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New families of q and $(q;p)-$Hermite polynomials

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Abstract. In this paper, we construct a new family of $q-$Hermite polynomials denoted by $H_n(x,s|q)$. Main properties and relations are established and proved. In addition, is deduced a sequence of novel polynomials, $\mathcal{L}_n(\cdot,\cdot|q)$, which appear to be connected with well known $(q,n)-$exponential functions $E_{q,n}(\cdot)$ introduced by Ernst in his work entitled: *A New Method for $q-$calculus,* (Uppsala Dissertations in Mathematics, Vol. **25**, 2002). Relevant results spread in the literature are retrieved as particular cases. Fourier integral transforms are explicitly computed and discussed. A $(q;p)-$extension of the $H_n(x,s|q)$ is also provided.

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§1. Introduction

The classical orthogonal polynomials and the quantum orthogonal polynomials, also called $q-$orthogonal polynomials, constitute an interesting set of special functions. Each family of these polynomials occupies different levels within the so-called Askey-Wilson scheme (Askey and Wilson, 1985; Koekoek and Swarttouw, 1998; Lesky, 2005; Koekoek et *al*, 2010). In this scheme, the Hermite polynomials $H_n(x)$ are the ground level and are characterized by a set of properties: (i) they are solutions of a hypergeometric second order differential equation, (ii) they are generated by a recursion relation, (iii) they are orthogonal with respect to a weight function and (iv) they obey the Rodrigues-type formula. Therefore, there are many ways to construct the Hermite polynomials. However, they are more commonly deduced from their generating function, i.e.,

$$\sum_{n=0}^{\infty} \frac{\mathbf{H}_n(x)}{n!} t^n = e^{2xt-t^2}, \tag{1.1}$$

giving rise to the so-called *physicists* Hermite polynomials [5]. Another family of Hermite polynomials, called the *probabilists* Hermite polynomials, is defined as [5]

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = e^{xt-\frac{t^2}{2}}. \tag{1.2}$$

The Hermite polynomials are at the bottom of a large class of hypergeometric polynomials to which most of their properties can be generalized [6], [11]-[16]. In [5], Cigler introduced another family of Hermite polynomials $H_n(x, s)$ generalizing the *physicists* and *probabilists* Hermite polynomials as

$$\sum_{n=0}^{\infty} \frac{H_n(x,s)}{n!} t^n = e^{xt-s\frac{t^2}{2}} \tag{1.3}$$

with $H_n(x, 1) = H_n(x)$ and $H_n(2x, 2) = \mathbf{H}_n(x)$.

In this work, we deal with a construction of two new families of q and $(q; p)-$Hermite polynomials.

The paper is organized as follows. In Section 2, we give a quick overview on the Hermite polynomials $H_n(x, s)$ introduced in [5]. Section 3 is devoted to the construction of a new family of $q-$Hermite polynomials $H_n(x, s|q)$ generalizing the discrete $q-$Hermite polynomials. The inversion formula and relevant properties of these polynomials are computed and discussed. Their Fourier integral transforms are performed in the Section 4. Doubly indexed Hermite polynomials and some concluding remarks are introduced in Section 5.

§2. On the Hermite polynomials $H_n(x, s)$

In [5], Cigler showed that the Hermite polynomials $H_n(x, s)$ satisfy

$$DH_n(x, s) = n\, H_{n-1}(x, s) \tag{2.1}$$

and the three term recursion relation

$$H_{n+1}(x, s) = x\, H_n(x, s) - s\, n\, H_{n-1}(x, s), \quad n \geq 1 \tag{2.2}$$

with $H_0(x, s) := 1$. $D := d/dx$ is the usual differential operator. Immediatly, one can see that

$$H_{2n}(0, s) = (-s)^n \prod_{k=1}^{n} (2k-1), \quad H_{2n+1}(0, s) = 0. \tag{2.3}$$

The computation of the first fourth polynomials gives:

$$\begin{aligned} H_1(x,s) &= x, \\ H_2(x,s) &= x^2 - s, \\ H_3(x,s) &= x^3 - 3\,s\,x, \\ H_4(x,s) &= x^4 - 6\,s\,x^2 + 3\,s^2. \end{aligned}$$

More generally, the explicit formula of $H_n(x,s)$ is written as [5]

$$H_n(x,s) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k\, s^k}{(2k)!!} \frac{x^{n-2k}}{(n-2k)!} = x^n\, {}_2F_0\left(\begin{matrix} -\frac{n}{2}, \frac{1-n}{2} \\ - \end{matrix} \;\Big|\; -\frac{2s}{x^2} \right), \tag{2.4}$$

where $\binom{n}{k} = n!/k!(n-k)!$ is a binomial coefficient, $n! := n(n-1)\cdots 2\cdot 1$, $(2n)!! := 2n(2n-2)\cdots 2$.

The symbol $\lfloor x \rfloor$ denotes the greatest integer in x and ${}_2F_0$ is called the hypergeometric series [2]. From (2.1) and (2.2), we have

$$H_n(x,s) = (x - sD)\, H_{n-1}(x,s), \tag{2.5}$$

where the operator $x - sD$ can be expressed as [5]

$$x - sD = e^{\frac{x^2}{2s}}\, (-sD)\, e^{-\frac{x^2}{2s}}. \tag{2.6}$$

The Rodrigues formula takes the form

$$e^{-\frac{x^2}{2s}}\, H_n(x,s) = (-sD)^n\, e^{-\frac{x^2}{2s}} \tag{2.7}$$

while the second order differential equation satisfied by $H_n(x,s)$ is

$$\big(sD^2 - xD + n\big)\, H_n(x,s) = 0. \tag{2.8}$$

Furthermore, from the relation (2.4) we derive the result

$$H_n(x + sD, s)\cdot(1) = x^n, \tag{2.9}$$

and the inverse formula for $H_n(x,s)$

$$x^n = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{s^k}{(2k)!!} \frac{H_{n-2k}(x,s)}{(n-2k)!}. \tag{2.10}$$

We then obtain

$$\sum_{k,\,n\,(even)} \frac{1}{(n-k)!\,k!} = \sum_{k,\,n\,(odd)} \frac{1}{(n-k)!\,k!}, \quad 0 \le k \le n, \quad n \ge 0. \tag{2.11}$$

From (2.4), it is also straighforward to note that the polynomials $H_n(x,s)$ have an alternative expression given by

$$H_n(x,s) = \exp\left(-s\frac{D^2}{2}\right)\cdot(x^n). \tag{2.12}$$

For any integer $k = 0, 1, ..., \lfloor n/2 \rfloor$, we have the following result

$$D^{2k}\, H_n(x,s) = \frac{n!}{(n-2k)!} H_{n-2k}(x,s). \tag{2.13}$$

Corollary 1. *The Hermite polynomials $H_n(x,s)$ obey*

$$\mathcal{T}_n(s,D)\, H_n(x,s) = x^n, \tag{2.14}$$

where the polynomial

$$\mathcal{T}_n(\alpha,\beta) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{(2k)!!}\alpha^k \beta^{2k}. \tag{2.15}$$

We are now in a position to formulate and prove the following.

Lemma 2.

$$\mathcal{T}_{2n}(\alpha,\beta) = \frac{(\alpha\beta^2)^n}{(2n)!!}\, {}_2F_0\left(\begin{array}{c} -n, 1 \\ - \end{array} \Big| -\frac{2}{\alpha\beta^2}\right) \tag{2.16}$$

and

$$\mathcal{T}_\infty(\alpha,\beta) = e^{\frac{\alpha\beta^2}{2}}. \tag{2.17}$$

Proof. From (2.15), we have

$$\begin{aligned} \mathcal{T}_{2n}(\alpha,\beta) &= \sum_{k=0}^{n} \frac{1}{(2k)!!}(\alpha\beta^2)^k \\ &= \frac{(\alpha\beta^2)^n}{(2n)!!}\sum_{k=n}^{\infty} \frac{(2n)!!}{(2k)!!}(\alpha\beta^2)^{k-n}. \end{aligned} \tag{2.18}$$

By substituting $m = n-k$ in the latter expression and using various identities, we arrive at

$$\mathcal{T}_{2n}(\alpha,\beta) = \frac{(\alpha\beta^2)^n}{(2n)!!}\sum_{m=0}^{\infty}(-n)_m \left(\frac{-2}{\alpha\beta^2}\right)^m, \tag{2.19}$$

where $(a)_j := a(a+1)\cdots(a+j-1),\ j \geq 1$ and $(a)_0 := 1$. When n goes to ∞, the polynomial (2.15) takes the form

$$\mathcal{T}_\infty(\alpha,\beta) = \sum_{k=0}^{\infty} \frac{\alpha^k \beta^{2k}}{(2k)!!} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{\alpha\beta^2}{2}\right)^k \tag{2.20}$$

where $(2k)!! = 2^k\, k!$ is used. □

To end this section, let us investigate the Fourier transform of the function $e^{-x^2/2s}H_n(x,s)$. In [5], Cigler has proven that

$$\frac{1}{\sqrt{2\pi s}}\int_{\mathbb{R}} e^{ixy-\frac{x^2}{2s}}\,dx = e^{-s\frac{y^2}{2}}. \tag{2.21}$$

Hence,

$$\frac{1}{\sqrt{2\pi\, s}}\int_{\mathbb{R}} e^{ixy+i(n-2k)\kappa x-\frac{x^2}{2s}}\,dx = e^{-s\frac{y^2}{2}-(n-2k)s\,y\,\kappa}, \tag{2.22}$$

where $e^{-2s\kappa^2} = 1$. By differentiating the relation (2.21) $2n-2k$ times with respect to y, one obtains

$$\frac{1}{\sqrt{2\pi\, s}}\int_{\mathbb{R}} (-1)^{n-k} x^{2n-2k} e^{ixy-\frac{x^2}{2s}}\,dx = D^{2n-2k} e^{-s\frac{y^2}{2}}. \tag{2.23}$$

Evaluating the latter expression at $y=0$ and by making use of (2.7), one gets

$$\begin{aligned}\frac{(-1)^{n-k}}{\sqrt{2\pi\, s}}\int_{\mathbb{R}} x^{2n-2k} e^{-\frac{x^2}{2s}}\,dx &= D^{2n-2k} e^{-s\frac{y^2}{2}}\Big|_{y=0} \\ &= (-s)^{2n-2k} H_{2n-2k}(y,s^{-1}) e^{-s\frac{y^2}{2}}\Big|_{y=0}.\end{aligned} \tag{2.24}$$

Theorem 3. *The Fourier transform of the function $e^{-x^2/2s}H_n(x,s)$ is given by*

$$\frac{1}{\sqrt{2\pi\, s}}\int_{\mathbb{R}} H_n(a\,e^{i\kappa x},s) e^{ixy-\frac{x^2}{2s}}\,dx = H_n(a\,e^{-s\,\kappa\,y},s) e^{-s\frac{y^2}{2}} \tag{2.25}$$

where a is an arbitrary constant factor. For $y=0$, we have

$$\frac{1}{\sqrt{2\pi\, s}}\int_{\mathbb{R}} H_n(x,s) e^{-\frac{x^2}{2s}}\,dx = 0. \tag{2.26}$$

Proof. Using (2.4) and (2.22), we obtain

$$\frac{1}{\sqrt{2\pi\, s}}\int_{\mathbb{R}} H_n(a\,e^{i\kappa x},s) e^{ixy-\frac{x^2}{2\,s}}\,dx$$

$$
\begin{aligned}
&= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k n!\, s^k\, a^{n-2k}}{(n-2k)!\,(2k)!!} \frac{1}{\sqrt{2\pi\, s}} \int_{\mathbb{R}} e^{ixy+i(n-2k)\kappa x-\frac{x^2}{2s}} dx \\
&= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k\, n!\, s^k\, a^{n-2k}}{(n-2k)!\,(2k)!!} e^{-\frac{s}{2}[\kappa(n-2k)+y]^2} \\
&= e^{-s\frac{y^2}{2}} H_n(a\, e^{-s\,\kappa\, y}, s).
\end{aligned}
$$

Combining (2.4) and (2.24) for $n = 2n$, we have

$$
\begin{aligned}
&\frac{1}{\sqrt{2\pi\, s}} \int_{\mathbb{R}} H_{2n}(x,s) e^{-\frac{x^2}{2\,s}} dx \\
&= \sum_{k=0}^{n} \frac{(-1)^k\,(2n)!\, s^k}{(2n-2k)!\,(2k)!!} \frac{1}{\sqrt{2\pi\, s}} \int_{\mathbb{R}} x^{2n-2k}\, e^{ixy-\frac{x^2}{2\,s}} dx\Big|_{y=0} \\
&= (-1)^n \sum_{k=0}^{n} \frac{(2n)!\, s^k}{(2n-2k)!\,(2k)!!} D^{2n-2k} e^{-s\frac{y^2}{2}}\Big|_{y=0} \\
&= (-1)^n\, s^{2n}\, e^{-s\frac{y^2}{2}} \sum_{k=0}^{n} \frac{(2n)!\, s^{-k}}{(2n-2k)!\,(2k)!!} H_{2n-2k}(y, s^{-1})\Big|_{y=0} \\
&= s^{2n}\,(2n)! \sum_{k=0}^{n} \frac{(-1)^k}{(2n-2k)!!\,(2k)!!} \\
&= 0
\end{aligned}
$$

where (2.11) is used. □

§3. New $q-$Hermite polynomials $H_n(x, s|q)$

In this section, we construct through the $q-$chain rule a new family of $q-$Hermite polynomials denoted by $H_n(x, s|q)$. We first introduce some standard $q-$notations. For $n \geq 1,\ q \in \mathbb{C}$, we denote the $q-$deformed number [10] by

$$
\{n\}_q := \sum_{k=0}^{n-1} q^k. \tag{3.1}
$$

In the same way, we define the $q-$factorials

$$
\{n\}_q! := \prod_{k=1}^{n} \{k\}_q, \quad \{2n\}_q!! := \prod_{k=1}^{n} \{2k\}_q, \quad \{2n-1\}_q!! := \prod_{k=1}^{n} \{2k-1\}_q \tag{3.2}
$$

and, by convention,

$$
\{0\}_q! := 1 =: \{0\}_q!! \quad \text{and} \quad \{-1\}_q!! = 1. \tag{3.3}
$$

For any positive number c, the $q-$Pochhammer symbol $\{c\}_{n,q}$ is defined as follows:

$$\{c\}_{n,q} := \prod_{k=0}^{n-1}\{c+k\}_q, \tag{3.4}$$

while the $q-$binomial coefficients are defined by

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_q := \frac{\{n\}_q!}{\{n-k\}_q!\{k\}_q!} = \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k}, \quad \text{for} \quad 0 \le k \le n, \tag{3.5}$$

and zero otherwise, where $(a;q)_n := \prod_{k=0}^{n-1}(1-a\,q^k),\ (a;q)_0 := 1$.

Definition 4. *[7, 8] The Hahn $q-$addition $\oplus_q$ is the function: $\mathbb{C}^3 \to \mathbb{C}^2$ given by:*

$$(x,y,q) \mapsto (x,y) \equiv x \oplus_q y, \tag{3.6}$$

where

$$\begin{aligned} (x \oplus_q y)^n : &= (x+y)(x+q\,y)\dots(x+q^{n-1}\,y) \\ &= \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_q q^{\binom{k}{2}}\, x^{n-k}\, y^k, \quad n \ge 1, \quad (x \oplus_q y)^0 := 1, \end{aligned} \tag{3.7}$$

while the $q-$subtraction $\ominus_q$ is defined as follows:

$$x \ominus_q y := x \oplus_q (-y). \tag{3.8}$$

Consider a function F

$$F : D_R \longrightarrow \mathbb{C}, \quad z \longmapsto \sum_{n=0}^{\infty} c_n\, z^n, \tag{3.9}$$

where D_R is a disc of radius R. We define $F(x \oplus_q y)$ to mean the formal series

$$\sum_{n=0}^{\infty} c_n (x \oplus_q y)^n \equiv \sum_{n=0}^{\infty}\sum_{k=0}^{n} c_n \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_q q^{\binom{k}{2}}\, x^{n-k}\, y^k. \tag{3.10}$$

Let e_q, E_q, $\cos_q$ and $\sin_q$ be the fonctions defined as follows:

$$\begin{aligned} e_q(x) : &= \sum_{n=0}^{\infty} \frac{1}{\{n\}_q!} x^n \\ E_q(x) : &= \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{\{n\}_q!} x^n \end{aligned} \tag{3.11}$$

$$\cos_q(x) := \frac{e_q(i\,x)+e_q(-i\,x)}{2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{\{2n\}_q!}x^{2n}, \tag{3.12}$$

$$\sin_q(x) := \frac{e_q(i\,x)-e_q(-i\,x)}{2\,i} = \sum_{n=0}^{\infty}\frac{(-1)^n}{\{2n+1\}_q!}x^{2n+1}. \tag{3.13}$$

We immediately obtain the following rules for the product of two exponential functions

$$e_q(x)E_q(y) = e_q(x\oplus_q y). \tag{3.14}$$

The new family of $q-$Hermite polynomials $H_n(x,s|q)$ can be determined by the generating function

$$e_q\big(tx\ominus_{q,q^2} st^2/\{2\}_q\big) = e_q(tx)E_{q^2}(-st^2/\{2\}_q) := \sum_{n=0}^{\infty}\frac{H_n(x,s|q)}{\{n\}_q!}t^n, \quad |t|<1, \tag{3.15}$$

where [8]

$$(a\ominus_{q,q^2} b)^n := \sum_{k=0}^{n}\frac{\{n\}_q!}{\{n-k\}_q!\,\{k\}_{q^2}!}(-1)^k q^{k(k-1)}a^{n-k}\,b^k,\ (a\ominus_{q,q^2} b)^0 := 1 \tag{3.16}$$

and

$$E_{q^2}(x) := \sum_{n=0}^{\infty}\frac{q^{n(n-1)}}{\{n\}_{q^2}!}x^n. \tag{3.17}$$

Performing the $q-$derivative D_x^q of both sides of (3.15) with respect to x, one obtains

$$D_x^q\,H_n(x,s|q) = \{n\}_q\,H_{n-1}(x,s|q), \tag{3.18}$$

where

$$D_x^q\,f(x) = \frac{f(x)-f(qx)}{(1-q)x} \tag{3.19}$$

satisfying

$$D_x^q(a\,x\oplus_q b)^n = a\{n\}_q\,(a\,x\oplus_q b)^{n-1}. \tag{3.20}$$

Recall [9] that the Al-Salam-Chihara polynomials $P_n(x;a,b,c)$ satisfy the following recursion relation:

$$P_{n+1}(x;a,b,c) = (x-a\,q^n)\,P_n(x;a,b,c) - (c+b\,q^{n-1})\,\{n\}_q\,P_{n-1}(x;a,b,c) \tag{3.21}$$

with $P_{-1}(x;a,b,c)=0$ and $P_0(x;a,b,c)=1$.
Performing the $q-$derivative of both sides of (3.15) with respect to t, we have

$$H_{n+1}(x,s|q)=x\,H_n(x,s|q)-s\,\{n\}_q\,q^{n-1}\,H_{n-1}(x,s|q),\quad n\geq 1 \tag{3.22}$$

with $H_0(x,s|q):=1$.
By setting $a=0=c$ and $b=s$ in (3.21), one obtains the recursion relation (3.22). From the latter equation, one can see that

$$H_{2n}(0,s|q)=(-s)^n\,q^{n(n-1)}\,\{2n-1\}_q!!,\quad H_{2n+1}(0,s|q)=0. \tag{3.23}$$

The first fourth new polynomials are given by

$$H_1(x,s|q) = x, \tag{3.24}$$
$$H_2(x,s|q) = x^2-s, \tag{3.25}$$
$$H_3(x,s|q) = x^3-\{3\}_q sx, \tag{3.26}$$
$$H_4(x,s|q) = x^4-(1+q^2)\{3\}_q sx^2+q^2\,\{3\}_q s^2. \tag{3.27}$$

More generally, we have the following.

Theorem 5. *The explicit formula for the new Hermite polynomials $H_n(x,s|q)$ is given by*

$$H_n(x,s|q) = \sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^k q^{k(k-1)}\{n\}_q!}{\{n-2k\}_q!\,\{2k\}_q!!}s^k x^{n-2k} \tag{3.28}$$
$$= x^n\,{}_2\phi_0\left(\begin{array}{c} q^{-n},q^{1-n}\\ -\end{array}\Big|\,q^2;\ \frac{sq^{2n-1}}{(1-q)x^2}\right), \tag{3.29}$$

where ${}_2\phi_0$ is the $q-$hypergeometric series [2].

Proof. Expanding the generation function given in (3.15) in Maclaurin series, we have

$$e_q(t\,x)E_{q^2}(-s\,t^2/\{2\}_q) = \sum_{k=0}^{\infty}\frac{(x\,t)^k}{\{k\}_q!}\sum_{m=0}^{\infty}\frac{(-1)^m q^{m(m-1)}}{\{m\}_{q^2}!}\left(\frac{s\,t^2}{\{2\}_q}\right)^m$$
$$= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m q^{m(m-1)}x^k}{\{k\}_q!\,\{m\}_{q^2}!}\left(\frac{s}{\{2\}_q}\right)^m t^{k+2m}. \tag{3.30}$$

By substituting

$$k+2m=n\ \Rightarrow\ m\leq\lfloor n/2\rfloor, \tag{3.31}$$

and

$$\{2\}_q\,\{m\}_{q^2}=\{2m\}_q \tag{3.32}$$

in (3.30), we have

$$e_q(t\,x)E_{q^2}(-s\,t^2/\{2\}_q) = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\lfloor n/2\rfloor}\frac{(-1)^m q^{m(m-1)} s^m\, x^{n-2m}}{\{n-2m\}_q!\,\{2m\}_q!!}\right)t^n, \tag{3.33}$$

which achieves the proof. □

In the limit case when $x \to \{2\}_q\, x,\ \ s \to (1-q)\,\{2\}_q$, the polynomials $H_n(x,s|q)$ are reduced to $H_n^q(x)$ investigated by Chung et *al* [8]. When $s \to 1-q$, they are reduced to the discrete $q-$Hermite I polynomials [2].
The relation (3.22) allows us to write

$$H_n(x,s|q) = (x - sq^N \circ D_x^q)\,H_{n-1}(x,s|q), \tag{3.34}$$

where the operator N acts on the polynomials $H_n(x,s|q)$ as follows:

$$NH_n(x,s|q) := n\,H_n(x,s|q), \quad q^N \circ D_x^q = D_x^q \circ q^{N-1}. \tag{3.35}$$

It is straightforward to show that the polynomials (3.28) satisfy the following $q-$difference equation

$$\big(s\,(D_x^q)^2 - x\,q^{2-n}\,D_x^q + q^{2-n}\,\{n\}_q\big)\,H_n(x,s|q) = 0. \tag{3.36}$$

In the limit case when q goes to 1, the $q-$difference equation (3.36) reduces to the well-known differential equation (2.8). For n even or odd, the polynomials $H_n(x,s|q)$ obey the following generating functions

$$\sum_{n=0}^{\infty}\frac{H_{2n}(x,s|q)}{\{2n\}_q!}(-t)^n = \cos_q(x\sqrt{t})\,E_{q^2}(s\,t/\{2\}_q),\ |t|<1 \tag{3.37}$$

or

$$\sum_{n=0}^{\infty}\frac{H_{2n+1}(x,s|q)}{\{2n+1\}_q!}(-t)^n = \frac{1}{\sqrt{t}}\sin_q(x\sqrt{t})\,E_{q^2}(s\,t/\{2\}_q),\ |t|<1, \tag{3.38}$$

respectively.

Theorem 6. *The polynomials $H_n(x,s|q)$ can be expressed as*

$$H_n(x,s|q) = \prod_{k=1}^{n}\big(x - s\,q^{n-1-k}\,D_x^q\big)\cdot(1) \tag{3.39}$$

and (3.34) takes the form

$$H_n\left(x + sq^N \circ D_x^q, s|q\right)\cdot(1) = x^n. \tag{3.40}$$

Proof. Since (3.18) and (3.22) are satisfied, we have

$$\begin{aligned} H_n(x,s|q) &= x\ H_{n-1}(x,s|q) - s\,q^{n-2}\,\{n-1\}_q\,H_{n-2}(x,s|q) \\ &= x\,H_{n-1}(x,s|q) - sq^{n-2}\,D_x^q\,H_{n-1}(x,s|q). \end{aligned} \tag{3.41}$$

The rest holds by induction on n.
To prove the relation (3.40) we replace x^{n-2k} in (3.28) by $(x + sq^N \circ D_x^q)^{n-2k}$ and apply the corresponding linear operator to 1. The relation (3.40) is true for $n = 0$ and $n = 1$. For $n = 2$, we have

$$\begin{aligned} H_2\left(x + sq^N \circ D_x^q, s|q\right)\cdot(1) &= \left(x + sq^N \circ D_x^q\right)^2\cdot(1) - s \\ &= \left(x + sq^N \circ D_x^q\right)\cdot(x) - s \\ &= x^2. \end{aligned} \tag{3.42}$$

Assume that (3.40) is true for $n-1,\ n \geq 3$. Then we must prove that

$$H_n\left(x + sq^N \circ D_x^q, s|q\right)\cdot(1) = x^n. \tag{3.43}$$

From (3.22), we have

$$\begin{aligned} H_n\left(x + sq^N \circ D_x^q, s|q\right)\cdot 1 &= \left(x + sq^N \circ D_x^q\right) H_{n-1}\left(x + sq^N \circ D_x^q, s|q\right)\cdot(1) \\ &- s\{n-1\}_q\,q^{n-2} H_{n-2}\left(x + sq^N \circ D_x^q, s|q\right)\cdot(1) \\ &= \left(x + sq^N \circ D_x^q\right)\cdot x^{n-1} - s\{n-1\}_q\,q^{n-2}x^{n-2} \\ &= x^n \end{aligned} \tag{3.44}$$

which achieves the proof. ☐

From the **Theorem** 6, we obtain the following.

Corollary 7. *The polynomials (3.28) have the following inversion formula*

$$x^n = \{n\}_q! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{q^{k(k-1)}\ s^k}{\{2k\}_q!!}\frac{H_{n-2k}(x,s|q)}{\{n-2k\}_q!}. \tag{3.45}$$

Proof. Let $h_n^q(x,s)$ be the polynomial defined by

$$h_n^q(x,s) = \left(x + sq^N \circ D_x^q\right)^n \cdot (1). \tag{3.46}$$

Note that $h_n^q(x,-s) = H_n(x,s|q)$. From (3.40), we have

$$\begin{aligned} x^n &= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k q^{k(k-1)}\{n\}_q!}{\{n-2k\}_q!\{2k\}_q!!} s^k \left(x + sq^N \circ D_x^q\right)^{n-2k}\cdot(1) \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{q^{k(k-1)}\{n\}_q!\,s^k}{\{n-2k\}_q!\{2k\}_q!!} h_{n-2k}^q(x,-s) \end{aligned} \tag{3.47}$$

which achives the proof. ☐

From (3.18), one readily deduces that, for integer powers $k = 0, 1, ..., \lfloor n/2 \rfloor$ of the operator D_x^q,

$$(D_x^q)^{2k} H_n(x, s|q) = \gamma_{n,k}(q) H_{n-2k}(x, s|q), \quad \gamma_{n,k}(q) = \frac{\{n\}_q!}{\{n-2k\}_q!}. \tag{3.48}$$

Therefore, we have the following decomposition of unity

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k q^{k(k-1)} s^k}{\{2k\}_q!!} (D_x^q)^{2k} \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{q^{m(m-1)} s^m}{\{2m\}_q!!} (D_x^q)^{2m} = \mathbf{1} \tag{3.49}$$

and the new $q-$Hermite polynomials $H_n(x, s|q)$ obey

$$\mathcal{L}_n(s, D_x^q|q) H_n(x, s|q) = x^n \tag{3.50}$$

where the polynomial $\mathcal{L}_n(\alpha, \beta|q)$ is defined as follows:

$$\mathcal{L}_n(\alpha, \beta|q) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{q^{k(k-1)}}{\{2k\}_q!!} \alpha^k \beta^{2k}. \tag{3.51}$$

This polynomial is essentially the $(q, n)-$exponential function $E_{q,n}(x)$ investigated by Ernst [10], i.e., $\mathcal{L}_{n-1}(\alpha, \beta|q) = E_{q^{-2}, \lfloor n/2 \rfloor}(\alpha\beta^2/\{2\}_q)$. We are now in a position to formulate and prove the following.

Lemma 8. *From the polynomial (3.51) we have*

$$\mathcal{L}_{2n}(\alpha, \beta|q) = \frac{(\alpha\beta^2)^n q^{n(n-1)}}{\{2n\}_q!!} {}_3\phi_2\left(\begin{matrix} q^{-n}, -q^{-n}, q \\ 0, 0 \end{matrix} \Big| q; -\frac{q^2}{(1-q)\alpha\beta^2} \right) \tag{3.52}$$

and

$$\mathcal{L}_\infty(\alpha, \beta|q) = E_{q^2}(\alpha\beta^2/\{2\}_q). \tag{3.53}$$

Proof. As it is defined in (3.51), we have

$$\begin{aligned} \mathcal{L}_{2n}(\alpha, \beta|q) &= \sum_{k=0}^{n} \frac{q^{k(k-1)}}{\{2k\}_q!!} (\alpha\beta^2)^k \\ &= \frac{(\alpha\beta^2)^n}{\{2n\}_q!!} \sum_{k=n}^{\infty} \frac{q^{k(k-1)} \{2n\}_q!!}{\{2k\}_q!!} (\alpha\beta^2)^{k-n}. \end{aligned} \tag{3.54}$$

By substituting $m = n - k$ in the latter expression, we arrive at

$$\mathcal{L}_{2n}(\alpha, \beta|q) = \frac{(\alpha\beta^2)^n}{\{2n\}_q!!} \sum_{m=0}^{\infty} \frac{q^{(n-m)(n-m-1)} \{2n\}_q!!}{\{2n-2m\}_q!!} (\alpha\beta^2)^{-m}$$

$$(3.55)\qquad = \frac{(\alpha\beta^2)^n q^{n(n-1)}}{\{2n\}_q!!}\sum_{m=0}^{\infty}(q^{-2n};q^2)_m\left(-\frac{q^2}{(1-q)\alpha\beta^2}\right)^m .$$

When $n\to\infty$, (3.51) takes the form

$$(3.56)\qquad \mathcal{L}_\infty(\alpha,\beta|q) = \sum_{k=0}^{\infty}\frac{q^{k(k-1)}}{\{2n\}_q!!}(\alpha\beta^2)^k = \sum_{k=0}^{\infty}\frac{q^{k(k-1)}}{\{k\}_{q^2}!}\left(\frac{\alpha\beta^2}{\{2\}_q}\right)^k$$

which achieves the proof. □

In the limit, when $q\to 1$, the polynomial $\mathcal{L}_n(\alpha,\beta|q)$ is reduced to the classical one's $\mathcal{T}_n(\alpha,\beta)$, i.e., $\lim_{q\to1}\mathcal{L}_n(\alpha,\beta|q) = \mathcal{T}_n(\alpha,\beta)$, $\forall\, n$.

§4. Fourier transforms of the new $q-$Hermite polynomials $H_n(x,s|q)$

In this section, we compute the Fourier integral transforms associated to the new $q-$Hermite polynomials $H_n(x,s|q)$.

4.1. $q^{-1}-$Hermite polynomials $H_n(x,s|q^{-1})$

Let us rewrite the new $q-$Hermite polynomials (3.28) in the following form

$$(4.1)\qquad H_n(x,s|q) = \sum_{k=0}^{\lfloor n/2\rfloor} c_{n,k}(q)\, s^k x^{n-2k},$$

where the associated coefficients $c_{n,\,k}(q)$ are given by

$$(4.2)\qquad c_{n,\,k}(q) := \frac{(-1)^k q^{k(k-1)}\,\{n\}_q!}{\{n-2k\}_q!\,\{2k\}_q!!}.$$

By a direct computation, one can easily check that these coefficients satisfy the following recursion relation

$$(4.3)\qquad c_{n+1,\,k}(q) = c_{n,\,k}(q) - q^{n-1}\{n\}_q\, c_{n-1,\,k-1}(q),$$

with $c_{0,\,k}(q)=\delta_{0,k}$, $c_{n,\,0}(q)=1$.
From the definition of the $q-$binomial coefficients in (3.5), it is not hard to derive an inversion formula

$$(4.4)\qquad \begin{Bmatrix} n\\ 2k\end{Bmatrix}_{q^{-1}} = q^{2k(2k-n)}\begin{Bmatrix} n\\ 2k\end{Bmatrix}_q, \qquad 0\le k\le\lfloor n/2\rfloor.$$

Then, one readily deduces that

$$c_{n,\,k}(q^{-1}) = q^{k(k+3-2n)} c_{n,\,k}(q), \tag{4.5}$$

allowing to define the $q^{-1}-$Hermite polynomials $H_n(x, s|q^{-1})$ in the following form

$$H_n(x, s|q^{-1}) := \sum_{k=0}^{\lfloor n/2 \rfloor} c_{n,\,k}(q^{-1})\, s^k x^{n-2k}. \tag{4.6}$$

The recursion relation

$$c_{n+1,\,k}(q^{-1}) = q^{-2k}\, c_{n,\,k}(q^{-1}) - q^{3-n-2k}\, \{n\}_q\, c_{n-1,\,k-1}(q^{-1}), \quad n \geq 1 \tag{4.7}$$

is valid for the coefficients (4.5) with $c_{0,\,k}(q^{-1}) = q^{k(k+3)}\, \delta_{0,k}$, $c_{n,\,0}(q^{-1}) = 1$. Since (4.7) is satisfied, the $q^{-1}-$Hermite polynomials $H_n(x, s|q^{-1})$ obey the relation

$$\begin{aligned} H_{n+1}(x, s|q^{-1}) &= xH_n(x, sq^{-2}|q^{-1}) \\ &- sq^{1-n}\{n\}_q\, H_{n-1}(x, sq^{-2}|q^{-1}), \quad n \geq 1, \end{aligned} \tag{4.8}$$

with $H_0(x, sq^{-2}|q^{-1}) := 1$.
The action of the operator D_x^q on the polynomials (4.6) is given by

$$D_x^q\, H_n(x, s|q^{-1}) = \{n\}_q\, H_{n-1}(x, sq^{-2}|q^{-1}). \tag{4.9}$$

Let ϵ denote the operator which maps $f(s)$ to $f(qs)$. Then, from (4.8) and (4.9) one can establish that

$$H_n(x, s|q^{-1}) = \prod_{k=1}^{n} \left(x\, \epsilon^{-2} - sq^{k+1-n} D_x^q\right) \cdot (1). \tag{4.10}$$

4.2. Fourier transforms of the new $q-$Hermite polynomials $H_n(x, s|q)$

Considering the well-known Fourier transforms (2.21) for the Gauss exponential function $e^{-x^2/2s}$, the Fourier integral transforms for the exponential function $\exp(i(n-2k)\kappa x - x^2/2s)$ is computed as follows:

$$\frac{1}{\sqrt{2\pi s}} \int_{\mathbb{R}} e^{ixy + i(n-2k)\kappa x - \frac{x^2}{2s}}\, dx = q^{\frac{n^2}{4} + k(k-n)} e^{-s\frac{y^2}{2} - (n-2k)sy\kappa}, \tag{4.11}$$

where $q = e^{-2s\kappa^2} \leq 1$ and $0 \leq \kappa < \infty$.

Theorem 9. *The new $q-$Hermite polynomials $H_n(x,s|q)$ and $H_n(x,s|q^{-1})$ defined in (4.1) and (4.6), respectively, are connected by the integral Fourier transform of the following form*

$$\frac{1}{\sqrt{2\pi s}}\int_{\mathbb{R}} H_n(be^{i\kappa x},s|q)e^{ixy-\frac{x^2}{2s}}dx = q^{\frac{n^2}{4}}H_n(be^{-s\kappa y},q^{n-3}s|q^{-1})\,e^{-s\frac{y^2}{2}} \tag{4.12}$$

where b is an arbitrary constant factor.

Proof. To prove this theorem, let us make use of (4.1) and evaluate the left hand side of (4.12). Then,

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi s}}\int_{\mathbb{R}} H_n(be^{i\kappa x},s|q)e^{ixy-\frac{x^2}{2s}}dx\\
&=\sum_{k=0}^{\lfloor n/2\rfloor} c_{n,k}(q)s^k b^{n-2k}\,\frac{1}{\sqrt{2\pi s}}\int_{\mathbb{R}} e^{ixy+i(n-2k)\kappa x-\frac{x^2}{2s}}dx\\
&=\sum_{k=0}^{\lfloor n/2\rfloor} c_{n,k}(q)s^k b^{n-2k}e^{-\frac{s}{2}[\kappa(n-2k)+y]^2}\\
&=q^{\frac{n^2}{4}}\sum_{k=0}^{\lfloor n/2\rfloor} c_{n,k}(q)q^{-k(n-k)}s^k b^{n-2k}e^{-s\frac{y^2}{2}-(n-2k)sy\kappa}\\
&=q^{\frac{n^2}{4}}\sum_{k=0}^{\lfloor n/2\rfloor} c_{n,k}(q^{-1})(q^{n-3}s)^k(be^{-sy\kappa})^{n-2k}e^{-s\frac{y^2}{2}}\\
&=q^{\frac{n^2}{4}}H_n(be^{-s\kappa y},q^{n-3}s|q^{-1})\,e^{-s\frac{y^2}{2}}.
\end{aligned}$$

□

§5. Doubly indexed Hermite polynomials $\mathcal{H}_{n,p}(x,s|q)$

In this section, we construct a novel family of Hermite polynomials called *doubly indexed Hermite polynomials*, $\mathcal{H}_{n,p}(x,s|q)$. First, let us defined the $(q;p)-$shifted factorials $(a;q)_{pk}$ and the $(q;p)-$number as follows:

$$(a;q)_0 := 1,\ (a;q)_{pk} := (a,aq,\cdots,aq^{p-1};q^p)_k,\ p\geq 1,\ k=1,2,3,\cdots \tag{5.1}$$

and

$$\{pk\}_q := \frac{1-q^{pk}}{1-q},\quad \{pk\}_q!! := \prod_{l=1}^{k}\{pl\}_q,\quad \{0\}_q!! := 1, \tag{5.2}$$

respectively.

Definition 10. *For a positive integer p, a class of doubly indexed Hermite polynomials* $\{\mathcal{H}_{n,p}\}_{n,p}$ *is defined such that*

$$\mathcal{H}_{n,p}(x,s|q) := E_{q^p}\left(-s\frac{(D_x^q)^p}{\{p\}_q}\right)\cdot(x^n). \tag{5.3}$$

If $p=2$, a subclass of the polynomials (5.3) is reduced to the class of polynomials (3.28). More generally, their explicit formula is given by

$$\begin{aligned}\mathcal{H}_{n,p}(x,s|q) &= \{n\}_q!\sum_{k=0}^{\lfloor n/p\rfloor}\frac{(-1)^k q^{p\binom{k}{2}}\,s^k}{\{pk\}_q!!}\frac{x^{n-pk}}{\{n-pk\}_q!}\\ &= x^n\,{}_p\phi_0\left(\begin{matrix}q^{-n},q^{-n+1},\cdots,q^{-n+p-1}\\ -\end{matrix}\;\Big|\;q^p;\;\frac{sq^{p(n+\frac{1-p}{2})}}{(1-q)^{p-1}x^p}\right),\end{aligned} \tag{5.4}$$

where ${}_p\phi_0$ is the $q-$hypergeometric series [2].

Since $D_x^q e_q(\omega\,x)=\omega\,e_q(\omega\,x)$, we derive the generating function of the polynomials (5.3) as

$$f_q(x,s;p) := e_q(tx)E_{q^p}(-st^p/\{p\}_q)=\sum_{n=0}^{\infty}\frac{\mathcal{H}_{n,p}(x,s|q)}{\{n\}_q!}t^n,\quad |t|<1. \tag{5.5}$$

These polynomials are the solutions of the $q-$analogue of the generalized heat equation [11]

$$(D_x^q)^p f_q(x,s;p) = -\{p\}_q D_s^q f_q(x,s;p),\quad f_q(x,0;p)=x^n. \tag{5.6}$$

For any real number c and a positive integer p, $|q|<1$, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{\{c\}_{n,q}\mathcal{H}_{n,p}(x,s|q)}{\{n\}_q!}t^n =\\ &\frac{1}{(xt;q)_c}\,{}_p\phi_p\left(\begin{matrix}q^c,q^{c+1},\cdots,q^{c+p-1}\\ xtq^c,xtq^{c+1},\cdots,xtq^{c+p-1}\end{matrix}\;\Big|\;q^p;\;\frac{s\,t^p}{(1-q)^{p-1}}\right),\ |xt|<1.\end{aligned} \tag{5.7}$$

Performing the $q-$derivative of both sides of (5.5) with respect to x and t, one obtains

$$D_x^q\,\mathcal{H}_{n,p}(x,s|q)=\{n\}_q\,\mathcal{H}_{n-1,p}(x,s|q) \tag{5.8}$$

and

$$\begin{aligned}\mathcal{H}_{n+1,p}(x,s|q) &= x\mathcal{H}_{n,p}(x,s|q)\\ &-sq^{n-p+1}\{n\}_q\{n-1\}_q\cdots\{n-p+2\}_q\mathcal{H}_{n-p+1}(x,s|q),\ n\geq 1,\end{aligned} \tag{5.9}$$

with $\mathcal{H}_{0,p}(x,s|q):=1$. The polynomials (5.3) obey the following $p-$th order difference equation

$$\Big(s\,(D_x^q)^p - q^{p-n}\,x\,D_x^q + q^{p-n}\,\{n\}_q\Big)\,\mathcal{H}_{n,p}(x,s|q)=0. \tag{5.10}$$

§6. Concluding remarks

In this paper, we have constructed a family of new $q-$Hermite polynomials $H_n(x,s|q)$. Several properties related to these polynomials have been computed and discussed. Finally, we have constructed a novel family of Hermite polynomials $\mathcal{H}_{n,p}(x,s|q)$ called doubly indexed Hermite polynomials.

In the limit cases, when q goes to 1 and s goes to $-py$, the polynomials $\mathcal{H}_{n,p}(x,s|q)$ are reduced to the higher-order Hermite polynomials, sometimes called the Kampé de Fériet or the Gould Hopper polynomials [11]-[15], i.e.,

$$\mathcal{H}_{n,p}(x,-py|1) \equiv g_n^p(x,y) := n! \sum_{k=0}^{\lfloor n/p \rfloor} \frac{y^k x^{n-pk}}{k!\,(n-pk)!}. \tag{6.1}$$

When q goes to 1, $x \to px$ and $s \to p$, the polynomials $\mathcal{H}_{n,p}(x,s|q)$ become the Hermite polynomials investigated by Habibullah and Shakoor [16], i.e.,

$$\mathcal{H}_{n,p}(px,p|1) \equiv S_{p,n}(x) := n! \sum_{k=0}^{\lfloor n/p \rfloor} \frac{(-1)^k (px)^{n-pk}}{k!\,(n-pk)!}. \tag{6.2}$$

For $p=2$, the doubly indexed polynomials $\mathcal{H}_{n,p}(x,s|q)$ are reduced to the new $q-$Hermite polynomials $H_n(x,s|q)$, i.e., $\mathcal{H}_{n,2}(x,s|q) \equiv H_n(x,s|q)$.

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