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On the generalized reduced Ostrovsky equation

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Abstract. We survey recent progress on the case of the Cauchy problem for the generalized reduced Ostrovsky equation $u_t = S(\partial_x) u + (f(u))_x$, where the operator $S(\partial_x)$ is defined through the Fourier transform as $S(\partial_x) = \mathcal{F}^{-1} \frac{1}{i\xi} \mathcal{F}$, and the nonlinear interaction is given by $f(u) = |u|^{\rho-1} u$ if $\rho > 1$ is not an integer and $f(u) = u^{\rho}$ if $\rho > 1$ is an integer.

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§1. Introduction

We survey our recent results on the Cauchy problem for the generalized reduced Ostrovsky equation

$$(1.1) \qquad \begin{cases} u_t = S(\partial_x) u + \partial_x f(u), \ x \in \mathbf{R}, \ t > 0, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}, \end{cases}$$

where the operator $S(\partial_x)$ is defined through the Fourier transform as $\mathcal{F}^{-1} \frac{1}{i\xi} \mathcal{F}$, and the nonlinear interaction is given by $f(u) = |u|^{\rho-1} u$ if $\rho > 1$ is not an integer and $f(u) = u^{\rho}$ if $\rho > 1$ is an integer. The Ostrovsky equation (1.1) with $S(\partial_x) = \mathcal{F}^{-1} \left(-ia\xi^3 - \frac{ib}{\xi} \right) \mathcal{F}$ and $f(u) = u^2$ was introduced in [33] for modelling the small-amplitude long waves in a rotating fluid of finite depth. It was studied by many authors (see, e.g., [28], [39], [40] and references cited therein). When $a = 0$, and $f(u) = u^2$, equation (1.1) is called the reduced Ostrovsky equation.

In order to survey the previous works on the Ostrovsky equations we introduce *Notation and Function Spaces.* We denote the Lebesgue space by $\mathbf{L}^p = \{\phi \in \mathbf{S}'; \|\phi\|_{\mathbf{L}^p} < \infty\}$, where the norm $\|\phi\|_{\mathbf{L}^p} = \left(\int_{\mathbf{R}} |\phi(x)|^p dx \right)^{\frac{1}{p}}$ for

$1 \le p < \infty$ and $\|\phi\|_{\mathbf{L}^\infty} = \sup_{x\in\mathbf{R}} |\phi(x)|$ for $p = \infty$. The weighted Sobolev space is

$$\mathbf{H}_p^{m,s} = \left\{\varphi \in \mathbf{S}'; \|\phi\|_{\mathbf{H}_p^{m,s}} = \|\langle x\rangle^s \langle i\partial_x\rangle^m \phi\|_{\mathbf{L}^p} < \infty\right\},$$

$m, s \in \mathbf{R}, 1 \le p \le \infty$, $\langle x\rangle = \sqrt{1+x^2}, \langle i\partial_x\rangle = \sqrt{1-\partial_x^2}$. We also use the notations $\mathbf{H}^{m,s} = \mathbf{H}_2^{m,s}$, $\mathbf{H}^m = \mathbf{H}^{m,0}, \mathbf{H}_p^m = \mathbf{H}_p^{m,0}$ shortly, if it does not cause any confusion. We denote the homogeneous Sobolev space by

$$\dot{\mathbf{H}}^m = \left\{\phi \in \mathbf{S}'/\mathbf{P}; \|\phi\|_{\dot{\mathbf{H}}^m} = \left\|\left(-\partial_x^2\right)^{\frac{m}{2}} \phi\right\|_{\mathbf{L}^2} < \infty\right\},$$

where $\mathbf{P}$ denotes the set of all polynomials. We also use the notation $D_x^m = \left(-\partial_x^2\right)^{\frac{m}{2}}$ for simplicity.

Local well-posedness for the Ostrovsky equation was shown in paper [40] in the case of the initial data

$$u_0 \in \mathbf{H}^s \cap \dot{\mathbf{H}}^{-1},\ s > \frac{3}{2}$$

by using the parabolic regularization technique and limiting arguments. Their method works also for the case of the generalized nonlinearity $f(u) = |u|^{\rho-1} u$ and also generalized reduced Ostrovsky equation (1.1), since the dispersive effects were not used in the proof. Thanks to the high frequency part u_{xxx}, the solutions to the linear equation $(u_t - \beta u_{xxx})_x = \gamma u$ obtain smoothing property. By using this property, in [28], the local well-posedness for the Ostrovsky equation was shown under the condition

$$u_0 \in \mathbf{H}^s \cap \dot{\mathbf{H}}^{-1},\ s > \frac{3}{4}.$$

The method on [28] depends on the linear part of the equation and also works for the nonlinearities of a general order. In [11], [25], [26], [39] the local well-posedness for the Ostrovsky equation was treated by the Fourier restriction norm method of [2] and in [39], the $\mathbf{H}^{-\frac{3}{4}+}$ local well-posedness was shown. We note here that the Sobolev space $\mathbf{H}^{-\frac{3}{4}+}$ is considered as a critical regularity compared to the work on Korteweg-de Vries. However, the Fourier restriction norm method does not work in the case of the fractional order nonlinearity.

Global well-posedness in the energy class was obtained for the Ostrovsky equation in [28] through the energy conservation law, when the initial data

$$u_0 \in \mathbf{H}^1 \cap \dot{\mathbf{H}}^{-1},$$

and $ab > 0$. After their work, the global well-posedness in

$$\mathbf{L}^2 \cap \dot{\mathbf{H}}^{-s}, 0 \le s \le 1,$$

was proved in [11], [39] due to the $\mathbf{L}^2$ - conservation law. The global well-posedness in the negative order Sobolev space $\mathbf{H}^{-\frac{3}{10}+}$, was shown in [26] by using the I method of [7].

We now turn to the reduced Ostrovsky equation (1.1). The local well-posedness was shown in the space $\mathbf{H}^2$ in [35] and after that in $\mathbf{H}^{\frac{3}{2}+}$ in [36]. Their methods work also in the case of the general nonlinear dispersive equations with different nonlinearities. We also refer [29] and [30] for the local well-posedness in the class

$$u_0 \in \mathbf{H}^m \cap \dot{\mathbf{H}}^{-1} \quad m \geq 2.$$

However there are few works on the global well-posedness for the reduced Ostrovsky equation (1.1) due to the lack of the smoothing property. The global well-posedness for the reduced Ostrovsky equation (1.1) with cubic nonlinearity $f(u) = u^3$ (which is called the short pulse equation) was obtained in the paper [34], when the initial data

$$u_0 \in \mathbf{H}^2, \|\partial_x u_0\|_{\mathbf{H}^1} < 1,$$

whereas for the quadratic nonlinearity $f(u) = u^2$ (which is called the reduced Ostrovsky equation or the Ostrovsky-Hunter equation, see [3], [24]), it was shown in [10] when the initial data

$$u_0 \in \mathbf{H}^3, 1 - 3\partial_x^2 u_0(x) < 0$$

for all $x \in \mathbf{R}$. The time decay properties of solutions to the corresponding linear problem can be studied if we assume that the initial data decay rapidly at infinity. So the global existence was shown in [36], for the nonlinearity $f(u) = u^\rho$ with integer $\rho \geq 4$, when the initial data are small and sufficiently regular

$$u_0 \in \mathbf{H}^5 \cap \mathbf{H}_1^3.$$

The rest of this review article is based on our papers [19], [21], [22] and is organized as follows. In Section 2, we consider the super critical nonlinearity in the sense of the scattering problem. Section 3 is devoted to the nonexistence of the usual scattering states in the case of sub critical or critical nonlinearities. We consider the critical case in the last section.

§2. Super Critical Case

Our first result is related to the work [36]. Denote by

$$\mathcal{U}(t) = \mathcal{F}^{-1} \exp\left(-\frac{it}{\xi}\right) \mathcal{F}$$

the free evolution group for the reduced Ostrovsky equation. We introduce the following operator $\mathcal{J} = \mathcal{U}(t)x\mathcal{U}(-t) = x - t\partial_x^{-2}$, where the anti-derivative ∂_x^{-1} is defined by

$$\left(\partial_x^{-1}\phi\right)(x) = \mathcal{F}^{-1}(i\xi)^{-1}\widehat{\phi} = \frac{1}{2}\left(\int_{-\infty}^{x}\phi(x')\,dx' - \int_{x}^{\infty}\phi(x')\,dx'\right).$$

It is known that the operator $\mathcal{J}$ is a useful tool for obtaining the $\mathbf{L}^\infty$ - time decay estimates of solutions. However, the operator $\mathcal{J}$ does not work well on the nonlinear terms. Then, instead of using the operator $\mathcal{J}$ we apply the following operator $\mathcal{P} = \mathcal{J}\partial_x - t\mathcal{L} = x\partial_x - t\partial_t$, where $\mathcal{L} = \partial_t - \partial_x^{-1}$ is a linear part of the reduced Ostrovsky equation. Note that $\mathcal{P}$ acts well on the nonlinear terms as the first order differential operator. To state the results, we introduce the function spaces

$$\mathbf{X}_T^m = \left\{u(t) \in \mathbf{C}([0,T];\mathbf{H}^m);\|u\|_{\mathbf{X}_T^m} < \infty\right\},$$

$$\mathbf{X}_0^m = \left\{\phi \in \mathbf{L}^2;\|\phi\|_{\mathbf{X}_0^m} < \infty\right\},$$

where

$$\|u\|_{\mathbf{X}_T^m} = \sup_{t\in[0,T]}\|u(t)\|_{\mathbf{H}^m} + \sup_{t\in[0,T]}\|\mathcal{J}\partial_x u(t)\|_{\mathbf{L}^2} + \sup_{t\in[0,T]}\|u(t)\|_{\dot{\mathbf{H}}^{-1}}$$

and

$$\|\phi\|_{\mathbf{X}_0^m} = \|\phi\|_{\mathbf{H}^m} + \|\partial_x\phi\|_{\mathbf{H}^{0,1}} + \|\phi\|_{\dot{\mathbf{H}}^{-1}}.$$

We consider the real-valued solutions, since one of the main tools to treat the so-called derivative loss of the nonlinear term is the energy method, which does not work in the case of quasi-linear nonlinearities if the solution is a complex-valued function.

Theorem 2.1. *Let the order ρ of the nonlinearity satisfy*

$$\rho > \max\left\{3 + \frac{2}{3}, m+1\right\}$$

or be an integer satisfying $\rho \geq 4$. Assume that the initial data $u_0 \in \mathbf{X}_0^m$, with $m > 2$. Then there exists a positive constant $\widetilde{\varepsilon}$ such that (1.1) has a unique global solution $u \in \mathbf{X}_\infty^m$ with the time decay

$$\|u(t)\|_{\mathbf{L}^\infty} \leq C\langle t\rangle^{-\frac{1}{2}}$$

for any u_0 *satisfying* $\|u_0\|_{\mathbf{X}_0^m} \leq \widetilde{\varepsilon}$. *Moreover for any* $u_0 \in \mathbf{X}_0^m$ *such that* $\|u_0\|_{\mathbf{X}_0^m} \leq \widetilde{\varepsilon}$, *there exists a unique scattering state* $u_+ \in \mathbf{H}^{m-\delta} \cap \dot{\mathbf{H}}^{-1}$, $\partial_x u_+ \in \mathbf{H}^{0,1-\delta}$ *satisfying*

$$\|\mathcal{U}(-t)\,u(t) - u_+\|_{\mathbf{H}^{m-\delta}} + \|\mathcal{U}(-t)\,u(t) - u_+\|_{\dot{\mathbf{H}}^{-1}} + \|\mathcal{U}(-t)\,\partial_x u(t) - \partial_x u_+\|_{\mathbf{H}^{0,1-\delta}} \to 0 \tag{2.1}$$

as $t \to \infty$, *where* $\delta > 0$ *is small.*

Next result states an almost global existence of small solutions to (1.1) with $\rho = 3$. We define a maximal existence time T^* by

$$T^* = \sup\left\{T > 0; \|u\|_{\mathbf{X}_T^m} < \infty\right\}.$$

Theorem 2.2. *Let* $\rho = 3$. *Assume the initial data* $u_0 \in \mathbf{X}_0^m$ *with* $m > 4$ *and* $\|u_0\|_{\mathbf{X}_0^m} = \widetilde{\varepsilon}$. *Then there exist positive constants* ε_0 *and* B *such that*

$$T^* \geq \exp\left(\frac{B}{\widetilde{\varepsilon}^2}\right)$$

for all $0 < \widetilde{\varepsilon} \leq \varepsilon_0$.

Remark. The proof of Theorem 2.2 works also for the Cauchy problem

$$\left\{\begin{array}{c} u_{tx} = u + a(t)\,(u^3)_{xx} \\ u(0) = u_0 \end{array}\right., \tag{2.2}$$

if the coefficient $a(t) \in \mathbf{C}^1(\mathbf{R})$ satisfies the following time decay estimate

$$\left|\partial_t^j a(t)\right| \leq C\,(1 + |t|)^{-j}\,(\log(2 + |t|))^{-1-\gamma}$$

for $j = 0, 1$ and $t > 0$, where $\gamma > 0$. We have the following result.

Theorem 2.3. *Let the initial data* $u_0 \in \mathbf{X}_0^m$, *where* $m > 4$. *Then there exists a positive constant* $\widetilde{\varepsilon}$ *such that (2.2) has a unique global solution* $u \in \mathbf{X}_\infty^m$ *with the time decay*

$$\|u(t)\|_{\mathbf{L}^\infty} \leq C\,\langle t\rangle^{-\frac{1}{2}}$$

for any u_0 *satisfying* $\|u_0\|_{\mathbf{X}_0^m} \leq \widetilde{\varepsilon}$. *Moreover for any* $u_0 \in \mathbf{X}_0^m$ *such that* $\|u_0\|_{\mathbf{X}_0^m} \leq \widetilde{\varepsilon}$, *there exists a unique scattering state* $u_+ \in \mathbf{H}^{m-\delta} \cap \dot{\mathbf{H}}^{-1}$, $\partial_x u_+ \in \mathbf{H}^{0,1-\delta}$ *satisfying (2.1) with a small* $\delta > 0$.

Remark. We improve the result of Theorem 2.2 in Section 4 below thanks to our recent work [19].

As it was stated before, the local well-posedness in the function space $\mathbf{H}^m \cap \dot{\mathbf{H}}^{-1}$ was treated in [29], [30]. However the local well-posedness for (1.1) in weighted Sobolev spaces is not known. For the convenience of the readers, we give a local existence result for (1.1) in the following Proposition 2.4, where we also justify the formal computation concerning the estimates of $\mathcal{P}u$, which was made in [19], [21].

Proposition 2.4. *Let the initial data $u_0 \in \mathbf{X}_0^m$ with $m \geq 2$, and the order ρ of the nonlinearity satisfy $\rho > m+1$, or be an integer $\rho > 1$. Then there exist a time $T(u_0) > 0$ and a unique solution*

$$\begin{aligned} u &\in \mathbf{C}\left([0,T]; \mathbf{H}^m \cap \dot{\mathbf{H}}^{-1}\right) \cap \mathbf{C}^1\left([0,T]; \mathbf{L}^2\right), \\ \mathcal{P}u &\in \mathbf{C}\left([0,T]; \mathbf{L}^2\right) \end{aligned}$$

to the Cauchy problem (1.1). Furthermore the estimate

$$\begin{aligned} &\|u(t)\|_{\mathbf{H}^m} + \|\mathcal{P}u(t)\|_{\mathbf{L}^2} + \|u(t)\|_{\dot{\mathbf{H}}^{-1}} \\ &\leq C\int_0^t \|u(s)\|_{\mathbf{H}^1_\infty}^{\rho-1}\left(\|u(s)\|_{\mathbf{H}^m} + \|\mathcal{P}u(s)\|_{\mathbf{L}^2} + \|u(s)\|_{\dot{\mathbf{H}}^{-1}}\right) ds \end{aligned}$$

is true for $t \in [0,T]$.

Proof. We use the parabolic regularization method to treat the derivative loss coming from the nonlinearity. We introduce the function spaces

$$\mathbf{Y}_T^m = \left\{u(t) \in \mathbf{C}([0,T]; \mathbf{H}^m); \|u\|_{\mathbf{Y}_T^m} < \infty\right\},$$

$$\mathbf{Y}_0^m = \left\{\phi \in \mathbf{L}^2; \|\phi\|_{\mathbf{Y}_0^m} < \infty\right\},$$

where the norms

$$\|u\|_{\mathbf{Y}_T^m} = \|u\|_{\mathbf{X}_T^m} + \sup_{t\in(0,T]} t^{\frac{1}{3}}\left\|D_x^{-2}u(t)\right\|_{\mathbf{L}^6} + \sup_{t\in(0,T]} t^{\frac{1}{3}}\|xu(t)\|_{\mathbf{L}^6}$$

and

$$\|\phi\|_{\mathbf{Y}_0^m} = \|\phi\|_{\mathbf{H}^m} + \left\|D_x^{-1}\phi\right\|_{\mathbf{L}^{\frac{6}{5}}} + \|x\phi\|_{\mathbf{H}^1},$$

with $D_x^\alpha = \mathcal{F}^{-1}|\xi|^\alpha \mathcal{F}$ for $\alpha \in \mathbf{R}$. Define a sequence $u_{0,j} \in \mathbf{Y}_0^m$ such that

$$\lim_{j\to\infty}\|u_{0,j} - u_0\|_{\mathbf{X}_0^m} = 0$$

and consider the local existence of solutions to the Cauchy problem

$$\left\{\begin{array}{c} u_{tx} - u - \nu u_{xxx} = (f(u))_{xx}, \ x \in \mathbf{R}, \ t > 0, \\ u(0,x) = u_{0,j}(x), \ x \in \mathbf{R}, \end{array}\right. \tag{2.3}$$

in $\mathbf{Y}_T^m$, where $\nu \in (0,1]$. The linearized integral equation associated with (2.3) is written as

$$u(t) = \mathcal{U}_\nu(t)\, u_0 + \int_0^t \mathcal{U}_\nu(t-s)\, \partial_x f(v(s))\, ds, \tag{2.4}$$

where

$$\mathcal{U}_\nu(t) = \mathcal{F}^{-1} \exp\left(-\frac{it}{\xi} - \nu t \xi^2\right) \mathcal{F}$$

and $\|v\|_{\mathbf{X}_T^m} \le M$. Next we use the time decay estimate for the free evolution group $\mathcal{F}^{-1} \exp\left(-\frac{it}{\xi}\right) \mathcal{F}$ (see paper [36] for the proof in the case $1 < p < \infty$ and paper [21] for $p = \infty$)

$$\left\| \mathcal{F}^{-1} \exp\left(-\frac{it}{\xi}\right) \mathcal{F}\phi \right\|_{\mathbf{L}^p} \le C t^{-\frac{1}{2}\left(1-\frac{2}{p}\right)} \left\| \mathcal{F}^{-1} |\xi|^{\frac{3}{2}\left(1-\frac{2}{p}\right)} \mathcal{F}\phi \right\|_{\mathbf{L}^{\frac{p}{p-1}}}$$

for $t > 0$. Also we use the estimate

$$\left\| \mathcal{F}^{-1} \xi^j \exp\left(-\nu t \xi^2\right) \right\|_{\mathbf{L}^1} \le C \nu^{-\frac{j}{2}} t^{-\frac{j}{2}}$$

for $j = 0, 1$, which can be obtained by an explicit computation

$$\sqrt{2\pi}\mathcal{F}^{-1} \exp\left(-\nu t \xi^2\right) = \int_{\mathbf{R}} e^{ix\xi - \nu t \xi^2} d\xi = \frac{\sqrt{\pi}}{\sqrt{\nu t}} e^{-\frac{x^2}{4\nu t}}.$$

Therefore by the Young inequality we find the following estimate

(2.5)

$$\begin{aligned}
\|\mathcal{U}_\nu(t)\, u_0\|_{\mathbf{L}^p} &= \left\| \mathcal{F}^{-1} \exp\left(-\frac{it}{\xi}\right) \mathcal{F}\mathcal{F}^{-1} \exp\left(-\nu t\xi^2\right) \mathcal{F} u_0 \right\|_{\mathbf{L}^p} \\
&\le C t^{-\frac{1}{2}\left(1-\frac{2}{p}\right)} \left\| \mathcal{F}^{-1} \exp\left(-\nu t \xi^2\right) \mathcal{F} D_x^{\frac{3}{2}\left(1-\frac{2}{p}\right)} u_0 \right\|_{\mathbf{L}^{\frac{p}{p-1}}} \\
&\le C t^{-\frac{1}{2}\left(1-\frac{2}{p}\right)} \left\| D_x^{\frac{3}{2}\left(1-\frac{2}{p}\right)} u_0 \right\|_{\mathbf{L}^{\frac{p}{p-1}}} \left\| \mathcal{F}^{-1} \exp\left(-\nu t \xi^2\right) \right\|_{\mathbf{L}^1} \\
&\le C t^{-\frac{1}{2}\left(1-\frac{2}{p}\right)} \left\| D_x^{\frac{3}{2}\left(1-\frac{2}{p}\right)} u_0 \right\|_{\mathbf{L}^{\frac{p}{p-1}}}
\end{aligned}$$

for $2 \le p \le \infty$ and similarly

$$\|\mathcal{U}_\nu(t)\, \partial_x u_0\|_{\mathbf{L}^p} \le C \nu^{-\frac{1}{2}} t^{-\frac{1}{2}\left(1-\frac{2}{p}\right)-\frac{1}{2}} \left\| D_x^{\frac{3}{2}\left(1-\frac{2}{p}\right)} u_0 \right\|_{\mathbf{L}^{\frac{p}{p-1}}}. \tag{2.6}$$

By virtue of (2.6), (2.6) with $p=2$ we obtain from (2.4)

$$\text{(2.7)} \qquad \begin{aligned} \|u\|_{\mathbf{H}^m} &\le \|u_0\|_{\mathbf{H}^m} + C\nu^{-\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} \|v\|_{\mathbf{H}^m}^{\rho} ds \\ &\le \|u_0\|_{\mathbf{H}^m} + C\nu^{-\frac{1}{2}} T^{\frac{1}{2}} \left(\sup_{t\in[0,T]} \|v(t)\|_{\mathbf{H}^m} \right)^{\rho} \\ &\le \|u_0\|_{\mathbf{H}^m} + C\nu^{-\frac{1}{2}} T^{\frac{1}{2}} M^{\rho} \end{aligned}$$

and

$$\text{(2.8)} \qquad \begin{aligned} \|u\|_{\dot{\mathbf{H}}^{-1}} &\le \|u_0\|_{\dot{\mathbf{H}}^{-1}} + C\int_0^t \|v^{\rho}\|_{\mathbf{L}^2} ds \\ &\le \|u_0\|_{\dot{\mathbf{H}}^{-1}} + CT \left(\sup_{t\in[0,T]} \|v(t)\|_{\mathbf{H}^1} \right)^{\rho} \\ &\le \|u_0\|_{\dot{\mathbf{H}}^{-1}} + CTM^{\rho}. \end{aligned}$$

Multiplying both sides of (2.4) by $D_x^{-2} = \mathcal{F}^{-1} |\xi|^{-2} \mathcal{F}$, taking the $\mathbf{L}^6$ - norm and using (2.6) with $p=6$, we obtain

$$\begin{aligned} \left\|D_x^{-2} u(t)\right\|_{\mathbf{L}^6} &\le \left\|\mathcal{U}_{\nu}(t) D_x^{-2} u_0\right\|_{\mathbf{L}^6} + \int_0^t \left\|\mathcal{U}_{\nu}(t-s) D_x^{-2}\partial_x f(v(s))\right\|_{\mathbf{L}^6} ds \\ &\le Ct^{-\frac{1}{3}} \left\|D_x^{-1} u_0\right\|_{\mathbf{L}^{\frac{6}{5}}} + C\int_0^t (t-s)^{-\frac{1}{3}} \left\|D_x^{-1}\partial_x f(v(s))\right\|_{\mathbf{L}^{\frac{6}{5}}} ds \\ &\le Ct^{-\frac{1}{3}} \left\|D_x^{-1} u_0\right\|_{\mathbf{L}^{\frac{6}{5}}} + C\int_0^t (t-s)^{-\frac{1}{3}} \|f(v(s))\|_{\mathbf{L}^{\frac{6}{5}}} ds, \end{aligned}$$

where we have used the fact that the Hilbert transformation $D_x^{-1}\partial_x$ is a bounded operator in $\mathbf{L}^p$, $1<p<\infty$. Hence

(2.9)

$$\begin{aligned} t^{\frac{1}{3}} \left\|D_x^{-2} u(t)\right\|_{\mathbf{L}^6} &\le C\left\|D_x^{-1} u_0\right\|_{\mathbf{L}^{\frac{6}{5}}} + Ct^{\frac{1}{3}} \int_0^t (t-s)^{-\frac{1}{3}} \|v(s)\|_{\mathbf{L}^{\frac{6}{5}\rho}}^{\rho} ds \\ &\le C\left\|D_x^{-1} u_0\right\|_{\mathbf{L}^{\frac{6}{5}}} + CTM^{\rho}. \end{aligned}$$

Next by a direct calculation we find

$$\|x\mathcal{U}_{\nu}(t)\phi\|_{\mathbf{L}^p} \le Ct\left\|D_x^{-2}\mathcal{U}_{\nu}(t)\phi\right\|_{\mathbf{L}^p} + Ct\nu \|\mathcal{U}_{\nu}(t)\partial_x\phi\|_{\mathbf{L}^p} + C\|\mathcal{U}_{\nu}(t)x\phi\|_{\mathbf{L}^p}.$$

Therefore by (2.6), we obtain from (2.4)

$$\begin{aligned} t^{\frac{1}{3}} \|xu(t)\|_{\mathbf{L}^6} \leq\ & Ct^{\frac{4}{3}} \left\| D_x^{-2} \mathcal{U}_\nu(t) u_0 \right\|_{\mathbf{L}^6} \\ & + Ct^{\frac{4}{3}} \nu \|\partial_x \mathcal{U}_\nu(t) u_0\|_{\mathbf{L}^6} + Ct^{\frac{1}{3}} \|\mathcal{U}_\nu(t) xu_0\|_{\mathbf{L}^6} \\ & + t^{\frac{1}{3}} \int_0^t (t-s) \left\| \mathcal{U}_\nu(t-s) D_x^{-2} \partial_x f(v(s)) \right\|_{\mathbf{L}^6} ds \\ & + t^{\frac{1}{3}} \int_0^t (t-s) \nu \left\| \mathcal{U}_\nu(t-s) \partial_x^2 f(v(s)) \right\|_{\mathbf{L}^6} ds \\ & + t^{\frac{1}{3}} \int_0^t \|\mathcal{U}_\nu(t-s) x \partial_x f(v(s))\|_{\mathbf{L}^6} ds. \end{aligned}$$

Applying (2.6) we have

$$\begin{aligned} & t^{\frac{1}{3}} \|xu(t)\|_{\mathbf{L}^6} \leq Ct \left\| D_x^{-1} u_0 \right\|_{\mathbf{L}^{\frac{6}{5}}} + C\nu^{\frac{1}{2}} t^{\frac{1}{2}} \|D_x u_0\|_{\mathbf{L}^{\frac{6}{5}}} + C \|xu_0\|_{\mathbf{H}^1} \\ & + Ct^{\frac{1}{3}} \int_0^t \left(\|v(s)\|_{\mathbf{L}^{\frac{6}{5}\rho}}^\rho + \|v(s)\|_{\mathbf{H}^2}^\rho + \|x\partial_x v(s)\|_{\mathbf{L}^2} \|v(s)\|_{\mathbf{H}^2}^{\rho-1} \right) ds \\ & \leq CT \left\| D_x^{-1} u_0 \right\|_{\mathbf{L}^{\frac{6}{5}}} + CT^{\frac{1}{2}} \|D_x u_0\|_{\mathbf{L}^{\frac{6}{5}}} + C \|xu_0\|_{\mathbf{H}^1} + CTM^\rho. \end{aligned} \tag{2.10}$$

Multiplying both sides of (2.4) by $x\partial_x$ and using the commutator

$$[x\partial_x, \mathcal{U}_\nu(t)] = -\mathcal{U}_\nu(t) \left(t \left(\partial_x^{-1} - 2\nu \partial_x^2 \right) \right)$$

we get

$$\begin{aligned} x\partial_x u(t) =\ & \mathcal{U}_\nu(t) \left(x\partial_x - t \left(\partial_x^{-1} - 2\nu\partial_x^2 \right) \right) u_0 \\ & + \int_0^t \mathcal{U}_\nu(t-s) \left(x\partial_x^2 - (t-s) \left(1 - 2\nu\partial_x^3 \right) \right) f(v(s)) ds. \end{aligned}$$

Then taking the $\mathbf{L}^2$ - norm, using the estimate

$$\left\| \mathcal{U}_\nu(t) \partial_x^j u_0 \right\|_{\mathbf{L}^2} \leq Ct^{-\frac{j-m}{2}} \nu^{-\frac{j-m}{2}} \|\partial_x^m u_0\|_{\mathbf{L}^2}$$

for $j \geq m \geq 0$, we get

$$\begin{aligned} & \|x\partial_x u\|_{\mathbf{L}^2} \leq \|x\partial_x u_0\|_{\mathbf{L}^2} + CT \|u_0\|_{\dot{\mathbf{H}}^{-1}} + C \|u_0\|_{\mathbf{L}^2} \\ & + C\nu^{-\frac{1}{2}} \int_0^T (t-s)^{-\frac{1}{2}} \left(\left\| |v|^{\rho-1} x\partial_x v \right\|_{\mathbf{L}^2} + \||v|^\rho\|_{\mathbf{L}^2} \right) ds \\ & + C \int_0^T \left(T \||v|^\rho\|_{\mathbf{L}^2} + \left\| |v|^{\rho-1} \partial_x v \right\|_{\mathbf{L}^2} \right) ds. \end{aligned}$$

Hence

$$
\begin{aligned}
(2.11)\ \|x\partial_x u\|_{\mathbf{L}^2} &\le \|x\partial_x u_0\|_{\mathbf{L}^2} + CT\|u_0\|_{\dot{\mathbf{H}}^{-1}} + C\|u_0\|_{\mathbf{L}^2} \\
&\quad + C\nu^{-\frac{1}{2}}T^{\frac{1}{2}} \sup_{t\in[0,T]} \|x\partial_x v\|_{\mathbf{L}^2} \left(\sup_{t\in[0,T]} \|v(t)\|_{\mathbf{H}^1} \right)^{\rho-1} \\
&\quad + C\left(T^2+T\right)\left(\sup_{t\in[0,T]} \|v(t)\|_{\mathbf{H}^1} \right)^{\rho} \\
&\le \|x\partial_x u_0\|_{\mathbf{L}^2} + CT\|u_0\|_{\dot{\mathbf{H}}^{-1}} + C\|u_0\|_{\mathbf{L}^2} \\
&\quad + C\left(\nu^{-\frac{1}{2}}T^{\frac{1}{2}} + T^2 + T\right) M^{\rho}.
\end{aligned}
$$

Next by the definition of the operator $\mathcal{J}$ we have $\mathcal{J}\partial_x = x\partial_x - t\partial_x^{-1}$. Hence by (2.9) and (2.12) we find

$$
\begin{aligned}
(2.12)\qquad \|\mathcal{J}\partial_x u\|_{\mathbf{L}^2} &\le C\|x\partial_x u\|_{\mathbf{L}^2} + Ct\left\|\partial_x^{-1}u\right\|_{\mathbf{L}^2} \\
&\le \|x\partial_x u_0\|_{\mathbf{L}^2} + CT\|u_0\|_{\dot{\mathbf{H}}^{-1}} + C\|u_0\|_{\mathbf{L}^2} \\
&+ C\left(\nu^{-\frac{1}{2}}T^{\frac{1}{2}} + T^2 + T\right) M^{\rho}.
\end{aligned}
$$

As in the proof of (2.8) we obtain

$$
\begin{aligned}
t\nu\left\|\partial_x^{m+2}u\right\|_{\mathbf{L}^2} &\le \|u_0\|_{\mathbf{H}^m} + C\nu^{-\frac{1}{2}}\int_0^t (t-s)^{-\frac{1}{2}}\|v\|_{\mathbf{H}^m}^{\rho}\,ds \\
&\le \|u_0\|_{\mathbf{H}^m} + C\nu^{-\frac{1}{2}}T^{\frac{1}{2}}M^{\rho},
\end{aligned}
$$

therefore we also can estimate $\mathcal{P} = x\partial_x - t\partial_t = \mathcal{J}\partial_x - t\left(\partial_t - \partial_x^{-1}\right)$ as follows

$$
\begin{aligned}
\|\mathcal{P}u\|_{\mathbf{L}^2} &\le C\|\mathcal{J}\partial_x u\|_{\mathbf{L}^2} + Ct\nu\|u_{xxx}\|_{\mathbf{L}^2} + Ct\|\partial_x f(v)\|_{\mathbf{L}^2} \\
&\le \|x\partial_x u_0\|_{\mathbf{L}^2} + CT\|u_0\|_{\dot{\mathbf{H}}^{-1}} \\
&+ C\|u_0\|_{\mathbf{L}^2} + C\left(\nu^{-\frac{1}{2}}T^{\frac{1}{2}} + T^2 + T\right) M^{\rho}.
\end{aligned}
$$

By virtue of (2.8)- (2.13) we find that there exists a time T_ν such that (2.3) has a unique solution $u = u^{(\nu)}$ such that

$$
u^{(\nu)} \in \mathbf{Y}_{T_\nu}^m.
$$

We next prove that the existence time T_ν can be taken independent of ν. We note that the estimates of $\|u\|_{\mathbf{H}^m}$, $\|x\partial_x u\|_{\mathbf{L}^2}$ and $\|\mathcal{J}\partial_x u\|_{\mathbf{L}^2}$ obtained above depend on ν. On the other hand, the estimates of $\|u\|_{\dot{\mathbf{H}}^{-1}}$, $t^{\frac{1}{3}}\left\|D_x^{-2}u(t)\right\|_{\mathbf{L}^6}$ and $t^{\frac{1}{3}}\|xu(t)\|_{\mathbf{L}^6}$ do not depend on ν. We need to prove that the estimates for $\|u\|_{\mathbf{H}^m}$, $\|x\partial_x u\|_{\mathbf{L}^2}$ and $\|\mathcal{J}\partial_x u\|_{\mathbf{L}^2}$ also do not depend on ν. We consider equation (2.3)

$$u_t - \partial_x^{-1}u - \nu u_{xx} = (f(u))_x \tag{2.13}$$

where $\partial_x^{-1} = \mathcal{F}^{-1}\frac{1}{i\xi}\mathcal{F}$. By (2.9), (2.10) and (2.11) we have

$$\sup t^{\frac{1}{3}} \|xu(t)\|_{\mathbf{L}^6} + \sup \|u(t)\|_{\dot{\mathbf{H}}^{-1}} + \sup t^{\frac{1}{3}} \left\|\partial_x^{-2}u(t)\right\|_{\mathbf{L}^6} \leq C, \tag{2.14}$$

therefore $\lim_{|x|\to\infty}\partial_x^{-2}u = \lim_{|x|\to\infty}\partial_x^{-1}u = 0$. Now we can apply the usual energy method to (2.13) for an integer m

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{\mathbf{H}^m}^2 + \nu\|u(t)\|_{\mathbf{H}^{m+1}}^2 \leq C\|u\|_{\mathbf{L}^\infty}^{\rho-2}\|\partial_x u\|_{\mathbf{L}^\infty}\|u\|_{\mathbf{H}^m}^2. \tag{2.15}$$

By Lemma 1 from [36] we find (2.15) also for the fractional order $m > 1$. We next consider the a-priori estimate of $\|\mathcal{P}u\|_{\mathbf{L}^2}$. We apply $\mathcal{P} = x\partial_x - t\partial_t$ to equation (1.1). In view of the commutation relations $[\mathcal{P},\mathcal{L}] = \mathcal{L}$, $[\mathcal{P},\partial_x] = -\partial_x$, we get

$$\mathcal{L}\mathcal{P}u = \nu(\mathcal{P}u)_{xx} + \mathcal{P}(f(u))_x - 3\nu u_{xx} + (f(u))_x.$$

The applying the energy method we obtain

$$\begin{aligned}\frac{d}{dt}\|\mathcal{P}u\|_{\mathbf{L}^2}^2 &= \int_{\mathbf{R}}\left(\partial_x\left(\partial_x^{-1}\mathcal{P}u\right)^2 + 2\nu\partial_x\left((\mathcal{P}u)_x\mathcal{P}u\right)\right)dx \\ &+2\left(\mathcal{P}(f(u))_x + (f(u))_x, \mathcal{P}u\right) - 2\nu\|(\mathcal{P}u)_x\|_{\mathbf{L}^2}^2 - 6\nu(u_{xx},\mathcal{P}u).\end{aligned}$$

Since $\partial_x^{-1}\mathcal{P} = \partial_x^{-1}(x\partial_x - t\partial_t) = x - \partial_x^{-1} - t\partial_x^{-1}\partial_t$ we have by equation (1.1)

$$\partial_x^{-1}\mathcal{P}u = \left(x - \partial_x^{-1}\right)u - t\partial_x^{-2}u + \nu u_x + (f(u)).$$

Therefore by (2.14), we know that $\lim_{|x|\to\infty}\partial_x^{-1}\mathcal{P}u = \lim_{|x|\to\infty}\mathcal{P}u = 0$ which implies the estimate

$$\begin{aligned}&\frac{d}{dt}\|\mathcal{P}u\|_{\mathbf{L}^2}^2 + \nu\|(\mathcal{P}u)_x\|_{\mathbf{L}^2}^2 \\ \leq\ & C\|u\|_{\mathbf{L}^\infty}^{\rho-2}\|\partial_x u\|_{\mathbf{L}^\infty}\left(\|\mathcal{P}u\|_{\mathbf{L}^2} + \|u\|_{\mathbf{L}^2}\right)\|\mathcal{P}u\|_{\mathbf{L}^2} + C\nu\|u\|_{\mathbf{H}^1}^2.\end{aligned} \tag{2.16}$$

By (2.15) and (2.17) we get

$$\begin{aligned}&\frac{d}{dt}\left(\|u\|_{\mathbf{H}^m} + \|\mathcal{P}u\|_{\mathbf{L}^2}\right) \\ \leq\ & C\|u\|_{\mathbf{L}^\infty}^{\rho-2}\|\partial_x u\|_{\mathbf{L}^\infty}\left(\|u\|_{\mathbf{H}^m} + \|\mathcal{P}u\|_{\mathbf{L}^2}\right).\end{aligned} \tag{2.17}$$

Integrating (2.18) we prove that the estimates for $\|u\|_{\mathbf{H}^m}$ and $\|\mathcal{P}u\|_{\mathbf{L}^2}$ are also independent of ν. From (2.18), (2.10), (2.11) and the estimate

$$\|\mathcal{J}\partial_x u\|_{\mathbf{L}^2} \leq \|\mathcal{P}u\|_{\mathbf{L}^2} + t\,\|u\|_{\mathbf{L}^\infty}^{\rho-1}\|\partial_x u\|_{\mathbf{L}^2}.$$

we find that the existence time T does not depends on ν. Therefore we obtain the local in time existence of solutions to (2.3) in the space $\mathbf{Y}_T^m$. To complete the proof of Proposition 2.4, we let $\nu \to 0$, and then $j \to \infty$. $\square$

We now explain our strategy of the proofs of Theorems 2.1-2.3. The operator $\mathcal{J} = \mathcal{U}(t)x\mathcal{U}(-t)$ was introduced in [8] first to study the scattering problem for the nonlinear Schrödinger equations and was used by many authors, see, e.g., [4]. However, the operator $\mathcal{J}$ does not work well on the nonlinear terms. To overcome this difficulty, we introduce the operator $\mathcal{P}$, which was used in [12] for studying the global existence of small solutions to quadratic nonlinear Schrödinger equations in three space dimensions. After that the operator $\mathcal{P}$ was used often for various equations appeared in fluid mechanics such as the modified Korteweg-de Vries equation [15], [16], the generalized Benjamin-Ono equation [17], and the generalized Kadomtsev-Petviashvili equation [20]. We use the set of operators $(\mathcal{P}, \partial_x, I)$ to get desired time decay estimates of solutions.

By the general theory of quasilinear hyperbolic equations we know that $\mathbf{H}^s$ - space with $s > \frac{3}{2}$ is necessary for the local well-posedness (see [36]). Hence it is reasonable to define our function space through the operators $\left(\mathcal{P}^2, \partial_x^2, \mathcal{P}\partial_x, \mathcal{P}, \partial_x, I\right)$. However the operator $\mathcal{P}^2$ is not acceptable for our equation since $\mathcal{P} = x\partial_x - t\partial_x^{-1} - t\mathcal{L}$ and $\mathcal{P}^2 \simeq (\mathcal{J}\partial_x)^2 = \left(x\partial_x - t\partial_x^{-1}\right)^2$ is equivalent to the use of ∂_x^{-2}. But we can not apply ∂_x^{-2} to the nonlinear term in our equation $u_t = \partial_x^{-1}u + (u^\rho)_x$. To avoid this difficulty, we use the fractional order operator $|\mathcal{J}|^\alpha = \mathcal{U}(t)\,|x|^\alpha\,\mathcal{U}(-t)$ (see [20]). A desired time decay of solutions is obtained by a-priori estimate of the norm $\|\partial_x\mathcal{U}(-t)u\|_{\mathbf{H}^{\frac{1}{2},\frac{1}{2}+\varepsilon}}$ (see Lemma 2.5 with $\phi = \mathcal{U}(-t)u$ and $\|\phi\|_{\mathbf{L}^1} \leq C\,\|\phi\|_{\mathbf{H}^{0,\frac{1}{2}+\varepsilon}}$, below). By Lemma 2.7 with $\phi = \mathcal{U}(-t)u$ and $l = 0$, the norm $\|\partial_x\mathcal{U}(-t)u\|_{\mathbf{H}^{\frac{1}{2},\frac{1}{2}+\varepsilon}}$ can be estimated by

$$C\left(\|\mathcal{J}\partial_x\phi\|_{\mathbf{L}^2} + \|\phi\|_{\mathbf{H}^{2+\varepsilon,0}}\right).$$

Thus we use the set of operators $\left(\mathcal{P}, \left(-\partial_x^2\right)^{\frac{m+\varepsilon}{2}}, I\right)$ to show a-priori estimates of the solutions. Here we encounter another difficulty. When we apply the energy method to estimate $\left\|\left(-\partial_x^2\right)^{\frac{m+\varepsilon}{2}} u\right\|_{\mathbf{L}^2}$, we need a time decay estimate of the norm $\|u\|_{\mathbf{H}^k_\infty}$, which requires the estimate of the norm $\|\mathcal{P}u\|_{\mathbf{H}^k}$. Whereas the application of the energy method for estimating the norm $\|\mathcal{P}u\|_{\mathbf{H}^k}$ leads to the estimate of the norm $\|\mathcal{P}u\|_{\mathbf{H}_\infty^{\frac{k}{2}+1}} \leq C\sum_{j=0}^2 \left\|\mathcal{P}^j u\right\|_{\mathbf{H}^{\frac{k}{2}+1}}$. So the higher order operator $\mathcal{P}^2$ appears. Thanks to Lemma 2.7, we can overcome this difficulty and consider the set $\left(\mathcal{P}, \left(-\partial_x^2\right)^{\frac{m+\varepsilon}{2}}, I\right)$, where the operators $\mathcal{P}$ and $\left(-\partial_x^2\right)^{\frac{1}{2}}$ have different orders.

Next we state the $\mathbf{L}^\infty$ - time decay estimate for the free evolution group $\mathcal{U}(t)$.

Lemma 2.5. *The estimate*

$$\|\mathcal{U}(t)\phi\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{1}{2}} \left\| D_x^{\frac{3}{2}} \phi \right\|_{\mathbf{L}^1}$$

is true for $t > 0$, *where* $D_x = \mathcal{F}^{-1} |\xi| \mathcal{F}\phi$.

The proof of Lemma 2.5 given in [21] is valid if we replace the right-hand side of the above estimate by the norm of the homogeneous Besov space $\dot{\mathbf{B}}_{1,1}^{\frac{3}{2}}$. However the norm $\|\phi\|_{\dot{\mathbf{B}}_{1,1}^{\frac{3}{2}}}$ can not be estimated by $\left\| D_x^{\frac{3}{2}} \phi \right\|_{\mathbf{L}^1}$ (see also [6]). Here we give a different proof of Lemma 2.5, which does not use the norm of the homogeneous Besov space $\dot{\mathbf{B}}_{1,1}^{\frac{3}{2}}$.

Proof. We have

$$\begin{aligned} \mathcal{U}(t)\phi &= \mathcal{F}^{-1} e^{-it\frac{1}{\xi}} \mathcal{F}\phi = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{ix\xi} e^{-it\frac{1}{\xi}} |\xi|^{-\frac{3}{2}} |\xi|^{\frac{3}{2}} \mathcal{F}\phi d\xi \\ &= \frac{1}{\sqrt{2\pi}} \lim_{\delta\to 0} \int_{|\xi|\geq\delta} e^{ix\xi} e^{-it\frac{1}{\xi}} |\xi|^{-\frac{3}{2}} |\xi|^{\frac{3}{2}} \mathcal{F}\phi d\xi. \end{aligned}$$

Hence changing the order of integration we get

$$\begin{aligned} \mathcal{U}(t)\phi &= \frac{1}{2\pi} \lim_{\delta\to 0} \int_{|\xi|\geq\delta} e^{ix\xi} e^{-it\frac{1}{\xi}} |\xi|^{-\frac{3}{2}} \int_{\mathbf{R}} e^{-iy\xi} \left(-\partial_y^2\right)^{\frac{3}{4}} \phi(y) dy d\xi \\ &= \frac{1}{2\pi} \lim_{\delta\to 0} \int_{\mathbf{R}} D_y^{\frac{3}{2}} \phi(y) dy \int_{|\xi|\geq\delta} e^{i(x-y)\xi} e^{-it\frac{1}{\xi}} |\xi|^{-\frac{3}{2}} d\xi \\ &= \lim_{\delta\to 0} \int_{\mathbf{R}} G_\delta(t, x-y) D_y^{\frac{3}{2}} \phi(y) dy = \int_{\mathbf{R}} \lim_{\delta\to 0} G_\delta(t, x-y) D_y^{\frac{3}{2}} \phi(y) dy \\ &= \int_{\mathbf{R}} G_0(t, x-y) D_y^{\frac{3}{2}} \phi(y) dy, \end{aligned}$$

where $G_0(t,x) = \lim_{\delta\to 0} G_\delta(t,x)$ and the kernel

$$G_\delta(t,x) = \frac{1}{2\pi} \int_{|\xi|\geq\delta} e^{ix\xi - it\frac{1}{\xi}} |\xi|^{-\frac{3}{2}} d\xi = \frac{1}{\pi} \mathrm{Re} \int_\delta^\infty e^{ix\xi - it\frac{1}{\xi}} \xi^{-\frac{3}{2}} d\xi.$$

Also changing $\xi^{-1}=\eta$ we get

$$\begin{aligned} G_0(t,x) &= \lim_{\delta\to 0} G_\delta(t,x) = \lim_{\delta\to 0}\frac{1}{\pi}\mathrm{Re}\int_\delta^\infty e^{ix\xi - it\frac{1}{\xi}}\xi^{-\frac{3}{2}}d\xi \\ &= -\lim_{\delta\to 0}\frac{1}{\pi}\mathrm{Re}\int_0^{\frac{1}{\delta}} e^{ix\eta^{-1}-it\eta}\eta^{-\frac{1}{2}}d\eta = -\frac{1}{\pi}\mathrm{Re}\int_0^\infty e^{ix\eta^{-1}-it\eta}\eta^{-\frac{1}{2}}d\eta. \end{aligned}$$

(this also justifies that the limit $\delta\to 0$ exists). We need to prove the estimate

$$|G_0(t,x)| = C\left|\int_0^\infty e^{ix\eta^{-1}-it\eta}\eta^{-\frac{1}{2}}d\eta\right| \le Ct^{-\frac{1}{2}}.$$

We change $\nu=\frac{x}{t}$, $\eta = y\sqrt{|\nu|}$, $\lambda = t\sqrt{|\nu|}$, $\sigma = \mathrm{sign}x$, then

$$\int_0^\infty e^{ix\eta^{-1}-it\eta}\eta^{-\frac{1}{2}}d\eta = |\nu|^{\frac{1}{4}}\int_0^\infty e^{i\lambda\left(\sigma y^{-1}-y\right)}y^{-\frac{1}{2}}dy.$$

The main advantage of the Littlewood-Paley decomposition is that they reduce the integral over $\mathbf{R}$ to the domain $\left(\frac{1}{2},2\right)$. However the tails $\int_2^\infty$ and $\int_0^{\frac{1}{2}}$ can be easily estimated by rotating the contour of integration and the integral $\int_{\frac{1}{2}}^2$ can be estimated by using the Van der Corput Lemma [37]: If μ is a real-valued function, smooth in (a,b), such that $\left|\mu^{(k)}(y)\right|\ge 1$ for some $k\ge 1$, then

$$\left|\int_a^b e^{i\lambda\mu(y)}\psi(y)\,dy\right| \le C\lambda^{-\frac{1}{k}}\left(\psi(b)+\int_a^b\left|\psi'(y)\right|dy\right).$$

Thus we get

$$\left|\int_{\frac{1}{2}}^2 e^{i\lambda\left(\sigma y^{-1}-y\right)}y^{-\frac{1}{2}}dy\right| \le C\lambda^{-\frac{1}{2}}.$$

In the integral $\int_2^\infty e^{i\lambda\left(\sigma y^{-1}-y\right)}y^{-\frac{1}{2}}dy$ we rotate the contour of integration $y=|y|\,e^{i\gamma}$ to show that it decays

$$\begin{aligned} &\left|\int_2^\infty e^{i\lambda\left(\sigma y^{-1}-y\right)}y^{-\frac{1}{2}}dy\right| \le \int_2^\infty e^{-\lambda\sin\gamma\left(|y|-\sigma|y|^{-1}\right)}\,|y|^{-\frac{1}{2}}\,d\,|y| \\ &+\left|\int_{C_\gamma} e^{i\lambda\left(\sigma y^{-1}-y\right)}y^{-\frac{1}{2}}dy\right| \le C\lambda^{-\frac{1}{2}}. \end{aligned}$$

The second integral is estimated in the same manner as in the Van der Corput Lemma. Finally the integral $\int_0^{\frac{1}{2}} e^{i\lambda\left(\sigma y^{-1}-y\right)}y^{-\frac{1}{2}}dy$ by the change $y=z^{-1}$ can be transformed to

$$\int_0^{\frac{1}{2}} e^{i\lambda\left(\sigma y^{-1}-y\right)}y^{-\frac{1}{2}}dy = \int_2^\infty e^{i\lambda\left(\sigma z - z^{-1}\right)}z^{-\frac{3}{2}}dz$$

and then we can rotate the contour of integration to show that it decays as $C\lambda^{-\frac{1}{2}}$. So we get the estimate

$$|G_0(t,x)| = C\left|\int_0^\infty e^{ix\eta^{-1}-it\eta}\eta^{-\frac{1}{2}}d\eta\right| \le Ct^{-\frac{1}{2}}.$$

Therefore by the Young inequality

$$\begin{aligned}|\mathcal{U}(t)\phi| &\le \int_{\mathbf{R}}\left|G_0(t,x-y)D_y^{\frac{3}{2}}\phi(y)\right|dy \\ &\le Ct^{-\frac{1}{2}}\int_{\mathbf{R}}\left|D_y^{\frac{3}{2}}\phi(y)\right|dy = Ct^{-\frac{1}{2}}\left\|D_y^{\frac{3}{2}}\phi\right\|_{\mathbf{L}^1}.\end{aligned}$$

This completes the proof of Lemma 2.5. □

The following lemma is necessary for considering the problem in the function space defined by the set of operators $\left(\mathcal{P},\left(-\partial_x^2\right)^{\frac{m+\varepsilon}{2}},I\right)$.

Lemma 2.6. *Let* $\mu \ge 2$, $0<\alpha<\beta<1$. *Then the estimate*

$$\|D_x^\mu\phi\|_{\mathbf{H}^{0,\alpha}} \le C\|\phi\|_{\mathbf{H}^{\frac{\mu-\beta}{1-\beta}}} + C\|x\partial_x\phi\|_{\mathbf{L}^2}$$

is true, provided that the right-hand side is finite.

From this lemma, we obtain

Lemma 2.7. *Let* $\varepsilon \in \left(0,\frac{1}{2}\right)$ *and* $l \ge 0$. *Then the estimate*

$$\left\|D_x^l\phi\right\|_{\mathbf{L}^\infty} \le Ct^{-\frac{1}{2}}\left(\|\phi\|_{\mathbf{H}^{\frac{2l+2-2\varepsilon}{1-2\varepsilon}}} + \|\mathcal{J}\partial_x\phi\|_{\mathbf{L}^2}\right)$$

is true, provided that the right-hand side is finite.

The following estimate was shown in [36] which is needed to consider the fractional order Sobolev spaces.

Lemma 2.8. *Let* u *be a smooth solution of*

$$u_{tx} = u + F(t,x)u_{xx} + G(t,x).$$

Then for any $s>1$, *there exists a constant* $C_s \simeq 1/(s-1)$, *and a positive constant* C *such that*

$$\begin{aligned}&\frac{d}{dt}\|D_x^s u(t)\|_{\mathbf{L}^2}^2 \le C_s\|\partial_x F(t)\|_{\mathbf{L}^\infty}\|D_x^s u(t)\|_{\mathbf{L}^2}^2 \\ &\quad + 2\|D_x^s u(t)\|_{\mathbf{L}^2}\left(\left\|D_x^{s-1}G(t)\right\|_{\mathbf{L}^2} + C\|\partial_x u(t)\|_{\mathbf{L}^\infty}\|D_x^s F(t)\|_{\mathbf{L}^2}\right).\end{aligned}$$

2.1. Proof of Theorem 2.1 (Global Existence)

We prove that for any $T > 0$

$$\|u\|_{\mathbf{X}_T^{2+\varepsilon}} < \sqrt{\widetilde{\varepsilon}}$$

by the contradiction argument. We assume that there exists a time T such that

$$\|u\|_{\mathbf{X}_T^{2+\varepsilon}} = \sqrt{\widetilde{\varepsilon}}.$$

We take in Lemma 2.8 $s = 2+\varepsilon$, $F = \rho u^{\rho-1}$, $G = \rho(\rho-1)u^{\rho-2}u_x^2$ if ρ is an integer and $F = \rho|u|^{\rho-1}$, $G = \rho(\rho-1)|u|^{\rho-3}uu_x^2$ if ρ is not integer and use the Sobolev inequality

$$\|u_x\|_{\mathbf{L}^\infty} \le C\|u\|_{\mathbf{L}^\infty}^{\frac{1+2\varepsilon}{3+2\varepsilon}}\|u\|_{\mathbf{H}^{2+\varepsilon}}^{\frac{2}{3+2\varepsilon}}, \tag{2.18}$$

to find that

$$\begin{aligned}\frac{d}{dt}\left\|\left(-\partial_x^2\right)^{\frac{s}{2}}u(t)\right\|_{\mathbf{L}^2}^2 &\le C\|u(t)\|_{\mathbf{L}^\infty}^{\rho-2+\frac{1+2\varepsilon}{3+2\varepsilon}}\|u(t)\|_{\mathbf{H}^s}^{\frac{2}{3+2\varepsilon}}\|u(t)\|_{\mathbf{H}^s}^2\\ &\le C\langle t\rangle^{-\frac{1}{2}\left(\rho-2+\frac{1+2\varepsilon}{3+2\varepsilon}\right)}\left(\|u\|_{\mathbf{H}^s}+\|\mathcal{J}\partial_x u\|_{\mathbf{L}^2}\right)^{\rho-2+\frac{1+2\varepsilon}{3+2\varepsilon}}\\ &\quad\times\|u(t)\|_{\mathbf{H}^s}^{\frac{2}{3+2\varepsilon}}\|u(t)\|_{\mathbf{H}^s}^2\end{aligned} \tag{2.19}$$

thanks to Lemma 2.8. Therefore

$$\|u(t)\|_{\mathbf{H}^s}^2 \le \widetilde{\varepsilon}^2 + C\widetilde{\varepsilon}^{\frac{\rho+1}{2}}\int_0^t \langle\tau\rangle^{-\frac{1}{2}\left(\rho-2+\frac{1+2\varepsilon}{3+2\varepsilon}\right)}d\tau \le \widetilde{\varepsilon}^2 + C\widetilde{\varepsilon}^{\frac{\rho+1}{2}} \le 2\widetilde{\varepsilon}^2 \tag{2.20}$$

since $\rho > 3 + \frac{2}{3}$ and $\varepsilon > 0$ is small. By the estimate of Proposition 2.4

$$\begin{aligned}&\|\mathcal{P}u(t)\|_{\mathbf{L}^2} + \|u(t)\|_{\dot{\mathbf{H}}^{-1}}\\ \le\ & C\int_0^t \|u\|_{\mathbf{L}^\infty}^{\rho-1}\|\partial_x u\|_{\mathbf{L}^\infty}\left(\|\mathcal{P}u(s)\|_{\mathbf{L}^2}+\|u\|_{\mathbf{L}^2}+\|u(s)\|_{\dot{\mathbf{H}}^{-1}}\right)ds.\end{aligned}$$

Then by (2.18)

$$\|\mathcal{P}u(t)\|_{\mathbf{L}^2} + \|u(t)\|_{\dot{\mathbf{H}}^{-1}} \le \sqrt{2}\widetilde{\varepsilon}. \tag{2.21}$$

By the identity

$$(\mathcal{P}-\mathcal{J}\partial_x)u = -t\left(u_t - \partial_x^{-1}u\right) = -t\left(|u|^{\rho-1}u\right)_x$$

we obtain

(2.22)

$$\begin{aligned}
\|\mathcal{J}\partial_x u\|_{\mathbf{L}^2} &\leq \|\mathcal{P}u\|_{\mathbf{L}^2} + t\|u\|_{\mathbf{L}^\infty}^{\rho-1}\|\partial_x u\|_{\mathbf{L}^2} \\
&\leq \|\mathcal{P}u\|_{\mathbf{L}^2} + C\langle t\rangle^{1-\frac{1}{2}(\rho-1)}\left(\|u\|_{\mathbf{H}^s} + \|\mathcal{J}\partial_x u\|_{\mathbf{L}^2}\right)^{\rho-1}\|\partial_x u\|_{\mathbf{L}^2} \\
&\leq \sqrt{2}\widetilde{\varepsilon} + C\widetilde{\varepsilon}^{\frac{\rho}{2}} \leq 2\widetilde{\varepsilon}.
\end{aligned}$$

By (2.21) and (2.23)

$$\|u\|_{\mathbf{X}_T^s} \leq 6\widetilde{\varepsilon} < \sqrt{\widetilde{\varepsilon}}.$$

This is the desired contradiction. Hence we have a global in time existence of the solution satisfying the estimate

$$\|u\|_{\mathbf{X}_\infty^s} \leq \sqrt{\widetilde{\varepsilon}}.$$

This completes the proof of the first part of Theorem 2.1.

Remark. For the proofs of Theorem 2.2 and Theorem 2.3, see [21].

§3. Sub Critical Case

To prove the nonexistence of the usual scattering states we need a lower bound for the time decay of solutions $w(t) = \mathcal{U}(t)\phi$ to the linear problem

$$\left\{\begin{array}{l} w_{tx} = w,\ t > 0,\ x \in \mathbf{R}, \\ w(0,x) = \phi(x),\ x \in \mathbf{R}, \end{array}\right. \tag{3.1}$$

which is given by

Theorem 3.1. *Let $\phi \in \mathbf{H}^1$ be such that $x\partial_x\phi \in \mathbf{H}^1$. Then the estimate*

$$\begin{aligned}
\|\mathcal{U}(t)\phi\|_{\mathbf{L}^r(-t,0)} \geq\ & \frac{1}{2}t^{-\frac{1}{2}\left(1-\frac{2}{r}\right)}\left(\left\|\widehat{\phi}\right\|_{\mathbf{L}^2\left(1,\sqrt{T}\right)} + \left\|\widehat{\phi}\right\|_{\mathbf{L}^2\left(-\sqrt{T},-1\right)}\right) \\
& - CAt^{-\frac{1}{4}-\frac{1}{2}\left(1-\frac{2}{r}\right)+\frac{\alpha}{4}}
\end{aligned}$$

is true for all $t \geq T > 1$, where $2 \leq r \leq \infty$, $\alpha \in \left(0, \frac{1}{2}\right)$ and

$$A = \|\phi\|_{\mathbf{H}^1} + \|x\partial_x\phi\|_{\mathbf{H}^1}.$$

Remark. The regularity assumptions on the data seems to be relaxed. Theorem 3.1 is related to Lemma 2.5 in which the assumption on the data is $\left\|D_x^{\frac{3}{2}}\phi\right\|_{\mathbf{H}^{0,1}} < \infty$.

Next we state the nonexistence of the usual scattering states for the Cauchy problem (1.1) as an application of Theorem 3.1.

Theorem 3.2. *Assume that there exists a solution*

$$u \in \mathbf{C}\left(\mathbf{R};\dot{\mathbf{H}}^{-1} \cap \mathbf{L}^2\right)$$

of the Cauchy problem (1.1) with $1 \leq \rho \leq 3$. *Furthermore, we assume that the time decay estimate*

$$\|u(t)\|_{\mathbf{L}^\infty} \leq C \langle t \rangle^{-\frac{1}{2}}$$

holds in the case of $2 < \rho \leq 3$. *Then, there does not exist any free solution* $w(t)$ *of the linear Cauchy problem (3.1) with the initial data*

$$\phi \in \mathbf{H}^2 \cap \dot{\mathbf{H}}^{-1}, \ x\partial_x \phi \in \mathbf{H}^1$$

and

$$\left\|\widehat{\phi}\right\|_{\mathbf{L}^2(1,T)} + \left\|\widehat{\phi}\right\|_{\mathbf{L}^2(-T,-1)} \neq 0$$

for some $T > 1$, *such that*

$$\lim_{t\to\infty} \|u(t) - w(t)\|_{\dot{\mathbf{H}}^{-1} \cap \mathbf{L}^2} = 0,$$

where $w(t) = \mathcal{U}(t)\phi$.

Remark. Since the local existence of solutions holds in $\dot{\mathbf{H}}^{-1} \cap \mathbf{H}^s$ with $s > \frac{3}{2}$, global solutions exist in $\mathbf{H}^2$ for $\rho = 3$ (see [34]) and in $\mathbf{H}^3$ for $\rho = 2$ (see [10]), so it is natural to expect the existence of the global solutions in $\dot{\mathbf{H}}^{-1} \cap \mathbf{H}^n$ with some $n \geq 2$. A formal computation implies that there are conserved quantities

$$E_0 = \int_{\mathbf{R}} u^2 dx$$

and

$$E_{-1} = \int_{\mathbf{R}} \left(\left(\partial_x^{-1} u\right)^2 - \frac{2}{\rho+1} |u|^{\rho+1} \right) dx.$$

Therefore the function space $\mathbf{C}\left(\mathbf{R};\dot{\mathbf{H}}^{-1} \cap \mathbf{L}^2\right)$ for the solutions in Theorem 3.2 is reasonable. However we have

$$\frac{d}{dt} E_1 = \frac{d}{dt} \int_{\mathbf{R}} \left(\sqrt{1 + 6u_x^2} - 1 \right) dx = 0$$

only for $\rho = 3$. Therefore for the case of fractional order nonlinearity, we do not have any result on the global existence and time decay of solutions to (1.1), when $\rho \leq 3 + \frac{2}{3}$ (see [20]).

Remark. The nonexistence of the scattering states for the nonlinear Klein-Gordon equations was studied by [9] for a real-valued solution and by [31] for a complex-valued solution. After their works, the idea by Glassey was used to prove the nonexistence of the scattering states for nonlinear Schrödinger equations in [1], [13], [38]. In their proofs, the lower bound of solutions to the linear problem was essential. Also note that for the case of the sub critical nonlinear Schrödinger equation $iu_t + \frac{1}{2}u_{xx} = |u|^{\rho-1} u$ with $\rho \leq 3$ the existence of the modified scattering states was proved in [14], along with the optimal time decay estimate $\|u(t)\|_{\mathbf{L}^\infty} \leq C \langle t\rangle^{-\frac{1}{2}}$. Recently in [19] we considered the cubic reduced Ostrovsky equation (the short-pulse equation) and proved the existence of the modified scattering states. Therefore we expect that the assumption on the time decay rate $\|u(t)\|_{\mathbf{L}^\infty} \leq C \langle t\rangle^{-\frac{1}{2}}$ in Theorem 3.2 is natural.

3.1. Proof of Theorem 3.2

We prove Theorem 3.2 by contradiction. Suppose that there exists a free solution $w(t) = \mathcal{U}(t)\phi$ of the linear Cauchy problem (3.1) with initial data ϕ such that

$$\lim_{t\to\infty} \left(\left\| \partial_x^{-1} (u(t) - w(t)) \right\|_{\mathbf{L}^2} + \|u(t) - w(t)\|_{\mathbf{L}^2} \right) = 0. \tag{3.2}$$

Define the functional

$$H_u(t) = \int_{\mathbf{R}} w(t,x)\, \partial_x^{-1} u(t,x)\, dx$$

as in [9] and [31]. In view of equations (1.1) and (3.1) we have $\partial_t \mathcal{U}(-t) w(t) = 0$ and $\partial_t \mathcal{U}(-t)\, \partial_x^{-1} u(t) = \mathcal{U}(-t) \left(|u|^{\rho-1} u \right)$. Also we can represent

$$H_u(t) = \int_{\mathbf{R}} \left(\mathcal{U}(-t) w(t) \right) \left(\mathcal{U}(-t)\, \partial_x^{-1} u(t) \right) dx.$$

Then by a direct calculation we find

$$\begin{aligned}
\frac{d}{dt} H_u(t) &= \int_{\mathbf{R}} \left(\partial_t \mathcal{U}(-t) w(t) \right) \left(\mathcal{U}(-t)\, \partial_x^{-1} u(t) \right) dx \\
&\quad + \int_{\mathbf{R}} \left(\mathcal{U}(-t) w(t) \right) \left(\partial_t \mathcal{U}(-t)\, \partial_x^{-1} u(t) \right) dx \\
&= \int_{\mathbf{R}} \left(\mathcal{U}(-t) w(t) \right) \left(\mathcal{U}(-t) \left(|u|^{\rho-1} u \right) \right) dx = \int_{\mathbf{R}} w\, |u|^{\rho-1} u dx \\
&= \int_{\mathbf{R}} |w|^{\rho+1} dx + \int_{\mathbf{R}} \left(w\, |u|^{\rho-1} u - |w|^{\rho+1} \right) dx.
\end{aligned}$$

For the case of $\rho > 2$ we have

$$\begin{aligned}
&\left|\int_{\mathbf{R}}\left(w\left|u\right|^{\rho-1}u-w\left|w\right|^{\rho-1}w\right)dx\right| \\
\leq\ & C\left\|w\right\|_{\mathbf{L}^{\infty}}\left(\left\|u\right\|_{\mathbf{L}^{2}}+\left\|w\right\|_{\mathbf{L}^{2}}\right)\left(\left\|w\right\|_{\mathbf{L}^{\infty}}^{\rho-2}+\left\|u\right\|_{\mathbf{L}^{\infty}}^{\rho-2}\right)\left\|u-w\right\|_{\mathbf{L}^{2}} \\
\leq\ & C\left(A+1\right)^{\rho}t^{-\frac{\rho-1}{2}}\left\|u-w\right\|_{\mathbf{L}^{2}},
\end{aligned}$$

where $A=\left\|\phi\right\|_{\mathbf{H}^{1}}+\left\|x\partial_{x}\phi\right\|_{\mathbf{H}^{1}}$. Here we applied the estimate $\left\|w\right\|_{\mathbf{L}^{\infty}}\leq Ct^{-\frac{1}{2}}$ from Lemma 2.5, also we have used that $\left\|u\right\|_{\mathbf{L}^{2}}$ does not depend on time and $\left\|u\right\|_{\mathbf{L}^{\infty}}\leq Ct^{-\frac{1}{2}}$, when $\rho > 2$. For the case of $1\leq\rho\leq 2$ we use the Hölder inequality

$$\begin{aligned}
&\left|\int_{\mathbf{R}}\left(w\left|u\right|^{\rho-1}u-w\left|w\right|^{\rho-1}w\right)dx\right| \\
\leq\ & C\left\|w\right\|_{\mathbf{L}^{\frac{2}{2-\rho}}}\left\|\left|u\right|^{\rho-1}u-\left|w\right|^{\rho-1}w\right\|_{\mathbf{L}^{\frac{2}{\rho}}} \\
\leq\ & C\left\|w\right\|_{\mathbf{L}^{\frac{2}{2-\rho}}}\left(\left\|u\right\|_{\mathbf{L}^{2}}+\left\|w\right\|_{\mathbf{L}^{2}}\right)^{\rho-1}\left\|u-w\right\|_{\mathbf{L}^{2}} \\
\leq\ & C\left(A+1\right)^{\rho}t^{-\frac{\rho-1}{2}}\left\|u-w\right\|_{\mathbf{L}^{2}}.
\end{aligned}$$

Then by Theorem 3.1 we estimate

$$\begin{aligned}
\frac{d}{dt}H_{u}\left(t\right) &\geq \int_{\mathbf{R}}\left|w\right|^{\rho+1}dx-C\left(A+1\right)^{\rho}t^{-\frac{\rho-1}{2}}\left\|u-w\right\|_{\mathbf{L}^{2}} \\
&\geq \frac{1}{2^{\rho+1}}t^{-\frac{\rho-1}{2}}\left(\left\|\widehat{\phi}\right\|_{\mathbf{L}^{2}\left(1,\sqrt{T}\right)}+\left\|\widehat{\phi}\right\|_{\mathbf{L}^{2}\left(-\sqrt{T},-1\right)}\right)^{\rho+1} \\
&\quad -CA^{\rho+1}t^{-\frac{\rho-1}{2}-\frac{1-\alpha}{4}\left(\rho+1\right)}-C\left(A+1\right)^{\rho}t^{-\frac{\rho-1}{2}}\left\|u-w\right\|_{\mathbf{L}^{2}}.
\end{aligned}$$

By the assumptions of Theorem 3.2, there exists $T > 1$ such that

$$\left\|u\left(t\right)-w\left(t\right)\right\|_{\mathbf{L}^{2}}<\varepsilon$$

for all $t\geq T$ and any $\varepsilon > 0$, from which it follows that

$$C\left(A+1\right)^{\rho}\varepsilon<\frac{1}{2^{\rho+1}}\left(\left\|\widehat{\phi}\right\|_{\mathbf{L}^{2}\left(1,\sqrt{T}\right)}+\left\|\widehat{\phi}\right\|_{\mathbf{L}^{2}\left(-\sqrt{T},-1\right)}\right)^{\rho+1}.$$

Hence

$$\left|H_{u}\left(2T\right)-H_{u}\left(T\right)\right|\geq C\int_{T}^{2T}t^{-\frac{\rho-1}{2}}dt\geq CT^{\frac{3-\rho}{2}} \tag{3.3}$$

for large T. On the other hand, by the definition of $H_u(t)$ and (3.2) we find

$$\begin{aligned} H_u(t) &= \int_{\mathbf{R}} w\partial_x^{-1}(u-w)\,dx \\ &\le C\|w(t)\|_{\mathbf{L}^2}\left\|\partial_x^{-1}(u(t)-w(t))\right\|_{\mathbf{L}^2} \\ &\le C\|u_0\|_{\mathbf{L}^2}\left\|\partial_x^{-1}(u(t)-w(t))\right\|_{\mathbf{L}^2} \to 0 \end{aligned} \tag{3.4}$$

for $t \to \infty$. From (3.3) and (3.4) we obtain a desired contradiction. This completes the proof of Theorem 3.2.

§4. Critical Case

We consider the Cauchy problem for the reduced Ostrovsky equation

$$\begin{cases} u_{tx} = u + \left(u^3\right)_{xx}, & (t,\ x) \in \mathbf{R}_+ \times \mathbf{R}, \\ u(0,x) = u_0(x), & x \in \mathbf{R}, \end{cases} \tag{4.1}$$

with real-valued initial data u_0. Equation (4.1) is called the short-pulse equation [35]. The short-pulse equation is derived as approximate solutions of Maxwell's equations describing the propagation of ultra-short optical pulses in nonlinear media, see [35], where the local well-posedness in $\mathbf{H}^2$ and non-existence of smooth traveling wave solutions were shown.

By changing the variables $t = \frac{1}{\sqrt{2}}(T-X)$, $x = \frac{1}{\sqrt{2}}(T+X)$ we have

$$\begin{aligned} \partial_T &= \frac{1}{\sqrt{2}}(\partial_t + \partial_x), \partial_X = \frac{1}{\sqrt{2}}(-\partial_t + \partial_x), \\ \partial_t &= \frac{1}{\sqrt{2}}(\partial_T - \partial_X), \partial_x = \frac{1}{\sqrt{2}}(\partial_T + \partial_X) \end{aligned}$$

from which it follows that

$$\left(\partial_T^2 - \partial_X^2 + 1\right)u = (-\partial_t\partial_x + 1)\,u.$$

Therefore (4.1) is transformed to the quasi linear Klein-Gordon equations

$$\left(\partial_T^2 - \partial_X^2 + 1\right)u = -\frac{1}{2}(\partial_T + \partial_X)^2\left(u^3\right) \tag{4.2}$$

with the cubic nonlinear terms. Vector field method is a powerful tool to study the large time existence of nonlinear evolution equations with critical nonlinearities in this field since the work by Klainerman [27]. To study the asymptotic behavior of solutions to the initial value problem for (4.2) with the data

$$u(0,X) = u_0, u_t(0,X) = u_1 \tag{4.3}$$

the vector $\Gamma = (\partial_T, \partial_X, X\partial_T + T\partial_X)$, hyperbolic coordinate and compact support conditions were used in [5]. However problem (4.1) differs from problem (4.2) with (4.3) since the data are given on the line of the light cone, namely the method of hyperbolic coordinate from [27] is not applicable. In this paper we adopt the method of the factorization technique for the free evolution group $\mathcal{U}(t) = \mathcal{F}^{-1}\exp\left(-\frac{it}{\xi}\right)\mathcal{F}$ which is similar to that developed in [18]. From the Kato theory, it is known that the Sobolev space $\mathbf{H}^s$ with $s > \frac{5}{2}$ is needed for the initial data u_0 to get a local existence theorem. It is also known that in order to obtain sharp $\mathbf{L}^\infty$ - time decay of $\partial_X u$, we need the condition $\Gamma^\alpha \partial_X u \in \mathbf{L}^2$ with $|\alpha| \leq 2$. Therefore when we use the space generated by Γ, it is natural to consider the problem in the space with a norm $\sum_{|\alpha|\leq 2} \|\Gamma^\alpha u\|_{\mathbf{H}^1}$. Though problem (4.1) is different from the problem (4.2) with (4.3) since the data are given on the line of the light cone, by the relation

$$X\partial_T + T\partial_X = x\partial_x - t\partial_t$$

one can expect that the function space with the norm $\sum_{|\alpha|\leq 2} \|\Lambda^\alpha u\|_{\mathbf{H}^1}$ is applicable to (4.1), where $\Lambda = (\partial_t, \partial_x, x\partial_x - t\partial_t)$. As was pointed out in our previous work [21], it seems difficult to derive a priori estimates of solutions in the norm $\sum_{|\alpha|\leq 2} \|\Lambda^\alpha u\|_{\mathbf{H}^1}$. To overcome this difficulty, we use the function space with the norm $\|(x\partial_x - t\partial_t)u\|_{\mathbf{L}^2} + \|u\|_{\mathbf{H}^m}$, where $m > 4$. This is the reason why we encounter the regularity assumption $m > 4$.

We are now in a position to state our main result of this section. Denote the dilation operator $\mathcal{D}_\omega \phi = |\omega|^{-\frac{1}{2}} \phi\left(x\omega^{-1}\right)$. Define the multiplication factor $M(t,x) = e^{-2it\sqrt{|x|}}$, the Heaviside function $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x \leq 0$, and $\left(\mathcal{B}^{-1}\phi\right)(x) = \frac{1}{\sqrt{2i}}\theta(-x)|x|^{-\frac{3}{4}}\phi\left(\frac{1}{\sqrt{|x|}}\right)$.

Theorem 4.1. *Assume that the initial data* $u_0 \in \dot{\mathbf{H}}^{-1} \cap \mathbf{H}^m$, $x\partial_x u_0 \in \mathbf{H}^l$, $m > \frac{5}{2} + l$, $l > \frac{3}{2}$, *and the norm* $\|u_0\|_{\dot{\mathbf{H}}^{-1}\cap\mathbf{H}^m} + \|x\partial_x u_0\|_{\mathbf{H}^l}$ *is sufficiently small. Then there exists a unique global solution* $u \in \mathbf{C}\left([0,\infty); \dot{\mathbf{H}}^{-1} \cap \mathbf{H}^m\right)$ *of the Cauchy problem (4.1) such that*

$$\|u(t)\|_{\mathbf{L}^\infty} \leq C(1+t)^{-\frac{1}{2}}.$$

Moreover there exists a unique modified final state $W_+ \in \mathbf{L}^\infty$ *such that the asymptotics*

$$u(t) = 2Re\mathcal{D}_t M\mathcal{B}^{-1}\left(W_+ \exp\left(\frac{3}{2}i\xi|W_+|^2\log t\right)\right) + O\left(t^{-\frac{1}{2}-\delta}\right) \tag{4.4}$$

is valid for $t \to \infty$ *uniformly with respect to* $x \in \mathbf{R}$, *where* $\delta \in \left(0, \frac{1}{4}\right)$ *is a small constant depending on* m.

Remark. After we have completed this work, we were informed by Dr. Niizato that he has got a similar result with $u_0 \in \dot{\mathbf{H}}^{-1} \cap \mathbf{H}^m$, $x\partial_x u_0 \in \mathbf{H}^n$, $m > n+7, n > 3$ by a different method (see [32]). His method strongly depends on our previous papers [15], [17] in which the factorization method was not used. This is the one of the reasons why undesirable additional regularity conditions on the data are required. Our method of the proof of Theorem 4.1 is based on the factorization technique (see [18]).

For the convenience of the readers we now state our strategy of the proof. The factorization formula for the free Schrödinger evolution group is represented by the multiplication factor $e^{\frac{i|x|^2}{2t}}$, the dilation operator $\mathcal{D}_t$ and Fourier transformation $\mathcal{F}$ such that $e^{\frac{it}{2}\partial_x^2} = i^{-\frac{1}{2}} e^{\frac{i|x|^2}{2t}} \mathcal{D}_t \mathcal{F} e^{\frac{i|x|^2}{2t}}$, see [23]. Similarly, in the present section we introduce the decomposition for the free Ostrovsky evolution group $\mathcal{U}(t)\mathcal{F}^{-1} = \mathcal{F}^{-1} e^{-\frac{it}{\xi}}$. Define the multiplication factors $M(t,x) = e^{-2it\sqrt{|x|}}$, $E(t,\xi) = e^{-\frac{it}{\xi}}$, and introduce the operator $\mathcal{Q}(t) = \overline{M}\mathcal{D}_t^{-1}\mathcal{F}^{-1}\theta E$. Denote $\widehat{\varphi} = \mathcal{F}\mathcal{U}(-t)u(t)$, then for the real-valued function $\mathcal{U}(t)\mathcal{F}^{-1}\widehat{\varphi}$ we find the factorization formula

$$\mathcal{U}(t)\mathcal{F}^{-1}\widehat{\varphi} = 2\mathrm{Re}\mathcal{F}^{-1}\theta E\widehat{\varphi} = 2\mathrm{Re}\mathcal{D}_t M \mathcal{Q}(t)\widehat{\varphi}. \tag{4.5}$$

It is known from [22], that solutions of the linear equation $u_{tx} = u$ decay in time rapidly for $x > 0$ comparing with the case of $x < 0$. Thus estimate of the solutions for the positive line is considered as a remainder. We introduce two operators

$$\left(\mathcal{B}^{-1}\phi\right)(x) = \frac{\theta(-x)}{\sqrt{2i}}|x|^{-\frac{3}{4}}\phi\left(\frac{1}{\sqrt{|x|}}\right)$$

and

$$(\mathcal{B}\phi)(\xi) = \sqrt{2i}\theta(\xi)|\xi|^{-\frac{3}{2}}\phi\left(-\frac{1}{\xi^2}\right).$$

We can easily see that the operator $\mathcal{B}$ is the inverse of $\mathcal{B}^{-1}$ for the functions defined on $\mathbf{R}_+$. In the same manner, $\mathcal{B}^{-1}$ is the inverse of $\mathcal{B}$ for the functions defined on $\mathbf{R}_-$. By virtue of the stationary phase method it is well-known that the main term of the large time asymptotics of solutions to the linear equation is given by $2\mathrm{Re}\mathcal{D}_t M\mathcal{B}^{-1}\widehat{\varphi}$. By (4.5) we write

$$\mathcal{U}(t)\varphi = 2\mathrm{Re}\mathcal{D}_t M\mathcal{B}^{-1}\widehat{\varphi} + 2\mathrm{Re}\mathcal{D}_t M\left(\mathcal{Q}(t) - \mathcal{B}^{-1}\right)\widehat{\varphi} \tag{4.6}$$

for $x < 0$ and

$$\mathcal{U}(t)\varphi = 2\mathrm{Re}\mathcal{D}_t M\mathcal{Q}(t)\widehat{\varphi}$$

for $x \geq 0$. In Lemma 4.2 below, we prove that $\|\mathcal{U}(t)\varphi\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \leq Ct^{-1}$ and in Lemma 4.3 below we obtain the estimate

$$\left\|2\mathrm{Re}\mathcal{D}_t M\left(\mathcal{Q}(t)-\mathcal{B}^{-1}\right)\widehat{\varphi}\right\|_{\mathbf{L}^\infty(\mathbf{R}_-)} \leq Ct^{-\frac{2}{3}}.$$

Thus we show that the main term of the large time asymptotics of the free Ostrovsky evolution group $\mathcal{U}(t)\mathcal{F}^{-1}\widehat{\varphi}$ is represented by $2\mathrm{Re}\mathcal{D}_t M\mathcal{B}^{-1}\widehat{\varphi}$ in the domain $\mathbf{R}_-$. By the identity $u(t)=\mathcal{U}(t)\mathcal{F}^{-1}\widehat{\varphi}$ we see that the $\mathbf{L}^\infty$ - norm of the solution $u(t)$ can be estimated as

$$\|u(t)\|_{\mathbf{L}^\infty(\mathbf{R})} \leq Ct^{-\frac{1}{2}}\left\||\xi|^{\frac{3}{2}}\widehat{\varphi}\right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} + Ct^{-\frac{2}{3}} + Ct^{-1}.$$

Therefore it is sufficient to obtain the uniform estimate of $\widehat{\varphi}=\mathcal{F}\mathcal{U}(-t)u(t)$ to prove the optimal time decay estimate of the solution $u(t)$ in the $\mathbf{L}^\infty$ - norm. We now define the operator $\mathcal{R}(t)=\overline{E}\mathcal{F}\mathcal{D}_t M$, so that we have the representation for the inverse evolution group

$$\mathcal{F}\mathcal{U}(-t)=\overline{E}\mathcal{F}=\mathcal{R}(t)\overline{M}\mathcal{D}_{\frac{1}{t}}. \tag{4.7}$$

Multiplying both sides of equation (4.1) by $\mathcal{F}\mathcal{U}(-t)$, using identity (4.7) and $u=\mathcal{D}_t M\mathcal{Q}(t)\widehat{\varphi}+\overline{\mathcal{D}_t M\mathcal{Q}(t)\widehat{\varphi}}$ with $\varphi=\mathcal{U}(-t)u$, we obtain

$$\widehat{\varphi}_t = i\xi\mathcal{F}\mathcal{U}(-t)u^3 = i\xi\mathcal{R}(t)\overline{M}\mathcal{D}_{\frac{1}{t}}\left(\mathcal{D}_t M\mathcal{Q}(t)\widehat{\varphi}+\overline{\mathcal{D}_t M\mathcal{Q}(t)\widehat{\varphi}}\right)^3.$$

We have four types of nonlinearities in the right-hand sides of the above identity. One of them is the resonance term given by

$$3i\xi\mathcal{R}(t)\overline{M}\mathcal{D}_{\frac{1}{t}}\left|\mathcal{D}_t M\mathcal{Q}(t)\widehat{\varphi}\right|^2\mathcal{D}_t M\mathcal{Q}(t)\widehat{\varphi} = 3i\xi t^{-1}\mathcal{R}(t)\left|\mathcal{Q}(t)\widehat{\varphi}\right|^2\mathcal{Q}(t)\widehat{\varphi}.$$

By virtue of Lemmas 4.4 and 4.5 below, the right-hand side of the above equality can be approximated by

$$3i\xi t^{-1}\mathcal{B}\left|\mathcal{Q}(t)\widehat{\varphi}\right|^2\mathcal{Q}(t)\widehat{\varphi}$$

in the domain $0 \leq \xi \leq \frac{1}{4}t^{\frac{1}{2}}$ and by Lemma 4.3 below we find that

$$3i\xi t^{-1}\mathcal{B}\left|\mathcal{Q}(t)\widehat{\varphi}\right|^2\mathcal{Q}(t)\widehat{\varphi} \simeq 3i\xi t^{-1}\mathcal{B}\left|\mathcal{B}^{-1}\widehat{\varphi}\right|^2\mathcal{B}^{-1}\widehat{\varphi} = \frac{3}{2}it^{-1}\xi^4\left|\widehat{\varphi}(\xi)\right|^2\widehat{\varphi}(\xi),$$

where the notation $A \simeq B$ means that $A=B+$ remainder terms. The estimates of the remainder terms are given in Lemma 4.6 below. Then we introduce the phase correction to remove the resonance term $\frac{3}{2}it^{-1}\xi^4\left|\widehat{\varphi}\right|^2\widehat{\varphi}$. Also we prove that the nonresonant terms in the nonlinearity have a better time decay rate through the integration by parts with respect to the time variable

t. Thus we obtain the desired uniform estimate of $\widehat{\varphi} = \mathcal{FU}(-t)\,u(t)$. In order to minimize m, we divide the estimates of $\widehat{\varphi} = \mathcal{FU}(-t)\,u(t)$ into the high-frequency part $\xi > \langle t\rangle^{\nu}$ and the low-frequency part $0 \le \xi \le \langle t\rangle^{\nu}$ with some $\nu > 0$. Lemma 4.6 is used for estimating $\mathcal{FU}(-t)\,u(t)$ in the low-frequency part $0 \le \xi \le \langle t\rangle^{\nu}$.

Next lemma is related to the estimate the operator $\mathcal{Q}(t) = \overline{M}\mathcal{D}_t^{-1}\mathcal{F}^{-1}\theta E$ in the domain $\mathbf{R}_+$.

Lemma 4.2. *Let $2 < p \le \infty$, $0 \le \alpha \le \min\left(\frac{1}{2}, 1 - \frac{2}{p}\right)$. Then the estimate*

$$\|\mathcal{Q}(t)\,\phi\|_{\mathbf{L}^p(\mathbf{R}_+)} \le Ct^{-\frac{\alpha}{2}-\frac{1}{p}} \left\| |\xi|^{\frac{3}{2}+\alpha}\,\phi\right\|_{\mathbf{H}^1}$$

is true for all $t > 0$, provided that the right-hand side is finite.

In the next lemma we estimate the difference $\mathcal{Q}(t) - \mathcal{B}^{-1}$.

Lemma 4.3. *Let $\alpha \in \left[0, \frac{1}{2}\right]$, $\beta \in \left[0, \frac{1}{4}\right]$ be such that $\frac{\alpha}{2} + \beta \le \frac{1}{4}$. Then the estimate*

$$\left\| |x|^{\beta}\left(\mathcal{Q}(t) - \mathcal{B}^{-1}\right)\phi\right\|_{\mathbf{L}^\infty(\mathbf{R}_-)} \le Ct^{-\frac{2}{3}\left(\frac{\alpha}{2}+\beta\right)} \left\| |\xi|^{\frac{3}{2}+\alpha}\,\phi\right\|_{\mathbf{H}^1}$$

is true for all $t \ge 1$, provided that the right-hand side is finite.

We estimate the difference $\mathcal{R}(t) - \mathcal{B}$.

Lemma 4.4. *Let ϕ be a real valued function. Then the estimate*

$$\begin{aligned} & \|(\mathcal{R}(t) - \mathcal{B})\,\phi\|_{\mathbf{L}^\infty(0,\frac{\sqrt{t}}{4})} \\ \le\ & Ct^{-\frac{1}{12}} \left\| \langle x\rangle^{\frac{1}{2}}\,\phi\right\|_{\mathbf{L}^2(\mathbf{R}_-)} + Ct^{-\frac{1}{12}} \left\| |x|^{\frac{7}{8}}\,\partial_x\phi\right\|_{\mathbf{L}^2(\mathbf{R}_-)} \\ +\ & Ct^{-\frac{1}{2}}\,\|\phi\|_{\mathbf{L}^\infty(\mathbf{R}_-)} + Ct^{\frac{1}{2}}\,\|\phi\|_{\mathbf{L}^1(\mathbf{R}_+)} \end{aligned}$$

is true for all $t \ge 1$, provided that the right-hand side is finite.

In the above lemma we do not need the assumption that ϕ is a real-valued function. We only consider real-valued functions here because this makes the proof shorter (see [19]) and suffices our purposes. Note that the local existence of complex-valued solutions is still an open problem.

In the next lemma we estimate the derivative $\partial_x\mathcal{Q}(t)$.

Lemma 4.5. *Let $\beta \in \left(\frac{3}{4}, 1\right)$. Then the estimate*

$$\left\| |x|^{\beta}\,\partial_x\mathcal{Q}(t)\,\phi\right\|_{\mathbf{L}^2(\mathbf{R}_-)} \le C\left\| \langle\xi\rangle^{\frac{3}{2}}\,\phi\right\|_{\mathbf{L}^\infty} + C\,\|\langle\xi\rangle\,\xi\phi_\xi\|_{\mathbf{L}^2}$$

is true for all $t \ge 1$, provided that the right-hand side is finite.

Next lemma is related to the asymptotic representation for $\mathcal{FU}(-t)\,u^3$. Denote $b_j = 2^{-1}e^{-\frac{\pi}{2}i(j-2)}\,|\omega_j|^{-\frac{1}{2}-3}\,a_j$, $\omega_j = 2j-3$, $0 \le j \le 3$, $a_0 = a_3 = 1$, $a_1 = a_2 = 3$.

Lemma 4.6. *The asymptotic representation*

$$\mathcal{FU}(-t)\,u^3 = t^{-1}\sum_{j=0}^{3} b_j e^{\frac{it}{\xi}(1-\omega_j)}\,|\xi|^3\left(\widehat{\varphi}\left(t,\frac{\xi}{\omega_j}\right)\right)^j\left(\overline{\widehat{\varphi}\left(t,\frac{\xi}{\omega_j}\right)}\right)^{3-j}$$
$$+O\left(t^{-\frac{13}{12}}\left\|\langle\xi\rangle^{\frac{3}{2}}\,\widehat{\varphi}\right\|_{\mathbf{L}^\infty}^3\right)+O\left(t^{-\frac{13}{12}}\left\|\langle\xi\rangle\,\xi\widehat{\varphi}\right\|_{\mathbf{H}^1}^3\right)$$

is true for all $t \ge 1$, $0 \le \xi \le \frac{\sqrt{t}}{4}$, *where* $\widehat{\varphi} = \mathcal{FU}(-t)\,u(t)$.

The following result is a consequence of Lemma 2.5. It says that the $\mathbf{L}^\infty$ - norm of solutions in higher order Sobolev spaces can be estimated through the $\mathbf{L}^2$ - norm of $\mathcal{J}\partial_x u$.

Lemma 4.7. *Let* $\rho \in \left(0,\frac{1}{2}\right)$ *and* $l \ge 1$. *Then the estimate*

$$\left\|\langle i\partial_x\rangle^l\,\phi\right\|_{\mathbf{L}^\infty} \le Ct^{-\frac{1}{2}}\left\|x\partial_x\mathcal{U}(-t)\,\phi\right\|_{\mathbf{L}^2}^{\frac{1}{2}+\rho}\left\|\mathcal{U}(-t)\,\phi\right\|_{\mathbf{H}^{\frac{2l-2\rho}{1-2\rho}}}^{\frac{1}{2}-\rho}$$
$$+Ct^{-\frac{1}{2}}\left\|\mathcal{U}(-t)\,\phi\right\|_{\mathbf{H}^{\frac{3}{2}+l}}$$

is true, provided that the right-hand side is finite.

4.1. The outline of the proof of Theorem 4.1.

Define the following norms

$$\|u\|_{\mathbf{X}_T} = \sup_{t\in[0,T]}\langle t\rangle^{\frac{1}{2}}\left\|u(t)\right\|_{\mathbf{H}^1_\infty},$$
$$\|u\|_{\mathbf{Y}_T} = \sup_{t\in[0,T]}\langle t\rangle^{-\gamma}\left(\left\|u(t)\right\|_{\dot{\mathbf{H}}^{-1}}+\left\|u(t)\right\|_{\mathbf{H}^m}+\left\|\partial_x\mathcal{J}u(t)\right\|_{\mathbf{H}^l}\right),$$

where $m > \frac{5}{2}+l, l > \frac{3}{2}$, $\mathcal{J} = x - t\partial_x^{-2}$. First we estimate the norm $\mathbf{Y}_T$ by supposing that the norm $\mathbf{X}_T$ is bounded.

Lemma 4.8. *Let the norm*

$$\|u\|_{\mathbf{X}_T} \le C\varepsilon.$$

Then the estimate

$$\|u\|_{\mathbf{Y}_T} \le C\varepsilon$$

is true.

Proof. By the local existence theorem Proposition 2.4 we get

$$\begin{aligned} & \|u\|_{\mathbf{H}^m} + \|\mathcal{P}u\|_{\mathbf{L}^2} + \|u\|_{\dot{\mathbf{H}}^{-1}} \\ \leq \quad & C\int_0^t \|u(s)\|^2_{\mathbf{H}^1_\infty} \left(\|u(s)\|_{\mathbf{H}^m} + \|\mathcal{P}u(s)\|_{\mathbf{L}^2} + \|u(s)\|_{\dot{\mathbf{H}}^{-1}} \right) ds. \end{aligned}$$

Hence we obtain

$$\|u\|_{\mathbf{H}^m} + \|\mathcal{P}u\|_{\mathbf{L}^2} + \|u\|_{\dot{\mathbf{H}}^{-1}} \leq 2\varepsilon \langle t \rangle^{\frac{\gamma}{2}}.$$

Then by the identity $(\mathcal{P} - \mathcal{J}\partial_x) u = -t \left(u^3\right)_x$ we get

$$\|\partial_x \mathcal{J}u\|_{\mathbf{L}^2} \leq \|\mathcal{P}u\|_{\mathbf{L}^2} + \|u\|_{\mathbf{L}^2} + t\|u\|^2_{\mathbf{L}^\infty} \|u_x\|_{\mathbf{L}^2} \leq 2\varepsilon \langle t \rangle^{\frac{\gamma}{2}}.$$

Next we consider $\partial_x \mathcal{P}u$. We have

$$\begin{aligned} \frac{d}{dt}\|\partial_x \mathcal{P}u\|_{\mathbf{L}^2} \quad \leq \quad & C\|u\|_{\mathbf{L}^\infty} \|u_x\|_{\mathbf{L}^\infty} \left(\|\partial_x \mathcal{P}u\|_{\mathbf{L}^2} + \|u\|_{\mathbf{H}^2}\right) \\ & + C\|u_x\|^2_{\mathbf{L}^\infty} \|\mathcal{P}u\|_{\mathbf{L}^2} + \|u\|_{\mathbf{L}^\infty} \|u_{xx}\|_{\mathbf{L}^\infty} \|\mathcal{P}u\|_{\mathbf{L}^2}. \end{aligned}$$

By Lemma 4.7 we find

$$\|u_{xx}\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{1}{2}} \left(\|\mathcal{P}u\|_{\mathbf{L}^2} + \|u\|_{\mathbf{H}^m}\right) \leq 2\varepsilon \langle t \rangle^{\frac{\gamma}{2}},$$

since $m > 4$. Therefore $\|\partial_x \mathcal{P}u\|_{\mathbf{L}^2} \leq 2\varepsilon \langle t \rangle^{\gamma}$. Then

$$\begin{aligned} \left\|\partial_x^2 \mathcal{J}u\right\|_{\mathbf{L}^2} \quad & \leq \quad \|\partial_x \mathcal{P}u\|_{\mathbf{L}^2} + \|u\|_{\mathbf{H}^1} + t\|u\|_{\mathbf{L}^\infty} \left(\|u\|_{\mathbf{L}^\infty} + \|u_x\|_{\mathbf{L}^\infty}\right) \|u\|_{\mathbf{H}^2} \\ & \leq \quad 2\varepsilon \langle t \rangle^{\gamma}. \end{aligned}$$

Next we consider $\partial_x D_x^s \mathcal{P}u$, where $D_x^s = \left(-\partial_x^2\right)^{\frac{s}{2}}, 0 < s < 1$. We have

$$\begin{aligned} & \frac{d}{dt}\|\partial_x D_x^s \mathcal{P}u\|_{\mathbf{L}^2} \\ \leq \quad & C\|u\|_{\mathbf{L}^\infty} \|u_x\|_{\mathbf{L}^\infty} \left(\|\partial_x D_x^s \mathcal{P}u\|_{\mathbf{L}^2} + \|u\|_{\mathbf{H}^{2+s}}\right) \\ & + C\left(\|u_x\|_{\mathbf{L}^\infty} \|D_x^s u\|_{\mathbf{L}^q} + \|u\|_{\mathbf{L}^\infty} \|D_x^s u_x\|_{\mathbf{L}^q}\right) \|\partial_x \mathcal{P}u\|_{\mathbf{L}^p} \\ & + C\left(\|u_x\|_{\mathbf{L}^\infty} \|D_x^s u_x\|_{\mathbf{L}^q} + \|u\|_{\mathbf{L}^\infty} \|D_x^s u_{xx}\|_{\mathbf{L}^q}\right) \|\mathcal{P}u\|_{\mathbf{L}^p} \\ & + C\left(\|u_x\|_{\mathbf{L}^\infty} \|u_x\|_{\mathbf{L}^q} + \|u\|_{\mathbf{L}^\infty} \|u_{xx}\|_{\mathbf{L}^q}\right) \|D_x^s \mathcal{P}u\|_{\mathbf{L}^p}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}, 2 < p, q < \infty$. By Lemma 4.7 we find

$$\|D_x^s u_{xx}\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{1}{2}} \left(\left\|D_x^s \partial_x^2 \mathcal{J}u\right\|^{\frac{2}{3}}_{\mathbf{L}^2} \|u\|^{\frac{1}{3}}_{\mathbf{H}^{\frac{7}{2}+s}} + \|u\|_{\mathbf{H}^{\frac{7}{2}+s}} \right).$$

Since

$$\left\|D_x^s\partial_x^2\mathcal{J}u\right\|_{\mathbf{L}^2} \le \|D_x^s\partial_x\mathcal{P}u\|_{\mathbf{L}^2} + C\varepsilon\langle t\rangle^{\frac{\gamma}{4}}$$

we obtain

$$\|D_x^s u_{xx}\|_{\mathbf{L}^\infty} \le Ct^{-\frac{1}{2}}\left(\|D_x^s\partial_x\mathcal{P}u\|_{\mathbf{L}^2}^{\frac{2}{3}}\varepsilon^{\frac{1}{3}}\langle t\rangle^{\frac{\gamma}{12}} + \varepsilon\langle t\rangle^{\frac{\gamma}{4}}\right).$$

We apply the Hölder inequality to obtain

$$\begin{aligned}
&\|D_x^s u_{xx}\|_{\mathbf{L}^{\frac{10}{\gamma}}}\\
\le\ & C\|D_x^s u_{xx}\|_{\mathbf{L}^\infty}^{1-\frac{\gamma}{5}}\|D_x^s u_{xx}\|_{\mathbf{L}^2}^{\frac{\gamma}{5}}\\
\le\ & Ct^{-\frac{1}{2}\left(1-\frac{\gamma}{5}\right)}\left(\|D_x^s\partial_x\mathcal{P}u\|_{\mathbf{L}^2}^{\frac{2}{3}\left(1-\frac{\gamma}{5}\right)}\varepsilon^{\frac{1}{3}\left(1+\frac{2}{5}\gamma\right)}\langle t\rangle^{\frac{\gamma}{12}\left(1+\frac{2}{5}\gamma\right)} + \varepsilon\langle t\rangle^{\frac{\gamma}{4}}\right),
\end{aligned}$$

since $m > \frac{7}{2} + s$. Therefore

$$\begin{aligned}
&\frac{d}{dt}\|\partial_x D_x^s\mathcal{P}u\|_{\mathbf{L}^2}\\
\le\ & C\varepsilon^2 t^{-1}\|\partial_x D_x^s\mathcal{P}u\|_{\mathbf{L}^2}\\
& + Ct^{-1+\frac{\gamma}{5}+\frac{\gamma^2}{30}}\varepsilon^2\varepsilon^{\frac{1}{3}\left(1+\frac{2}{5}\gamma\right)}\|D_x^s\partial_x\mathcal{P}u\|_{\mathbf{L}^2}^{\frac{2}{3}\left(1-\frac{\gamma}{5}\right)} + C\varepsilon^3\langle t\rangle^{-1+\gamma}\\
=\ & C\varepsilon^2 t^{-1}\|\partial_x D_x^s\mathcal{P}u\|_{\mathbf{L}^2}\\
& + Ct^{-\frac{1}{3}\left(1+\frac{2}{5}\gamma\right)+\frac{\gamma}{5}+\frac{\gamma^2}{30}}\varepsilon^{1+\frac{2}{5}\gamma}\left(\varepsilon^2 t^{-1}\|D_x^s\partial_x\mathcal{P}u\|_{\mathbf{L}^2}\right)^{\frac{2}{3}\left(1-\frac{\gamma}{5}\right)} + C\varepsilon^3\langle t\rangle^{-1+\gamma}\\
\le\ & C\varepsilon^2 t^{-1}\|\partial_x D_x^s\mathcal{P}u\|_{\mathbf{L}^2} + C\varepsilon^3\langle t\rangle^{-1+\gamma}
\end{aligned}$$

from which it follows that $\|\partial_x D_x^s\mathcal{P}u\|_{\mathbf{L}^2} \le 2\varepsilon\langle t\rangle^{\gamma}$. Then

$$\begin{aligned}
\|\partial_x D_x^s\mathcal{J}u\|_{\mathbf{L}^2} \le\ & \|\partial_x D_x^s\mathcal{P}u\|_{\mathbf{L}^2} + \|u\|_{\mathbf{H}^{2+s}}\\
+\ & t\|u\|_{\mathbf{L}^\infty}\left(\|u\|_{\mathbf{L}^\infty} + \|D_x^s u_x\|_{\mathbf{L}^\infty}\right)\|u\|_{\mathbf{H}^{2+s}}\\
\le\ & 2\varepsilon\langle t\rangle^{\gamma}.
\end{aligned}$$

Lemma 4.8 is proved. □

We next estimate the norm $\mathbf{X}_T$ by supposing that the norm $\mathbf{Y}_T$ is bounded.

Lemma 4.9. *Let the norm*

$$\|u\|_{\mathbf{Y}_T} \le C\varepsilon.$$

Then the estimate

$$\|u\|_{\mathbf{X}_T} \le C\varepsilon$$

is true.

Proof. We estimate $\langle i\partial_x\rangle u(t)$. By (4.6) and Lemma 4.2 we find for $x>0$

$$|\langle i\partial_x\rangle u(t)| = |2\mathrm{Re}\mathcal{D}_t M\mathcal{Q}(t)\langle\xi\rangle\widehat{\varphi}(t,\xi)| \le Ct^{-\frac{3}{4}+\gamma}\|u\|_{\mathbf{Y}_T}.$$

By (4.6) and Lemma 4.4 with $\beta=0$, $\alpha=l-\frac{3}{2}$, we find for $x<0$

$$\begin{aligned}
&\langle i\partial_x\rangle u(t) = 2\mathrm{Re}\mathcal{D}_t M\mathcal{Q}(t)\langle\xi\rangle\widehat{\varphi}(t,\xi)\\
=\;& 2\mathrm{Re}\mathcal{D}_t M\mathcal{B}^{-1}\langle\xi\rangle\widehat{\varphi} + 2\mathrm{Re}\mathcal{D}_t M\left(\mathcal{Q}(t)-\mathcal{B}^{-1}\right)\langle\xi\rangle\widehat{\varphi}(t,\xi)\\
=\;& \mathrm{Re}\left(\frac{\sqrt{2}}{\sqrt{it}}e^{-2i\sqrt{|xt|}}\left|\frac{t}{x}\right|^{\frac{3}{4}}\left\langle\sqrt{\left|\frac{t}{x}\right|}\right\rangle\widehat{\varphi}\left(\sqrt{\left|\frac{t}{x}\right|}\right)\right)\\
+\;& O\left(t^{-\frac{1}{2}-\frac{1}{3}\left(l-\frac{3}{2}\right)}\left(\left\||\xi|^{l}\langle\xi\rangle\widehat{\varphi}\right\|_{\mathbf{L}^2(\mathbf{R}_+)}+\left\||\xi|^{l}\langle\xi\rangle\partial_\xi\widehat{\varphi}\right\|_{\mathbf{L}^2(\mathbf{R}_+)}\right)\right)\\
=\;& \mathrm{Re}\left(\frac{\sqrt{2}}{\sqrt{it}}e^{-2i\sqrt{|xt|}}\left|\frac{t}{x}\right|^{\frac{3}{4}}\left\langle\sqrt{\left|\frac{t}{x}\right|}\right\rangle\widehat{\varphi}\left(\sqrt{\left|\frac{t}{x}\right|}\right)\right)\\
+\;& O\left(\varepsilon t^{-\frac{1}{2}-\frac{1}{3}\left(l-\frac{3}{2}\right)+\gamma}\right),
\end{aligned}$$

where $l-\frac{3}{2}>3\gamma$. In the domain $|\xi|\ge\langle t\rangle^{\nu}$ we get by the Sobolev embedding theorem

$$\begin{aligned}
&\left\||\xi|^{\frac{3}{2}}\langle\xi\rangle\widehat{\varphi}(t,\xi)\right\|_{\mathbf{L}^\infty(|\xi|\ge\langle t\rangle^{\nu})}\\
\le\;& \langle t\rangle^{-\left(l-\frac{3}{2}\right)\nu}\left\|\langle\xi\rangle^{l+1}\widehat{\varphi}(t,\xi)\right\|_{\mathbf{L}^\infty(|\xi|\ge\langle t\rangle^{\nu})}\\
\le\;& C\langle t\rangle^{-\frac{1}{2}\left(l-\frac{3}{2}\right)\nu}\left(\|\xi\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^2}+\|\langle\xi\rangle^{m}\widehat{\varphi}\|_{\mathbf{L}^2}\right)\le C\varepsilon\langle t\rangle^{-\frac{1}{2}\left(l-\frac{3}{2}\right)\nu+\gamma},
\end{aligned}$$

if $\nu>\frac{\gamma}{\frac{1}{2}\left(l-\frac{3}{2}\right)}$, so we need to estimate the function $|\xi|^{\frac{3}{2}}\langle\xi\rangle\widehat{\varphi}(t,\xi)$ in the domain $|\xi|\le\langle t\rangle^{\nu}$. Applying the operator $\mathcal{FU}(-t)$ to equation (4.1) $\mathcal{L}u=\left(u^3\right)_x$, we get

$$\widehat{\varphi}_t(t,\xi)=\mathcal{FU}(-t)\left(u^3\right)_x=i\xi\mathcal{FU}(-t)u^3.$$

By Lemma 4.6 we find

$$\begin{aligned}
\mathcal{FU}(-t)u^3 \;=\;& |t|^{-1}\sum_{j=0}^{3}b_j e^{\frac{it}{\xi}(1-\omega_j)}|\xi|^3\left(\widehat{\varphi}\left(t,\frac{\xi}{\omega_j}\right)\right)^{j}\left(\overline{\widehat{\varphi}\left(t,\frac{\xi}{\omega_j}\right)}\right)^{3-j}\\
+\;& O\left(\varepsilon^3 t^{-\frac{13}{12}+3\gamma}\right)
\end{aligned}$$

for all $t\ge1$, $|\xi|\le\langle t\rangle^{\frac{1}{2}}$, where $b_j=2^{-1}e^{-\frac{\pi}{2}i(j-2)}|\omega_j|^{-\frac{1}{2}-3}a_j$, $\omega_j=2j-3$,

$a_0 = a_3 = 1$, $a_1 = a_2 = 3$. Multiplying this formula by $i\xi\left|\xi\right|^{\frac{3}{2}}\langle\xi\rangle$ we get

$$\begin{aligned} & i\xi\left|\xi\right|^{\frac{3}{2}}\langle\xi\rangle\mathcal{F}\mathcal{U}(-t)u^3 \\ = \; & i\left|t\right|^{-1}\sum_{j=0}^{3}b_j e^{\frac{it}{\xi}(1-\omega_j)}\xi\left|\xi\right|^{\frac{9}{2}}\langle\xi\rangle\left(\widehat{\varphi}\left(t,\frac{\xi}{\omega_j}\right)\right)^j\left(\overline{\widehat{\varphi}\left(t,\frac{\xi}{\omega_j}\right)}\right)^{3-j} \\ & +O\left(\varepsilon^3 t^{\frac{7}{2}\nu-\frac{13}{12}+3\gamma}\right) \end{aligned}$$

in the domain $|\xi| \leq \langle t\rangle^{\nu}, \nu \leq \frac{1}{2}$. Define the cut-off function $\chi \in \mathbf{C}^1(\mathbf{R})$, such that $\chi(x) = 1$ for $|x| < 1$ and $\chi(x) = 0$ for $|x| > 2$, and define $\widehat{\varphi}_1(t,\xi) = \chi\left(\xi\langle t\rangle^{-\nu}\right)\widehat{\varphi}(t,\xi)$. Thus we get for the function $\psi(t,\xi) = |\xi|^{\frac{3}{2}}\langle\xi\rangle\widehat{\varphi}_1(t,\xi)$

$$\begin{aligned} \psi_t(t,\xi) \; = \; & i\left|t\right|^{-1}\sum_{j=0}^{3}b_j\left|\omega_j\right|^{\frac{9}{2}}e^{\frac{it}{\xi}(1-\omega_j)}\xi\left|\xi\right|^{\frac{9}{2}}\langle\xi\rangle\chi\left(\xi\langle t\rangle^{-\nu}\right) \\ & \times\left(\widehat{\varphi}_1\left(t,\frac{\xi}{\omega_j}\right)\right)^j\left(\overline{\widehat{\varphi}_1\left(t,\frac{\xi}{\omega_j}\right)}\right)^{3-j} \\ & + \; |\xi|^{\frac{3}{2}}\langle\xi\rangle\xi\langle t\rangle^{-1-\nu}\chi'\left(\xi\langle t\rangle^{-\nu}\right)\widehat{\varphi}(t,\xi) \\ & + \; O\left(\varepsilon^3 t^{-1-\beta}\right) \end{aligned}$$

for all $t \geq 1$ with some $\delta > 0$ if $\nu < \frac{1}{42}$. The second term is estimated by $C\varepsilon\langle t\rangle^{-1-\frac{1}{2}\left(l-\frac{3}{2}\right)\nu+\gamma}$. To exclude the resonant term with $j = 2$, we make a change $\psi(t,\xi) = y(t,\xi)\Psi(t,\xi)$, where

$$\Psi(t,\xi) = \exp\left(\frac{3}{2}i\xi^4\int_1^t\left|\widehat{\varphi}_1(\tau,\xi)\right|^2\frac{d\tau}{\tau}\right).$$

Then we get

$$\begin{aligned} y_t(t,\xi) \; = \; & it^{-1}\sum_{j\neq 2}b_j\left|\omega_j\right|^{\frac{9}{2}}e^{\frac{it}{\xi}(1-\omega_j)}\xi\left|\xi\right|^{\frac{9}{2}}\langle\xi\rangle\chi\left(\xi\langle t\rangle^{-\nu}\right)\overline{\Psi(t,\xi)} \\ & \times\left(\widehat{\varphi}\left(t,\frac{\xi}{\omega_j}\right)\right)^j\left(\overline{\widehat{\varphi}\left(t,\frac{\xi}{\omega_j}\right)}\right)^{3-j} \\ & + \; O\left(\varepsilon^3 t^{-1-\delta}\right). \end{aligned}$$

Integrating by parts we obtain

$$
\begin{aligned}
y(t,\xi) - y(0,\xi) &= i \sum_{j\neq 2} b_j |\omega_j|^{\frac{9}{2}} \xi |\xi|^{\frac{9}{2}} \langle \xi \rangle \int_0^t e^{\frac{i\tau}{\xi}(1-\omega_j)} \chi\left(\xi \langle \tau \rangle^{-\nu}\right) \overline{\Psi(\tau,\xi)} \\
&\times \left(\widehat{\varphi}\left(\tau, \frac{\xi}{\omega_j}\right)\right)^j \left(\overline{\widehat{\varphi}\left(\tau, \frac{\xi}{\omega_j}\right)}\right)^{3-j} \frac{d\tau}{\tau} + O\left(\varepsilon^3\right) \\
= \quad & \sum_{j\neq 2} \frac{b_j |\omega_j|^{\frac{9}{2}}}{1-\omega_j} \xi |\xi|^{\frac{9}{2}} \langle \xi \rangle e^{\frac{i\tau}{\xi}(1-\omega_j)} \overline{\Psi(\tau,\xi)} \\
&\times \chi\left(\xi \langle \tau \rangle^{-\nu}\right) \left(\widehat{\varphi}_1\left(\tau, \frac{\xi}{\omega_j}\right)\right)^j \left(\overline{\widehat{\varphi}_1\left(\tau, \frac{\xi}{\omega_j}\right)}\right)^{3-j} \frac{1}{\tau} \Bigg|_{\tau=0}^{\tau=t} \\
&+ \sum_{j\neq 2} \frac{b_j |\omega_j|^{\frac{9}{2}}}{1-\omega_j} \xi |\xi|^{\frac{9}{2}} \langle \xi \rangle \int_0^t e^{\frac{i\tau}{\xi}(1-\omega_j)} \partial_\tau \left(\tau^{-1} \chi\left(\xi \langle \tau \rangle^{-\nu}\right) \overline{\Psi(\tau,\xi)}\right. \\
&\times \left. \left(\widehat{\varphi}_1\left(\tau, \frac{\xi}{\omega_j}\right)\right)^j \left(\overline{\widehat{\varphi}_1\left(\tau, \frac{\xi}{\omega_j}\right)}\right)^{3-j} \right) d\tau + O\left(\varepsilon^3\right) = O\left(\varepsilon^3\right).
\end{aligned}
$$

Thus we get the estimate $|y(t,\xi)| \leq |y(0,\xi)| + O\left(\varepsilon^3\right)$, and $|\xi|^{\frac{3}{2}} \langle \xi \rangle |\widehat{\varphi}_1(\xi)| \leq C\varepsilon$ in the domain $|\xi| \leq \langle t \rangle^\nu$. Therefore we find the desired estimate

$$
\|\langle i\partial_x \rangle u(t)\|_{\mathbf{L}^\infty} \leq C\varepsilon \langle t \rangle^{-\frac{1}{2}}.
$$

Lemma 4.9 is proved. □

By Lemma 4.8 we see that a priori estimate of $\|u\|_{\mathbf{X}_T}$ implies a priori estimate of $\|u\|_{\mathbf{Y}_T}$. On the other hand by Lemma 4.9 a priori estimate of $\|u\|_{\mathbf{Y}_T}$ yields a priori estimate of $\|u\|_{\mathbf{X}_T}$. Therefore the global existence of solutions of the Cauchy problem (4.1) satisfying estimates

$$
\|u\|_{\mathbf{X}_\infty} \leq C\varepsilon, \;\; \|u\|_{\mathbf{Y}_\infty} \leq C\varepsilon
$$

follows by a standard continuation argument and the local existence theorem. Thus we obtain the global in time existence of solutions to the Cauchy problem (1.1). Then the asymptotics of solutions is proved in a standard way (see [19]).

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