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Enumeration of unlabeled graphs such that both the graph and its complement are 2-connected

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Abstract. It is well-known that the complement of a disconnected graph is connected, that is, the number of disconnected unlabeled graphs whose complement is also disconnected is zero. By this fact, we can easily express the number of connected unlabeled graphs whose complement is also connected, by the numbers of graphs and connected graphs. The generating functions of them are obtained by Harary [5]. In this paper, we express the number of unlabeled graphs such that both the graph and its complement are 2-connected, by the numbers of graphs whose generating functions are known.

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§1. Introduction

In this paper, we consider finite simple unlabeled graphs, which have neither loops nor multiple edges. For terminology and notation not defined in this paper, we refer the readers to [4]. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. For $n \geq 1$, let $\mathcal{G}_n$, $\mathcal{D}_n$, $\mathcal{C}_n$, $\mathcal{S}_n$ and $\mathcal{B}_n$ be the sets of graphs, disconnected graphs, connected graphs, graphs with connectivity one and 2-connected graphs of order n, respectively. For a graph G, $\overline{G}$ denotes the complement of G. For sets $\mathcal{H}_n$ and $\mathcal{K}_n$ of graphs, we let $\mathcal{HK}_n = \{G \in \mathcal{H}_n \mid \overline{G} \in \mathcal{K}_n\}$.

Ramsey theory discusses graphs in which either the graph or its complement has a specified property. As a variation, in 1979–1981, Akiyama and Harary [1, 2, 3] have researched graphs such that both the graph and its complement have a common specified property. In this paper, we focus the connectivity as the specified property, and enumerate such graphs. It is well-known that the complement of a disconnected graph is connected, that is, $|\mathcal{DD}_n| = 0$. By this fact, it is easy to obtain the number of graphs such that both the graph and its complement are connected.

Proposition 1.1. $|\mathcal{CC}_n| = 2|\mathcal{C}_n| - |\mathcal{G}_n|$ *holds for* $n \geq 1$.

Proof. Since $|\mathcal{DD}_n| = 0$ and $|\mathcal{D}_n| = |\mathcal{G}_n| - |\mathcal{C}_n|$, we have $|\mathcal{CD}_n| = |\mathcal{DC}_n| = |\mathcal{D}_n| - |\mathcal{DD}_n| = |\mathcal{D}_n| = |\mathcal{G}_n| - |\mathcal{C}_n|$. Since $|\mathcal{CC}_n| = |\mathcal{C}_n| - |\mathcal{CD}_n|$, it follows that $|\mathcal{CC}_n| = 2|\mathcal{C}_n| - |\mathcal{G}_n|$. □

The numbers of graphs and connected graphs are obtained by Harary [5]. By this theorem, we can obtain the following table.

n	1	2	3	4	5	6	7	8	9
$\lvert\mathcal{CC}_n\rvert$	1	0	0	1	8	68	662	9888	247492

In this paper, we enumerate graphs such that both the graph and its complement are 2-connected. To do it, we need to obtain the cardinalities of $\mathcal{SD}_n$ and $\mathcal{SS}_n$. In 1979, Akiyama and Harary obtained a necessary and sufficient condition for a graph to belong to $\mathcal{SD}_n$ and $\mathcal{SS}_n$. (In 2002, Kawarabayashi, Nakamoto, Oda, Ota, Tazawa and Watanabe rediscovered this result.) For a graph G and $v \in V(G)$, $N_G(v)$ and $\deg_G(v)$ denote the neighborhood and the degree of a vertex v in G, respectively. Moreover, $G - v$ denotes the graph obtained from G by deleting v and the edges incident with v.

Theorem 1.2 (Akiyama and Harary [1], Kawarabayashi et al. [7])**.** *Let* G *be a separable graph with a cut vertex* u.

(i) $\overline{G}$ *is disconnected if and only if* $\deg_G(u) = |V(G)| - 1$.

(ii) $\overline{G}$ *is separable if and only if either* (a) *or* (b) *holds:*

  (a) $\deg_G(u) = |V(G)| - 2$.

  (b) $\deg_G(u) \leq |V(G)| - 3$ *and* G *has a vertex* v *such that* $N_G(v) = \{u\}$ *and* $G - v$ *has a spanning complete bipartite subgraph.*

By using this theorem, we determine the cardinalities of $\mathcal{SD}_n$ and $\mathcal{SS}_n$. We first determine the structure of $\mathcal{SD}_n$ and obtain the cardinality of it. Let $\mathcal{S}_n^0$ be the set of graphs G of order n such that G has exactly one cut vertex u and $\deg_G(u) = n - 1$ (see Figure 1). Then by Theorem 1.2 (i), it is easy to see that $\mathcal{SD}_n = \mathcal{S}_n^0$. Moreover, it is easily observed that there exists a one-to-one correspondence between $\mathcal{S}_n^0$ and $\mathcal{D}_{n-1}$. Hence we obtain the following proposition.

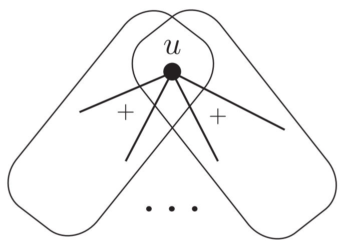

Figure 1: The structure of graphs in $\mathcal{S}_n^0$.

Proposition 1.3. *$|\mathcal{SD}_n| = |\mathcal{D}_{n-1}|$ holds for $n \geq 3$.*

We next determine the structure of $\mathcal{SS}_n$ and obtain the cardinality of it. Let $\mathcal{S}_n^1$ be the set of graphs G of order n such that G has exactly one cut vertex v and $\deg_G(v) = n-2$ (see Figure 2 (i)). Let $\mathcal{S}_n^2$ be the set of graphs G of order n such that G has exactly two cut vertices u_1 and u_2 such that $\deg_G(u_1) = n-2$ and $\deg_G(u_2) = n-2$, moreover and G has a vertex v_i $(i = 1, 2)$ such that $N_G(v_i) = \{u_i\}$ (see Figure 2(ii)). Note that $G \in \mathcal{S}_n^2$ has no cut vertex except for u_1 and u_2. Let $\mathcal{S}_n^3$ be the set of graphs G of order n with a cut vertex u such that G and u satisfy Theorem 1.2 (ii)-(b) (see Figure 2(iii)).

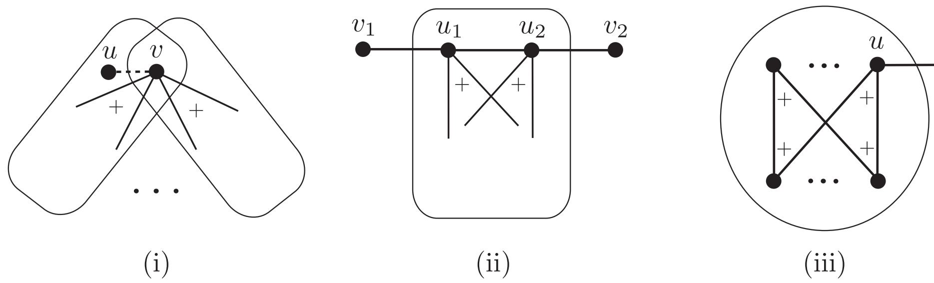

Figure 2: The structures of graphs in $\mathcal{S}_n^1$, $\mathcal{S}_n^2$ and $\mathcal{S}_n^3$

Theorem 1.4. *Let n be an integer with $n \geq 5$. Then $\mathcal{SS}_n = \mathcal{S}_n^1 \cup \mathcal{S}_n^2 \cup \mathcal{S}_n^3$, and $\mathcal{S}_n^i \cap \mathcal{S}_n^j = \emptyset$ for $1 \leq i < j \leq 3$.*

Here we show the relationship among $\mathcal{S}_n^1$, $\mathcal{S}_n^2$ and $\mathcal{S}_n^3$, and obtain the cardinalities of $\mathcal{S}_n^1$ and $\mathcal{S}_n^2$. For a set $\mathcal{H}$ of graphs let $\overline{\mathcal{H}} := \{\overline{H} : H \in \mathcal{H}\}$. A *rooted* graph is a graph in which one vertex has been distinguished as the root. Let $\mathcal{D}_n^r$ be the set of rooted disconnected graphs of order n. The generating functions of rooted graphs and rooted connected graphs have been obtained by Harary [5] (see [6] p.100 for the detail). Notice that the number of rooted disconnected graphs of order n is equal to the difference between the number of rooted graphs of order n and that of rooted connected graphs of order n.

Theorem 1.5. (i) $\overline{\mathcal{S}_n^1} = \mathcal{S}_n^3$ *and* $|\mathcal{S}_n^1| = |\mathcal{D}_{n-1}^r| - |\mathcal{D}_{n-2}^r| - |\mathcal{G}_{n-2}|$ *holds for* $n \geq 3$.

(ii) $\overline{\mathcal{S}_n^2} = \mathcal{S}_n^2$ *and* $|\mathcal{S}_n^2| = |\mathcal{G}_{n-4}|$ *holds for* $n \geq 5$.

We postpone the proofs of Theorems 1.4 and 1.5 to the next section. By Theorems 1.4 and 1.5, we can obtain the cardinality of $\mathcal{SS}_n$.

Theorem 1.6. $|\mathcal{SS}_n| = 2(|\mathcal{D}_{n-1}^r| - |\mathcal{D}_{n-2}^r| - |\mathcal{G}_{n-2}|) + |\mathcal{G}_{n-4}|$ *holds for* $n \geq 5$.

By Proposition 1.3 and Theorem 1.6, we obtain the cardinality of $\mathcal{BB}_n$. The generating function of 2-connected graphs is obtained by Robinson [8] (see [6] p.187 for the detail).

Theorem 1.7. *For* $n \geq 5$*, the following equality holds.*

$$|\mathcal{BB}_n| = 2|\mathcal{B}_n| - |\mathcal{G}_n| + 2(|\mathcal{D}_{n-1}| + |\mathcal{D}_{n-1}^r| - |\mathcal{D}_{n-2}^r| - |\mathcal{G}_{n-2}|) + |\mathcal{G}_{n-4}|.$$

Proof. Note that $|\mathcal{S}_n| = |\mathcal{C}_n| - |\mathcal{B}_n|$. By Proposition 1.3 and Theorem 1.6, we deduce

$$\begin{aligned}
|\mathcal{BB}_n| &= |\mathcal{B}_n| - |\mathcal{BS}_n| - |\mathcal{BD}_n| \\
&= |\mathcal{B}_n| - (|\mathcal{S}_n| - |\mathcal{SS}_n| - |\mathcal{SD}_n|) - (|\mathcal{D}_n| - |\mathcal{DS}_n| - |\mathcal{DD}_n|) \\
&= |\mathcal{B}_n| - (|\mathcal{C}_n| - |\mathcal{B}_n|) - |\mathcal{D}_n| + 2|\mathcal{SD}_n| + |\mathcal{SS}_n| \\
&= 2|\mathcal{B}_n| - |\mathcal{G}_n| + 2(|\mathcal{D}_{n-1}| + |\mathcal{D}_{n-1}^r| - |\mathcal{D}_{n-2}^r| - |\mathcal{G}_{n-2}|) + |\mathcal{G}_{n-4}|.
\end{aligned}$$

□

By this theorem, we can obtain the following table.

n	1	2	3	4	5	6	7	8	9
$\lvert\mathcal{BB}_n\rvert$	0	0	0	0	1	8	126	3287	125838

§2. Proofs of Theorems 1.4 and 1.5.

In this section, we give proofs of Theorem 1.4 and Theorem 1.5. Before that, we prepare some notation. Let G be a graph, and let u, v, w and x be vertices with $u \notin V(G)$, $v, w, x \in V(G)$, $vw \notin E(G)$ and $vx \in E(G)$. We denote by

$G+u$, $G-v$, $G+vw$, $G-vx$ and $G\vee u$ the graph obtained from G by adding u, the graph obtained from G by deleting v and all the edges of v, the graph obtained from G by adding vw, the graph obtained from G by deleting vx, and the graph obtained from G by adding u and by joining u and all vertices of $V(G)$, respectively.

We first give a proof of Theorem 1.5.

Proof of Theorem 1.5. (i) We first show that $\overline{\mathcal{S}_n^1}=\mathcal{S}_n^3$. Suppose that $G\in\mathcal{S}_n^1$ (see Figure 2(i)). Let v be the unique cut vertex of G. Since $G-v$ is a disconnected graph, $\overline{G-v}$ has a spanning complete bipartite subgraph. Let u be the vertex with $N_G(v)=V(G)\setminus\{u,v\}$. Then note that $N_{\overline{G}}(v)=\{u\}$, and hence u is a cut vertex in $\overline{G}$. Since G has exactly one cut vertex, we have $\deg_G(u)\geq 2$, and hence $\deg_{\overline{G}}(u)\leq n-3$. Therefore $\overline{G}\in\mathcal{S}_n^3$. Conversely, suppose that $G\in\mathcal{S}_n^3$ (see Figure 2(iii)). Since $\deg_G(v)=1$, it follows that $\deg_{\overline{G}}(v)=n-2$. Since $G-v$ has a spanning complete bipartite subgraph, $\overline{G-v}$ is a disconnected graph. Hence v is a cut vertex of $\overline{G}$. Since $\deg_G(u)\leq n-3$, we have $\deg_{\overline{G}}(u)\geq 2$. Hence $\overline{G}$ has no cut vertex except for v. Therefore $\overline{G}\in\mathcal{S}_n^1$.

We next show that $|\mathcal{S}_n^1|=|\mathcal{D}_{n-1}^r|-|\mathcal{D}_{n-2}^r|-|\mathcal{G}_{n-2}|$. Let $\mathcal{D}_n^{r(\geq 2)}$ be the set of rooted disconnected graphs of order n in which the degree of the root is at least 2.

Claim 1. There exists a one-to-one correspondence between $\mathcal{S}_n^1$ and $\mathcal{D}_{n-1}^{r(\geq 2)}$.

Proof. Suppose that $G\in\mathcal{S}_n^1$. Let v be the unique cut vertex of G. Let $H=G-v$. Let u be the vertex with $N_G(v)=V(G)\setminus\{u,v\}$. Then H is disconnected and $\deg_H(u)=\deg_G(u)\geq 2$. Therefore if we regard u as the root of H then $H\in\mathcal{D}_{n-1}^{r(\geq 2)}$. Conversely, suppose that $H\in\mathcal{D}_{n-1}^{r(\geq 2)}$. Let u be the root of H with $\deg_H(u)\geq 2$. Let v be a vertex with $v\notin V(H)$, and let $G=(H\vee v)-uv$. Since $\deg_G(u)=\deg_H(u)\geq 2$, v is the unique cut vertex of G. Therefore, we obtain $G\in\mathcal{S}_n^1$. □

By this claim, it suffices to enumerate the order of $\mathcal{D}_{n-1}^{r(\geq 2)}$. Let $\mathcal{D}_n^{r(0)}$ and $\mathcal{D}_n^{r(1)}$ be the sets of rooted disconnected graphs of order n in which the degree of the root is 0 and 1, respectively. Then it is obvious that

$$|\mathcal{D}_{n-1}^{r(\geq 2)}|=|\mathcal{D}_{n-1}^r|-|\mathcal{D}_{n-1}^{r(0)}|-|\mathcal{D}_{n-1}^{r(1)}|. \qquad (2.1)$$

Claim 2. There exists a one-to-one correspondence between $\mathcal{D}_{n-1}^{r(0)}$ and $\mathcal{G}_{n-2}$.

Proof. Suppose that $G \in \mathcal{D}^{r(0)}_{n-1}$. Let x be the root of G. Then $G - x \in \mathcal{G}_{n-2}$. Conversely, suppose that $H \in \mathcal{G}_{n-2}$. Let x be a vertex with $x \notin V(H)$. Then if we regard x as the root of $H + x$, then $H + x \in \mathcal{D}^{r(0)}_{n-1}$. □

Claim 3. There exists a one-to-one correspondence between $\mathcal{D}^{r(1)}_{n-1}$ and $\mathcal{D}^{r}_{n-2}$.

Proof. Suppose that $G \in \mathcal{D}^{r(1)}_{n-1}$. Let x be the root of G, let y be the vertex with $N_G(x) = \{y\}$ and let $H = G - x$. If we regard y as the root of H, then $H \in \mathcal{D}^{r}_{n-2}$. Conversely, suppose that $H \in \mathcal{D}^{r}_{n-2}$. Let y be the root of H, let x be a vertex with $x \notin V(H)$, and let $G = (H + x) + xy$. If we regard x as the root of G, then $G \in \mathcal{D}^{r(1)}_{n-1}$. □

By the equality (2.1) and Claims 1–3, we deduce

$$\begin{aligned} |\mathcal{S}^1_n| &= |\mathcal{D}^{r(\geq 2)}_{n-1}| \\ &= |\mathcal{D}^{r}_{n-1}| - |\mathcal{D}^{r(0)}_{n-1}| - |\mathcal{D}^{r(1)}_{n-1}| \\ &= |\mathcal{D}^{r}_{n-1}| - |\mathcal{G}_{n-2}| - |\mathcal{D}^{r}_{n-2}|. \end{aligned}$$

(ii) We show that $\overline{\mathcal{S}^2_n} = \mathcal{S}^2_n$. Suppose that $G \in \mathcal{S}^2_n$. Let u_1 and u_2 be the cut vertices such that $\deg_G(u_1) = \deg_G(u_2) = n-2$. Let v_1 and v_2 be the vertices such that $N_G(v_1) = \{u_1\}$ and $N_G(v_2) = \{u_2\}$. Note that $N_{\overline{G}}(u_1) = \{v_2\}$, $N_{\overline{G}}(u_2) = \{v_1\}$ and $\deg_{\overline{G}}(v_1) = \deg_{\overline{G}}(v_2) = n-2$. Furthermore, the other vertices are not cut vertices and have degree at most $n-3$ in $\overline{G}$. Hence $\overline{G} \in \mathcal{S}^2_n$. We show that $|\mathcal{S}^2_n| = |\mathcal{G}_{n-4}|$. Suppose that $G \in \mathcal{S}^2_n$. We construct a graph H from $(G-u_1)-u_2$ by deleting vertices v_1 and v_2. Then $H \in \mathcal{G}_{n-4}$. Conversely, suppose that $H \in \mathcal{G}_{n-4}$. Let $P_4 = v_1u_1u_2v_2$ be a path of order 4. We construct a graph G from H and P_4 by joining each of u_1 and u_2 to all vertices of H. Then $G \in \mathcal{S}^2_n$. Therefore $|\mathcal{S}^2_n| = |\mathcal{G}_{n-4}|$. □

We next prove Theorem 1.4.

Proof of Theorem 1.4. By Theorem 1.5, we obtain $\mathcal{S}^1_n \cup \mathcal{S}^2_n \cup \mathcal{S}^3_n \subseteq \mathcal{SS}_n$. Suppose that $G \in \mathcal{SS}_n$. If there exists a cut vertex u with $\deg_G(u) \leq n-3$, then Theorem 1.2 (ii) implies $G \in \mathcal{S}^3_n$. Hence, by Theorem 1.2, we may assume that $\deg_G(u) = n-2$ for each cut vertex u. If G has exactly one cut vertex, then $G \in \mathcal{S}^1_n$. Therefore, we may assume that G has at least two cut vertices.

Let u_1 be a cut vertex in G. Since $\deg_G(u_1) = n-2$, there exists a vertex v_2 such that $N_G(u_1) = V(G) \setminus \{u_1, v_2\}$. Since G has at least two cut vertices, there exists a vertex u_2 with $N_G(v_2) = \{u_2\}$, and u_2 is the unique cut vertex except for u_1. Since $\deg_G(u_2) = n-2$, there exists a vertex v_1 such that $N_G(u_2) = V(G) \setminus \{u_2, v_1\}$. Since u_1 is a cut vertex, we have $N_G(v_1) = \{u_1\}$. Therefore $G \in \mathcal{S}^2_n$. □

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