1 and 2, the CS model holds if and only if both the PS and MM-t models hold.*

**Theorem 10.** *The PS model holds if and only if both the GPS and EMH models hold.*

**Theorem 11.** *The CS model holds if and only if both the GPS and MM-3 models hold.*

Kateri and Papaioannou [28] described the following theorem.

**Theorem 12.** *The S model holds if and only if both the QS[f] and MH models hold.*

Yamamoto, Ando and Tomizawa [94] gave the following Theorems 13, 14 and 15.

**Theorem 13.** *The S model holds if and only if both the CQS and MH models hold.*

**Theorem 14.** *The CLDPS model holds if and only if both the CQS and EMH models hold.*

**Theorem 15.** *The C2RPS model holds if and only if both the CEQS and EMH models hold.*

Yamamoto et al. [107] gave the following Theorems 16 and 17.

**Theorem 16.** *For a fixed m ($m = 1, \dots, r-1$), the CLDPS(m) model holds if and only if both the CQS and MH(m) models hold.*

**Theorem 17.** *For a fixed m ($m = 1, \dots, r-1$), the C2RPS(m) model holds if and only if both the CEQS and MH(m) models hold.*

Note that when $m = 1$, Theorems 16 and 17 are identical to Theorems 14 and 15, respectively.

Yamamoto and Tomizawa [103] gave the following theorem.

**Theorem 18.** *The S model holds if and only if both the CLDPS and ME models hold.*

Tahata, Yamamoto and Tomizawa [58] gave the following theorem.

**Theorem 19.** *The S model holds if and only if all the C2RPS, GS and ME models hold.*

Tomizawa, Miyamoto and Ouchi [84] proposed the cumulative subsymmetry (CSS) model as follows:

$$G_{i,i+2} = G_{i+2,i} \quad (i = 1, \ldots, r-2).$$

Tahata, Yamamoto and Tomizawa [57] gave the following Theorems 20 and 21.

**Theorem 20.** *The S model holds if and only if all the PS, ME and CSS models hold.*

**Theorem 21.** *The S model holds if and only if all the GPS, EMH, ME and CSS models hold.*

Iki et al. [24] gave the following theorem.

**Theorem 22.** *The S model holds if and only if all the MPS, ME and CSS models hold.*

Tomizawa [78] gave the following theorem.

**Theorem 23.** *The MH model holds if and only if both the EMH and ME models hold.*

Denote the marginal variance equality model, $Var(X_1) = Var(X_2)$, by MV. Tomizawa [78] also gave the following theorem.

**Theorem 24.** *The MH model holds if and only if all the GMH, ME and MV models hold.*

Tahata and Tomizawa [50] gave the following theorem.

**Theorem 25.** *For a given $m$ ($m = 1, \ldots, r-1$), the MH model holds if and only if both the MH(m) and MME$_m$ models hold.*

Note that this theorem with $m = 1$ and 2 are Theorems 23 and 24, respectively.

Miyamoto et al. [38] gave the following theorem.

**Theorem 26.** *The MH model holds if and only if both the L (or CL) and ME models hold.*

Kurakami et al. [30] gave the following theorem.

**Theorem 27.** *For a given $m$ ($m = 1, \ldots, r-1$), the MH model holds if and only if both the L(m) (or CL(m)) and MME$_m$ models hold.*

Note that this theorem with $m = 1$ is Theorem 26.

Tomizawa [64] gave the following theorem (also see Tahata and Tomizawa [51]).

**Theorem 28.** *The P model holds if and only if both the QP and MP models hold.*

Tomizawa [65] gave the following theorem (also see Yamamoto et al. [105]).

**Theorem 29.** *The DS model holds if and only if both the QDS and MDS models hold.*

Tahata and Tomizawa [53] considered the double mean equalities (DME) model defined by

$$E(X_1) = E(X_2) = E(X_1^*) = E(X_2^*),$$

where $E(X_t^*) = E(r + 1 - X_t)$ for $t = 1$ and 2. Thus the DME model is expressed as

$$E(X_1) = E(X_2) = \frac{r+1}{2}.$$

Tahata and Tomizawa [53] gave the following theorem.

**Theorem 30.** *The DS model holds if and only if both the D-LDPS and DME models hold.*

Yamamoto and Tomizawa [101] gave the following theorem.

**Theorem 31.** *The SQU model holds if and only if both the QU and MH models hold.*

### 3.2. Orthogonality of test statistic

Let $n_{ij}$ denote the observed frequency in the $(i, j)$th cell of the $r \times r$ table $(i = 1, \ldots, r; j = 1, \ldots, r)$. Assume that a multinomial distribution is applied to the $r \times r$ table. Each model (say M) can be tested for goodness-of-fit by e.g., the likelihood ratio chi-squared statistic with the corresponding degrees of freedom (df). The likelihood ratio statistic for testing goodness-of-fit of model M is given by

$$G^2(M) = 2 \sum_{i=1}^{r} \sum_{j=1}^{r} n_{ij} \log \left( \frac{n_{ij}}{\hat{m}_{ij}} \right),$$

where $\hat{m}_{ij}$ is the maximum likelihood estimate of expected frequency $m_{ij}$ under model M.

We point out that for each theorem in Section 3.1, for example, when model $M_0$ is decomposed into models $M_1$, $M_2$ and $M_3$, the number of df for $M_0$ is equal to the sum of numbers of df for $M_1$, $M_2$ and $M_3$ (although the details are omitted).

Lang and Agresti [32] and Lang [31] considered the simultaneous modeling of the joint distribution and the marginal distribution. Aitchison [8] discussed the asymptotic separability, which is equivalent to the orthogonality in Read [40] and the independence in Darroch and Silvey [17], of test statistic for the goodness-of-fit of two models (also see Land and Agresti [32]; Lang [31]; Tomizawa and Tahata [89]; Tahata and Tomizawa [51]).

As described in Tomizawa and Tahata [89], for Theorem 1 the orthogonality of test statistic holds; namely, the test statistic $G^2(S)$ is asymptotically equivalent to the sum of $G^2(QS)$ and $G^2(MH)$. In addition, we point out that the orthogonality of test statistic holds for Theorems 3, 5, 7, 28, 29, 30 and 31 (for details, see the corresponding articles).

## §4. Relationships among models for square tables

As described in Section 3, many models of symmetry and asymmetry are considered. Therefore it would be meaningful to show the relationships among models. In Figures 1, 2, 3 and 4, we shall show them. In Figure, A→B indicates that model A implies model B.

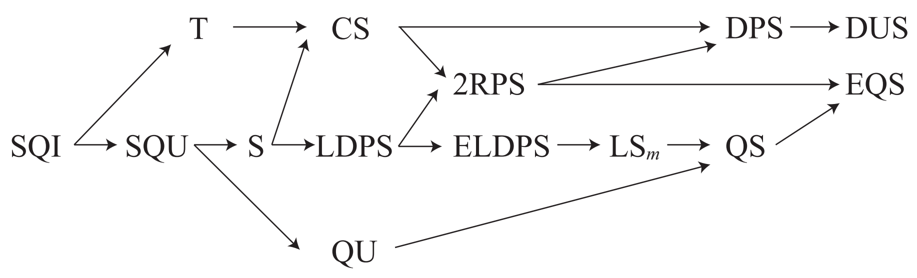


Figure 1: Relationships among models (I).

## §5. New decomposition of symmetry model

From Theorems 8 and 13, we are interested in whether we can decompose the S model into the CQS model and $MM_0$-t (t=1, 2) model. We now obtain the

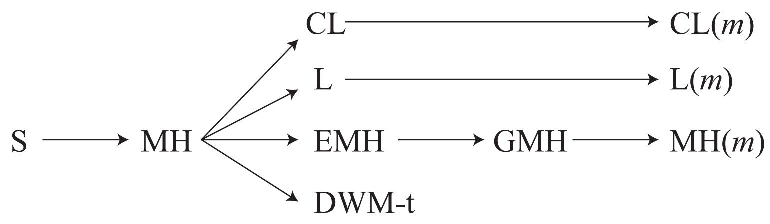


Figure 2: Relationships among models (II).

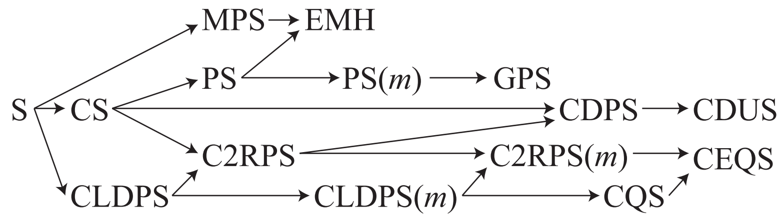


Figure 3: Relationships among models (III).

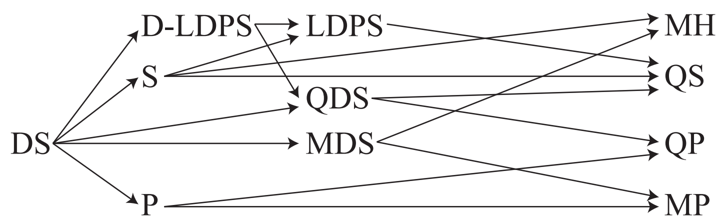


Figure 4: Relationships among models (IV).

following theorem.

**Theorem 32.** *For t=1 and 2, the S model holds if and only if both the CQS and $MM_0$-t models hold.*

*Proof.* If the S model holds, both the CQS and $MM_0$-t models hold. Assume that the CQS and $MM_0$-t models hold, and then we shall show that the S model holds. We consider the case of t=1. The CQS model can be expressed as

$$\frac{G_{ij}}{G_{ji}} = \frac{\gamma_j}{\gamma_i} \quad (i < j),$$

where $\gamma_1 = 1$ without loss of generality (see Yamamoto et al. [94]). We see

$$p_{i\cdot}^{+} = G_{i,i+1} - G_{i-1,i+1}$$

and

$$p_{\cdot i}^{-} = G_{i+1,i} - G_{i+1,i-1} \quad (i = 1, \ldots, r-1),$$

where $G_{02} = G_{20} = 0$. From the CQS model, we see

$$p_{i\cdot}^{+} = \frac{\gamma_{i+1}}{\gamma_i} G_{i+1,i} - \frac{\gamma_{i+1}}{\gamma_{i-1}} G_{i+1,i-1} \quad (i = 1, \ldots, r-1).$$

Also $p_{1\cdot}^{+} = G_{12}$ and $p_{\cdot 1}^{-} = G_{21}$. Since the CQS model holds, we obtain

$$\frac{G_{12}}{G_{21}} = \frac{\gamma_2}{\gamma_1}.$$

Since the $MM_0$-1 model holds, $p_{1\cdot}^{+} = p_{\cdot 1}^{-}$. Noting $\gamma_1 = 1$, we see $\gamma_2 = 1$. Also we see

$$p_{2\cdot}^{+} = \frac{\gamma_3}{\gamma_2} G_{32} - \frac{\gamma_3}{\gamma_1} G_{31}$$

and

$$p_{\cdot 2}^{-} = G_{32} - G_{31}.$$

From $\gamma_1 = \gamma_2 = 1$ and $p_{2\cdot}^{+} = p_{\cdot 2}^{-}$, we obtain $\gamma_3 = 1$. By similar way, we obtain $\gamma_1 = \gamma_2 = \cdots = \gamma_r$. Therefore we see $G_{ij} = G_{ji}$ $(i < j)$. Namely the S model holds. The case of t=2 can be proved in a similar way to the case of t=1. The proof is completed. □

## §6. Models of multi-way tables

This section reviews briefly various models of symmetry or asymmetry. Consider the $r^T$ contingency table $(T \geq 2)$. Let $X_k$ be the $k$th random variable $(k = 1, \ldots, T)$. Let $p_i$ denote the probability that an observation will fall in the $i = (i_1, \ldots, i_T)$th cell of the table $(i_k = 1, \ldots, r; k = 1, \ldots, T)$.

### 6.1. Symmetry models

The symmetry ($S^T$) model is defined by

$$p_i = p_j \quad \text{for any } i,$$

where $j = (j_1, \ldots, j_T)$ is any permutation of $i = (i_1, \ldots, i_T)$; see Bhapkar [12], Bhapkar and Darroch [13], Lovison [33], and Agresti [5, p.440].

For a fixed $h$ ($h = 1, \ldots, T-1$), the $h$th-order quasi-symmetry ($Q_h^T$) model is defined by

$$\log p_i = \lambda + \sum_{k=1}^{T} \lambda_k(i_k) + \sum\sum_{1 \le k_1 < k_2 \le T} \lambda_{k_1 k_2}(i_{k_1}, i_{k_2}) + \cdots + \sum\cdots\sum_{1 \le k_1 < \cdots < k_h \le T} \lambda_{k_1 \ldots k_h}(i_{k_1}, \ldots, i_{k_h}) + \lambda(i),$$

for any $i$, where $\lambda(i) = \lambda(j)$ for any permutation $j = (j_1, \ldots, j_T)$ of $i = (i_1, \ldots, i_T)$; see Bhapkar and Darroch [13]. Note that the $S^T$ model implies the $Q_h^T$ model.

Denote the $h$th-order ($1 \le h < T$) marginal probability by $p_i^s$, i.e., $p_i^s = P(X_{s_1} = i_1, \ldots, X_{s_h} = i_h)$, where $s = (s_1, \ldots, s_h)$ and $i = (i_1, \ldots, i_h)$ with $1 \le s_1 < \cdots < s_h \le T$ and $i_k = 1, \ldots, r$ ($k = 1, \ldots, h$). The $h$th-order marginal symmetry ($M_h^T$) model is defined by

$$p_i^s = p_j^s = p_i^t$$

for any permutation $j = (j_1, \ldots, j_h)$ of $i = (i_1, \ldots, i_h)$ and for any $s = (s_1, \ldots, s_h)$ and $t = (t_1, \ldots, t_h)$; see Bhapkar and Darroch [13] and Agresti [5, p.440].

### 6.2. Asymmetry models

For the $r^T$ contingency table ($T \ge 2$), Tahata and Tomizawa [54] proposed the $k$th-order linear asymmetry (denoted by $LS_k^T$) model ($k = 1, \ldots, r-1$), defined by

$$p_i = \mu \left( \prod_{s=1}^{T} \alpha_{1(s)}^{i_s} \right) \left( \prod_{s=1}^{T} \alpha_{2(s)}^{i_s^2} \right) \cdots \left( \prod_{s=1}^{T} \alpha_{k(s)}^{i_s^k} \right) \psi_i,$$

where $\psi_i = \psi_j$ for any permutation $j = (j_1, \ldots, j_T)$ of $i = (i_1, \ldots, i_T)$. Especially when $k = 1$ and 2, the $LS_1^T$ and $LS_2^T$ models are the linear diagonals-parameter symmetry model and the extended linear diagonals-parameter symmetry model, respectively, for $r^T$ table, which are considered by Tahata et al.

[60]. As described in Tahata and Tomizawa [54], the $Q_1^T$ model is equivalent to the $LS_{r-1}^T$ model. Therefore the $LS_k^T$ $(k < r-1)$ model is a special case of the $Q_1^T$ model.

Tahata et al. [61] proposed the $h$th-order linear ordinal quasi-symmetry ($LQ_h^T$) model ($h = 1, \ldots, T-1$), defined by

$$\log p_i = \lambda + \sum_{k=1}^{T} i_k \lambda_k + \sum\sum_{1 \le k_1 < k_2 \le T} i_{k_1} i_{k_2} \lambda_{k_1 k_2} + \cdots + \sum\cdots\sum_{1 \le k_1 < \cdots < k_h \le T} i_{k_1} \ldots i_{k_h} \lambda_{k_1 \ldots k_h} + \lambda(i),$$

where $\lambda(i) = \lambda(j)$ for any permutation $j = (j_1, \ldots, j_T)$ of $i = (i_1, \ldots, i_T)$. The $LQ_h^T$ model can be expressed in a multiplicative form

$$p_i = \mu \left( \prod_{k=1}^{T} \alpha_k^{i_k} \right) \left( \prod\prod_{1 \le k_1 < k_2 \le T} \alpha_{k_1 k_2}^{i_{k_1} i_{k_2}} \right) \cdots \left( \prod\cdots\prod_{1 \le k_1 < \cdots < k_h \le T} \alpha_{k_1 \ldots k_h}^{i_{k_1} \ldots i_{k_h}} \right) \gamma(i),$$

where $\gamma(i) = \gamma(j)$ for any permutation $j = (j_1, \ldots, j_T)$ of $i = (i_1, \ldots, i_T)$. Especially, when $h = 1$, the $LQ_1^T$ model is the $LS_1^T$ model. Note that Agresti [5, p.440] refers to the $LQ_1^T$ ($LS_1^T$) model (with the score) as the ordinal quasi-symmetry model. The $LQ_h^T$ model is a special case of the $Q_h^T$ model.

Yamamoto et al. [93] and Tahata et al. [60] considered the generalized $LS_1^T$ (denoted by $GLS^T$) model defined by

$$p_i = \left( \prod_{s=1}^{T} \alpha_s^{i_s} \right) \left( \prod_{t=1}^{T} \beta_t^{i_t^2} \right) \left( \prod_{s=1}^{T-1} \prod_{t=s+1}^{T} \gamma_{st}^{i_s i_t} \right) \psi_i,$$

where $\psi_i = \psi_j$ for any permutation $j = (j_1, \ldots, j_T)$ of $i = (i_1, \ldots, i_T)$. Note that the $LS_2^T$ model implies the $GLS^T$ model, and the $GLS^T$ model implies the $Q_2^T$ model.

For the $r^T$ contingency table, Agresti [5, p.442] considered the marginal cumulative logistic ($L^T$) model. Tahata, Katakura and Tomizawa [46] considered the conditional marginal cumulative logistic ($CL^T$) model, although the details are omitted. Kurakami et al. [30] considered the $k$th-order generalized cumulative logistic ($L^T(k)$) model and the $k$th-order generalized conditional marginal cumulative logistic ($CL^T(k)$) model ($k = 1, \ldots, r-1$), although the details are omitted.

### 6.3. Point-symmetry model

Consider the $r^T$ contingency table. The point-symmetry ($P^T$) model is defined by

$$p_i = p_{i^*} \quad \text{for any } i = (i_1, \ldots, i_T),$$

where $i^* = (i_1^*, \ldots, i_T^*)$ and $i_k^* = r + 1 - i_k$ (Wall and Lienert [92]).

Although the details are omitted, Tahata and Tomizawa [51] proposed the $h$th-order quasi point-symmetry ($QP_h^T$) model and the $h$th-order marginal point-symmetry ($MP_h^T$) model ($h = 1, \ldots, T-1$).

Yamamoto et al. [105] proposed the double symmetry ($DS^T$) model defined by

$$p_i = p_j = p_{i^*} = p_{j^*},$$

where $j = (j_1, \ldots, j_T)$ is any permutation of $i = (i_1, \ldots, i_T)$. Yamamoto et al. [105] also proposed the $h$th-order quasi double symmetry ($QDS_h^T$) model and the $h$th-order marginal double symmetry ($MDS_h^T$) model ($h = 1, \ldots, T-1$), although the details are omitted.

## §7. Decompositions of models for multi-way tables

This section reviews briefly the decompositions of models for multi-way $r^T$ contingency tables. Bhapkar and Darroch [13] gave the following theorem.

**Theorem 33.** *For a fixed $h$ ($h = 1, \ldots, T-1$), the $S^T$ model holds if and only if both the $Q_h^T$ and $M_h^T$ models hold.*

For a fixed $k$ ($k = 1, \ldots, r-1$), consider the marginal $k$th moment equality ($MME_k^T$) model defined by

$$E(X_1^l) = \cdots = E(X_T^l) \quad (l = 1, \ldots, k).$$

Tahata and Tomizawa [54] gave the following theorem.

**Theorem 34.** *For a fixed $k$ ($k = 1, \ldots, r-1$), the $S^T$ model holds if and only if both the $LS_k^T$ and $MME_k^T$ models hold.*

For a fixed $h$ ($h = 1, \ldots, T-1$), consider the $h$th moment equality ($ME_h^T$) model defined by

$$E(X_{k_1} \cdots X_{k_l}) = E(X_1 \cdots X_l) \quad (l = 1, \ldots, h; 1 \le k_1 < \cdots < k_l \le T).$$

Tahata et al. [61] gave the following theorem.

**Theorem 35.** *For a fixed $h$ ($h = 1, \ldots, T-1$), the $S^T$ model holds if and only if both the $LQ_h^T$ and $ME_h^T$ models hold.*

Consider the mean, variance and correlation equality ($MVC^T$) model defined by

$$E(X_1) = \cdots = E(X_T), \quad Var(X_1) = \cdots = Var(X_T),$$

and

$$Corr(X_i, X_j) = c \quad (i < j),$$

where $Corr(X_i, X_j)$ is the correlation of $X_i$ and $X_j$ and $c$ is a constant. Tahata et al. [60] gave the following theorem.

**Theorem 36.** *The $S^T$ model holds if and only if both the $GLS^T$ and $MVC^T$ models hold.*

Kurakami et al. [30] gave the following theorem.

**Theorem 37.** *For a fixed $k$ ($k = 1, \dots, r-1$), the $M_1^T$ model holds if and only if both the $L^T(k)$ and $MME_k^T$ models hold.*

Note that when $k = r-1$, the $L^T(r-1)$ model is saturated model and the $MME_{r-1}^T$ model is equivalent to the $M_1^T$ model (see Kurakami et al. [30]).

Tahata and Tomizawa [51] gave the following theorem.

**Theorem 38.** *For a fixed $h$ ($h = 1, \dots, T-1$), the $P^T$ model holds if and only if both the $QP_h^T$ and $MP_h^T$ models hold.*

Yamamoto et al. [105] gave the following theorem.

**Theorem 39.** *For a fixed $h$ ($h = 1, \dots, T-1$), the $DS^T$ model holds if and only if both the $QDS_h^T$ and $MDS_h^T$ models hold.*

We point out that in Section 7, the orthogonality of test statistic hold for Theorems 33, 34, 35, 36, 38 and 39 (for details, see the corresponding articles).

## §8. Relationships among models for multi-way tables

For the multi-way $r^T$ contingency table, there are many models of symmetry and asymmetry. For example, for the $Q_h^T$ model there are $(T-1)$ kinds of quasi-symmetry models, as $Q_1^T$, $Q_2^T$, $\dots$, $Q_{T-1}^T$. Also, the $LQ_h^T$, $LS_k^T$ and $GLS^T$ models are special quasi-symmetry models. Therefore, it would be meaningful to give the figure which indicates the relationships among various quasi-symmetry models for the $r^T$ table. In Figure 5 we shall show them. Since the relationships among the other models are similar to Figures 2 and 4, we omit their figures.

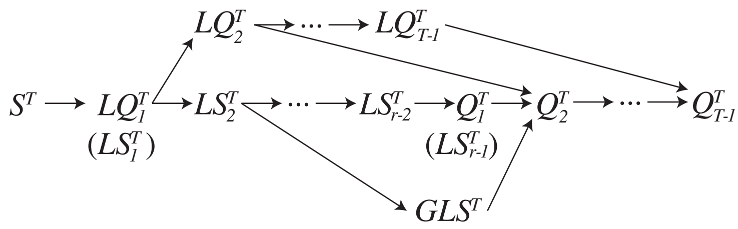


Figure 5: Relationships among models (V).

## §9. Concluding remarks

In Sections 2 and 3, we have reviewed various models of symmetry and asymmetry for the $r \times r$ contingency table and the decompositions of models. In Section 4, we have given the figures which indicate the relationships among various models for the $r \times r$ table. Since there are many models of symmetry or asymmetry, it would be meaningful to give these figures. In Section 5, we have given a new decomposition of symmetry model. The CQS model for cumulative probabilities is similar to the structure of QS model for cell probabilities. Therefore, many readers would be interested in whether Theorem 8 with the QS model replaced by the CQS model holds. The new decomposition (Theorem 32) indicates that it holds.

In Sections 6 and 7, we have reviewed various models of symmetry and asymmetry and the decompositions of models for the multi-way $r^T$ contingency table. In Section 8, we have given the figure which indicates the relationships among various quasi-symmetry models for the $r^T$ tables. It would be meaningful to give the figure (Figure 5) for the $r^T$ tables since there are many models.

## §10. Discussion

For analyzing the data of square contingency tables, one applies various models of symmetry. If the symmetry model does not hold, the extended model, e.g., the asymmetry model, is applied. Also we are interested in measuring the degree of departure from the symmetry when the symmetry model does not hold. Various measures are proposed to represent the degree of departure from the model. For the measures of the S model, see, e.g., Tomizawa [75], Tomizawa, Seo and Yamamoto [88], Tomizawa, Miyamoto and Hatanaka [83], Tahata, Yamamoto, Nagatani and Tomizawa [59], Tahata, Miyazawa and Tomizawa

[49], and Tahata, Akinaga and Tomizawa [43], etc. For the measures of the QS model, see, e.g., Tahata, Miyamoto and Tomizawa [48], and Tahata, Kozai and Tomizawa [47], etc. For the measures of the MH model, see, e.g., Tomizawa [76], Tomizawa and Makii [80], Tomizawa, Miyamoto and Ashihara [82], and Tahata, Iwashita and Tomizawa [44], etc. For the measures of some symmetry or asymmetry models, see, e.g., Tomizawa, Miyamoto and Yamane [85], Yamamoto and Tomizawa [100], Yamamoto, Furuya and Tomizawa [95], Tahata, Iwashita and Tomizawa [45], Iki, Tahata and Tomizawa [25], and Yamamoto, Tahata, Suzuki and Tomizawa [104], etc.

By the way, some decompositions of models of symmetry for the discrete multivariate distribution may be considered for the continuous multivariate distribution (i.e., the multivariate probability density function). Iki, Tahata and Tomizawa [26], and Iki and Tomizawa [22] gave the decompositions of symmetric multivariate probability density function.

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