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Even vertex odd mean labeling of transformed trees

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Abstract. A graph G with p vertices and q edges is said to have an even vertex odd mean labeling if there exists an injective function $f : V(G) \to \{0, 2, 4, \ldots, 2q-2, 2q\}$ such that the induced map $f^*E(G) \to \{1, 3, 5, \ldots, 2q-1\}$ defined by $f^*(uv) = \frac{f(u)+f(v)}{2}$ is a bijection. A graph that admits an even vertex odd mean labeling is called an even vertex odd mean graph. In this paper, we prove that every T_p-tree T, $T@P_n, T@2P_n, T \odot \overline{K_n}, T@C_n$ and $T\hat{o}C_n$ are even vertex odd mean graphs.

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§1. Introduction

Throughout this paper, by a graph we mean a finite, undirected and simple graph. Let $G(V, E)$ be a graph with p vertices and q edges. For notations and terminology we follow [4].

Path on n vertices is denoted by P_n and a cycle on n vertices is denoted by C_n. The corona $G_1 \odot G_2$ of the graphs G_1 and G_2 is obtained by taking one copy of G_1 with p vertices and p copies of G_2 and joining the i^{th} vertex of G_1 to every vertex of the i^{th} copy of G_2.

Let T be a tree and u_0 and v_0 be two adjacent vertices in $V(T)$. Let there be two pendant vertices u and v in T such that the length of $u_0 - u$ path is equal to the length of $v_0 - v$ path. If the edge u_0v_0 is deleted from T and u, v are joined by an edge uv, then such a transformation of T is called an elementary parallel transformation (or an EPT) and the edge u_0v_0 is called a transformable edge. If by a sequence of EPT's T can be reduced to a path, then T is called a T_p-tree (transformed tree) and any such sequence regarded as a composition of mappings (EPT's) denoted by P, is called a parallel transformation of T. The path, the image of T under P is denoted as $P(T)$. A T_p-tree and a sequence of two EPT's reducing it to a path are shown in Figure 1.

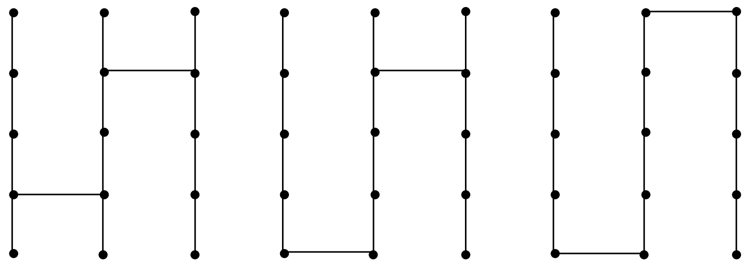

Figure 1. A T_p-tree and a sequence of two EPT's reducing it to a path.

Let T be a T_p-tree with m vertices. Let $T@P_n$ be the graph obtained from T and m copies of P_n by identifying a pendant vertex of i^{th} copy of P_n with i^{th} vertex of T. Let $T@2P_n$ be the graph obtained from T by identifying the pendant vertices of two vertex disjoint paths of equal lengths $n-1$ at each vertex of the T_p-tree T. Let $T@C_n$ be a graph obtained from T and m copies of C_n by identifying a vertex of i^{th} copy of C_n with i^{th} vertex of T. Let $T\widehat{o}C_n$ be a graph obtained from T and m copies of C_n by joining a vertex of i^{th} copy of C_n with i^{th} vertex of T by an edge.

The graceful labelings of graphs was first introduced by Rosa, in 1967 [1] and R.B. Gnanajothi introduced odd graceful graphs [3]. The concept of mean labeling was introduced and meanness of some standard graphs was studied by S. Somasundaram and R. Ponraj [7]. Further some more results on mean graphs are discussed in [6, 8, 9]. A graph G is said to be a mean graph if there exists an injective function f from $V(G)$ to $\{0, 1, 2, \ldots, q\}$ such that the induced map f^* from $E(G)$ to $\{1, 2, 3, \ldots, q\}$ defined by $f^*(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ is a bijection.

In [5], K. Manickam and M. Marudai introduced odd mean labeling of a graph. A graph G is said to be odd mean if there exists an injective function f from $V(G)$ to $\{0, 1, 2, 3, \ldots, 2q-1\}$ such that the induced map f^* from $E(G)$ to $\{1, 3, 5, \ldots, 2q-1\}$ defined by $f^*(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ is a bijection. Further some new families of odd mean graphs are discussed in [11, 12]. The concept of even mean labeling was introduced and studied by B. Gayathri and R. Gopi [2]. A function f is called an even mean labeling of a graph G with p vertices and q edges, if f is an injection from the vertices of G to the set $\{2, 4, \ldots, 2q\}$ such that when each edge uv is assigned the label $\frac{f(u)+f(v)}{2}$, then the resulting edge labels are distinct. A graph which admits an even mean labeling is said to be even mean graph.

Motivated by these, R. Vasuki et al. introduced the concept of even vertex odd mean labeling [10] and discussed the even vertex odd mean behaviour of some standard graphs. A graph G with p vertices and q edges is said to have an even vertex odd mean labeling if there exists an injective function $f : V(G) \to \{0, 2, 4, \ldots, 2q-2, 2q\}$ such that the induced map $f^*E(G) \to \{1, 3, 5, \ldots, 2q-1\}$ defined by $f^*(uv) = \frac{f(u)+f(v)}{2}$ is a bijection. A graph that admits an even vertex odd mean labeling is called an even vertex odd mean graph.

An even vertex odd mean labeling of $P_6 \odot K_1$ is shown in Figure 2.

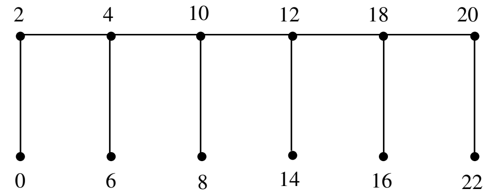

Figure 2. An even vertex odd mean labeling of $P_6 \odot K_1$.

In this paper, we prove that every T_p-tree T, $T@P_n, T@2P_n, T \odot \overline{K_n}, T@C_n$ and $T\hat{\circ}C_n$ are even vertex odd mean graphs.

§2. Even vertex odd mean graphs

Theorem 2.1. *Every T_p-tree T is an even vertex odd mean graph.*

Proof. Let T be a T_p-tree with m-vertices. By the definition of a T_p-tree there exists a parallel transformation P of T such that for the path $P(T)$, we have (i) $V(P(T)) = V(T)$ and (ii) $E(P(T)) = (E(T) - E_d) \cup E_p$, where E_d is the set of edges deleted from T and E_p is the set of edges newly added through the sequence $P = (P_1, P_2, \ldots, P_k)$ of the EPT's P used to arrive at the path $P(T)$. Clearly E_d and E_p have the same number of edges. Now, denote the vertices of $P(T)$ successively as $v_1, v_2, v_3, \ldots, v_m$ starting from one pendant vertex of $P(T)$ right upto the other.

Define $f : V(T) \to \{0, 2, \ldots, 2q-2, 2q = 2(m-1)\}$ as follows:

$$f(v_i) = 2i - 2, \quad 1 \le i \le m.$$

Let v_iv_j be an edge of T for some indices i and $j, 1 \le i \le j \le m$ and let P_1 be the EPT that deletes this edge and adds the edge $v_{i+t}v_{j-t}$ where t is the distance from v_i to v_{i+1} and also the distance from v_j to v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent EPT's. Since $v_{i+t}v_{j-t}$ is an edge of the path $P(T)$, it follows that $i + t + 1 = j - t$, which implies $j = i + 2t + 1$. The induced label of the edge $v_i v_j$ is given by

$$\begin{aligned} f^*(v_i v_j) &= f^*(v_i v_{i+2t+1}) \\ &= \frac{f(v_i) + f(v_{i+2t+1})}{2} \\ &= 2(i+t) - 1 \text{ and} \\ f^*(v_{i+t} v_{j-t}) &= f^*(v_{i+t} v_{i+t+1}) \\ &= \frac{f(v_{i+t}) + f(v_{i+t+1})}{2} \\ &= 2(i+t) - 1 \\ \text{Therefore, } f^*(v_i v_j) &= f^*(v_{i+t} v_{j-t}). \end{aligned}$$

For each vertex label f, the induced edge label f^* is defined as follows:

$$f^*(v_i v_{i+1}) = 2i - 1, \quad 1 \le i \le m-1.$$

It can be verified that f is an even vertex odd mean labeling. Hence, every T_P-tree T is an even vertex odd mean graph.

For example, an even vertex odd mean labeling of a T_p-tree with 18 vertices is given in Figure 3.

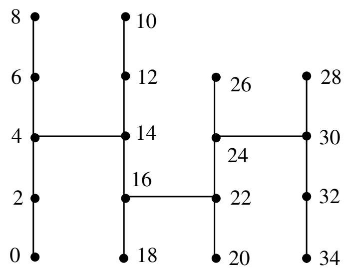

Figure 3. An even vertex odd mean labeling of a T_p-tree.

□

Theorem 2.2. *Let T be a T_p-tree on m-vertices. Then the graph $T@P_n$ is an even vertex odd mean graph.*

Proof. Let T be a T_p-tree with m-vertices. By the definition of a T_p-tree there exists a parallel transformation P of T such that for the path $P(T)$ we have (i) $V(P(T)) = V(T)$ and (ii) $E(P(T)) = (E(T) - E_d) \cup E_p$, where E_d is the set of edges deleted from T and E_p is the set of edges newly added through the sequence $P = (P_1, P_2, \ldots, P_k)$ of the EPT's P used to arrive at the path $P(T)$.

Clearly E_d and E_p have the same number of edges. Now denote the vertices of $P(T)$ successively as $v_1, v_2, v_3, \ldots, v_m$ starting from one pendant vertex of

$P(T)$ right upto other. Let $u_1^j, u_2^j, u_3^j, \ldots, u_n^j (1 \leq j \leq m)$ be the vertices of j^{th} copy of P_n. Then $V(T@P_n) = \{u_i^j : 1 \leq i \leq n, 1 \leq j \leq m \text{ with } u_n^j = v_j\}$

The graph $T@P_n$ has mn vertices and $mn-1$ edges. Define $f : V(T@P_n) \to \{0, 2, 4, \ldots, 2q-2, 2q = 2(mn-1)\}$ as follows:
For $1 \leq j \leq m$,

$$f(u_i^j) = \begin{cases} 2n(j-1) + 2i - 2, & 1 \leq i \leq n \text{ and } j \text{ is odd} \\ 2nj - 2i, & 1 \leq i \leq n \text{ and } j \text{ is even.} \end{cases}$$

Let $v_i v_j$ be a transformed edge in T for some indices i and $j, 1 \leq i \leq j \leq m$ and let P_1 be the EPT that deletes the edge $v_i v_j$ and adds the edge $v_{i+t} v_{j-t}$ where t is the distance of v_i from v_{i+t} and also the distance of v_j from v_{j-t}.

Let P be a parallel transformation of T that contains P_1 as one as the constituent EPT's. Since $v_{i+t} v_{j-t}$ is an edge in the path $P(T), i+t+1 = j-t$, which implies $j = i + 2t + 1$. The induced label of the edge $v_i v_j$ is given by

$$\begin{aligned} f^*(v_i v_j) &= f^*(v_i v_{i+2t+1}) \\ &= \frac{f(v_i) + f(v_{i+2t+1})}{2} \\ &= 2n(i+t) - 1 \text{ and} \\ f^*(v_{i+t} v_{j-t}) &= f^*(v_{i+t} v_{i+t+1}) \\ &= \frac{f(v_{i+t}) + f(v_{i+t+1})}{2} \\ &= 2n(i+t) - 1 \\ \text{Therefore, } f^*(v_i v_j) &= f^*(v_{i+t} v_{j-t}). \end{aligned}$$

For each vertex label f, the induced edge label f^* is obtained as follows:
For $1 \leq j \leq m$,

$$f^*(u_i^j u_{i+1}^j) = \begin{cases} 2n(j-1) + 2i - 1, & 1 \leq i \leq n-1 \text{ and } j \text{ is odd} \\ 2nj - (2i+1), & 1 \leq i \leq n-1 \text{ and } j \text{ is even} \end{cases}$$
$$f^*(v_j v_{j+1}) = 2nj - 1 \text{ for } 1 \leq j \leq m-1.$$

It can be verified that f is an even vertex odd mean labeling of $T@P_n$. Hence, $T@P_n$ is an even vertex odd mean graph.

For example, an even vertex odd mean labeling of $T@P_5$, where T is a T_p-tree with 14-vertices is given in Figure 4.

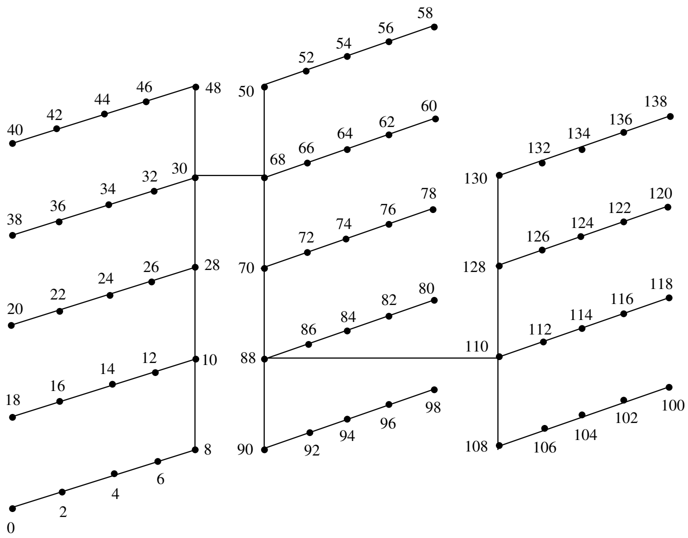

Figure 4. An even vertex odd mean labeling of $T@P_5$.

□

Theorem 2.3. *Let T be a T_p-tree on m-vertices. Then the graph $T@2P_n$ is an even vertex odd mean graph.*

Proof. Let T be a T_p-tree with m-vertices. By the definition of a T_p-tree there exists a parallel transformation P of T such that for the path $P(T)$ we have (i) $V(P(T)) = V(T)$ and (ii) $E(P(T)) = (E(T) - E_d) \cup E_p$, where E_d is the set of edges deleted from T and E_p is the set of edges newly added through the sequence $P = (P_1, P_2, \ldots, P_k)$ of the EPT's P used to arrive at the path $P(T)$. Clearly E_d and E_p have the same number of edges.

Now denote the vertices of $P(T)$ successively as $v_1, v_2, v_3, \ldots, v_m$ starting from one pendant vertex of $P(T)$ right upto other. Let $u^j_{1,1}, u^j_{1,2}, u^j_{1,3}, \ldots, u^j_{1,n}$ and $u^j_{2,1}, u^j_{2,2}, u^j_{2,3}, \ldots, u^j_{2,n} (1 \leq j \leq m)$ be the vertices of the two vertex disjoint paths joined with j^{th} vertex of T such that $v_j = u^j_{1,n} = u^j_{2,n}$. Then $V(T@2P_n) = \{v_j, u^j_{1,i} u^j_{2,i} : 1 \leq i \leq n, 1 \leq j \leq m \text{ with } v_j = u^j_{1,n} = u^j_{2,n}\}$.

Define $f : V(T@2P_n) \to \{0, 2, 4, \ldots, 2q-2, 2q = 2m(2n-1)-2\}$ as follows:

$$f(u^j_{1,i}) = (4n-2)(j-1) + 2i - 2, \quad 1 \leq i \leq n \text{ and } 1 \leq j \leq m$$
$$f(u^j_{2,i}) = (4n-2)j - 2i, \quad 1 \leq i \leq n-1 \text{ and } 1 \leq j \leq m.$$

Let $v_i v_j$ be a transformed edge in T for some indices i and $j, 1 \leq i \leq j \leq m$ and let P_1 be the EPT that deletes the edge $v_i v_j$ and adds the edge $v_{i+t} v_{j-t}$ where t is the distance of v_i from v_{i+t} and also the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one as the constituent

EPT's. Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T), i+t+1 = j-t$, which implies $j = i+2t+1$. The induced label of the edge v_iv_j is given by

$$\begin{aligned} f^*(v_iv_j) &= f^*(v_iv_{i+2t+1}) \\ &= \frac{f(v_i)+f(v_{i+2t+1})}{2} \\ &= (4n-2)(i+t)-1 \text{ and} \\ f^*(v_{i+t}v_{j-t}) &= f^*(v_{i+t}v_{i+t+1}) \\ &= \frac{f(v_{i+t})+f(v_{i+t+1})}{2} \\ &= (4n-2)(i+t)-1 \\ \text{Therefore, } f^*(v_iv_j) &= f^*(v_{i+t}v_{j-t}). \end{aligned}$$

For each vertex label f, the induced edge label f^* is obtained as follows:

$$\begin{aligned} f^*(v_jv_{j+1}) &= (4n-2)j-1, \quad 1 \le j \le m-1 \\ f^*(u^j_{1,i}u^j_{1,i+1}) &= (4n-2)(j-1)+2i-1, \quad 1 \le i \le n-1 \text{ and } 1 \le j \le m \\ f^*(u^j_{2,i}u^j_{2,i+1}) &= (4n-2)j-(2i+1), \quad 1 \le i \le n-1 \text{ and } 1 \le j \le m. \end{aligned}$$

It can be verified that f is an even vertex odd mean labeling of $T@2P_n$. Hence, $T@2P_n$ is an even vertex odd mean graph.

For example, an even vertex odd mean labeling of $T@2P_4$, where T is a T_p-tree with 10-vertices is given in Figure 5.

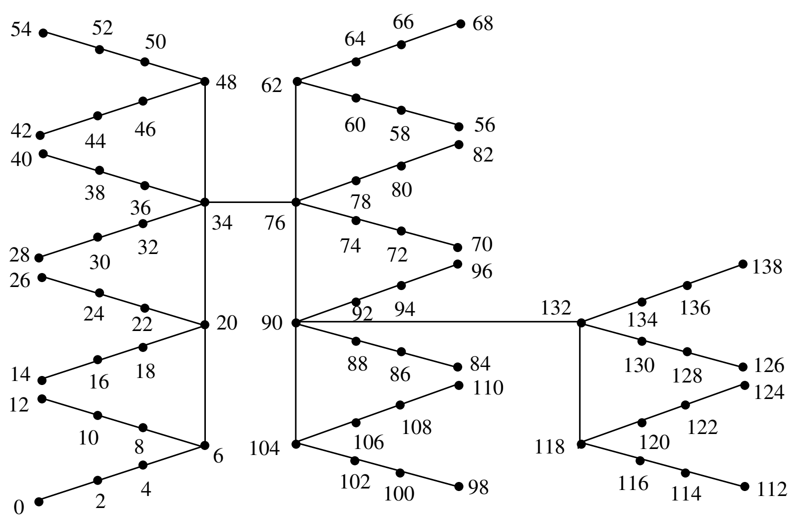

Figure 5. An even vertex odd mean labeling of $T@2P_4$.

□

Theorem 2.4. *Let T be a T_p-tree on m-vertices. Then the graph $T \odot \overline{K_n}$ is an even vertex odd mean graph if m is even.*

Proof. Let T be a T_p-tree with m-vertices with the vertex set $V(T) = \{v_1, v_2, \ldots, v_m\}$. Let $u_1^j, u_2^j, \ldots, u_n^j$ be the pendant vertices joined with $v_j (1 \leq j \leq m)$ by an edge. Then $V(T \odot \overline{K_n}) = \{v_j, u_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$.

By the definition of a T_p-tree there exists a parallel transformation P of T such that for the path $P(T)$ we have (i) $V(P(T)) = V(T)$ and (ii) $E(P(T)) = (E(T) - E_d) \cup E_p$, where E_d is the set of edges deleted from T and E_p is the set of edges newly added through the sequence $P = (P_1, P_2, \ldots, P_k)$ of the EPT's P used to arrive at the path $P(T)$. Clearly E_d and E_p have the same number of edges.

Now denote the vertices of $P(T)$ successively as $v_1, v_2, v_3, \ldots, v_m$ starting from one pendant vertex of $P(T)$ right upto the other.

We define $f : V(T \odot \overline{K_n}) \to \{0, 2, 4, \ldots, 2q-2, 2q = 2m(n+1) - 2\}$ as follows:

$$f(v_j) = \begin{cases} 2n(j-1) + 2j & \text{for } 1 \leq j \leq m \text{ and } j \text{ is odd} \\ 2j(n+1) - 4 & \text{for } 1 \leq j \leq m \text{ and } j \text{ is even} \end{cases}$$

$$f(u_i^j) = \begin{cases} 2(n+1)(j-1) + 4i - 4 & \text{for } j \text{ is odd, } 1 \leq j \leq m \text{ and } 1 \leq i \leq n \\ 2(n+1)(j-2) + 4i + 2 & \text{for } j \text{ is even, } 1 \leq j \leq m \text{ and } 1 \leq i \leq n \end{cases}$$

Let $v_i v_j$ be an edge of T for some indices i and $j, 1 \leq i \leq j \leq n$ and let P_1 be the EPT that deletes this edge and adds the edge $v_{i+t} v_{j-t}$ where t is the distance from v_i to v_{i+t} and also the distance from v_j to v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent EPT's. Since $v_{i+t} v_{j-t}$ is an edge in the path $P(T)$, it follows that $i + t + 1 = j - t$, which implies $j = i + 2t + 1$. The induced label of the edge $v_i v_j$ is given by

$$\begin{aligned} f^*(v_i v_j) &= f^*(v_i v_{i+2t+1}) \\ &= \frac{f(v_i) + f(v_{i+2t+1})}{2} \\ &= (2n+2)(i+t) - 1 \text{ and} \\ f^*(v_{i+t} v_{j-t}) &= f^*(v_{i+t} v_{i+t+1}) \\ &= \frac{f(v_{i+t}) + f(v_{i+t+1})}{2} \\ &= (2n+2)(i+t) - 1 \\ \text{Therefore, } f^*(v_i v_j) &= f^*(v_{i+t} v_{j-t}). \end{aligned}$$

For each vertex label f, the induced edge labeling f^* is obtained as follows:

$$f^*(v_j u_i^j) = 2(n+1)(j-1) + 2i - 1, \quad 1 \leq j \leq m \text{ and } 1 \leq i \leq n$$
$$f^*(v_j v_{j+1}) = 2j(n+1) - 1, \quad 1 \leq j \leq m - 1.$$

It can be verified that f is an even vertex odd mean labeling. Hence, $T \odot \overline{K_n}$ is an even vertex odd mean graph.

For example, an even vertex odd mean labeling of $T \odot \overline{K_5}$, where T is a T_p-tree with 12-vertices is given in Figure 6.

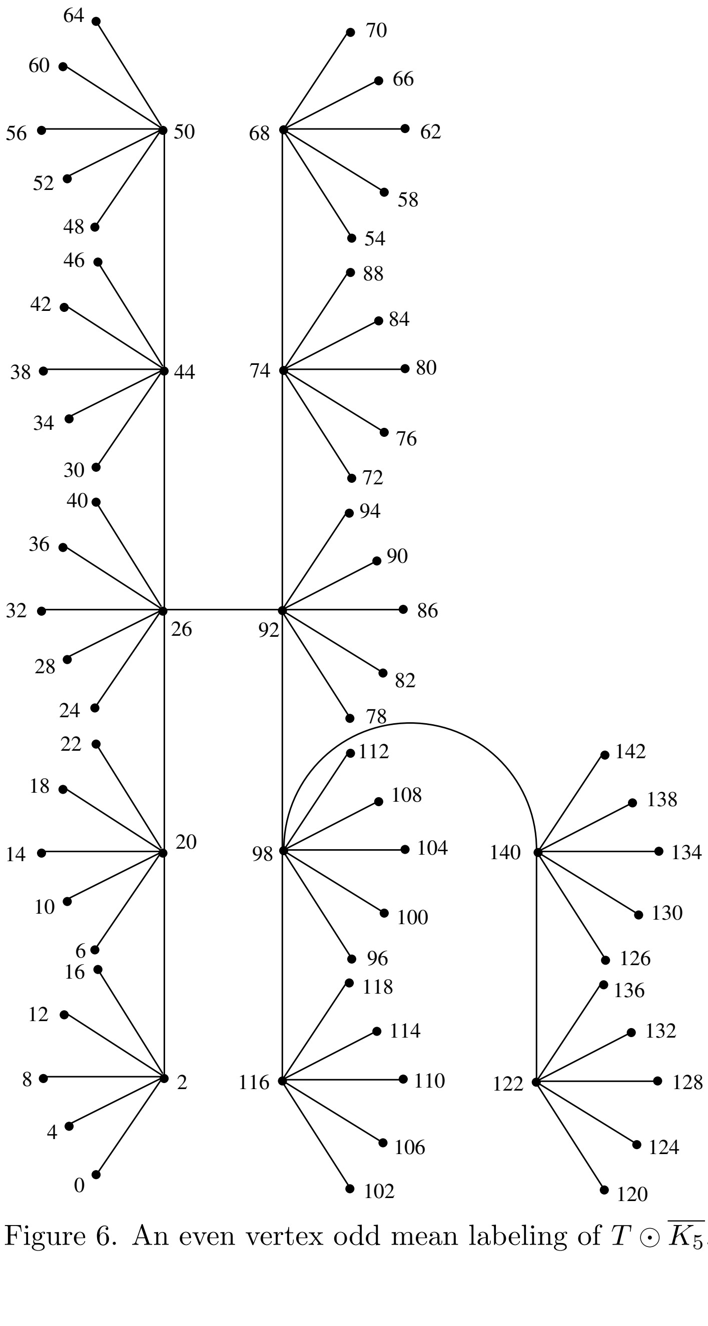

Figure 6. An even vertex odd mean labeling of $T \odot \overline{K_5}$.

□

Theorem 2.5. *Let T be a T_p-tree on m-vertices. Then the graph $T@C_n$ is an even vertex odd mean graph if $n \equiv 0(mod\ 4)$.*

Proof. Let T be a T_p-tree with m-vertices. By the definition of a transformed tree there exists a parallel transformation P of T such that for the path $P(T)$ we have (i) $V(P(T)) = V(T)$ and (ii) $E(P(T)) = (E(T) - E_d) \cup E_p$, where E_d is the set of edges deleted from T and E_p is the set of edges newly added through the sequence $P = (P_1, P_2, \ldots, P_k)$ of the EPT's P used to arrive at the path $P(T)$. Clearly, E_d and E_p have the same number of edges.

Now denote the vertices of $P(T)$ successively by $v_1, v_2, v_3, \ldots, v_m$ starting from one pendant vertex of $P(T)$ right upto the other one.

Let $u_1^j, u_2^j, u_3^j, \ldots, u_n^j (1 \leq j \leq m)$ be the vertices of j^{th} copy of P_n. Then $V(T@C_n) = \{u_i^j : 1 \leq i \leq n, 1 \leq j \leq m \text{ with } u_1^j = v_j\}$.

Define $f : V(T@C_n) \to \{0, 2, 4, \ldots, 2q-2, 2q = 2m(n+1)-2\}$ as follows:
Case (i). j is odd.

$$f(u_i^j) = \begin{cases} 2(n+1)(j-1)+2i-2, & 1 \leq j \leq m \text{ and } 1 \leq i \leq \frac{n}{2} \\ 2(n+1)(j-1)+2i+2, & 1 \leq j \leq m, \frac{n}{2}+1 \leq i \leq n \text{ and } i \text{ is odd} \\ 2(n+1)(j-1)+2i-2, & 1 \leq j \leq m, \frac{n}{2}+2 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

Case (ii). j is even.

$$f(u_i^j) = \begin{cases} 2(n+1)j-2i, & 1 \leq j \leq m \text{ and } 1 \leq i \leq \frac{n}{2} \\ 2(n+1)j-2(i+2), & 1 \leq j \leq m, \frac{n}{2}+1 \leq i \leq n \\ & \quad \text{and } i \text{ is odd} \\ 2(n+1)j-2i, & 1 \leq j \leq m, \frac{n}{2}+2 \leq i \leq n \\ & \quad \text{and } i \text{ is even.} \end{cases}$$

Let v_iv_j be a transformed edge in T for some indices i and $j, 1 \leq i \leq j \leq m$ and let P_1 be the EPT that deletes the edge v_iv_j and adds the edge $v_{i+t}v_{j-t}$ where t is the distance of v_i from v_{i+t} and also the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent EPT's. Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that, $i+t+1 = j-t$, which implies $j = i+2t+1$. Therefore i and j are of opposite parity, that is i is odd and j is even or vice-versa.

The induced label of the edge v_iv_j is given by

$$\begin{aligned} f^*(v_iv_j) &= f^*(v_iv_{i+2t+1}) \\ &= \frac{f(v_i)+f(v_{i+2t+1})}{2} \end{aligned}$$

$$\begin{aligned}
&= 2(n+1)(i+t)-1 \\
\text{and } f^*(v_{i+t}v_{j-t}) &= f^*(v_{i+t}v_{i+t+1}) \\
&= \frac{f(v_{i+t})+f(v_{i+t+1})}{2} \\
&= 2(n+1)(i+t)-1 \\
\text{Therefore, } f^*(v_iv_j) &= f^*(v_{i+t}v_{j-t}).
\end{aligned}$$

For each vertex label f, the induced edge label f^* is defined as follows:

$$f^*(v_jv_{j+1}) = 2(n+1)j-1, \quad 1 \le j \le m-1.$$

For $1 \le j \le m$ and j is odd,

$$f^*(u_i^ju_{i+1}^j) = \begin{cases} 2(n+1)(j-1)+2i-1, & 1 \le i \le \frac{n}{2}-1 \\ 2(n+1)(j-1)+2i+1, & \frac{n}{2} \le i \le n-1 \end{cases}$$
$$f^*(u_n^ju_1^j) = 2(n+1)j-(n+3).$$

For $1 \le j \le m$ and j is even,

$$f^*(u_i^ju_{i+1}^j) = \begin{cases} 2(n+1)j-2i-1, & 1 \le i \le \frac{n}{2}-1 \\ 2(n+1)j-2i-3, & \frac{n}{2} \le i \le n-1 \end{cases}$$
$$f^*(u_n^ju_1^j) = 2(n+1)j-(n+1).$$

It can be verified that f is an even vertex odd mean labeling of $T@C_n$ if $n \equiv 0(mod\,4)$. Hence, $T@C_n$ is an even vertex odd mean graph if $n \equiv 0(mod\,4)$.

For example, an even vertex odd mean labeling of $T@C_8$, where T is a T_p-tree with 13 vertices is shown in Figure 7.

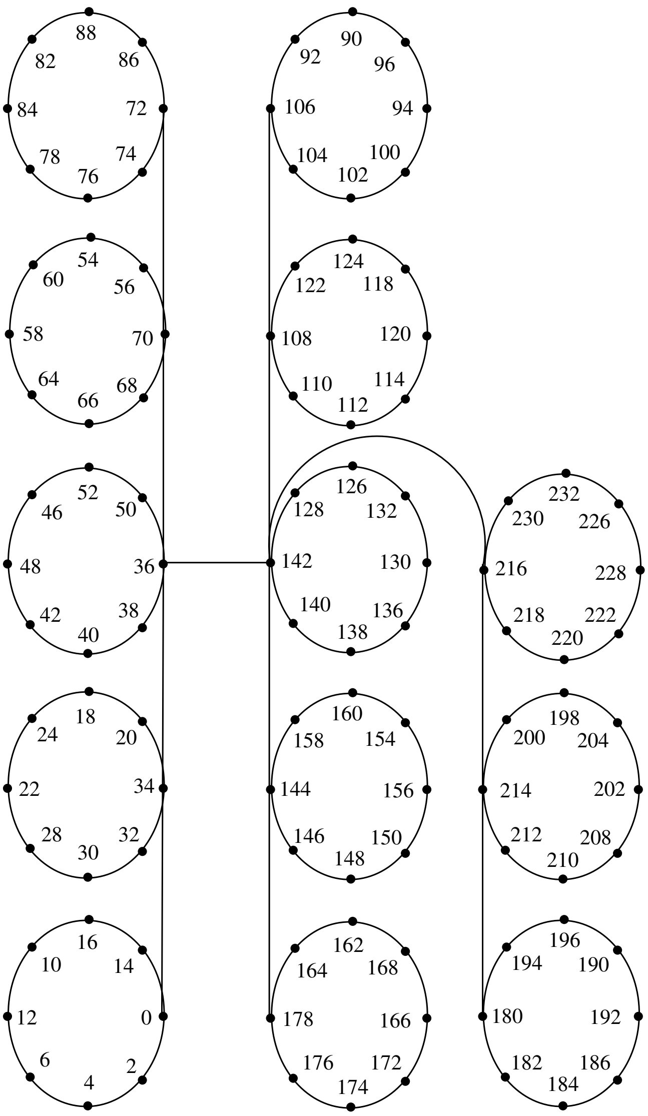

Figure 7. An even vertex odd mean labeling of $T@C_8$.

□

Theorem 2.6. *Let T be a T_p-tree on m-vertices. Then the graph $T\check{o}C_n$ is an even vertex odd mean graph if $n \equiv 0(mod\ 4)$.*

Proof. Let T be a T_p-tree with m-vertices. By the definition of a T_p-tree there exists a parallel transformation P of T such that for the path $P(T)$ we have (i) $V(P(T)) = V(T)$ and (ii) $E(P(T)) = (E(T) - E_d) \cup E_p$, where E_d is the set of edges deleted from T and E_p is the set of edges newly added through the sequence $P = (P_1, P_2, \ldots, P_k)$ of the EPT's P used to arrive at the path $P(T)$. Clearly E_d and E_p have the same number of edges.

Now denote the vertices of $P(T)$ successively as $v_1, v_2, v_3, \ldots, v_m$ starting from one pendant vertex of $P(T)$ right upto other. Let $u_1^i, u_2^i, \ldots, u_n^i$ be the vertices of the i^{th} copy of C_n for $1 \le i \le n$. Then

$$V(T\check{o}C_n) = \{v_j, u_i^j : 1 \le i \le n, 1 \le j \le m\}$$
$$\text{and } E(T\check{o}C_n) = E(T) \cup E(C_n) \cup \{v_j u_1^j : 1 \le j \le m\}.$$

Define $f : V(T\check{o}C_n) \to \{0, 2, 4, \ldots, 2q-2, 2q = 2m(n+2)-2\}$ as follows:

$$f(v_j) = \begin{cases} 2(n+2)(j-1), & 1 \le j \le m \text{ and } j \text{ is odd} \\ 2(n+2)j - 2, & 1 \le j \le m \text{ and } j \text{ is even.} \end{cases}$$

For j is odd,

$$f(u_i^j) = \begin{cases} 2(n+2)(j-1) + 2i, & 1 \le j \le m \text{ and } 1 \le i \le \frac{n}{2} \\ 2(n+2)(j-1) + 2i + 4, & 1 \le j \le m,\ \frac{n}{2} + 1 \le i \le n \\ & \quad \text{and } i \text{ is odd} \\ 2(n+2)(j-1) + 2i, & 1 \le j \le m,\ \frac{n}{2} + 2 \le i \le n \\ & \quad \text{and } i \text{ is even.} \end{cases}$$

For j is even,

$$f(u_i^j) = \begin{cases} 2(n+2)j - 2(i+1), & 1 \le j \le m \text{ and } 1 \le i \le \frac{n}{2} \\ 2(n+2)j - 2(i+3), & 1 \le j \le m,\ \frac{n}{2} + 1 \le i \le n \text{ and } i \text{ is odd} \\ 2(n+2)j - 2(i+1), & 1 \le j \le m,\ \frac{n}{2} + 2 \le i \le n \text{ and } i \text{ is even.} \end{cases}$$

Let $v_i v_j$ be a transformed edge in T for some indices i and $j, 1 \le i \le j \le m$ and let P_1 be the EPT that deletes the edge $v_i v_j$ and adds the edge $v_{i+t} v_{j-t}$ where t is the distance of v_i from v_{i+t} and also the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent EPT's. Since $v_{i+t} v_{j-t}$ is an edge in the path $P(T)$, $i + t + 1 = j - t$, which implies $j = i + 2t + 1$. The induced label of the edge $v_i v_j$ is given by

$$\begin{aligned} f^*(v_i v_j) &= f^*(v_i v_{i+2t+1}) \\ &= \frac{f(v_i) + f(v_{i+2t+1})}{2} \\ &= 2(n+2)(i+t) - 1 \text{ and} \\ f^*(v_{i+t} v_{j-t}) &= f^*(v_{i+t} v_{i+t+1}) \\ &= \frac{f(v_{i+t}) + f(v_{i+t+1})}{2} \\ &= 2(n+2)(i+t) - 1 \\ \text{Therefore, } f^*(v_i v_j) &= f^*(v_{i+t} v_{j-t}). \end{aligned}$$

The induced edge label f^* is defined as follows:

$$f^*(v_j v_{j+1}) = 2j(n+2) - 1, \quad 1 \le j \le m - 1.$$

For j is odd,

$$f^*(u_i^j u_{i+1}^j) = \begin{cases} 2(n+2)(j-1)+2i+1, & 1 \leq j \leq m \text{ and } 1 \leq i \leq \frac{n}{2}-1 \\ 2(n+2)(j-1)+2i+3, & 1 \leq j \leq m \text{ and } \frac{n}{2} \leq i \leq n-1 \end{cases}$$

$$f^*(u_n^j u_1^j) = 2(n+2)(j-1)+n+1, \quad 1 \leq j \leq m$$

$$f^*(v_j u_1^j) = 2(n+2)(j-1)+1, \quad 1 \leq j \leq m.$$

For j is even,

$$f^*(u_i^j u_{i+1}^j) = \begin{cases} 2(n+2)j-(2i+3), & 1 \leq j \leq m \text{ and } 1 \leq i \leq \frac{n}{2}-1 \\ 2(n+2)j-(2i+5), & 1 \leq j \leq m \text{ and } \frac{n}{2} \leq i \leq n-1 \end{cases}$$

$$f^*(u_n^j u_1^j) = 2(n+2)j-n-3, \quad 1 \leq j \leq m$$

$$f^*(v_j u_1^j) = 2(n+2)j-3, \quad 1 \leq j \leq m.$$

It can be verified that f is an even vertex odd mean labeling of $T\check{o}C_n$ if $n \equiv 0(mod\, 4)$. Hence, $T\check{o}C_n$ is an even vertex odd mean graph if $n \equiv 0(mod\, 4)$.

For example, an even vertex odd mean labeling of $T\check{o}C_8$ where T is a T_p-tree with 8-vertices is given in Figure 8.

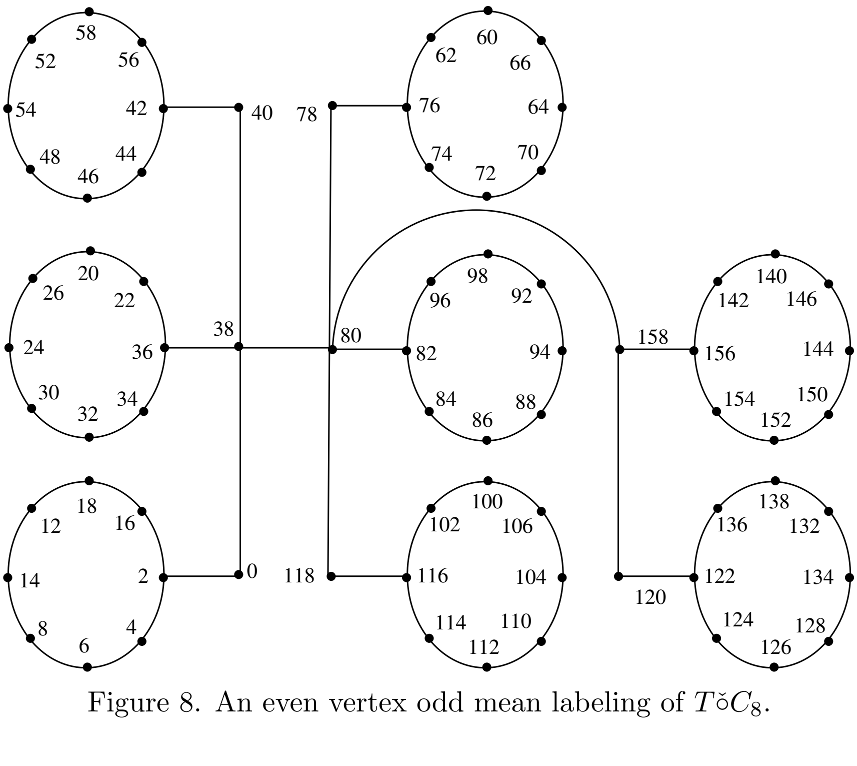

Figure 8. An even vertex odd mean labeling of $T\check{o}C_8$.

□

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