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Ricci pseudosymmetric generalized quasi-Einstein manifolds

Shyamal Kumar Hui and Richard S. Lemence

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Abstract. As a generalization of quasi-Einstein manifold, De and Ghosh introduced the notion of generalized quasi-Einstein manifold. The object of the present paper is to study Ricci pseudosymmetric generalized quasi-Einstein manifolds (briefly, $G(QE)_n$) in the framework of pseudo-Riemannian geometry. Specifically, we study the concircular Ricci pseudosymmetric $G(QE)_n$, projective Ricci pseudosymmetric $G(QE)_n$, W_3-Ricci pseudosymmetric $G(QE)_n$, conharmonic Ricci pseudosymmetric $G(QE)_n$, conformal Ricci pseudosymmetric $G(QE)_n$ and quasi-conformal Ricci pseudosymmetric $G(QE)_n$.

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§1. Introduction

It is well known that a pseudo-Riemannian manifold $(M^n, g)(n > 2)$ is Einstein if its Ricci tensor S of type (0,2) is of the form $S = \alpha g$, where α is a constant, which turns into $S = \frac{r}{n} g$, r being the scalar curvature (constant) of the manifold. Let $(M^n, g)(n \geq 3)$ be a pseudo-Riemannian manifold. Let $U_S = \{x \in M : S \neq \frac{r}{n} g \text{ at } x\}$. Then the manifold (M^n, g) is said to be quasi-Einstein manifold ([2],[8],[10], [11],[13]–[17],[19],[21],[22]) if on $U_S \subset M$, we have

$$S - \alpha g = \beta A \otimes A, \tag{1.1}$$

where A is a unit 1-form on U_S and α, β are some functions on U_S. It is clear that the 1-form A as well as the function β are non-zero at every point on U_S. From the above definition, it follows that every Einstein manifold is quasi-Einstein. In particular, every Ricci-flat manifold (e.g. Schwarzschild spacetime) is quasi-Einstein. The scalars α, β are known as the associated scalars of the manifold. Also, the unit 1-form A is called the associated 1-form of the manifold defined by $g(X,\rho) = A(X)$ for any vector field X; ρ being a unit vector field, called the generator of the manifold. Such an n-dimensional quasi-Einstein manifold is denoted by $(QE)_n$. The quasi-Einstein manifolds has also been studied among other by De and De [3], De and Ghosh [4], Deszcz, Hotloś and Sentürk [18], Deszcz, Glogowska, Hotloś and Sawicz [12], Shaikh, Yoon and Hui [30], Shaikh, Kim and Hui [31], Shaikh and Patra [32].

As a generalization of quasi-Einstein manifold, in [5], De and Ghosh introduced and studied the notion of generalized quasi-Einstein manifold. A pseudo-Riemannian manifold $(M^n, g)(n \geq 3)$ is said to be generalized quasi-Einstein manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the following:

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma B(X)B(Y), \tag{1.2}$$

where α, β, γ are scalars of which $\beta \neq 0, \gamma \neq 0$ and A, B are orthonormal system of 1-forms such that $g(X,\rho) = A(X)$, $g(X,\mu) = B(X)$ for all vector fields X. The unit vectors ρ and μ corresponding to the 1-forms A and B are orthogonal to each other. Also, ρ and μ are known as the generators of the manifold. Such an n-dimensional manifold is denoted by $G(QE)_n$. The generalized quasi-Einstein manifolds are also studied by De and Ghosh [6], Shaikh and Hui [29] and many others.

Again, as a generalization of quasi-Einstein manifold, recently Shaikh [28] introduced the notion of pseudo quasi-Einstein manifolds. A pseudo-Riemannian manifold $(M^n, g)(n \geq 3)$ is said to be pseudo quasi-Einstein manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the following:

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma D(X,Y), \tag{1.3}$$

where α, β, γ are scalars of which $\beta \neq 0, \gamma \neq 0$ and D is a trace free symmetric tensor of type (0,2) such that $D(X,\rho) = 0$ for any vector field X. It follows that every quasi-Einstein manifold is a pseudo quasi-Einstein manifold but not conversely as follows by various examples given in [28].

It is known that the outer product of two covariant tensors is a tensor of type (0,2) but the converse is not true, in general [7]. Consequently, the tensor D can not be decomposed into product of two 1-forms. In particular, if $D = B \otimes B$, B being a non-zero 1-form, then a pseudo quasi-Einstein manifold reduces to generalized quasi-Einstein manifold by De and Ghosh [5].

An n-dimensional pseudo-Riemannian manifold (M^n, g) is called Ricci pseudosymmetric [9] if the tensor $R{\cdot}S$ and the Tachibana tensor $Q(g, S)$ are linearly dependent, where

$$(R(X,Y)\cdot S)(Z,U) = -S(R(X,Y)Z,U) - S(Z,R(X,Y)U), \tag{1.4}$$

$$Q(g,S)(Z,U;X,Y) = -S((X\wedge_g Y)Z,U) - S(Z,(X\wedge_g Y)U), \tag{1.5}$$

and

$$(X\wedge_g Y)Z = g(Y,Z)X - g(X,Z)Y \tag{1.6}$$

for all vector fields X, Y, Z, U of M, R denotes the curvature tensor of M. Then (M^n, g) is Ricci pseudosymmetric if and only if

$$(R(X,Y)\cdot S)(Z,U) = L_S Q(g,S)(Z,U;X,Y) \tag{1.7}$$

holds on $U_S = \{x \in M : S \neq \frac{r}{n}g \text{ at } x\}$, where L_S is some function on U_S. If $R\cdot S = 0$, then M^n is called Ricci semisymmetric. Every Ricci semisymmetric manifold is Ricci pseudosymmetric but the converse is not true [9]. In [5] De and Ghosh studied Ricci semisymmetric $G(QE)_n$ and in [23], Shaikh and Hui studied Ricci pseudosymmetric $G(QE)_n$.

The object of the present paper is to study Ricci pseudosymmetric $G(QE)_n$. The paper is organized as follows. Section 2 is concerned with preliminaries including the known examples of $G(QE)_n$. These examples ensured the existence of $G(QE)_n$. In sections 3-8, we investigate, respectively, the concircular Ricci pseudosymmetric $G(QE)_n$, the projective Ricci pseudosymmetric $G(QE)_n$, W_3-Ricci pseudosymmetric $G(QE)_n$, conharmonic Ricci pseudosymmetric $G(QE)_n$, conformal Ricci pseudosymmetric $G(QE)_n$ and quasi-conformal Ricci pseudosymmetric $G(QE)_n$. In each of the case, we obtained that either the associated scalars β and γ are equal or the curvature tensor R satisfies a definite condition. Finally, in the last section, we gave the geometrical significance of the paper.

§2. Preliminaries

In this section, we will obtain some formulas of $G(QE)_n$, which will be required in the sequel. Let $\{e_i : i = 1, 2, \cdots, n\}$ be an orthonormal frame field at any point of the manifold. Then setting $X = Y = e_i$ in (1.2) and taking summation over $i, 1 \le i \le n$, we obtain

$$r = n\alpha + \epsilon_A\beta + \epsilon_B\gamma, \tag{2.1}$$

where r is the scalar curvature of the manifold and $\epsilon_A = g(\rho,\rho)(=\pm 1)$ and $\epsilon_B = g(\mu,\mu)(=\pm 1)$. Also, from (1.2), we have

$$S(X,\rho) = (\alpha + \epsilon_A\beta)A(X), \quad S(\rho,\rho) = \epsilon_A\alpha + \beta, \tag{2.2}$$

$$S(X,\mu) = (\alpha + \epsilon_B\gamma)B(X), \quad S(\mu,\mu) = \epsilon_B\alpha + \gamma \tag{2.3}$$

and

$$S(\rho,\mu) = 0. \tag{2.4}$$

Let Q be the Ricci-operator. Then $g(QX,Y) = S(X,Y)$ for all X, Y.
Below are some known examples of $G(QE)_n$.

Example 2.1. An n-dimensional hypersurface M, $n \geq 3$, in a Riemannian manifold $\tilde{M}$ is said to be quasi-umbilical [20] at a point $x \in M$ if at the point x its second fundamental tensor H satisfies the relation

$$H = ag + b\omega \otimes \omega,$$

where ω is an 1-form and a and b are some functions on M. If $a = 0$ (respectively, $b = 0$ or $a = b = 0$) holds at x then it is called cylindrical (respectively, umbilical or geodesic) at x.

It is proved that [5] a 2-quasi umbilical hypersurface of a Euclidean space is a generalized quasi-Einstein manifold.

Example 2.2. In contact metric geometry, a Kenmotsu manifold with constant ϕ-holomorphic sectional curvature c is called Kenmotsu-space-form and the curvature tensor of such a manifold is given by [33]

$$\begin{aligned}\tilde{R}(X,Y)Z &= \frac{c-3}{4}\big\{g(Y,Z)X - g(X,Z)Y\big\}\\ &+ \frac{c+1}{4}\big\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\big\}\\ &+ \frac{c+1}{4}\big[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\big].\end{aligned}$$

Let M be a quasi-umbilical hypersurface of a Kenmotsu-space-form $\tilde{M}^n(c)$, $n = 2m+1$. Then M is a generalized quasi-Einstein manifold [33].

Example 2.3. [29] Let (M^4, g) be a Riemannian manifold endowed with the metric given by

$$ds^2 = g_{ij}dx^i dx^j = (1+2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2], (i,j = 1,2,3,4),$$

where $p = \frac{e^{x^1}}{k^2}$ and k is a non-zero constant. Then (M^4, g) is a $G(QE)_4$ with non-vanishing scalar curvature which is not quasi-Einstein.

Example 2.4. [29] Let (M^4, g) be a Riemannian manifold endowed with the metric given by

$$ds^2 = e^{2x^1}(dx^1)^2 + \sin^2 x^1[(dx^2)^2 + (dx^3)^2 + (dx^4)^2],$$

where $0 < x^1 < \frac{\pi}{2}$ but $x^1 \neq \frac{\pi}{4}$. Then (M^4, g) is a $G(QE)_4$ with non-vanishing scalar curvature which is not quasi-Einstein.

§3. Concircular Ricci pseudosymmetric $G(QE)_n$

A transformation of an n-dimensional pseudo-Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation [34]. The interesting invariant of a concircular transformation is the concircular curvature tensor $\tilde{C}$, which is defined by [34]

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}\big[g(Y,Z)X - g(X,Z)Y\big], \tag{3.1}$$

where r is the scalar curvature of the manifold.

An n-dimensional pseudo-Riemannian manifold (M^n, g) is said to be concircular Ricci pseudosymmetric if its concircular curvature tensor $\tilde{C}$ satisfies

$$(\tilde{C}(X,Y) \cdot S)(Z,U) = L_S Q(g,S)(Z,U;X,Y) \tag{3.2}$$

holds on $U_S = \{x \in M : S \neq \frac{r}{n} g \text{ at } x\}$, where L_S is some function on U_S. We now state our result on concircular Ricci pseudosymmetric $G(QE)_n$.

Theorem 3.1. *In a concircular Ricci pseudosymmetric $G(QE)_n$, either the associated scalars β and γ satisfy $\beta = \epsilon_A\epsilon_B\gamma$ or the curvature tensor R of the manifold satisfies the relation*

$$R(X,Y,\rho,\mu) = \big[L_S + \frac{r}{n(n-1)}\big]\{A(Y)B(X) - A(X)B(Y)\},$$

where $\epsilon_A = g(\rho,\rho)$ and $\epsilon_B = g(\mu,\mu)$.

Proof. Let us take a concircular Ricci pseudosymmetric $G(QE)_n$. Then we get (3.2). From (3.2), we get

$$\begin{aligned} S(\tilde{C}(X,Y)Z,U) + S(Z,\tilde{C}(X,Y)U) = L_S[g(Y,Z)S(X,U) \\ -g(X,Z)S(Y,U) + g(Y,U)S(X,Z) - g(X,U)S(Y,Z)]. \end{aligned} \tag{3.3}$$

By virtue of (1.2) and (3.1) it follows from (3.3) that

$$\begin{aligned} &\beta[A(\tilde{C}(X,Y)Z)A(U) + A(Z)A(\tilde{C}(X,Y)U)] + \gamma[B(\tilde{C}(X,Y)Z)B(U) \\ &+B(Z)B(\tilde{C}(X,Y)U)] = L_S[\beta\{g(Y,Z)A(X)A(U) - g(X,Z)A(Y)A(U) \\ &+g(Y,U)A(X)A(Z) - g(X,U)A(Y)A(Z)\} + \gamma\{g(Y,Z)B(X)B(U) \\ &-g(X,Z)B(Y)B(U) + g(Y,U)B(X)B(Z) - g(X,U)B(Y)B(Z)\}]. \end{aligned}$$

Setting $Z = \rho$ and $U = \mu$ in the equation above, we get

$$(3.4) \qquad (\epsilon_A\epsilon_B\gamma - \beta)[\tilde{C}(X, Y, \rho, \mu) - L_S\{A(Y)B(X) - A(X)B(Y)\}] = 0,$$

which yields either $\beta = \epsilon_A\epsilon_B\gamma$ or

$$(3.5) \qquad \tilde{C}(X, Y, \rho, \mu) = L_S\{A(Y)B(X) - A(X)B(Y)\}.$$

Using (3.1) in (3.5), we get

$$(3.6) \qquad R(X, Y, \rho, \mu) = \big[L_S + \frac{r}{n(n-1)}\big]\{A(Y)B(X) - A(X)B(Y)\}.$$

□

If the scalar curvature r is identically equal to zero then $\tilde{C}(X, Y)Z = R(X, Y)Z$ for all X, Y, Z, and hence we can state the following:

Corollary 3.1. In a Ricci semisymmetric $G(QE)_n$, either the associated scalars β and γ satisfy $\beta = \epsilon_A\epsilon_B\gamma$ or the curvature tensor R of the manifold satisfies the relation

$$R(X, Y, \rho, \mu) = L_S\{A(Y)B(X) - A(X)B(Y)\}.$$

Note that the above corollary is similar to the result of Shaikh and Hui in [29].

Now plugging $X = \rho$ and $Y = \mu$ in (3.6), we get

$$(3.7) \qquad L_S = -\big[\epsilon_A\epsilon_B R(\rho, \mu, \rho, \mu) + \frac{r}{n(n-1)}\big].$$

This leads to the following:

Theorem 3.2. *In a concircular Ricci pseudosymmetric $G(QE)_n$ with $\beta \neq \epsilon_A\epsilon_B\gamma$, L_S is determined by the relation* (3.7)*.*

§4. Projective Ricci pseudosymmetric $G(QE)_n$

The projective transformation on a pseudo-Riemannian manifold is a transformation under which geodesic transforms into geodesic. The Weyl projective curvature tensor is given by [7]

$$(4.1) \qquad P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}\big[S(Y, Z)X - S(X, Z)Y\big].$$

An n-dimensional pseudo-Riemannian manifold (M^n, g) is said to be projective Ricci pseudosymmetric if its projective curvature tensor P satisfies

$$(4.2) \qquad (P(X, Y) \cdot S)(Z, U) = L_S Q(g, S)(Z, U; X, Y).$$

holds on $U_S = \{x \in M : S \neq \frac{r}{n}g \text{ at } x\}$, where L_S is some function on U_S.

We now state our result on projective Ricci pseudosymmetric $G(QE)_n$.

Theorem 4.1. *In a projective Ricci pseudosymmetric $G(QE)_n$, either the associated scalars β and γ satisfy $\beta = \epsilon_A \epsilon_B \gamma$ or the curvature tensor R of the manifold satisfies the relation*

$$R(X, Y, \rho, \mu) = \big[L_S + \frac{1}{n-1}\big]\{A(Y)B(X) - A(X)B(Y)\}.$$

Proof. Consider a projective Ricci pseudosymmetric $G(QE)_n$. Then we get (4.2), i.e.,

$$\begin{aligned}(4.3)\qquad S(P(X,Y)Z,U) + S(Z,P(X,Y)U) &= L_S[g(Y,Z)S(X,U) \\ -g(X,Z)S(Y,U) + g(Y,U)S(X,Z) &- g(X,U)S(Y,Z)].\end{aligned}$$

By virtue of (1.2) and (4.1) it follows from (4.3) that

$$\begin{aligned}&\alpha\big[g(P(X,Y)Z,U) + g(Z,P(X,Y)U)\big] \\ &+\beta[A(P(X,Y)Z)A(U) + A(Z)A(P(X,Y)U)] + \gamma[B(P(X,Y)Z)B(U) \\ &+B(Z)B(P(X,Y)U)] = L_S[\beta\{g(Y,Z)A(X)A(U) - g(X,Z)A(Y)A(U) \\ &+g(Y,U)A(X)A(Z) - g(X,U)A(Y)A(Z)\} + \gamma\{g(Y,Z)B(X)B(U) \\ &-g(X,Z)B(Y)B(U) + g(Y,U)B(X)B(Z) - g(X,U)B(Y)B(Z)\}].\end{aligned}$$

Setting $Z = \rho$ and $U = \mu$ in the equation above, we get

$$\begin{aligned}(4.4)\qquad &(\alpha + \epsilon_B\gamma)P(X,Y,\rho,\mu) + (\alpha + \epsilon_A\beta)P(X,Y,\mu,\rho) \\ = \;&L_S(\epsilon_B\gamma - \epsilon_A\beta)\{A(Y)B(X) - A(X)B(Y)\}.\end{aligned}$$

In view of (4.1), (4.4) yields

$$(4.5)\qquad (\epsilon_A\epsilon_B\gamma - \beta)[P(X,Y,\rho,\mu) - L_S\{A(Y)B(X) - A(X)B(Y)\}] = 0,$$

which yields either $\beta = \epsilon_A\epsilon_B\gamma$ or

$$(4.6)\qquad R(X,Y,\rho,\mu) = \left(L_S + \frac{1}{n-1}\right)\{A(Y)B(X) - A(X)B(Y)\}.$$

□

Now putting $X = \rho$ and $Y = \mu$ in (4.6), we get

$$(4.7)\qquad L_S = -\epsilon_A\epsilon_B R(\rho,\mu,\rho,\mu).$$

This leads to the following:

Theorem 4.2. *In a projective Ricci pseudosymmetric $G(QE)_n$ with $\beta \neq \epsilon_A\epsilon_B\gamma$, L_S is determined by the relation* (4.7).

§5. W_3-Ricci pseudosymmetric $G(QE)_n$

In 1973 Pokhariyal [27] introduced the notion of a new curvature tensor, denoted by W_3 and studied its relativistic significance. The W_3-curvature tensor of type (1,3) is defined by

$$W_3(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}\big[g(Y,Z)QX - S(X,Z)Y\big], \tag{5.1}$$

where R is the curvature tensor and Q is the Ricci-operator, i.e., $g(QX,Y) = S(X,Y)$ for all X, Y.

An n-dimensional pseudo-Riemannian manifold (M^n, g) is said to be W_3-Ricci pseudosymmetric if it satisfies

$$(W_3(X,Y)\cdot S)(Z,U) = L_S Q(g,S)(Z,U;X,Y). \tag{5.2}$$

holds on $U_S = \{x \in M : S \neq \frac{r}{n} g \text{ at } x\}$, where L_S is some function on U_S. We now state our result on W_3-Ricci pseudosymmetric $G(QE)_n$.

Theorem 5.1. *In a W_3-Ricci pseudosymmetric $G(QE)_n$, either the associated scalars β and γ satisfy $\beta = \epsilon_A\epsilon_B\gamma$ or the curvature tensor R of the manifold satisfies the relation*

$$R(X,Y,\rho,\mu) = \big(L_S - \frac{\alpha+\epsilon_B\gamma}{n-1}\big)A(Y)B(X) - \big(L_S - \frac{\alpha+\epsilon_A\beta}{n-1}\big)A(X)B(Y).$$

Proof. Consider a W_3-Ricci pseudosymmetric $G(QE)_n$. Then we get (5.2), i.e.,

$$\begin{aligned} S(W_3(X,Y)Z,U) + S(Z,W_3(X,Y)U) &= L_S[g(Y,Z)S(X,U) \\ &\quad -g(X,Z)S(Y,U) + g(Y,U)S(X,Z) - g(X,U)S(Y,Z)]. \end{aligned} \tag{5.3}$$

Using (1.2) in (5.3), we get by virtue of (5.1) that

$$\begin{aligned} &\alpha\big[g(W_3(X,Y)Z,U) + g(Z,W_3(X,Y)U)\big] \\ &+\beta[A(W_3(X,Y)Z)A(U) + A(Z)A(W_3(X,Y)U)] + \gamma[B(W_3(X,Y)Z)B(U) \\ &+B(Z)B(W_3(X,Y)U)] = L_S[\beta\{g(Y,Z)A(X)A(U) - g(X,Z)A(Y)A(U) \\ &+g(Y,U)A(X)A(Z) - g(X,U)A(Y)A(Z)\} + \gamma\{g(Y,Z)B(X)B(U) \\ &-g(X,Z)B(Y)B(U) + g(Y,U)B(X)B(Z) - g(X,U)B(Y)B(Z)\}]. \end{aligned}$$

Since $g(W_3(X,Y)Z,U) = -g(W_3(X,Y)U,Z)$, we get the following

$$\begin{aligned} &\beta[A(W_3(X,Y)Z)A(U) + A(Z)A(W_3(X,Y)U)] + \gamma[B(W_3(X,Y)Z)B(U) \\ &+B(Z)B(W_3(X,Y)U)] = L_S[\beta\{g(Y,Z)A(X)A(U) - g(X,Z)A(Y)A(U) \\ &+g(Y,U)A(X)A(Z) - g(X,U)A(Y)A(Z)\} + \gamma\{g(Y,Z)B(X)B(U) \\ &-g(X,Z)B(Y)B(U) + g(Y,U)B(X)B(Z) - g(X,U)B(Y)B(Z)\}]. \end{aligned}$$

Setting $Z = \rho$ and $U = \mu$ in the above equation, we get

$$(\epsilon_A \epsilon_B \gamma - \beta)[W_3(X, Y, \rho, \mu) - L_S\{A(Y)B(X) - A(X)B(Y)\}] = 0, \tag{5.4}$$

which yields either $\beta = \epsilon_A \epsilon_B \gamma$ or

$$W_3(X, Y, \rho, \mu) = L_S\{A(Y)B(X) - A(X)B(Y)\}. \tag{5.5}$$

By virtue of (2.2), (2.3) and (5.1) it follows from (5.5) that

$$R(X, Y, \rho, \mu) = \big(L_S - \frac{\alpha + \epsilon_B \gamma}{n-1}\big)A(Y)B(X) - \big(L_S - \frac{\alpha + \epsilon_A \beta}{n-1}\big)A(X)B(Y). \tag{5.6}$$

□

We now put $X = \rho$ and $Y = \mu$ in (5.6), we get

$$L_S = \frac{\alpha + \epsilon_A \beta}{n-1} - \epsilon_A \epsilon_B R(\rho, \mu, \rho, \mu). \tag{5.7}$$

This leads to the following:

Theorem 5.2. *In a W_3-Ricci pseudosymmetric $G(QE)_n$ with $\beta \neq \epsilon_A \epsilon_B \gamma$, L_S is determined by the relation* (5.7)*.*

§6. Conharmonic Ricci pseudosymmetric $G(QE)_n$

As a special subgroup of the conformal transformation group, Ishii [24] introduced the notion of conharmonic transformation under which a harmonic function transform into a harmonic function. The conharmonic curvature tensor of type (1,3) on a Riemannian manifold (M^n, g), $n > 3$, is given by [24].

$$\overline{C}(X, Y)Z = R(X, Y)Z - \frac{1}{n-2}\big[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY\big], \tag{6.1}$$

which is invariant under conharmonic transformation, where S is the Ricci tensor of the manifold of type (0,2).

An n-dimensional pseudo-Riemannian manifold (M^n, g) is said to be conharmonic Ricci pseudosymmetric if its conharmonic curvature tensor $\overline{C}$ satisfies

$$(\overline{C}(X, Y) \cdot S)(Z, U) = L_S Q(g, S)(Z, U; X, Y). \tag{6.2}$$

holds on $U_S = \{x \in M : S \neq \frac{r}{n} g \text{ at } x\}$, where L_S is some function on U_S. We now state our result on conharmonic Ricci pseudosymmetric $G(QE)_n$.

Theorem 6.1. *In a conharmonic Ricci pseudosymmetric $G(QE)_n$, either the associated scalars β and γ satisfy $\beta = \epsilon_A\epsilon_B\gamma$ or the curvature tensor R of the manifold satisfies the relation*

$$R(X,Y,\rho,\mu) = \Big[L_S + \frac{2\alpha + \epsilon_A\beta + \epsilon_B\gamma}{n-2}\Big]\big[A(Y)B(X) - A(X)B(Y)\big].$$

Proof. Suppose we have a manifold which is a conharmonic Ricci pseudosymmetric $G(QE)_n$. Then we get (6.2), which implies that

$$\text{(6.3)} \qquad S(\overline{C}(X,Y)Z,U) + S(Z,\overline{C}(X,Y)U) = L_S[g(Y,Z)S(X,U) \\ -g(X,Z)S(Y,U) + g(Y,U)S(X,Z) - g(X,U)S(Y,Z)].$$

Using (1.2) and (6.1) in (6.3), we obtain

$$\beta[A(\overline{C}(X,Y)Z)A(U) + A(Z)A(\overline{C}(X,Y)U)] + \gamma[B(\overline{C}(X,Y)Z)B(U) \\ +B(Z)B(\overline{C}(X,Y)U)] = L_S[\beta\{g(Y,Z)A(X)A(U) - g(X,Z)A(Y)A(U) \\ +g(Y,U)A(X)A(Z) - g(X,U)A(Y)A(Z)\} + \gamma\{g(Y,Z)B(X)B(U) \\ -g(X,Z)B(Y)B(U) + g(Y,U)B(X)B(Z) - g(X,U)B(Y)B(Z)\}].$$

Setting $Z = \rho$ and $U = \mu$ in the above equation, we get

$$\text{(6.4)} \qquad (\epsilon_A\epsilon_B\gamma - \beta)[\overline{C}(X,Y,\rho,\mu) - L_S\{A(Y)B(X) - A(X)B(Y)\}] = 0,$$

which yields either $\beta = \epsilon_A\epsilon_B\gamma$ or

$$\text{(6.5)} \qquad \overline{C}(X,Y,\rho,\mu) = L_S\{A(Y)B(X) - A(X)B(Y)\}.$$

In view of (2.2), (2.3) and (6.1), (6.5) yields

$$\text{(6.6)} \qquad R(X,Y,\rho,\mu) = \Big[L_S + \frac{2\alpha + \epsilon_A\beta + \epsilon_B\gamma}{n-2}\Big]\big[A(Y)B(X) - A(X)B(Y)\big].$$

□

Setting $X = \rho$ and $Y = \mu$ in (6.6), we get

$$\text{(6.7)} \qquad L_S = -\epsilon_A\epsilon_B R(\rho,\mu,\rho,\mu) - \frac{2\alpha + \epsilon_A\beta + \epsilon_B\gamma}{n-2}.$$

This leads to the following:

Theorem 6.2. *In a conharmonic Ricci pseudosymmetric $G(QE)_n$ with $\beta \neq \epsilon_A\epsilon_B\gamma$, L_S is determined by the relation* (6.7).

§7. Conformal Ricci pseudosymmetric $G(QE)_n$

The conformal transformation on a pseudo Riemannian manifold is a transformation under which the angle between two curves remains invariant. The Weyl conformal curvature tensor C of type (1,3) of an n-dimensional pseudo-Riemannian manifold $(M^n, g)(n > 3)$ is defined by [7]

$$(7.1) \qquad \begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY] \\ &+ \frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned}$$

An n-dimensional pseudo-Riemannian manifold (M^n, g) is said to be conformal Ricci pseudosymmetric if its conformal curvature tensor C satisfies

$$(7.2) \qquad (C(X,Y)\cdot S)(Z,U) = L_S Q(g,S)(Z,U;X,Y).$$

holds on $U_S = \{x \in M : S \neq \frac{r}{n}g \text{ at } x\}$, where L_S is some function on U_S. We now state our result on conformal Ricci pseudosymmetric $G(QE)_n$.

Theorem 7.1. *In a conformal Ricci pseudosymmetric $G(QE)_n$, either the associated scalars β and γ satisfy $\beta = \epsilon_A\epsilon_B\gamma$ or the curvature tensor R of the manifold satisfies the relation*

$$\begin{aligned} &R(X,Y,\rho,\mu) \\ &= \Big[L_S + \frac{2\alpha+\epsilon_A\beta+\epsilon_B\gamma}{n-2} - \frac{r}{(n-1)(n-2)}\Big]\big[A(Y)B(X) - A(X)B(Y)\big]. \end{aligned}$$

Proof. Consider a conformal Ricci pseudosymmetric $G(QE)_n$. Then we get (7.2), which yields

$$(7.3) \qquad \begin{aligned} &S(C(X,Y)Z,U) + S(Z,C(X,Y)U) = L_S[g(Y,Z)S(X,U) \\ &-g(X,Z)S(Y,U) + g(Y,U)S(X,Z) - g(X,U)S(Y,Z)]. \end{aligned}$$

Using (1.2) and (7.1) in (7.3), we obtain

$$\begin{aligned} &\beta[A(C(X,Y)Z)A(U) + A(Z)A(C(X,Y)U)] + \gamma[B(C(X,Y)Z)B(U) \\ &+B(Z)B(C(X,Y)U)] = L_S[\beta\{g(Y,Z)A(X)A(U) - g(X,Z)A(Y)A(U) \\ &+g(Y,U)A(X)A(Z) - g(X,U)A(Y)A(Z)\} + \gamma\{g(Y,Z)B(X)B(U) \\ &-g(X,Z)B(Y)B(U) + g(Y,U)B(X)B(Z) - g(X,U)B(Y)B(Z)\}]. \end{aligned}$$

Setting $Z = \rho$ and $U = \mu$ in the equation above, we get

$$(7.4) \qquad (\epsilon_A\epsilon_B\gamma - \beta)[C(X,Y,\rho,\mu) - L_S\{A(Y)B(X) - A(X)B(Y)\}] = 0,$$

which yields either $\beta = \epsilon_A\epsilon_B\gamma$ or

$$C(X,Y,\rho,\mu) = L_S\{A(Y)B(X) - A(X)B(Y)\}. \tag{7.5}$$

Using (7.1) in (7.5), we have

$$\begin{aligned} &R(X,Y,\rho,\mu) \\ &= \Big[L_S + \frac{2\alpha+\epsilon_A\beta+\epsilon_B\gamma}{n-2} - \frac{r}{(n-1)(n-2)}\Big]\big[A(Y)B(X) - A(X)B(Y)\big]. \end{aligned} \tag{7.6}$$

□

Plugging $X = \rho$ and $Y = \mu$ in (7.6), we get

$$L_S = \frac{r}{(n-1)(n-2)} - \frac{2\alpha+\epsilon_A\beta+\epsilon_B\gamma}{n-2} - \epsilon_A\epsilon_B R(\rho,\mu,\rho,\mu). \tag{7.7}$$

This leads to the following:

Theorem 7.2. *In a conformal Ricci pseudosymmetric $G(QE)_n$ with $\beta \neq \epsilon_A\epsilon_B\gamma$, L_S is determined by the relation* (7.7).

§8. Quasi-conformal Ricci pseudosymmetric $G(QE)_n$

In 1968, Yano and Sawaki [35] defined and studied a curvature tensor W of type (1,3) which includes both the conformal curvature tensor C and the concircular curvature tensor $\tilde{C}$ as special cases and is called quasi-conformal curvature tensor. The quasi-conformal curvature tensor W of type (1,3) of a pseudo-Riemannian manifold $(M^n, g)(n > 3)$ is defined by

$$W(X,Y)Z = -(n-2)bC(X,Y)Z + [a + (n-2)b]\tilde{C}(X,Y)Z, \tag{8.1}$$

where a and b are arbitrary constants not simultaneously zero. In particular, if $a = 1$, $b = 0$ then W reduces to the concircular curvature tensor and if $a = 1$ and $b = -\frac{1}{(n-2)}$, then W reduces to the conformal curvature tensor. Using the expression of the conformal and the concircular curvature tensor in (8.1), the quasi-conformal curvature tensor W of type (1,3) can be written as

$$\begin{aligned} W(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY] \\ &- \frac{r}{n}(\frac{a}{n-1} + 2b)\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{8.2}$$

An n-dimensional pseudo-Riemannian manifold (M^n, g) is said to be quasi-conformal Ricci pseudosymmetric if its quasi-conformal curvature tensor W satisfies

$$(W(X,Y)\cdot S)(Z,U) = L_S Q(g,S)(Z,U;X,Y). \tag{8.3}$$

holds on $U_S = \{x \in M : S \neq \frac{r}{n} g \text{ at } x\}$, where L_S is some function on U_S. We now state our final result which is on quasi-conformal Ricci pseudosymmetric $G(QE)_n$.

Theorem 8.1. *In a quasi-conformal Ricci pseudosymmetric $G(QE)_n$, either the associated scalars β and γ satisfy $\beta = \epsilon_A \epsilon_B \gamma$ or the curvature tensor R of the manifold satisfies the relation*

$$R(X,Y,\rho,\mu) = \frac{1}{a}\Big[L_S - (2\alpha + \epsilon_A\beta + \epsilon_B\gamma)b + \frac{r}{n}\big(\frac{a}{n-1} + 2b\big)\Big]\big[A(Y)B(X) - A(X)B(Y)\big].$$

Proof. Consider a quasi-conformal Ricci pseudosymmetric $G(QE)_n$. Consequently, we have (8.3), which implies that

$$S(W(X,Y)Z,U) + S(Z,W(X,Y)U) = L_S[g(Y,Z)S(X,U) - g(X,Z)S(Y,U) + g(Y,U)S(X,Z) - g(X,U)S(Y,Z)]. \tag{8.4}$$

Using (1.2) and (8.1) in (8.4), we obtain

$$\begin{aligned}&\beta[A(W(X,Y)Z)A(U) + A(Z)A(W(X,Y)U)] + \gamma[B(W(X,Y)Z)B(U)\\&+B(Z)B(W(X,Y)U)] = L_S[\beta\{g(Y,Z)A(X)A(U) - g(X,Z)A(Y)A(U)\\&+g(Y,U)A(X)A(Z) - g(X,U)A(Y)A(Z)\} + \gamma\{g(Y,Z)B(X)B(U)\\&-g(X,Z)B(Y)B(U) + g(Y,U)B(X)B(Z) - g(X,U)B(Y)B(Z)\}].\end{aligned}$$

Setting $Z = \rho$ and $U = \mu$ in the above equation, we get

$$(\epsilon_A\epsilon_B\gamma - \beta)[W(X,Y,\rho,\mu) - L_S\{A(Y)B(X) - A(X)B(Y)\}] = 0, \tag{8.5}$$

which yields either $\beta = \epsilon_A\epsilon_B\gamma$ or

$$W(X,Y,\rho,\mu) = L_S\{A(Y)B(X) - A(X)B(Y)\}. \tag{8.6}$$

In view of (2.2), (2.3) and (8.2), (8.6) yields

$$aR(X,Y,\rho,\mu) = \Big[L_S - (2\alpha + \epsilon_A\beta + \epsilon_B\gamma)b + \frac{r}{n}\big(\frac{a}{n-1} + 2b\big)\Big]\big[A(Y)B(X) - A(X)B(Y)\big]. \tag{8.7}$$

□

Plugging $X = \rho$ and $Y = \mu$ in (8.7), we get

$$L_S = (2\alpha + \epsilon_A\beta + \epsilon_B\gamma)b - \frac{r}{n}\big(\frac{a}{n-1} + 2b\big) - a\epsilon_A\epsilon_B R(\rho, \mu, \rho, \mu). \tag{8.8}$$

This leads to the following:

Theorem 8.2. *In a quasi-conformal Ricci pseudosymmetric $G(QE)_n$ with $\beta \neq \epsilon_A\epsilon_B\gamma$, L_S is determined by the relation* (8.8).

§9. Significance of the study

In differential geometry and mathematical physics, an Einstein manifold is a pseudo-Riemannian manifold whose Ricci tensor is proportional to the metric [1]. This name was given after A. Einstein because this condition is equivalent to saying that the metric is a solution of the vacuum Einstein field equations (with cosmological constant), although the dimension, as well as the signature, of the metric can be arbitrary, unlike the four-dimensional Lorentzian manifolds usually studied in general relativity. Einstein manifolds has many applications in mathematical physics such as string theory and supergravity.

The generalizations of Einstein manifolds help us to have a deeper understanding of the global characteristics of the universe including its topology. Quasi-Einstein manifold is a simple and natural generalization of an Einstein manifold. The generalized quasi-Einstein manifold is a further generalization of quasi-Einstein manifold. Also in Cosmology, space-time models are studied in order to represent the different phases in the evolution of the Universe which can be divided into three phases:
Initial Phase: The initial phase is just after the big bang when the effects of both viscosity and heat flux were quite pronounced.
Intermediate Phase: The effect of viscosity was no longer significant but the heat flux was still not negligible.
Final Phase: This phase extends to the present state of the universe. In this phase, both the effects of viscosity and the heat flux have become negligible and the matter content of the universe may be assumed to be a perfect fluid.

The significance of the study of $G(QE)_n$ and $(QE)_n$ lies in the fact that $G(QE)_n$ space-time manifold represents the second phase while $(QE)_n$ the space-time manifold correspond to the third phase in the evolution of the universe [23]. One way of understanding the geometric properties of such manifolds is by studying the tensors these manifolds admit.

In this paper, we have studied the concircular Ricci pseudosymmetric $G(QE)_n$, projective Ricci pseudosymmetric $G(QE)_n$, W_3-Ricci pseudosymmetric $G(QE)_n$, conharmonic Ricci pseudosymmetric $G(QE)_n$, conformal

Ricci pseudosymmetric $G(QE)_n$ and quasi-conformal Ricci pseudosymmetric $G(QE)_n$. Here, each curvature tensor has geometrical significance and hence each type of Ricci pseudosymmetries has different geometrical interpretation. For instances, concircular curvature tensor is an interesting invariant of a concircular transformation. A transformation of an n-dimensional pseudo-Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation [34]. A concircular transformation is always a conformal transformation [26]. Here geodesic circle means a curve in M whose first curvature is constant and whose second curvature is identically zero. Thus the geometry of concircular transformations, that is, the concircular geometry, is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism. Also pseudo-Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus, the concircular curvature tensor is a measure of the failure of a pseudo-Riemannian manifold to be of constant curvature.

The projective curvature tensor is an important tensor from the differential point of view. Let M be an n-dimensional pseudo-Riemannian manifold. If there exists a one to one correspondence between each coordinate neighbourhood of M and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 3$, M is locally projectively flat if and only if the projective curvature tensor vanishes. Here the projective curvature tensor P is given by (4.1). In fact M is projectively flat if and only if it is of constant curvature [7]. Thus the projective curvature tensor is the measure of the failure of a pseudo-Riemannian manifold to be of constant curvature. Again the W_3-curvature tensor has many relativistic significance, see [27].

In differential geometry, the Weyl curvature tensor, named after Hermann Weyl, is a measure of the curvature of spacetime or, more generally, a pseudo-Riemannian manifold. Like the Riemann curvature tensor, the Weyl tensor expresses the tidal force that a body feels when moving along a geodesic. The Weyl tensor differs from the Riemann curvature tensor in that it does not convey information on how the volume of the body changes, but rather only how the shape of the body is distorted by the tidal force. The Ricci curvature, or trace component of the Riemann tensor contains precisely the information about how volumes change in the presence of tidal forces, so the Weyl tensor is the traceless component of the Riemann tensor. It is a tensor that has the same symmetries as the Riemann tensor with the extra condition that it be trace-free: metric contraction on any pair of indices yields zero.

In general relativity, the Weyl curvature is the only part of the curvature that exists in free space, a solution of the vacuum Einstein equation and it governs the propagation of gravitational radiation through regions of space devoid of matter. More generally, the Weyl curvature is the only component of curvature for Ricci-flat manifolds and always governs the characteristics of the field equations of an Einstein manifold. In dimensions 2 and 3 the Weyl curvature tensor vanishes identically. In dimensions ≥ 4, the Weyl curvature is generally nonzero. If the Weyl tensor vanishes in dimension ≥ 4, then the metric is locally conformally flat: there exists a local coordinate system in which the metric tensor is proportional to a constant tensor. This fact was a key component of Nordström's theory of gravitation, which was a precursor of general relativity.

The Weyl tensor has the special property that it is invariant under conformal changes to the metric. For this reason the Weyl tensor is also called the conformal tensor. It follows that a necessary condition for a pseudo-Riemannian manifold to be conformally flat is that the Weyl tensor vanish. In dimensions ≥ 4 this condition is sufficient as well. In dimension 3 the vanishing of the Cotton tensor is a necessary and sufficient condition for the Riemannian manifold being conformally flat. Any 2-dimensional pseudo-Riemannian manifold is conformally flat, a consequence of the existence of isothermal coordinates. Conformal transformations of a pseudo-Riemannian structures are an important object of study in differential geometry. Of considerable interest in a special type of conformal transformations, conharmonic transformations, which are conformal transformations are preserving the harmonicity property of smooth functions. This type of transformation was introduced by Ishii [24] in 1957 and is now studied from various points of view. It is well known that such transformations have a tensor invariant, the so-called conharmonic curvature tensor. It is easy to verify that this tensor is an algebraic curvature tensor; that is, it possesses the classical symmetry properties of the pseudo Riemannian curvature tensor. It is known that a harmonic function is defined as a function whose Laplacian vanishes. A harmonic function is not invariant, in general. The conditions under which a harmonic function remains invariant have been studied by Ishii [24] who introduced the conharmonic transformation as a subgroup of the conformal transformation. A pseudo Riemannian manifold whose conharmonic curvature tensor vanishes at every point of the manifold is called conharmonically flat manifold. Thus this tensor represents the deviation of the manifold from conharmonic flatness. Similarly quasi-conformal curvature tensor has geometrical significance in physics.

A geometrical interpretation of Ricci pseudosymmetric manifolds, in the Riemannian case, is given in [25]. Due to importance of each type of Ricci pseudosymmetries in physics we motivate to study this topic.

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Shyamal Kumar Hui
Department of Mathematics, Sidho Kanho Birsha University
Purulia - 723104, West Bengal, India
E-mail: shyamal_hui@yahoo.co.in

Richard S. Lemence
Institute of Mathematics, College of Science, University of the Philippines
Diliman, Quezon City 1101 Philippines
E-mail: rslemence@math.upd.edu.ph