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Uncertainty Balance Principle

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Abstract

The objective of this paper is to present new and simple mathematical approach to deal with uncertainty transformation for fuzzy to random or random to fuzzy data. In particular we present a method to describe fuzzy (possibilistic) distribution in terms of a pair (or more) of related random (probabilistic) events, both fixed and variable. Our approach uses basic properties of both fuzzy and random distributions, and it assumes data is both possibilistic and probabilistic. We show that the data fuzziness can be viewed as a non uniqueness of related random events, and prove our Uncertainty Balance Principle. We also show how Zadeh's fuzzy-random Consistency Principle can be given precise mathematical meaning. Various types of fuzzy distributions are examined and several numerical examples presented.

Keywords*:* Fuzzy Distributions; Cumulative Distributions; Fuzzy to Random Transformation; Random to Fuzzy Transformation; Uncertainty Balance Principle; Uncertainty Change Law; Big Data

1. Introduction

In studies of uncertain phenomena, several methods are employed. Two most widely used are random and fuzzy data approaches. They are typically described in terms of random and fuzzy distributions [7]. These two methods look at the uncertainty from different points of view. In literature one can find various terms for fuzzy data, such as possibilistic, soft and subjective [3], as opposed to random called probabilistic, hard and objective [21]. These terms are somewhat arbitrary and there are authors who used probability distributions to represent subjective information [42, 43, 44]. Similarly, other authors used fuzzy sets and possibility distributions to describe objective imprecise information either about constants or about probability distributions [41]. Historically, probability is defined in the context of some physical measurement and mathematically in terms of probability axioms by Kolmogorov [34] where probability space, events and associated probabilities are defined. Related notion of random variables are defined in terms of mappings from probabilistic space of events to real line, carrying an underlying probabilities from the original event space. Indexing with some independent variable, such as time, one can define random processes as dynamic versions of random variables. On the other hand fuzzy, possibilistic approach relates to some intuitive uncertain notion (often of human nature) of an underlying uncertain event with some confidence (presumption) levels defined. Often in fuzzy data there is no reference, at least not directly, to any experiment or hard measurement. It is more representation of our confidence level in an uncertain phenomenon. If a need arises to combine fuzzy and random data, such as in soft/hard data fusion, [7],[21], each distribution is typically handled separately for a specific problem at hand, and to our knowledge no rigorous mathematical methodologies exist for a practical uncertainty alignment between two types of data. In a fundamental paper by Zadeh [2], a concept of possibilistic fuzzy distributions was introduced as opposed to random and probabilistic distributions. The possibilistic distribution is shown to be equivalent numerically to fuzzy membership function. In classic fuzzy references [7],[8], various algebraic operations on fuzzy data are dscribed, as well as the methods as how to combine fuzzy and random data in meaningful ways. One obvious method is to normalize random data distribution to unity and combine it with the fuzzy data. Mathematically correct in principle, this method can be considered as a sort of uncertainty alignment from random to fuzzy data. Unfortunately the method is not practical because of loss of information in the process [7]. Also, in our opinion this method does not have any strong conceptual ground. Another approach is to define hybrid data which retains both fuzzy and random properties of original data. One can define random fuzzy data where fuzzy distribution argument is "randomized" according to a probabilistic distribution density. Or, one can consider fuzzy random data where the value of random distribution density is fuzzified according to fuzzy distribution. From these original ideas, there was very extensive development last two decades, [9]-[19] in the area of "random fuzzy sets" and "fuzzy random variables". Neither is the focus of our paper. The subject of our paper is to consider fuzzy to random uncertainty alignment (i.e. starting with fuzzy and generating random data, or vice versa) using very basic properties of fuzzy and random distributions. Our motivation is to produce a common

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fuzzy or random data base to process the data further for either decision making process for a given application or a possible data filtering. In our approach we employ three step methodology, i.e.:

(I) Decompose any fuzzy distribution via cumulative (probabilistic) distribution functions (CDF). We do not use probabilistic density functions (PDF), a derivative of CDF, for two reasons. First, it may not always exist [34], and second is that CDF is normalized to unity by the definition, similar to fuzzy distributions.

(II) Use basic probabilistic axioms whereas the CDFs are defined in terms of random event probabilities of the form $P(X \leq x)$, [34],[35], and combined with (I) above resulting in probability differences $\Delta P(A_i)$ for some TBA events A_i.

(III) Use Big Data or some other statistical methodologies to produce best $\Delta P(A_i)$ choices in (II). a

The result of our approach is that for any unimodal fuzzy data, fuzzy distribution can be thought of as a combination of fixed and variable probability events. In the case of multimodal fuzzy data, this representation consists of a number of fixed and variable random events. We believe our approach can bring about new avenues in aligning fuzzy and random data, in particular in very important area of soft-hard (human-machine) data fusion [21]. In our previous introductory paper [40] we presented the basics of our uncertainty alignment methodology. This paper extends these results with additional fuzzy to random alignment methods, and it presents a unifying Uncertainty Balance Principle of the general form $\Pi(x) + \sum\Delta P(A_i,A_j) = 1$ and $\Pi(x) = \sum\Delta P(A_i,A_j)$ for all alignment cases. The paper is organized as follows. Section 2 summarize, very briefly, basic probabilistic and possibilistic results employed in this paper. We do not aim to be complete with these summary, just what is of interest for the current paper. In Section 4 we introduce our main fuzzy to random uncertainty alignment arguments using standard triangular fuzzy distribution (TFN). The approach applies to any fuzzy distribution. We describe steps which result in fuzzy distribution as a combination of fixed and variable probabilities. In Section 4 we formalize and prove key results:

(i) Probabilistic decomposition of an n-modal fuzzy distribution

(ii) Universal Uncertainty Balance Principle which is presumption and x-invariant, and related

(iii) Uncertainty Change Law.

In Section 4.7 we point to a potential use of our methodology in Data Fusion and Decision Making situations. Section 5 presents numerical examples showing fuzzy distributions in terms of fixed (unique) and variable (non unique) random events and related probabilities. Symmetric and non symmetric TFN, convex and non convex distributions are illustrated. The numerical examples confirm results of Section 4. Section 6 has the pseudo code for the uncertainty alignment algorithms, Conclusion is in Section 7, and key references are included in Section 8.

2. Random (Probabilistic) Distributions And Fuzzy (Possibilistic) Distributions

For the purposes of this paper we recall few basic classic probability and cumulative distribution facts as well as elementary fuzzy distributions results used in this paper.

2.1. Probability and Cumulative Distribution

The random events A, A_1, A_2, etc. are the subsets of a certain event S and they are assigned probabilities P. From classic references [34], [35] we have:

$$0 \leq P(A) \leq 1, \quad P(S) = 1, P(O) = 0 \tag{1}$$
$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \tag{2}$$

where O is an impossible event. If $A_1 \cup A_2 = S$ and two events are mutually exclusive, then $P(A_1 \cup A_2) = P(A_1)+P(A_2)=1$, hence A_1 and A_2 are complementary with $P(A_1) = 1 - P(A_2) = P(A_2^*)$ and A_2^* indicates complementary event to A_2. For any event A, the following holds:

$$P(A) + P(A^*) = 1 \tag{3}$$

If the events are independent, then we have:

$$P(A_1 \cap A_2) = P(A_1)P(A_2) \tag{4}$$

Mutual exclusivity and independency do not imply each other. A random variable $X(\xi)$ is a function that assigns a real number to each outcome ξ in the sample space S of a random experiment [34],[35]. If an event A is given in S such that $A=\{\xi: X(\xi)\epsilon B\}$, where B is a subset of real line R, then A and B are equivalent events with the same probability:

$$P(X\epsilon B) = P(A) = P(\xi: X(\xi)\, \epsilon B) \tag{5}$$

Cumulative distribution function (CDF) of X is defined as:

$$F_X(x) = F(x) = P(X \leq x), \ -\infty \leq x \leq +\infty \tag{6}$$

which is a probability that X has a value in $(-\infty, x]$, and hence it is a function of x. Figure 1 shows uniform CDF and related PDF, which is a derivative of CDF. The properties of CDF and PDF can be found in any classic probability theory reference, such as [34] and [35].

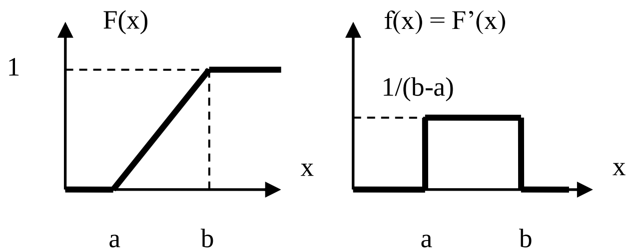


Figure 1. Uniform CDF and PDF

In this paper we deal with the cumulative rather than density functions (which may not exist in some cases), for mathematical as well as conceptual and practical reasons.

2.2. Possibility and Fuzzy Distributions

Possibility theory was developed early on by Zadeh [2]-[5] as an extension of fuzzy sets theory, in the context of information meaning, in particular in the context of semantic variables and human soft (fuzzy) data. Possibility was associated with fuzziness, either due to lack of knowledge or related to the subset for which possibility is defined. Since its inception possibility theory was developed in an axiomatic framework. We do not aim to discuss recent theoretical developments in possibility theory which there are many [9]-[19]. We simply recall possibility distribution $\Pi_X(\xi)$ as a fuzzy restriction on the values assigned to an uncertain variable X and numerically equivalent to fuzzy membership function $\mu_A(\xi)$, i.e. $\Pi_X(\xi) = \mu_A(\xi)$. For simplicity of the notation we use $\Pi(x)$, x representing specific choice of fuzzy variable X. Here we recall just a few fuzzy distribution properties which are used in this paper.

$$0 \le \Pi(A) \le 1, \quad \Pi(S) = 1, \Pi(O) = 0 \tag{7}$$

where O is an empty set and S is a universe of discourse, with all subsets to which we can assign possibilities. For any A we also have:

$$\Pi(A) + \Pi(A^*) \ge 1 \tag{8}$$

where A* indicates complementary event to A. The full axiomatic description can be found in [18].

2.3. Consistency Principle

We find it useful for our paper to recall Consistency Principle between fuzzy and random variable X defined in Zadeh's classic paper [2] as:

$$\Gamma_X = \sum P_i\Pi_i = P_1\Pi_1 + P_2\Pi_2 + \ldots + P_n\Pi_n \tag{9}$$

where i=1,…,n and variable X can be interpreted both as probabilistic and possibilistic, with the corresponding distributions consisting of the same number of choices in the interval of interest. Consistency Principle carries an intuitive observation that reducing the possibility of an event tends to reduce its probability. The opposite may not hold. If there is a precise (point wise) match between possibilistic and probabilistic distributions, $P_i = \Pi_i$, then:

$$\Gamma_X = \sum \Pi^2_i = \sum P^2_i \tag{10}$$

Consistency Principle as given in (9) may be useful when possibility is known about uncertain event X but not the probability. Our paper expands this idea via Uncertainty Balance Principle which produces variable probability from given possibility. In that sense our approach is similar to (10) rather than (9), as shown in Section 4. Our approach also lends itself to a precise mathematical and quantitative treatment.

3. Fuzzy To Random Alignment

Methodology described in this paper can be of good use in decision making process where data is inherently mixed, both soft (fuzzy) and hard (random). Often in literature one finds terms such as objective, sensor based, or machine for hard data, and subjective, human based for soft data. One particular area of interest is human-machine (soft-hard) data fusion [21]. The main contribution of our work can be understood as two fold:

(i) If fuzzy data is available, we can produce variable random data with variable probabilities reflecting original fuzziness.

(ii) On the other hand, if a variable random data is available we can produce corresponding fuzzy distribution, both contributions per our Uncertainty Balance Principle.

3.1. General Considerations

We proceed by considering a typical triangular fuzzy distribution number (TFN) given in Figure 2 with the interval of interest {a,b,c} for any a, b and c, and the corresponding fuzzy distribution $\Pi(x)$ numerically equivalent to the fuzzy membership function $\mu(x)$, [2]. At this point we will not write equations for the segments of $\Pi(x)$. This is done in Section 6 with numerical examples. Note that our approach can be applied to any other fuzzy variable, unimodal or multimodal, symmetric or not, normal or non normal, convex or non convex, trapezoidal or arbitrary shaped fuzzy distribution. Section 5.5 shows additional examples of fuzzy distributions. The key is that the CDF properties [35] are satisfied. Next step is to define a pair of CDF's such as in Figure 1 to "decompose" TFN distribution $\Pi(x)$:

$$\Pi(x) = F_1(x) - F_2(x) \tag{11}$$

where $F_1(x)$ and $F_2(x)$ are shown in Figure 3. They are both uniform probabilistic distributions. The purpose of the decomposition (11) is a first step in relating fuzzy to random variables. Our first idea to define (11) came from an obvious fact that CDF is maximum at 1 similar to $\Pi(x)$. Next step is to find a way how to describe both rising and falling part of $\Pi(x)$, and hence (11) came as a natural solution. Recall that a CDF is a probability of an event A = "$X \le x$" as given in (6). It is critical we assume the uncertain variable X is <u>both possibilistic and probabilistic</u>.

Next we take another "probabilistic" step to refine (11) using basic probability relation in (2). The key is that equation (1) has negative term in it which we can associate with $F_2(x)$ in (11). This negative term can take different form depending how we move different terms around in (2). Three methods are described.

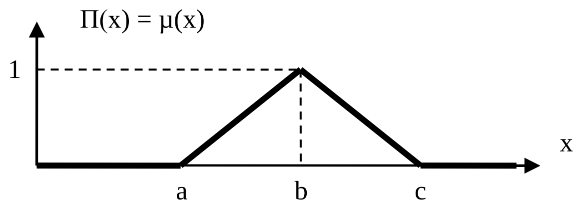
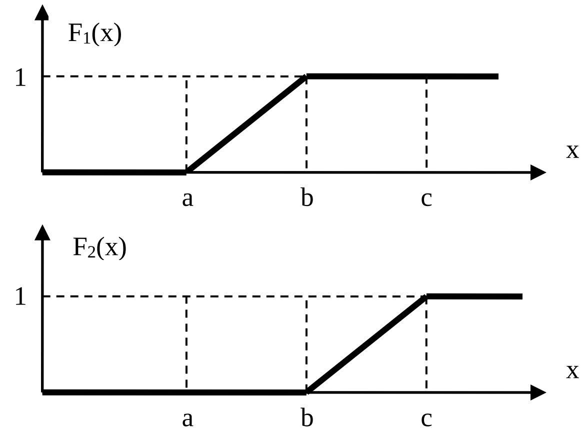


Figure 2. Triangular fuzzy number (TFN)



Figures 3. CDF's $F_1(x)$ and $F_2(x)$

3.2. Method 1

This method is described in full details in our earlier paper [40]. It is summarized here. We rewrite (2) as:

$$P(A_1) - P(A_1 \cap A_2) = P(A_1 \cup A_2) - P(A_2) \qquad (12)$$

Each side of Equation (12) is a probability and it satisfies basic probabilistic axioms in (1) and (2). Nice property of (12) is that both sides have negative terms, as does Equation (11). See also Figure 4. We proceed in two Π(x) parts, rising and falling.

<u>Rising Part.</u> We equate left side of (12) with (11):

$$\Pi(x) = F_1(x) - F_2(x) = P(A_1) - P(A_1 \cap A_2) \qquad (13)$$

from where $F_1(x)$ and $F_2(x)$ could be uniquely associated with the corresponding probabilities in (13). The aim of this step is to formally align $F_1(x)$ with $P(A_1)$ and $F_2(x)$ with $P(A_1 \cap A_2)$, with an idea that an intersection of two events $A_1 \cap A_2$ can produce probability non uniqueness, whereas A_1 would be fixed. We again recall the fact that CDF is a probability of an event per (6). Next, the events A_2 and $A_1 \cap A_2$ (their probabilities $P(A_2)$ and $P(A_1 \cap A_2)$ given via $F_1(x)$ and $F_2(x)$) are to be determined. What is not uniquely determined is A_2 because different A_2 can produce the same intersection $A_1 \cap A_2$. A little reflection on set theory brings us to:

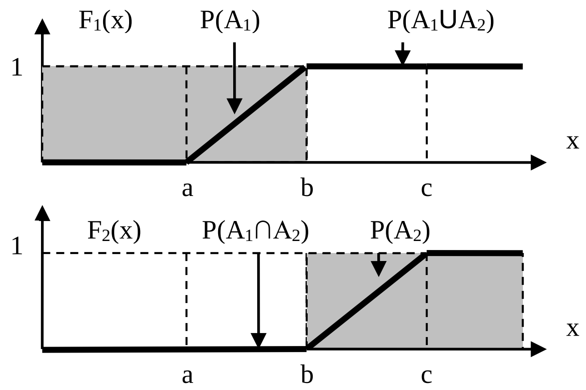
$$P(A_1 \cap A_2) \le P(A_2) \le 1 - \Pi(x) \qquad (14)$$

producing the same $P(A_1 \cap A_2)$. The right side $1 - \Pi(x) \le \Pi^*(x)$ represents complementary fuzzy distribution to Π(x), which upholds the condition in (10). Note that non unique A_2 corresponds to $x \le b$, while A_2 is unique for $b \le x$, due to a simultaneous action of conditions in (14). One can consider that the interplay of unique $P(A_1)$ and non unique $P(A_2)$ in $P(A_1 \cap A_2)$, produces "fuzziness" on the left hand side of Π(x). See also Figures 5.

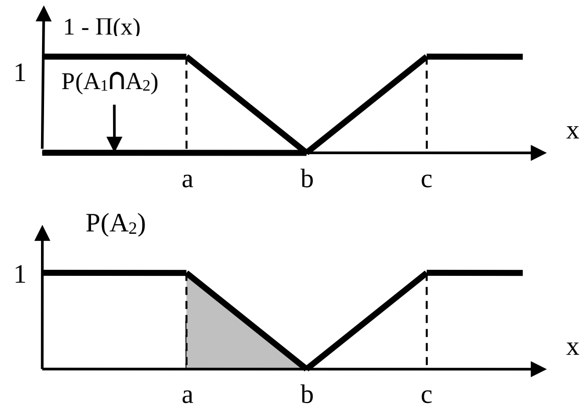
<u>Falling Part</u>. We equate right side of (12) with (11):

$$\Pi(x) = F_1(x) - F_2(x) = P(A_1 \cup A_2) - P(A_2) \qquad (15)$$

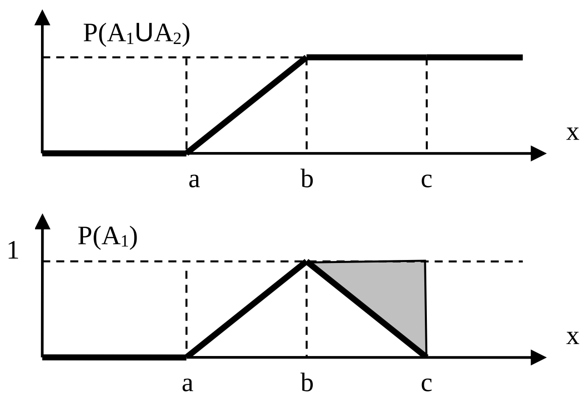
where $P(A_2)$ is uniquely defined in (15). The events $A_1 \cup A_2$ and A_1 are to be determined.



Figures 4. Method 1: CDF's $F_1(x)$ and $F_2(x)$



Figures 5. Rising part bounds on $P(A_2)$



Figures 6. Falling part bounds on $P(A_1)$

Note that the forms of F_1 and F_2 are same as before but we interpret them differently, i.e. as $P(A_1 \cup A_2)$

and $P(A_2)$. To further clarify A_1, we consider union $A_1 \cup A_2$, which produces the following condition on $P(A_1)$:

$$\Pi(x) \leq P(A_1) \leq P(A_1 \cup A_2) \qquad (16)$$

Figures 6 summarize $P(A_1)$ and bounds given in (16) for $\Pi(x)$ using $P(A_1 \cup A_2)$. The gray area in $P(A_1)$ indicates its non unique choices. They will all generate the same $F_1 = P(A_1 \cup A_2)$. Note that the grey area corresponds to $b \leq x$, while $P(A_1)$ is uniquely defined for $x \leq b$, due to a simultaneous action of conditions (16). As in Rising Part, one can consider that the interplay of unique A_2 and $P(A_2)$ and non unique A_1 and $P(A_1 \cup A_2)$ is equivalent to "fuzziness" of the right hand side of $\Pi(x)$, when $b \leq x$. By combining two parts, we conclude that non unique choices for A_1 and A_2 and their corresponding probabilities $P(A_1)$ and $P(A_2)$, correspond to the non zero part of the distribution $\Pi(x)$. Outside of that, when $\Pi(x)=0$, they can be considered independent for the trivial cases of probabilities 0 or 1, per Table 1, where we used the notation $P_1=P(A_1)$ and $P_2=P(A_2)$. Note that the non zero $\Pi(x)$ corresponds to the gray shaded areas in Table 1.

Table 1. Π(x), Method 1. Equations (16) and (19)

x	A_1	A_2	A_1 vs. A_2
$0 \leq x < a$	$P_1 = 0$	$P_2 = 0$	$P_1 + P_2 = 0$
$a \leq x < b$	$P_1 = \Pi$	Non unique	$P_1 + P_2 \leq 1$
$b \leq x < c$	Non unique	$P_2 = 1 - \Pi$	$P_1 + P_2 \leq 1$
$c \leq x$	$0 = P_1$	$P_2 = 1$	$P_1 + P_2 = 1$

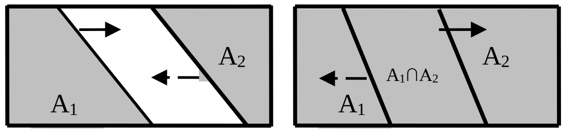
Final note is that if we choose (13) for the Falling Part instead of (15) we end up with $P(A_1)$ and $P(A_2)$ as constant probabilities, given $\Pi(x)$, and there will be no "fuzziness" induced by variable probabilities.

3.3. Method 2

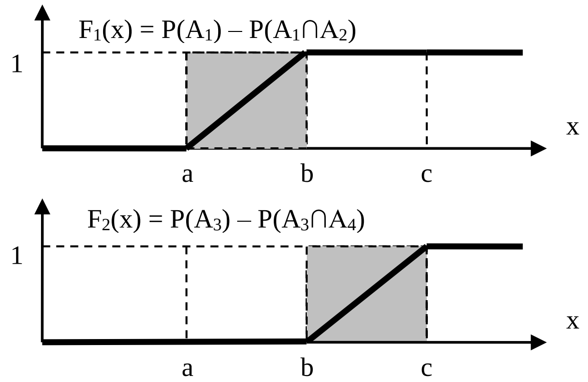
Now we use Equation (2) and define fuzzy distribution (11), with the following choices for CDF's $F_1(x)$ and $F_2(x)$:

$$F_1(x) = P(A_1) - P(A_1 \cap A_2) \qquad (17a)$$
$$F_2(x) = P(A_3) - P(A_3 \cap A_4) \qquad (17b)$$

Venn diagrams in Figures 7 are for $x \leq b$ with the arrows indicating two evens A_1 and A_2 "extending" to eventually form a certain event with the probability 1, with either $P(A_1 \cap A_2) = 0$ (A_1 and A_2 meet) or $P(A_1 \cap A_2) \neq 0$ (A_1 and A_2 overlap), with $P(A_1) + P(A_2) - P(A_1 \cap A_2) = 1$ in either case. This corresponds to the presumption (possibility) level of $\Pi(x) = 1$.



Figures 7. Formation of fuzzy distribution Π(x) in (17a)



Figures 8 Method 2: CDF's $F_1(x)$ and $F_2(x)$

For $b \leq x$, the same process starts with two new events, A_3 and A_4 with the probabilities in (17b) for F_2. For $c \leq x$ we have $\Pi = 0$, when the events (A_1, A_2) and (A_3, A_4) form $P_1 + P_2 - P_{12} = 1$ and $P_3 + P_4 - P_{34} = 1$ canceling each other. See Table 2. Figures 8 show probability diagrams. The shaded areas indicate non unique probabilities. The probability limits are obtained for $x \leq b$ from Figures 7, 8, and from (17a):

$$\Pi(x) = P(A_1) - P(A_1 \cap A_2) \qquad (18)$$

with A_1 is chosen first and $A_1 \cap A_2$ and A_2 follow, to obtain:

$$\Pi(x) \leq P(A_1) \leq 1$$
$$0 \leq P(A_1 \cap A_2) \leq 1 - \Pi(x)$$
$$0 \leq P(A_2) \leq 1 - \Pi(x) \qquad (19)$$

Similarly for A_3 chosen then $A_3 \cap A_4$ and A_4 follow on $b \leq x$, and from (17b):

$$\Pi(x) = 1 - [P(A_3) - P(A_3 \cap A_4)] \qquad (20)$$

producing:

$$1 - \Pi(x) \leq P(A_3) \leq 1$$
$$0 \leq P(A_3 \cap A_4) \leq \Pi(x)$$
$$0 \leq P(A_4) \leq \Pi(x) \qquad (21)$$

Table 2 has Method 2 summary, with $P_i=P(A_i)$, $P_{ij}=P(A_i \cap A_j)$, i,j=1,2. Note that the role of events A_1 and A_2 can be reversed, and similarly for A_3 and A_4, i.e. either pair of events can be chosen first. There may be some TBD probabilistic connection between events (A_1, A_2) and (A_3, A_4).

Table 2. Π(x) using Method 2, Equations (17a,b)

x	A_1, A_3	A_2, A_4	A_1 vs. A_2, A_3 vs A_4

$0 \le x \le a$	$P_i = 0$	$P_j = 0$	$P_k + P_{k+1} = 0$, k=1,3
$a \le x \le b$ $P_3=P_4=0$	A_1 non unique	A_2 non unique	$\Pi = P_1 - P_{12}$ $0 \le P_1 + P_2 - P_{12} \le 1$
$b \le x \le c$ $P_1=P_2=0$	A_3 non unique	A_4 non unique	$\Pi = P_3 - P_{34}$ $0 \le P_3 + P_4 - P_{34} \le 1$
$c \le x$	$P_i = 1$	$P_j = 1$	$P_k+P_{k+1}-P_{k,k+1}=1$, k=1,3

3.4. Method 3

We use (1) and define fuzzy distribution decomposed as in (11), with the following choices for $F_1(x)$ and $F_2(x)$:

$$F_1(x)=P(A_1 \cup A_2)=P(A_1)+P(A_2) - P(A_1 \cap A_2) \qquad (22a)$$
$$F_2(x) = P(A_3 \cup A_4)=P(A_3) + P(A_4) - P(A_3 \cap A_4) \qquad (22b)$$

Figures 7 applies here as well. For $b \le x$, the same process starts with two new events (A_3,A_4) with the corresponding $F_2(x)$ probabilities (22b). For $c \le x$ we have $\Pi(x) = 0$, when (A_1,A_2) and (A_3,A_4) form $P_1+P_2-P_{12}=1$ and $P_3+P_4-P_{34}=1$ compensating each other. See also Table 3. Figures 10 show two diagrams with the corresponding probabilities. The shaded areas indicate non unique probabilities. The probability limits similar to (14) and (16) can be obtained from Figures 7 and 8, as we assume A_1 is chosen first and then A_2 and $A_1 \cap A_2$ follow for $x \le b$:

$$0 \le P(A_1) \le \Pi(x)$$
$$0 \le \Pi(x) - P(A_1) \le P(A_2) \le \Pi(x)$$
$$0 \le P(A_1 \cap A_2) \le P(A_1) \le \Pi(x) \qquad (23)$$

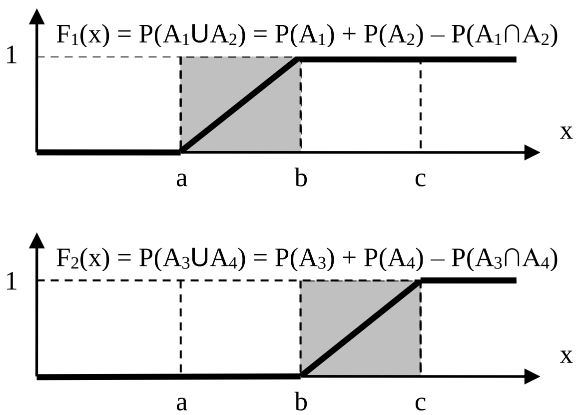
which is equivalent to:

$$0 \le P(A_1),\ P(A_2),\ P(A_1 \cap A_2) \le \Pi(x) \qquad (24)$$

Similarly for A_3, A_4 and $A_3 \cap A_4$ for $b \le x$ we have:

$$0 \le P(A_3),\ P(A_4),\ P(A_3 \cap A_4) \le \Pi(x) \qquad (25)$$

Table 3 has Method 3 summary. Note that the role of A_1 and A_2 can be reversed, and also for A_3 and A_4. We assume that the events A_1, A_2 are not related to A_3, A_4.



Figures 9. Method 3: CDF's $F_1(x)$ and $F_2(x)$

Table 3. Π(x) using Method 3, Equations (22a,b)

x	A_1, A_3	A_2, A_4	A_1 vs. A_2, A_3 vs A_4
$0 \le x < a$	$P_i = 0$	$P_j = 0$	$P_k + P_{k+1} = 0$, k=1,3
$a \le x < b$ $P_3=P_4=0$	A_1 non unique	A_2 non unique	$\Pi = P_1 + P_2 - P_{12}$ $0 \le P_1 + P_2 - P_{12} \le 1$
$b \le x < c$ $P_1=P_2=0$	A_3 non unique	A_4 non unique	$\Pi = P_3 + P_4 - P_{34}$ $0 \le P_3 + P_4 - P_{34} \le 1$
$c \le x$	$P_i \le 1$	$P_j \le 1$	$P_k+P_{k+1}-P_{k,k+1}=1$,k=1,3

3.5. Other Methods

Before we state the main results in the next Section 4, few comments are in order. By considering Section 3.1 and Equation (11) we can attempt to decompose $\Pi(x)$ in other ways. For example an "obvious" choice is $\Pi(x) = F_1(x) - F_2(x) = P(A_1) - P(A_2)$. The problem with this choice is that it does not offer any probability variations (non uniqueness). Similarly if we choose $\Pi(x) = F_1(x) - F_2(x) = [P(A_1) + P(A_2)] - [P(A_3) + P(A_4)]$, the same comment applies, i.e. once we choose say $P(A_1)$, then $P(A_2)$ follows uniquely, and the same for $P(A_3)$ and $P(A_4)$. We comment on this further in Section 4.5.

4. Uncertainty Balance Principle

This section advances Section 4 results and states several general results. The main goal is to produce a usable and practical result to relate fuzzy and variable random data.

4.1. General Considerations

For simplicity we assume TFN within the intervals {a,b} and {b.c} non zero $\Pi(x)$. The results are general for any fuzzy distribution $\Pi(x)$ which can be represented by repeated procedure (11) for increasing x values. These distributions can be non convex, non normalized, and of other shapes, symmetric, non symmetric, unimodal and multimodal. Figure 10 shows a bimodal fuzzy distribution consisting of two non-overlapping TFNs. They can also overlap.

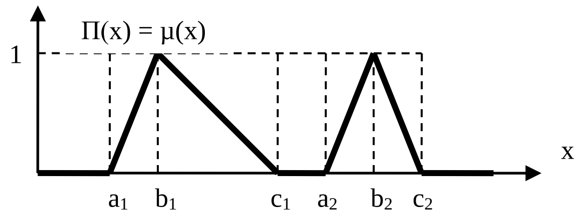


Figure 10. Bimodal TFN

The $\Pi(x)$ decomposition consists of two pairs of cumulative probabilistic distributions:

$$\Pi(x) = F_1(x) - F_2(x) + F_3(x) - F_4(x) \qquad (26)$$

By an induction extension of (2), for "n" modal TFN we have the following general result:

Theorem 1. Fuzzy n-modal distribution function $\Pi(x)$ can be decomposed as a difference of sums of probabilistic cumulative distributions:

$$\Pi(x) = \sum F_i(x) - \sum F_j(x) \qquad (27)$$

with $i = 2k-1$, $j = 2k$, $k = 1,2,\ldots,n$, for any x, where the odd functions amount for rising portion of fuzzy distribution and even for the falling portion.

For an unimodal distribution, n = 1, (27) reduces to (11), and for a bimodal one, n = 2, and (27) reduces to (26). In terms of expressing (27) as specific probabilities, we have three possibilities depending on how we use (1).

4.2. Method 1

Using (26) and Section 3.2 results for bimodal case, we have:

$$\Pi(x) = P(A_1)–P(A_1\cap A_2)+P(A_3)–P(A_3\cap A_4) \qquad (28a)$$

for two rising parts of $\Pi(x)$ and:

$$\Pi(x) = P(A_1\cup A_2)–P(A_2)+P(A_3\cup A_4)–P(A_4) \qquad (28b)$$

for two falling parts of $\Pi(x)$. With (27) and (28a,b) we have the following:

Corollary 1.1. Fuzzy n-modal distribution function given in Theorem 1 can be further expressed as a difference of sums of probabilities:

$$\Pi(x) = \sum P(A_i) – \sum P(A_i\cap A_j) \qquad (29a)$$

for two rising parts of $\Pi(x)$, and:

$$\Pi(x) = \sum P(A_i\cup A_j) – \sum P(A_j) \qquad (29b)$$

for two falling parts of $\Pi(x)$, with $i = 2k-1$ and $j = 2k$, $k = 1,2,\ldots,n$.

When n=1, for unimodal fuzzy distribution, (29a,b) reduce to (13) and (15), and for n=2, bimodal fuzzy distribution, (29a,b) reduce to (28a,b), respectively. Next, we define:

$$\sum\Delta P(A_i,A_j) = \sum\Delta P(A_i) + \sum\Delta P(A_j) \qquad (30)$$

as total probability change, with:

$$\Delta P(A_k) = P(A_k)_M – P(A_k)_m \qquad (31)$$

as probability range for event A_k, where "M" stands for maximum value, and "m" is for minimum value. We now state the following general result which relates a fuzzy distribution and a set of changes in related random event probabilities.

Theorem 2. Any multimodal fuzzy distribution $\Pi(x)$ can be expressed in terms of fuzzy presumption-invariant and x-invariant universal fuzzy-random Uncertainty Balance Principle:

$$\Pi(x) + \sum\Delta P(A_i,A_j) = 1 \qquad (32a)$$
$$\Pi^*(x) \geq \sum\Delta P(A_i,A_j) \qquad (32b)$$

for any x, with $i = 2k-1$, $j = 2k$, $k = 1,2,\ldots,n$.

Note that practical implication of this Theorem is to be able to express fuzzy data distribution as a combination of a number of variable random events and corresponding probabilities. We illustrate this notion in Section 5 with numerical examples, and in particular in Example of Section 5.5 which discusses a specific fuzzy distribution and specific resulting variable probabilities. To continue, note that for simplicity, we did not burden the notation in Theorem 2 with stating dependency of ΔP's on x. The key feature of Theorem 2 is that it holds for any x and any presumption level of $\Pi(x)$. We prove the unimodal case when n=1, for TFN in Figure 2. The proof for any n and $\Pi(x)$ is straightforward, by repeating the procedure n times. From (14) and (16) we obtain (see also Section 5 examples):

For $x \leq b$: (33)

$$P(A_1)_m = P(A_1)_M$$
$$\Delta P(A_1) = P(A_1)_M – P(A_1)_m = 0$$
$$P(A_2)_M = 1- \Pi(x),\ P(A_2)_m = 0$$
$$\Delta P(A_2) = P(A_2)_M – P(A_2)_m = 1 – \Pi(x)$$

For $b \leq x$: (34)

$$P(A_1)_m = \Pi(x),\ P(A_1)_M = 1$$
$$\Delta P(A_1) = P(A_1)_M – P(A_1)_m = 1 – \Pi(x)$$
$$P(A_2)_m = P(A_2)_X$$
$$\Delta P(A_2) = P(A_2)_M – P(A_2)_m = 0$$

Note that the point "b" is maximum $\Pi(x)$ point of a TFN, or any other unimodal fuzzy distribution. Replacing (33) and (34) into (32), for n = 1, we obtain:

$$\Pi(x) + \Delta P(A_1) + \Delta P(A_2) = 1 \qquad (35)$$

or:

$$\Pi(x) + \Delta P(A_1,A_2) = 1 \qquad (36a)$$
$$\Pi^*(x) \geq \Delta P(A_1,A_2) \qquad (36b)$$

holding across the full range of argument x and $\Pi(x)$, and (36b) follows from (10).

One can interpret Theorem 2 result as "randomness" pool left to form fuzzy distribution to a random certainty. This also means that for higher "presumption" levels, near 1, corresponding randomness pool is smaller (less uncertainty to adjust) and for lower "presumption" levels it is larger (more uncertainty to adjust). Examples in Section 7 and Figures 10 show that clearly. We have the following result based on Theorem 2:

Corollary 2. Any fuzzy distribution derivative $d\Pi(x)/dx$ can be expressed for any argument x as a universal fuzzy-random Uncertainty Change Law:

$$d\Pi(x)/dx = - \sum d[\Delta P(A_i,A_j)]dx \qquad (38a)$$
$$d\Pi^*(x)/dx \geq \sum d[\Delta P(A_i,A_j)]dx \qquad (38b)$$

and $i = 2k-1$, $j = 2k$, $k = 1,2,\ldots,n$. For n = 1, we have

$$d\Pi(x)/dx = - d[\Delta P(A_1,A_2)]/dx \qquad (39a)$$
$$d\Pi^*(x)/dx \geq d[\Delta P(A_1,A_2)]/dx \qquad (39b)$$

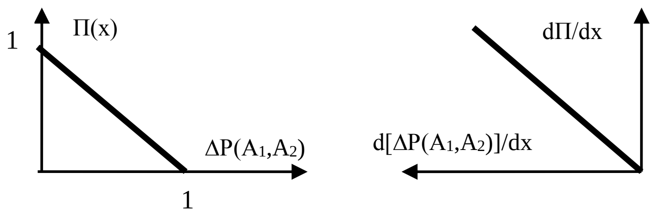
Due to the fact that the changes in two probabilities $\Delta P(A_1)$ and $\Delta P(A_2)$ are not zero at different arguments x (39a) reduces to a very simple fact:

$$d\Pi(x)/dx = - d[\Delta P(A_i)]/dx \qquad (40)$$

where $\Delta P(A_i)$ is $\Delta P(A_1)$ or $\Delta P(A_2)$ for i=2, depending on x value. Simply stated, (40) says that the change in fuzzy distribution is the opposite of probability change. This is also shown in Section 6 with numerical examples and in Figures 11. The first diagram shows $\Pi(x)$ changes with ΔP, for any fuzzy distribution. This is a consequence of Theorem 2 and presumption and x-invariant nature of it. The second diagram in Figure 11 indicates how $d\Pi/dx$ and $d(\Delta P)/dx$ relate, based on Corollary 2.1. The diagrams are universal for any $\Pi(x)$, for any $F_i(x)$. This is also illustrated in Section 6 with various distributions $\Pi(x)$.

4.3. Method 2

In this case Theorem 1 still holds as stated above. Instead of Corollary 1.1 and using (17ab), we have:



Figures 11: Method 1: Theorem 2 and Corollary 2

Corollary 1.2. Fuzzy n-modal distribution function given in Theorem 1 can be expressed as a difference of sums of probabilities:

$$\Pi(x) = \sum P(A_i) - \sum P(A_i \cap A_j) \qquad (41)$$

with i = 2k-1 and j = 2k, k = 1,2,…,n.

For n=1, we obtain $\Pi(x) = P(A_i) - P(A_i \cap A_j)$. Using (17a,b)-(21), with total probability change:

$$\sum \Delta P(A_i,A_j) = \sum \Delta P(A_i) - \sum \Delta P(A_i \cap A_j) \qquad (42)$$

we state the following general result similar to Theorem 2.

Theorem 3. Any multimodal fuzzy distribution $\Pi(x)$ can be expressed in terms of fuzzy presumption-invariant and x-invariant universal fuzzy-random Uncertainty Balance Principle. For any $x \leqq b$, we have:

$$\Pi(x) + \sum \Delta P(A_i,A_j) = 1 \qquad (43a)$$
$$\Pi^*(x) \geq \sum \Delta P(A_i,A_j) \qquad (43b)$$

and for any $b \leq x$:

$$\Pi(x) = \sum \Delta P(A_i,A_j) \qquad (44a)$$
$$\Pi^*(x) + \sum \Delta P(A_i,A_j) \geq 1 \qquad (44b)$$

Similar to Theorem 2, the result above holds for any x and any presumption level of $\Pi(x)$. We prove the case when n=1, for TFN in Figure 2. The proof for any n and any $\Pi(x)$ is straightforward. From (18) through (21) we obtain:

For $x \leq b$: (45)

$$P(A_1)_m = \Pi(x),\; P(A_1)_M = 1$$
$$\Delta P(A_1) = P(A_1)_M - P(A_1)_m = 1 - \Pi(x)$$
$$P(A_1 \cap A_2)_m = 0,\; P(A_1 \cap A_2)_M = 1 - \Pi(x)$$
$$\Delta P(A_1 \cap A_2) = P(A_1 \cap A_2)_M - P(A_1 \cap A_2)_m = 1 - \Pi(x)$$
$$P(A_2)_m = 0,\; P(A_2)_M = 1 - \Pi(x)$$
$$\Delta P(A_2) = P(A_2)_M - P(A_2)_m = 1 - \Pi(x)$$

Replacing (45) into (43), for n=1, we obtain:

$$\Pi(x) + \Delta P(A_1) + \Delta P(A_2) - \Delta P(A_1 \cap A_2) = 1 \qquad (46)$$

or:

$$\Pi(x) + \Delta P(A_1,A_2) = 1 \qquad (47a)$$
$$\Pi^*(x) \geq \Delta P(A_1,A_2) \qquad (47b)$$

For $b \leq x$: (48)

$$P(A_3)_m = 1 - \Pi(x),\; P(A_3)_M = 1$$
$$\Delta P(A_3) = P(A_3)_M - P(A_3)_m = \Pi(x)$$
$$P(A_3 \cap A_4)_m = 0,\; P(A_3 \cap A_4)_M = \Pi(x)$$
$$\Delta P(A_3 \cap A_4) = P(A_3 \cap A_4)_M - P(A_3 \cap A_4)_m = \Pi(x)$$
$$P(A_4)_m = 0,\; P(A_4)_M = \Pi(x)$$
$$\Delta P(A_4) = P(A_4)_M - P(A_4)_m = \Pi(x)$$

Replacing (48) into (43), for n = 1, we obtain:

$$1 - \Pi(x) + \Delta P(A_3) + \Delta P(A_4) - \Delta P(A_3 \cap A_4) = 1 \qquad (49)$$

or:

$$\Pi(x) = \Delta P(A_3,A_4) \qquad (50a)$$
$$\Pi^*(x) + \Delta P(A_3,A_4) \geq 1 \qquad (50a)$$

Next, we have the following result based on Theorem 3:

Corollary 3. Any n-modal fuzzy distribution derivative $d\Pi(x)/dx$ can be expressed for any argument x as a universal fuzzy-random Uncertainty Change Law:

$$d\Pi(x)/dx = - \sum d[\Delta P(A_i,A_j)]/dx \qquad (51a)$$
$$d\Pi(x)/dx = \sum d[\Delta P(A_i,A_j)]/dx \qquad (51b)$$

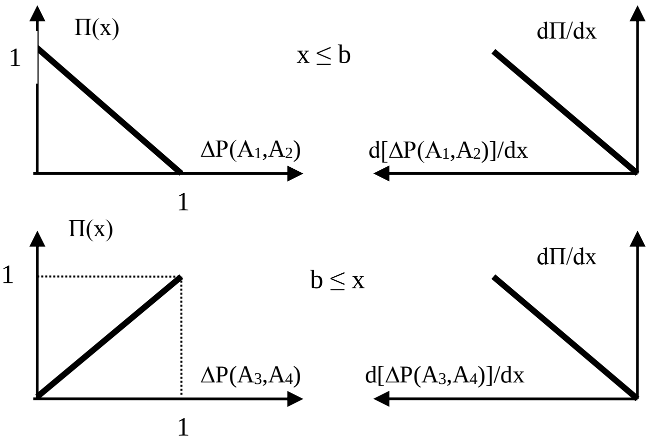
for $x \leq b$ and $b \leq x$ respectively, i = 2k-1, j = 2k, k = 1,2,…,n.

For unimodal distribution, n = 1, Figures 12, we have:

$$d\Pi(x)/dx = - d[\Delta P(A_1,A_2)]/dx \qquad (52a)$$
$$d\Pi(x)/dx = d[\Delta P(A_3A_4)]/dx \qquad (52b)$$

4.4. Method 3

Theorem 1 still holds. Using (22a) and (22b), we have:



Figures 12: Method 2: Theorem 3 and Corollary 3

Corollary 1.3. Fuzzy n-modal distribution function given in Theorem 1 can be further expressed as a difference of sums of probabilities:

$$\Pi(x)=\sum P(A_i \cup A_j)=\sum P(A_i)+\sum P(A_j)-\sum P(A_i \cap A_j) \quad (54)$$

with i = 2k-1 and j = 2k, k = 1,2,…,n.

Using (23)-(25), as well as (18) and (21), and with:

$$\sum \Delta P(A_i,A_j)=\sum \Delta P(A_i)+\sum \Delta P(A_j)-\sum \Delta P(A_i \cap A_j) \quad (55)$$

for total probability change, we state the following:

Theorem 4. Any multimodal fuzzy distribution Π(x) can be expressed in terms of fuzzy presumption-invariant and x-invariant universal fuzzy-random Uncertainty Balance Principle. For any $x \leq b$, we have:

$$\Pi(x)=\sum \Delta P(A_i,A_j) \quad (56a)$$
$$\Pi^*(x)+\sum \Delta P(A_i,A_j) \geq 1 \quad (56b)$$

and for any $b \leq x$:

$$\Pi(x)+\sum \Delta P(A_i,A_j)=1 \quad (57a)$$
$$\Pi^*(x) \geq \sum \Delta P(A_i,A_j) \quad (57b)$$

We prove for n = 1, TFN in Figure 2. The proof for any n and any Π(x) is straightforward. From (23)-(25) we obtain:

For $x \leq b$: (58)

$$P(A_1)_m=0,\ P(A_1)_M=\Pi(x)$$
$$\Delta P(A_1)=P(A_1)_M-P(A_1)_m=\Pi(x)$$
$$P(A_1 \cap A_2)_m=0,\ P(A_1 \cap A_2)_M=\Pi(x)$$
$$\Delta P(A_1 \cap A_2)=P(A_1 \cap A_2)_M-P(A_1 \cap A_2)_m=\Pi(x)$$
$$P(A_2)_m=0,\ P(A_2)_M=\Pi(x)$$
$$\Delta P(A_2)=P(A_2)_M-P(A_2)_m=\Pi(x)$$

Replacing (58) into (56a), for n = 1, we obtain:

$$\Pi(x)=\Delta P(A_1)+\Delta P(A_2)-\Delta P(A_1 \cap A_2) \quad (59)$$

or:

$$\Pi(x)=\Delta P(A_1,A_2) \quad (60a)$$
$$\Pi^*(x)+\Delta P(A_1,A_2) \geq 1 \quad (60b)$$

with $\Delta P(A_1,A_2)$ representing total probability change for two events A_1 and A_2.

For $b \leq x$: (61)

$$P(A_3)_m=0,\ P(A_3)_M=\Pi(x)$$
$$\Delta P(A_3)=P(A_3)_M-P(A_3)_m=\Pi(x)$$
$$P(A_3 \cap A_4)_m=0,\ P(A_3 \cap A_4)_M=\Pi(x)$$
$$\Delta P(A_3 \cap A_4)=P(A_3 \cap A_4)_M-P(A_3 \cap A_4)_m=\Pi(x)$$
$$P(A_4)_m=0,\ P(A_4)_M=\Pi(x)$$
$$\Delta P(A_4)=P(A_4)_M-P(A_4)_m=\Pi(x)$$

Replacing (61) into (56b), for n = 1, we obtain:

$$\Pi(x)+\Delta P(A_3)+\Delta P(A_4)-\Delta P(A_3 \cap A_4)=1 \quad (62)$$

or:

$$\Pi(x)+\Delta P(A_3,A_4)=1 \quad (63a)$$
$$\Pi^*(x) \geq \Delta P(A_3,A_4) \quad (63b)$$

with:

$$\Delta P(A_3,A_4)=\Delta P(A_3)+\Delta P(A_4)-\Delta P(A_3 \cap A_4) \quad (64)$$

representing total probability change of random events A_3 and A_4. We have the following result based on Theorem 4:

Corollary 4. Any n-modal fuzzy distribution derivative dΠ(x)/dx can be expressed for any argument x as a universal fuzzy-random Uncertainty Change Law:

$$d\Pi(x)/dx=\sum d[\Delta P(A_i,A_j)]/dx \quad (65a)$$
$$d\Pi(x)/dx=-\sum d[\Delta P(A_i,A_j)]/dx \quad (65b)$$

for $x \leq b$ and $b \leq x$ respectively, i=2k-1, j=2k, k=1,2,…,n. For unimodal distribution n=1, Figure 13, we have:

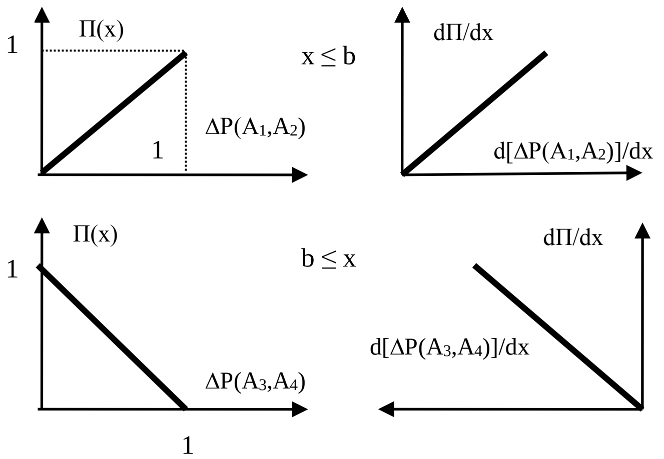
$$d\Pi(x)/dx=d[\Delta P(A_1,A_2)]/dx \quad (66a)$$
$$d\Pi(x)/dx=-d[\Delta P(A_3,A_4)]/dx \quad (66b)$$

and:

$$\Pi(x)=\Delta P(A_1,A_2) \quad (67b)$$
$$\Pi^*(x)+\Delta P(A_1,A_2) \leq 1 \quad (67a)$$

for $x \leq b$ and $b \leq x$ respectively.

Note that three methods (Theorems 2, 3, and 4) produce similar but not quite equivalent results. They offer different choices for variable probabilities. The common feature is that all state Uncertainty Balance Principle in which fuzzy distribution is either equal to probability change or offset by it, adding up to 1, i.e. certain event in random and presumption level 1 in fuzzy distribution. Corresponding Corollaries state Uncertainty Change Laws. <u>Hence we have more than one option to transform fuzzy to random data</u>, and vice versa. Section 7 illustrates all results with several numerical examples and suggests ideas how to generate fuzzy distributions using variable random events.



Figures 13. Method 3: Theorem 4 and Corollary 4

We also state the reverse result to Theorems 2, 3 and 4, i.e. random to fuzzy uncertainty alignment, as:

Corollary 5. Given a probabilistic and possibilistic uncertain variable X, with defined range of probabilities $\sum\Delta P(A_i,A_j)$ defined for random events A_i and A_j, $i = 2k-1$, $j = 2k$, $k = 1,2,\ldots,n$, a possibilistic distribution $\Pi(x)$ can be formed observing Uncertainty Balance Principle in either of the forms:

$$\Pi(x) = 1 - \sum\Delta P(A_i,A_j) \quad (68a)$$
$$\Pi(x) = \sum\Delta P(A_i,A_j) \quad (68b)$$

depending on the uncertainty alignment method used, and the rising or falling side of Π (x) represented.

Note that the results of this section can be also used in hard-soft data fusion where the staring point are probabilistic rather than possibilistic data. This gives our results an universal applicability in either fuzzy-random or random-fuzzy uncertainty data alignment.

4.5. Other Methods

As described in Section 4.5 we may consider to use other probability decomposition methods in (14) such as $\Pi(x) = F_1(x) - F_2(x) = [P(A_1) + P(A_2)] - [P(A_3) + P(A_4)]$. These choices do not offer any variability of probabilities, once one is chosen, the other follow uniquely. Hence there is no “fuzziness” involved, i.e. all $\Delta P(A_i,A_{i+1})$ are zero. For such cases Uncertainty Balance Principle reduces to “Certainty Principle” of the form:

$$\Pi(x) = P(A_1) + P(A_2) \quad (69a)$$
$$\Pi(x) + P(A_3) + P(A_4) = 1 \quad (69b)$$

for $x \leq b$ and $b \leq x$ respectively and it looks to be of less practical use. Note that for probability variability we need presence of non zero intersection of the random events and non zero probability $P(A_i \cap A_j)$. This is because specific $P(A_i \cap A_j)$ can be generated by a variety of random events A_i and A_j and their corresponding probabilities. This is illustrated in Section 5.5.

4.6. Consistency Principle as Uncertainty Balance Principle

Referring back to Zadeh’s Consistency Principle [2], as given in (7a) for an unimodal fuzzy distribution, one can re interpret it in the light of our Theorems 2, 3 and 4 which hold for any multimodal fuzzy distribution, any presumption level and any argument x. In this reinterpretation the Principle has a clear conceptual and numeric meaning, as well as an intuitive rationale. We can consider it as a “Fuzzy-Random Uncertainty Balance Principle” and it can be an alternative to Consistancy Principle given in (9). Instead of multiplying P_i and Π_i, we can use the sum of ΔP_i‘s and Π_i in the spirit of Theorems 2, 3 and 4 where “i” now points to a different x, that is x_i. Recall that all Theorems hold for any presumption level Π(x) as well as any x. For example, from Theorem 2 we can redefine Consistence Principle using our Uncertainty Balance Principle general form $\Pi_i + \Delta P_i = 1$, equivalent to $\Pi_i = 1 - \Delta P_i = \Delta P^*_i$ or $\Pi^*_i \geq \Delta P_i$ which hold for any x_i, (i.e. on both sides of Π(x)) as:

$$\Gamma_X = \sum(\Pi_i + \Delta P_i) = \sum(\Delta P_i \Delta P^*_i) = n$$
$$\Gamma_X \leq \sum(\Pi_i \Pi^*_i) \quad (70)$$

where $i=1,\ldots,n$ and ΔP_i is the corresponding total probability change in the x range where Π_i’s are non zero. Complementary values are indicated with “*”. Theorems 3 and 4 produce similar result, with slight modification:

$$\Gamma_X = \sum(\Pi_i \Pi^*_i) + \sum(\Delta P_i \Delta P^*_i) = n \quad (71)$$

where i and j can go from 1 to n/2 or some other ratio. We see how Consistency Principle as defined above is a reflection of our Uncertainty Balance Principle, and besides an intuitiveness it has a definite numerical meaning as well. For example we can agree that $\Gamma_X = 10$ is better consistency than $\Gamma_X = 5$, if 10 and 5 are number of arguments x_i for which we have the agreement (or knowledge) that $\Pi_i = \Delta P_i$. This can be used in decision making situations when we need to combine soft (fuzzy) with hard (random) data, starting from either one. The assumption is, as stated earlier, that the uncertain variable X is both possibilistic as well as probabilistic. See also Table 4 bellow. We will elaborate on various applications of Uncertainty Balance Principle in our currently going research work on soft-hard or hard-soft data fusion.

4.7. Note On Soft-Hard Data Fusion

One of the practical motivations for this work, is to have a methodology to transform fuzzy data to random, and vice versa, so we can apply unique approach and available tools to both. In the first case it is probabilistic methodology, once fuzzy data are described in terms of certain variable probabilities. At the same time we can reverse the process, and given probabilistic description of some phenomenon, where there is a natural variability of probabilities, we can transform random to fuzzy and use all the fuzzy tools available. In either case our approach can enhance decision making process where both soft and hard data are present, and both are to be used to make some decision. Section 3.6 points to one way to judge level of alignment of fuzzy and random data, in the situations where both are generated for a phenomenon which can be treated both as a fuzzy and as random. This phenomenon may come from a system of sensors or soft valuations such as coming from a human operator [21]. Table 4 has an intuitive summary of various equivalent descriptions and attributes found in literature on soft (human generated) and hard (sensor or machine) generated data. Other views on what is hard and soft and when to apply fuzzy vs. random are possible as well [41]-[44].

Table 4. Intuitive Soft and Hard Data Designations

Soft Data ↔ Human Operator	Hard Data ↔ Sensor (Sensors - Machine)
Subjective	Objective
Expresses Valuation	Expresses Measure
Qualitative	Quantitative
Possibilistic Methodology	Probabilistic Methodology
Fuzzy Models	Random Models
Distribution $\Pi_X(x) = \mu_X(x)$	Distribution $F_X(x)$

4.8. Other Fuzzy Distributions

Figures 14 show other types of fuzzy distributions which can be handled by our approach. The first one is a trapezoidal distribution which can be decomposed using a pair of CDFs. The second one is a bimodal and a combination of two distributions put together (gray area can belong to either). It can be decomposed by using two pairs of CDFs. The next one is a convex distribution with the maximum at "b". It can be decomposed by a pair of CDFs, with the break at "b". The last one is a concave distribution. First two fuzzy distributions consist of uniform random distributions, and the last two are not uniform. Any combination of the above distributions is possible too. Uncertainty Balance Principle and Uncertainty Change Law hold in any case, for uniform or non uniform distributions. In Section 6 we show four numerical examples, two uniform, two non uniform fuzzy distributions. Note that in every case the continuity conditions for CDFs are observed when Π(x) is expressed in terms of Theorem 1.

5. Numerical Examples

5.1. Examples of Π(x) Distributions

In this section we consider four numerical examples which illustrate the main results of the paper. Figure 2 and Equation (11) give a simple TFN decomposition with the CDFs as:

$$\begin{aligned} F_1(x) &= 0, \quad x < a \\ &= (x-a)/(b-a), \quad a \le x < b \\ &= 1, \quad b \le x \\ F_2(x) &= 0, \quad x < b \\ &= (x-b)/(c-b), \quad b \le x < c \\ &= 1, \quad c \le x \end{aligned} \tag{72}$$

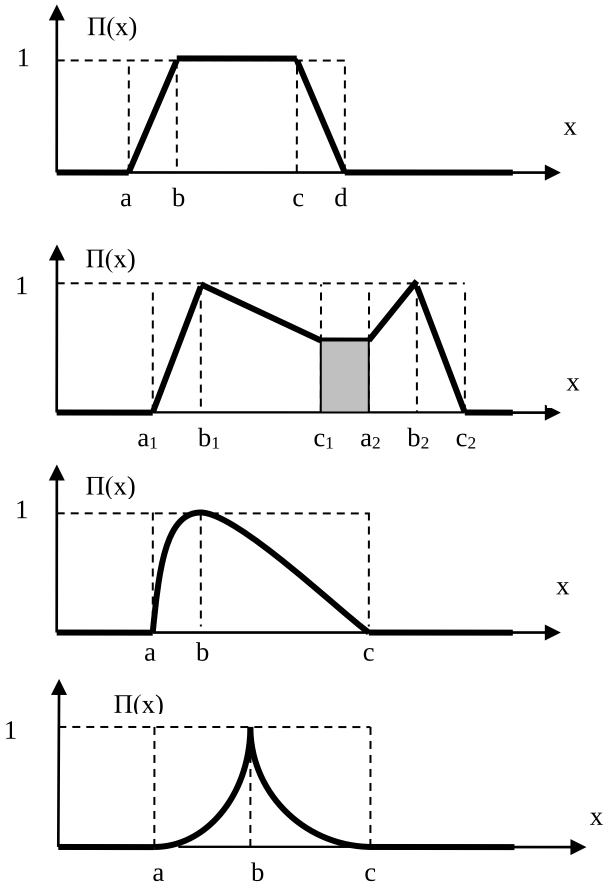


Figure 14. Various fuzzy distributions

which is used in Examples 1 and 2 bellow. Last two distributions in Figures 14 are used for Examples 3 and 4. Recall that F_1 and F_2 are equal to various probabilities as described in Methods 1, 2 and 3.

5.2. Method 1

Example 1 The symmetric TFN triplet {a, b, c} in Figure 2 is {2,3,4}. Table 5 has the values for x and the corresponding fuzzy "presumption" Π(x) levels. The gray areas show Π(x) and $\Delta P(A_2)$ and $\Delta P(A_1)$. For $x \le b$, the probability $P(A_1)$ is fixed for a fixed x. On the other hand, $P(A_2)$ resides in $\Delta P(A_2)$. We observe that for small Π(x) values (low fuzzy "presumption") the corresponding range of $P(A_2)$ is wider (more uncertainty), and for bigger values of Π(x) (high fuzzy "presumption" level), range of $P(A_2)$ is narrower (less uncertainty). This makes intuitive sense. We have the same situation for $b \leq x$, except that the non unique probability is now $P(A_1)$ residing in $\Delta P(A_1)$.

Example 2. We change the triplet {a,b,c} in Figure 5 to {10,15,30}, a non symmetric TFN with a larger spread of x. Table 6 shows numerical values. The same comments apply as in Example 1. Note that $\Pi(x)$ values and probability ranges $\Delta P(A)$ are as in Example 1 (confirming x-and fuzzy presumption invariance).

Example 3. For Example 3 we choose a fuzzy distribution described by a half circle with {a,b,c} = {1,2,3} where "b" is at the circle center, with radius 1, and outside of {1,2,3} distribution is 0. For $1 \leq x < 3$:

$$\Pi(x) = \sqrt{[1 - (x - 2)^2]} \qquad (73)$$

Table 6 has the results. Note that the $\Pi(x)$ values are not uniformly distributed. The distribution changes the fastest right from x=1 and left from x=3, as in Table 7. Still linear Uncertainty Balance Principle holds.

Example 4. This example is two quarter circles of radius 1, centered at (1,1) and (3,1):

$$\Pi(x) = 1 - \sqrt{[1 - (x - 1)^2]}, \quad 1 \leq x < 2$$
$$= 1 - \sqrt{[1 - (x - 3)^2]}, \quad 2 \leq x < 3 \qquad (74)$$

$\Pi(x)$ values are not uniformly distributed. The distribution changes faster near x=2, on both sides. For all examples Figures 12, 13 and 14 as illustrated in Tables 5, 6, 7 and 8, confirm linear relationships of any $\Pi(x)$ (or $\Pi^*(x)$) with total probability change $\Delta P(A_i,A_j)$, as well as linear relationship of $d\Pi(x)/dx$ and $d[\Delta P(A_1,A_2)]/dx$.

Table 5. Example 1

x	F_1	F_2	Π	P_{1M}	P_{1m}	ΔP_1	P_{2m}	P_{2M}	ΔP_2
1.5	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	0	1	1
2.2	0.2	0	0.2	0.2	0.2	0	0	0.8	0.8
2.4	0.4	0	0.4	0.4	0.4	0	0	0.6	0.6
2.6	0.6	0	0.6	0.6	0.6	0	0	0.4	0.4
2.8	0.8	0	0.8	0.8	0.8	0	0	0.2	0.2
3	1	0	1	1	1	0	0	0	0
3.1	1	0.1	0.9	1	0.9	0.1	0.1	0.1	0
3.3	1	0.3	0.7	1	0.7	0.3	0.3	0.3	0
3.5	1	0.5	0.5	1	0.5	0.5	0.5	0.5	0
3.7	1	0.7	0.3	1	0.3	0.7	0.7	0.7	0
4	1	1	0	1	0	1	1	1	0
4.5	1	1	0	1	0	1	1	1	0

Table 6. Example 2

x	F_1	F_2	Π	P_{1M}	P_{1m}	ΔP_1	P_{2m}	P_{2M}	ΔP_2
8	0	0	0	0	0	0	0	1	1
10	0	0	0	0	0	0	0	1	1
11	0.2	0	0.2	0.2	0.2	0	0	0.8	0.8
12	0.4	0	0.4	0.4	0.4	0	0	0.6	0.6
13	0.6	0	0.6	0.6	0.6	0	0	0.4	0.4
14	0.8	0	0.8	0.8	0.8	0	0	0.2	0.2
15	1	0	1	1	1	0	0	0	0
18	1	0.2	0.8	1	0.8	0.2	0.2	0.2	0
21	1	0.4	0.6	1	0.6	0.4	0.4	0.4	0
27	1	0.8	0.2	1	0.2	0.8	0.8	0.8	0
30	1	1	0	1	0	1	1	1	0
35	1	1	0	1	0	1	1	1	0

Table 7. Example 3

x	F_1	F_2	Π	P_{1M}	P_{1m}	ΔP_1	P_{2m}	P_{2M}	ΔP_2
.5	0	0	0	0	0	0	0	1	1
1	0	0	0	0	0	0	0	1	1
1.2	0.6	0	0.6	0.6	0.6	0	0	0.4	0.4
1.4	0.8	0	0.8	0.8	0.8	0	0	0.2	0.2
1.6	0.9	0	0.9	0.9	0.9	0	0	0.1	0.1
1.8	.98	0	.98	.98	.98	0	0	0.02	0.02
2	1	0	1	1	1	0	0	0	0
2.2	1	0.6	.98	1	0.98	0.02	0.6	0.6	0
2.4	1	0.8	0.9	1	0.9	0.1	0.8	0.8	0
2.8	1	.98	0.6	1	0.4	0.4	.98	.98	0
3	1	1	0	1	0	1	1	1	0
3.5	1	1	0	1	0	1	1	1	0

Table 8. Example 4

x	F_1	F_2	Π	P_{1M}	P_{1m}	ΔP_1	P_{2m}	P_{2M}	ΔP_2
.5	0	0	0	0	0	0	0	1	1
1	0	0	0	0	0	0	0	1	1
1.2	.02	0	.02	.02	.02	0	0	.98	.98
1.4	.08	0	.08	.08	.08	0	0	.92	.92
1.6	.2	0	.2	.2	.2	0	0	.8	.8
1.8	.4	0	.4	.4	.4	0	0	.6	.6
2	1	0	1	1	1	0	0	0	0
2.2	1	.02	.98	1	.98	.02	.02	.02	0
2.4	1	.08	.92	1	.92	.08	.08	.08	0
2.6	1	.2	.8	1	.8	.2	.2	.2	0
2.8	1	.4	.6	1	.6	.4	.4	.4	0
3	1	1	0	1	0	1	1	1	0
3.5	1	1	0	1	0	1	1	1	0

5.3. Method 2

We use Example 1 from Method 1 again. Now we have more complexity due to more variable probabilities, $P(A_1)$ and $P(A_2)$ but also $P(A_1 \cap A_2)$, which is not always zero. This gives more options to form fuzzy distribution $\Pi(x)$. Conditions (19), (20) and (21) are used to determine values of various probabilities, as summarized in Tables 9.1 and 9.2 bellow. For simplicity we only included minimum number of values of fuzzy distribution $\Pi(x)$, due to many different combinations of individual probabilities $P(A_1)$, $P(A_2)$, and $P(A_1 \cap A_2)$. Table 9.1 corresponds to $x \leq b$, rising part of $\Pi(x)$ and $F_1(x)$, and Table 9.2 corresponds to $b \leq x$ and the falling part of $\Pi(x)$ and $F_2(x)$. Table 9.1 confirms Theorem 3 and Corollary 3. Of many options in Tables 9 we can simplify things by choosing, for example, various probabilities to be 1 or 0 at the critical points [a,b,c] (**boldfaced**). Also, between the critical points we can reduce number of options. Double lines indicate breaks in Tables 9 where probability values in between are obvious. Examples 2, 3 and 4, are not repeated for simplicity.

5.4. Method 3

For simplicity we will not repeat all the details for Method 3. The key is for the probabilities to follow conditions (23) and (25), given in Theorem 4 proof. Other comments given for Method 3 apply for Method 2 as well. As in previous two Methods, we can see a variety of probability choices which generate the same presumption level $\Pi(x)$. Depending on the specific application we can choose specific ΔP_i's.

Table 9.1 Example 1 (F_1, P_{1m}, P_{2m} and P_{12m} are zero)

x	Π	Π*	F_1	P_{1M}	ΔP_1	P_{2M}	ΔP_2	P_{12M}	ΔP_{12}
0.5	0	1	0	0	0	0	0	0	0
1	**0**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
1.2	0.2	0.8	0.2	0	0	0.2	0.2	0	0
				0.1	0.1	0.1	0.1	0	0
				0.2	0.2	0	0	0	0
				0.2	0.2	0.1	0.1	0.1	0.1
				0.2	0.2	0.2	0.2	0.2	0.2
1.5	0.5	0.5	0.5	0.0	0.0	0.5	0.5	0	0
				0.1	0.1	0.4	0.4	0	0
				0.2	0.2	0.3	0.3	0	0
				0.4	0.4	0.1	0.1	0	0
				0.5	0.5	0.0	0.0	0	0
				0.5	0.5	0.1	0.1	0.1	0.1
				0.5	0.5	0.5	0.5	0.5	0.5
2	**1**	**0**	**1**	**0.0**	**0.0**	**1.0**	**1.0**	**0**	**0**
				0.1	0.1	0.9	0.9	0	0
				0.2	0.2	0.8	0.8	0	0
				0.3	0.3	0.7	0.7	0	0
				0.8	0.8	0.2	0.2	0	0
				0.9	0.9	0.1	0.1	0	0
				1.0	**1.0**	**0.0**	**0.0**	**0**	**0**
				1.0	1.0	0.1	0.1	0.1	0.1
				1.0	1.0	0.9	0.9	0.9	0.9
				1.0	**1.0**	**1.0**	**1.0**	**1.0**	**1.0**

Table 9.2 Example 1 (F_2, P_{3m}, P_{4m} and P_{34m} are zero)

x	Π	Π*	F_2	P_{3M}	ΔP_3	P_{4M}	ΔP_4	P_{34M}	ΔP_{34}
2.1	0.9	0.1	0.1	0.1	0.1	0	0	0	0
				0.1	0.1	0.1	0.1	0.1	0.1
2.2	0.8	0.2	0.2	0	0	0.2	0.2	0	0
				0.1	0.1	0.1	0.1	0	0
				0.2	0.2	0.1	0.1	0.1	0.1
				0.2	0.2	0.2	0.2	0.2	0.2
2.5	0.5	0.5	0.5	0	0	0.5	0.5	0	0
				0.1	0.1	0.4	0.4	0	0
				0.1	0.1	0.5	0.5	0.1	0.1
				0.2	0.2	0.3	0.3	0	0
				0.2	0.2	0.5	0.5	0.2	0.2
				0.3	0.3	0.2	0.2	0	0
				0.3	0.3	0.3	0.3	0.1	0.1
				0.3	0.3	0.5	0.5	0.3	0.3
				0.4	0.4	0.1	0.1	0	0
				0.4	0.4	0.1	0.1	0.1	0.1
				0.4	0.4	0.4	0.4	0.4	0.4
				0.5	0.5	0	0	0	0
				0.5	0.5	0.0	0.0	0.1	0.1
				0.5	0.5	0.5	0.5	0.5	0.5
3.0	**0**	**1**	**1**	**0**	**0**	**1.0**	**1.0**	**0**	**0**
				0.1	0.1	0.9	0.9	0	0
				0.1	0.1	1.0	1.0	0.1	0.1
				0.2	0.2	0.8	0.8	0	0
				0.2	0.2	0.9	0.9	0.1	0.1
				0.2	0.2	1.0	1.0	0.2	0.2
				0.3	0.3	0.7	0.7	0	0
				0.4	0.4	0.9	0.9	0.3	0.3
				0.4	0.4	1.0	1.0	0.4	0.4
				0.5	0.5	0.5	0.5	0	0
				0.8	0.8	0.2	0.2	0	0
				0.9	0.9	0.1	0.1	0	0
				1.0	**1.0**	**0**	**0**	**0**	**0**
				1.0	1.0	0.1	0.1	0.1	0.1
				1.0	1.0	0.9	0.9	0.9	0.9
				1.0	**1.0**	**1.0**	**1.0**	**1.0**	**1.0**
3.5	0	1	1	1.0	1.0	1.0	1.0	1.0	1.0

5.5. Practical Example of Π(x)

Finally, we illustrate our methodology by a specific TFN fuzzy distribution describing people productivity, and other similar fuzzy applications, [23]-[29]. The triplet {a,b,c} is {10,45,90} expressed in years. Assumption is that an average person is the most productive around age of 45 which corresponds to $\Pi(x) = 1$. On the opposite end, it is assumed person has zero productivity at age of 10 and 95, hence $\Pi(x) = 0$. Obviously this is just an approximation but it serves our purposes here. Using (71) we obtain Table 10 where the second row will be described shortly.

Table 10. People Productivity Example, Method 2

x, Age	Π	Π*	ΔP_1	ΔP_2	ΔP_{12}	ΔP_3	ΔP_4	ΔP_{34}
			Educ	Empl	Edu. x Empl	Healt	Marri	Healt x Marri
10	0	1	1	1	1	0	0	0
27.5	0.5	0.5	0.5	0.5	0.5	0	0	0
36.25	0.75	0.25	0.25	0.25	0.25	0	0	0
45	1	0	0	0	0	1	1	1
57.5	0.75	0.25	0.25	0.25	0.25	0.75	0.75	0.75
70	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
95	0	1	0	0	0	0	0.5	0

If we use Method 2, the following relations hold:

For $x \leq 45$: $\Pi(x) + \Delta P(A_1) + \Delta P(A_2) - \Delta P(A_1 \cap A_2) = 1$

For $45 \leq x$: $\Pi^*(x) + \Delta P(A_1) + \Delta P(A_2) - \Delta P(A_1 \cap A_2) = 1$

So how do we interpret the results ? It is assumed the variable X is both possibilistic (fuzzy) and probabilistic (random). Data produced by different sources related to peoples productivity may be soft (subjective, expert opinions) or hard (statistics, random analysis, objective). Hence we can assume that the fuzzy presumption level Π(x) is generated by an interplay of $P(A_i)$ and $P(A_i \cap A_{i+1})$ per of Corollary 1.2, i.e.:

For $x \leq 45$: $\Pi(x) = P(A_1) - P(A_1 \cap A_2)$

For $45 \leq x$: $\Pi(x) = P(A_3) - P(A_3 \cap A_4)$

From Theorem 3 we know that all of the probabilities vary for a given presumption level Π(x) hence producing "fuzziness" of uncertain variable X. Table 10 indicates that as well. Next we can assume that the random (probabilistic) events A_1 through A_2 are related to level of people's productivity. For example, given the age x and presumption level Π(x) we can assume there are other factors playing the role in productivity for the given age. One possibility may be as given in Table 11. There are other posibilities as well, per specific interest. Second row in Table 10 indicates choices from Table 11.

Table 11. Random event categories

	A_1 ΔP_1	A_2 ΔP_2	A_{12} ΔP_{12}	A_3 ΔP_3	A_4 ΔP_4	A_{34} ΔP_{34}
Education	X					
Employment		X				
Educ. vs. Employm.			X			
Health				X		
Marriage					X	
Health vs. Marriage						X

All of the A_i's and their probabilities ΔP's can be described by some CDF, uniform or non uniform, obtained analytically or statistically. Once A_i are defined we can interpret variable probabilities accordingly. For example for x=36.25 years in Table 11, presumption level is $\Pi(x) = 0.75$. Corresponding ΔP's are all 0.25 on rising part of $\Pi(x)$ and that can be interpreted as variability of education, employment and their intersection for that age. On the other hand for the falling part of $\Pi(x)$, when $\Pi(x)=0.75$ we note larger variability of 0.75 of ΔP's, and this refers to health, marriage and their intersection. Methods 1 and 3 could have been used as well. Which Method do we use may depend on a specific fuzzy distribution and application, and more research needs to be done in this subject, looking into specific applications. Note also that Big Data methodology or various statistical methods can be used to have specific choices for the probabilities in Table 11, which exhibit strongest correlations with the original uncertain soft data.

Per Corollary 5, one can reverse the problem. Namely, instead of starting with the possibilistic description and produce (or interpret it via) a probabilistic one, we can start with the probabilistic and produce possibilistic description using essentially the same methodology described in this paper, just in a reverse order. This may be advantageous in some specific cases where we have the fuzzy tools or other fuzzy applications available, but the initial data is random and probabilistic in nature.

6. Uncertainty Alignment Algorithms

In this Section we present pseudo codes for the algorithms for generating various probability values in Methods 1, 2 and 3, once fuzzy distribution $\Pi(x)$ is specified. We assume in each case that the uncertain variable X is both possibilistic and probabilistic, hence Uncertainty Balance and Consistency Principles apply. We do not elaborate on how to generate various probabilities, which is specific for an application at hand, such as illustrated in Section 5.5

Method 1 (Section 3.2)

START Method 1
- For Rising Part: $(x \le b)$
  - Given $\Pi(x)$, $0 \le \Pi(x) \le 1$
  - $\Pi(x) = F_1(x) – F_2(x) = P(A_1) - P(A_1 \cap A_2)$
    - $P(A_1 \cap A_2) = 0$
  - Choose Random Event A_1
    - Set $P(A_1) = \Pi(x)$
  - Choose Random Event A_2
    - $P(A_2)$, $P(A_1 \cap A_2) \le P(A_2) \le 1 - \Pi(x)$
- End Rising Part
- For Falling Part: $(b \le x)$
  - Given $\Pi(x)$, $0 \le \Pi(x) \le 1$
  - $\Pi(x) = F_1(x) – F_2(x) = P(A_1 \cup A_2) - P(A_2)$
    - $P(A_1 \cup A_2) = 1$
  - Choose Random Event A_2
    - $P(A_2) = 1 - \Pi(x)$
  - Choose Random Event A_1
    - $P(A_1) \rightarrow \Pi(x) \le P(A_1) \le P(A_1 \cup A_2)$
- End Falling Part

END Method 1

Method 2 (Section 3.3)

START Method 2
- For Rising Part: $(x \le b)$
  - Given $\Pi(x)$, $0 \le \Pi(x) \le 1$
  - $\Pi(x) = F_1(x) = P(A_1) - P(A_1 \cap A_2)$
    - $P(A_1 \cap A_2) \neq 0$
  - Choose Random Event A_1
    - $P(A_1) \rightarrow \Pi(x) \le P(A_1) \le 1$
  - Choose Random Event A_2
    - $P(A_2) \rightarrow 0 \le P(A_2) \le 1 - \Pi(x)$
    - $P(A_1 \cap A_2) \rightarrow 0 \le P(A_1 \cap A_2) \le 1 - \Pi(x)$
- End Rising Part
- For Falling Part: $(b \le x)$
  - Given $\Pi(x)$, $0 \le \Pi(x) \le 1$
  - $\Pi(x) = 1 – F_2(x) = P(A_3) - P(A_3 \cap A_4)$
    - $P(A_3 \cap A_4) \neq 0$
  - Choose Random Event A_3
    - $P(A_3) \rightarrow \Pi(x) \le P(A_3) \le 1$
  - Choose Random Event A_4
    - $P(A_4) \rightarrow 0 \le P(A_4) \le 1 - \Pi(x)$
    - $P(A_3 \cap A_4) \rightarrow 0 \le P(A_3 \cap A_4) \le 1 - \Pi(x)$
- End Falling Part

END Method 2

Method 3 (Section 3.4)

START Method 3
- For Rising Part: $(x \le b)$
  - Given $\Pi(x)$, $0 \le \Pi(x) \le 1$
  - $\Pi(x) = F_1(x) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$
    - $P(A_1 \cap A_2) \neq 0$
  - Choose Random Event A_1
    - $P(A_1) \rightarrow 0 \le P(A_1) \le \Pi(x)$
  - Choose Random Event A_2
    - $P(A_2) \rightarrow 0 \le P(A_2) \le \Pi(x)$
    - $P(A_1 \cap A_2) \rightarrow 0 \le P(A_1 \cap A_2) \le \Pi(x)$
- End Rising Part
- For Falling Part: $(b \le x)$
  - Given $\Pi(x)$, $0 \le \Pi(x) \le 1$
  - $\Pi(x) = 1 - F_1(x) = P(A_3) + P(A_4) - P(A_3 \cap A_4)$
    - $P(A_3 \cap A_4) \neq 0$
  - Choose Random Event A_3
    - $P(A_3) \rightarrow 0 \le P(A_3) \le \Pi(x)$
  - Choose Random Event A_4
    - $P(A_4) \rightarrow 0 \le P(A_4) \le \Pi(x)$
    - $P(A_3 \cap A_4) \rightarrow 0 \le P(A_3 \cap A_4) \le \Pi(x)$
- End Falling Part

END Method 3

7. Conclusion

In this paper we define new fuzzy to random uncertainty alignment methodology, in which fuzziness can be described as precisely defined non unique randomness. We employ the most basic properties of random and fuzzy distributions for this result, starting from fuzzy distributions decomposed as a combination of probabilistic cumulative distribution functions, CDFs, rather than probabilistic density functions, PDF's, which may not always exist. We also give precise both upper and lower bounds of changes in random distributions, required to produce data fuzziness. The range of randomness of the corresponding probabilistic events is a function of fuzzy distribution presumption levels and it holds for any fuzzy distribution. The main results is a universal fuzzy-random (possibilistic-probabilistic) uncertainty alignment law which we named Uncertainty Balance Principle, for its simple statement $\Pi(x) + \sum\Delta P(A_i,A_j) = 1$, which is a linear law, fuzzy presumption-invariant and fuzzy argument x invariant for any fuzzy distribution. Another byproduct of this Principle is also a linear law, Uncertainty Change Law which relates changes in fuzzy distribution against corresponding changes in probabilities. Our results hold for any fuzzy distributions, triangular, trapezoidal, convex, non convex, symmetric or not, normalized or not, uni modal or multi modal alike. This universal range of applicability comes as a result of employing CDF rather than PDF. The results of this paper can be employed effectively in a variety of data fusion and decision problems where both objective (hard, random, probabilistic, sensor based) data are to be fused with subjective (soft, fuzzy, possibilistic, human based) data [9], [10], [21]. They can be also used to generate random from fuzzy data for other applications. Additional feature of our approach is a reverse applicability of the results, i.e. going from random to fuzzy. If the range of probabilities is given we can form corresponding fuzzy distribution satisfying Uncertainty Balance principle. Another way to interpret the results is as precise mathematical description of Consistency Principle first introduced by Zadeh in his classic "possibility" paper [2], as a loose and intuitive notion connecting fuzzy and random data. As defined in [2] this Principle relates fuzzy and random distributions in an intuitive way. In our paper, a precise mathematical definition is given for the modified Consistency Principle in terms of Uncertainty Balance Principle of this paper, as a measure of agreement between fuzzy and random data distributions. This can be effectively used to measure level of agreement between related fuzzy and random data in decision making process. In our future work we will extend the results in decision making areas such as machine-human data fusion where using both types of data is crucial for the fusion usefulness. Also, further properties of both random and fuzzy data will be analyzed in the light of the paper's main results. In particular we will consider relationship between probabilistic and possibilistic axioms in the light of this paper results, as well as special cases such as random events independence or dependence.

8. References

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