The Right Setting of the Quaternion Calculus

El cálculo cuaterniónico, como debe ser

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quaternionic analysis, it is usual to persist in pointing out to their distinguished characteristics. It is our aim in this paper, by contrast, to set forth a natural (*i.e.* canonical) and rather comprehensive account of the quaternion calculus. Accordingly, we show that a proper notion of quaternionic derivative leads to the fundamental integral theorem which generalizes straightforwardly the better-known complex and real cases.

Key words: Quaternion and other division algebras, exterior differential systems, harmonic functions.

Resumen. A pesar del parecido entre los cálculos real, complejo y cuaterniónico, se suele insistir en sus diferencias. En este artículo, por el contrario, queremos realizar una presentación canónica del cálculo cuaterniónico. Así pues, la noción correcta de derivada cuaterniónica conduce naturalmente al teorema integral fundamental que generaliza lo que se conoce en los cálculos real y complejo.

Palabras clave: Cuaterniones y otras álgebra con división, sistemas diferenciales exteriores, funciones armónicas.

1 INTRODUCTION

Even though the works of Fueter and his school proclaim a common construction for both quaternion and complex calculus, a comprehensive polished account of the subject has yet to come. A handful of well-known papers ((Deavours, 1973),

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(Sudbery, 1979) and even the amazing original (Fueter, 1935)) must suppose the continuity of the derivative in order to reach quickly Cauchy-Fueter theorem. On the contrary, here we follow closely the modern theory of complex functions (*cf.* Levinson & Redheffer (1970), Remmert (1991)) and discard any redundant assumption from our exposition. Differential forms provide the appropriate language to depict quaternion infinitesimal calculus, owing to the difficulties arising from the four-dimensional character of the space. In a way, we may still say “wir wollen am Beispiel der Potentialfunktionen in einem Linearsystem den Zusammenhang mit dem *äußern Differentialcalcul* aufzeigen” (Haefeli, 1947, p. 382).

The elements of the algebra $\mathbb{H}$ of real quaternions are denoted by $h = t+ix+jy+kz$ or $h = \mathfrak{R}(h)+\mathfrak{I}(h)$. $\mathfrak{R}(h) = t$ is called the real part of h and $\mathfrak{I}(h) = ix+jy+kz$ is termed its imaginary part. Products of quaternions are often easily performed by the rule

$$hh' = \mathfrak{R}(h)\mathfrak{R}(h') - \mathfrak{I}(h) \cdot \mathfrak{I}(h') + \mathfrak{R}(h)\mathfrak{I}(h') + \mathfrak{R}(h')\mathfrak{I}(h) + \mathfrak{I}(h) \times \mathfrak{I}(h').$$

We make use of the notations $h/h' = h(h')^{-1}$ and $h' \setminus h = (h')^{-1}h$. For each $h = \mathfrak{R}(h) + \mathfrak{I}(h) \in \mathbb{H}$, $\overline{h} = \mathfrak{R}(h) - \mathfrak{I}(h) \in \mathbb{H}$ is the conjugate of h. The modulus $|h| = \sqrt{h\overline{h}}$ measures the euclidean length of h. We assume familiarity with the most prominent properties of the skew field of real quaternions. In particular, any ingenuous attempt to mimic complex calculus is inconsequential, *cf.* (Subery, 1979) and (Hayek & Rivera, 2010).

Proposition 1.1. *Suppose $U \subset \mathbb{H}$ is a domain and the function $f : U \to \mathbb{H}$ is such that $\lim_{h\to 0}(f(a + h) - f(a))/h$ exists for all $a \in U$. Then, f is right-affine in U. In other words, $f(h) = mh + b$, for some constants $m, b \in \mathbb{H}$.*

The correct notion of quaternionic differential relies *au contraire* upon the following basic geometric facts.

Proposition 1.2. *Let h_1, h_2 and h_3 be quaternions.*

1. $\mathfrak{A}(h_1, h_2) = \frac{1}{4}(h_1h_2 - h_2h_1)$ *is a quaternion that is perpendicular to* $\mathfrak{I}(h_1)$ *and* $\mathfrak{I}(h_2)$. *Also,* $|\mathfrak{A}(h_1, h_2)|$ *equals the area of the triangle having edges* $\mathfrak{I}(h_1)$ *and* $\mathfrak{I}(h_2)$.
2. $\mathfrak{V}(h_1, h_2, h_3) = \frac{1}{4}(h_3\overline{h_1}h_2 - h_2\overline{h_1}h_3)$ *is a quaternion which is normal to* h_1, h_2 *and* h_3. *Its modulus* $|\mathfrak{V}(h_1, h_2, h_3)|$ *gives the volume of the tetrahedron whose edges are these three quaternions.*

Our treatment of topology and integration is truly naive. Nevertheless, there should be no difficulty to restate our claims in a more general background.

In section 1, we propose and motivate a notion of holomorphic function. It leads to the necessity of the Cauchy-Fueter equations. The notion proves to be fitting in section 2, where the fundamental integral theorem is established via Goursat's dissection. Section 3 is devoted to the integral formula and its principal consequences. They comprise the continuity of the derivative, Liouville-Fueter theorem and a kind of Morera's holomorphy condition.

2 DIFFERENTIABILITY AND HOLOMORPHY

Let $f : U \to \mathbb{H}$ be a function defined on a domain $U \subseteq \mathbb{H}$. Since we are dealing with the behavior of f at point a in a 3-submanifold of U, we shall make use of the set H^3 of all ordered triples (h_1, h_2, h_3) of linearly independent quaternions such that $a+h_1, a+h_2, a+h_3 \in U$. The "3-increment" of f will be given by map $\mathfrak{D}f : H^3 \to \mathbb{H}$,

$$\begin{aligned}\mathfrak{D}f(h_1, h_2, h_3) =& (f(a+h_3) - f(a))\mathfrak{A}(h_1, h_2)+\\ &(f(a+h_1) - f(a))\mathfrak{A}(h_2, h_3) + (f(a+h_2) - f(a))\mathfrak{A}(h_3, h_1).\end{aligned}$$

Definition 2.1. *A function $f : U \to \mathbb{H}$ is said to be right-differentiable at $a \in U$ if the limit*

$$\lim_{(h_1,h_2,h_3)\to(0,0,0)} \mathfrak{D}f(h_1, h_2, h_3)/\mathfrak{V}(h_1, h_2, h_3) = f'(a)$$

exists. When this is the case, we speak of $f'(a)$ as the right derivative of f at a.

Equivalently, right-differentiability is a type of right-linear approximation. In fact, f is right-differentiable at a if and only if there is a quaternion $f'(a)$ such that $\mathfrak{D}f(h_1, h_2, h_3) = f'(a)\mathfrak{V}(h_1, h_2, h_3) + e(h_1, h_2, h_3)\mathfrak{V}(h_1, h_2, h_3)$, where $e : H^3 \to \mathbb{H}$ is a continuous function satisfying $e(0, 0, 0) = 0$. Then, the right-differentiability of f at a point a entails the continuity-evoking condition $\mathfrak{D}f(h_1, h_2, h_3) \to 0$ as $(h_1, h_2, h_3) \to (0, 0, 0)$. By the way, f is $\mathbb{R}^4$-differentiable and so, continuous. Definition 2.1 also produces at once the expected differentiation rules. If $f, g : U \to \mathbb{H}$ are right-differentiable at $a \in U$, then $(f + g)'(a) = f'(a) + g'(a)$ and $(cf)'(a) = cf'(a)$, for all $c \in \mathbb{H}$.

Our definition holds for any $(h_1, h_2, h_3) \in H^3$ approaching $(0, 0, 0)$. In particular, $f'(a) = \lim_{(t,x,y)\to(0,0,0)} \mathfrak{D}f(t, ix, jy)/\mathfrak{V}(t, ix, jy) = -\partial f/\partial t(a)$. Also, $f'(a)$ is equal to $\lim_{(x,y,z)\to(0,0,0)} \mathfrak{D}f(ix, jy, kz)/\mathfrak{V}(ix, jy, kz) = i\partial f/\partial x(a) + j\partial f/\partial y(a) + k\partial f/\partial z(a)$. Let $\Box = \frac{\partial}{\partial t} + i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$. If f is right-differentiable at a, then f is $\mathbb{R}^4$-differentiable at a and $f\Box(a) = 0$. As expected, the converse is not true. However, if $f = p + iq + jr + ks$ is $\mathbb{R}^4$-differentiable at a, the real-valued functions p, q, r, s are continuously differentiable in a neighborhood of a and $f\Box(a) = 0$, then f is right-differentiable at a. With the aid of the vector calculus operator $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$, the right Cauchy-Fueter differential system $f\Box = 0$ is written

$$\frac{\partial}{\partial t}\,\mathfrak{R}(f) = \mathfrak{I}(f)\cdot\nabla;\ \mathfrak{R}(f)\nabla = -\frac{\partial}{\partial t}\,\mathfrak{I}(f) - \mathfrak{I}(f)\times\nabla. \tag{1}$$

Pretty much the same as in the complex case, these equations imply that the components of any smooth enough right-differentiable function are harmonic.

Proposition 2.2. *If $f : U \to \mathbb{H}, f = p + iq + jr + ks$, is a right-differentiable function and $p, q, r, s : U \to \mathbb{R}$ are twice continuously differentiable in U, then $\Delta p = \Delta q = \Delta r = \Delta s = 0$ in U, where $\Delta = \frac{\partial}{\partial t^2} + \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2}$.*

Proof. On the one hand, $(\Re(f)\nabla) \cdot \nabla =$

$$\Big(- \frac{\partial}{\partial t}\, \Im(f) - \Im(f) \times \nabla\Big) \cdot \nabla = -\frac{\partial}{\partial t}(\Im(f) \cdot \nabla) - (\Im(f) \times \nabla) \cdot \nabla = -\frac{\partial^2 p}{\partial^2 t}.$$

Hence, $\Delta p = 0$. On the other hand, $\partial^2 \Im(f)/\partial^2 t =$

$$-\frac{\partial}{\partial t}(p\nabla) - \frac{\partial}{\partial t}(\Im(f) \times \nabla) = -\Big(\frac{\partial p}{\partial t}\Big)\nabla - \Im(f)(\nabla \cdot \nabla) + (\Im(f) \cdot \nabla)\nabla = -\Im(f)(\nabla \cdot \nabla).$$

Therefore, $\Delta q = \Delta r = \Delta s = 0$. □

As a matter of fact, the suitable notion for the quaternion calculus is the following.

Definition 2.3. *A function $f : U \to \mathbb{H}$ is called right-holomorphic in the domain U if f is right-differentiable at every point of U.*

Similarly, by setting $f\mathfrak{D}(h_1, h_2, h_3) = \mathfrak{A}(h_1, h_2)(f(a + h_3) - f(a)) + \mathfrak{A}(h_2, h_3)$ $(f(a+h_1) - f(a)) + \mathfrak{A}(h_3, h_1)(f(a+h_2) - f(a))$, f is said left-differentiable at a if $\lim_{(h_1,h_2,h_3)\to(0,0,0)} \mathfrak{V}(h_1, h_2, h_3) \setminus f\mathfrak{D}(h_1, h_2, h_3)$ exists. Also, f is left holomorphic in U if it is left-differentiable at every $a \in U$.

Examples 2.4. *If $u(x, y) + iv(x, y)$ is a complex-valued holomorphic function of a complex variable defined in a complex domain, $\hat{\imath} = \Im(h)/|\Im(h)|$ and $f(h) = u(\Re(h), |\Im(h)|) + \hat{\imath}v(\Re(h), |\Im(h)|)$, then $\Delta f(h)$ is right- and left- holomorphic in its domain of definition. In addition, $\Delta\Delta f = 0$. It is not hard to see that*

$$f_{\mathfrak{F}}(h) := \Delta f(h) = \frac{2}{|\Im(h)|}\ \hat{\imath}\ \Big(\frac{\partial f}{\partial t} - \frac{v(\Re(h), |\Im(h)|)}{|\Im(h)|}\Big).$$

This important result, which provides a whole bunch of non-trivial examples, is due to (Fueter, 1935) himself.

If we assume momentarily that f' is continuous about a, we shall be able to justify its definition. In a 3-submanifold $V \subset U$ containing a, we consider any simply connected differentiable 3-submanifold $M \subset V$ of U with $a \in M$. We may also assume that the boundary $\partial M \subset V$ is a smooth 2-submanifold of U. Quaternionic differential forms are built upon familiar real differential forms. A quaternionic l-form, $l \in \{0, 1, 2, 3\}$, is in truth a combination $\alpha_1 + i\alpha_2 + j\alpha_3 + k\alpha_4$ of four real

l-forms $\alpha_i,\ i \in \{1,2,3,4\}$. Their wedge product is performed as the product of two quaternions by using the wedge product of real forms. The exterior derivative is computed by taking the quaternions $1, i, j, k$ as constants. Some of these forms are particularly helpful to describe quaternion geometry and so, to elucidate the meaning of the quaternionic right derivative. A 0-form is just an $\mathbb{R}^4$-differentiable function, say $f = p + iq + jr + ks$. Hence,

$$df = dp + idq + jdr + kds = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

is a 1-form and, when f is the identity function, it provides the quaternionic line element $\mathfrak{d}h = dt + idx + jdy + kdz$. We have that $\mathfrak{d}h(h_1) = dt(h_1) + idx(h_1) + jdy(h_1) + kdz(h_1) = h_1$. The striking area element $\delta h = \mathfrak{d}h \wedge \mathfrak{d}h = idy \wedge dz + jdz \wedge dx + kdx \wedge dy$ and $f\delta h$ are 2-forms. Clearly, $\delta h(h_1, h_2) = idy \wedge dz(h_1, h_2) + jdz \wedge dx(h_1, h_2) + kdx \wedge dy(h_1, h_2) = \mathfrak{A}(h_1, h_2)$. The quaternionic volume element $dh = dx \wedge dy \wedge dz - idt \wedge dy \wedge dz - jdt \wedge dz \wedge dx - kdt \wedge dx \wedge dy$ is a 3-form on M. Besides, $dh(h_1, h_2, h_3) = \mathfrak{V}(h_1, h_2, h_3)$. The 3-form $df \wedge \delta h$ is also relevant. It is indeed the "limit" of $\mathfrak{D}f$, for

$$\begin{aligned} df \wedge \delta h = & \Big(\frac{\partial f}{\partial t}dt\Big) \wedge \delta h + \Big(\frac{\partial f}{\partial x}dx\Big) \wedge i(dy \wedge dz) \\ & + \Big(\frac{\partial f}{\partial y}dy\Big) \wedge j(dz \wedge dx) + \Big(\frac{\partial f}{\partial z}dz\Big) \wedge k(dx \wedge dy). \end{aligned}$$

In addition,

$$\begin{aligned} df \wedge \delta h = & -\Big(\frac{\partial q}{\partial t}dt + \frac{\partial q}{\partial x}dx\Big) \wedge dy \wedge dz - \Big(\frac{\partial r}{\partial t}dt + \frac{\partial r}{\partial y}dy\Big) \wedge dz \wedge dx \\ & - \Big(\frac{\partial s}{\partial t}dt + \frac{\partial s}{\partial z}dz\Big)dx \wedge dy + i\Big(\Big(\frac{\partial p}{\partial t}dt + \frac{\partial p}{\partial x}dx\Big)dy \wedge dz \\ & - \Big(\frac{\partial s}{\partial t}dt + \frac{\partial s}{\partial y}dy\Big)dz \wedge dx + \Big(\frac{\partial r}{\partial t}dt + \frac{\partial r}{\partial z}dz\Big)dx \wedge dy\Big) \\ & + j\Big(\Big(\frac{\partial s}{\partial t}dt + \frac{\partial s}{\partial x}dx\Big)dy \wedge dz + \Big(\frac{\partial p}{\partial t}dt + \frac{\partial p}{\partial y}dy\Big)dz \wedge dx \\ & - \Big(\frac{\partial q}{\partial t}dt + \frac{\partial q}{\partial z}dz\Big)dx \wedge dy\Big) + k\Big(- \Big(\frac{\partial r}{\partial t}dt + \frac{\partial r}{\partial x}dx\Big)dy \wedge dz \\ & + \Big(\frac{\partial q}{\partial t}dt + \frac{\partial q}{\partial y}dy\Big)dz \wedge dx + \Big(\frac{\partial p}{\partial t}dt + \frac{\partial p}{\partial z}dz\Big)dx \wedge dy\Big) \\ = & -(\mathfrak{I}(f) \cdot \nabla)\mathfrak{R}(dh) + \frac{\partial}{\partial t}\ \mathfrak{I}(f) \cdot \mathfrak{I}(dh) \\ & - (\mathfrak{I}(f) \cdot \nabla)\mathfrak{I}(dh) - \mathfrak{R}(dh)\frac{\partial}{\partial t}\ \mathfrak{I}(f) - \frac{\partial}{\partial t}\ \mathfrak{I}(f) \times \mathfrak{I}(dh). \end{aligned}$$

Consequently, by virtue of Stokes' theorem and the mean value theorem for integrals,

$$\lim_{M \to a} \frac{\int_{\partial M} f\ \delta h}{\int_M\ dh} = \lim_{M \to a} \frac{\int_M df \wedge \delta h}{\int_M\ dh} = \lim_{M \to a} \frac{\int_M f'\ dh}{\int_M\ dh} = f'(a),$$

where, as a result of (1), we must have

$$f' = -\Im(f) \cdot \nabla - \partial \Im(f)/\partial t = -\partial \Re(f)/\partial t + \Re(f)\nabla + \Im(f) \times \nabla. \qquad (2)$$

The limit indicates that M is collapsing to point a, *i.e.*, any (h_1, h_2, h_3) tends to $(0, 0, 0)$. To sum up, f is right-differentiable at a if $d(f\delta h) = df \wedge \delta h = f'dh$ about a. Just like with complex-differentiable functions, the continuity of f' is entirely adequate for the theory, but it happens to be redundant. By the way, (2) yields immediately the product-rule

$$(fg)' = f'g + fg' - 2\Im(g) \cdot (\Im(f) \times \nabla).$$

So, it is also possible to establish a quotient-rule.

3 THE CAUCHY-FUETER INTEGRAL THEOREM∎

A function $F : U \to \mathbb{H}$ is a right primitive of a continuous function $f : U \to \mathbb{H}$ in U if F is right-differentiable in U and $F' = f$. It is clear that, if f has a primitive F in U and M is a 3-submanifold of U with boundary ∂M, then

$$\int_M f dh = \int_M F' dh = \int_M dF \wedge \delta h = \int_{\partial M} F \delta h.$$

Therefore, if M is closed (*i.e.*, compact with no boundary), $\int_M f dh = 0$.

Since we do not have the continuity of f', Stokes' theorem does not lead to the fundamental integral theorem right away. Luckily, we can turn to a clever well-known device, namely Goursat's dissection.

Lemma 3.1. *If $f : U \to \mathbb{H}$ is right-holomorphic in the domain U, then, for the boundary ∂T of every closed 4-pentahedron $T \subset U$, we have*

$$\int_{\partial T} f dh = 0.$$

Proof. The boundary ∂T is composed of five 3-tetrahedra. We divide T *à la Goursat* into sixteen smaller congruent 4-pentahedra T_i. Then,

$$\int_{\partial T} f dh = \sum_{i=1}^{16} \int_{\partial T_i} f dh.$$

From among the integrals on the right-hand side, we choose one with the largest modulus and denote the associated 4-pentahedron by T_1. Thus,

$$\left| \int_{\partial T} f dh \right| \leq 16 \left| \int_{\partial T_1} f dh \right|.$$

Proceeding with T_1 just as we have done with T, we find a 4-pentahedron T_2 that satisfies $\left|\int_{\partial T} fdh\right| \leq 16^2 \left|\int_{\partial T_2} fdh\right|$. Continuing in this way yields a decreasing sequence of compact 4-pentahedra $T \supset T_1 \supset T_2 \supset \cdots$ such that

$$\left|\int_{\partial T} fdh\right| \leq 16^n \left|\int_{\partial T_n} fdh\right|, \; n \in \mathbb{N}. \tag{3}$$

As a result of the nested interval principle, there is a unique $a \in T$ such that $\cap_{i=1}^{\infty} T_i = \{a\}$. Since f is $\mathbb{R}^4$-differentiable at a, it holds that $f(h) = f(a) + Df(a)(h-a)+e(h)(h-a)$, where $Df(a)$ denotes the linear Jacobian map and $e(h)$ is a continuous function such that $e(h) = 0$. Now, $\int_{\partial T_n} f(a)\, dh = \int_{\partial T_n} Df(a)(h-a)\, dh = 0$ because the integrands possess primitives. It follows that

$$\int_{\partial T_n} fdh = \int_{\partial T_n} e(h)(h-a)dh, \; n \in \mathbb{N}.$$

Since e is continuous, $\max\{|e(h)| : h \in \partial T_n\} \leq \mathfrak{m}$, for some $\mathfrak{m} > 0$. From the standard estimate, we get the inequality

$$\left|\int_{\partial T_n} fdh\right| \leq \frac{\mathfrak{m}\mathfrak{c}}{16^n},$$

where $\mathfrak{c}$ is a constant. Combining this with (3) and noting that $\mathfrak{m} \to 0$ as $n \to \infty$, we find that $\int_{\partial T} fdh$ vanishes. □

Corollary 3.2. *Let $U \subset \mathbb{H}$ be a domain and $c \in U$. Let also $f, g : U \to \mathbb{H}$ be continuous in U. If f is right-holomorphic in $U \setminus \{c\}$ and g is left-holomorphic in $U \setminus \{c\}$, then $\int_{\partial T} fdhg = 0$ for every closed 4-pentahedron $T \subset U$ which has a vertex at c.*

Under proper assumptions on the topology of U, it is possible to establish the fundamental integral theorem.

Theorem 3.3 (Cauchy-Fueter)**.** *Let U be a star domain and let $f : U \to \mathbb{H}$ be right-holomorphic in U. Then,*

$$\int_M f(h)\; dh = 0$$

for any 3-dimensional closed (smooth) subset M of U.

Proof. Let o be a center of U. First off, with the aid of the 1-form $\mathsf{d}h = -dt - \frac{i}{3}dx - \frac{j}{3}dy - \frac{k}{3}dz$, we define $F : U \to \mathbb{H}$ by

$$F(h) = \int_o^h f(\eta)\mathsf{d}\eta.$$

Then, $dF = f\mathsf{d}h$ and $dF \wedge \delta h = f\mathsf{d}h \wedge \delta h = fdh$. Now, if $a \in U$ and $|h_1|, |h_2|$ and $|h_3|$ are sufficiently small, then $o, a, a+h_1, a+h_2, a+h_3$ are the vertices of a 4-pentahedron T lying entirely in U. Let $\tau_1 = [o, a, a+h_1, a+h_2]$, $\tau_2 = [o, a+h_2, a+h_1, a+h_3]$, $\tau_3 = [o, a+h_3, a+h_1, a]$, $\tau_4 = [o, a, a+h_2, a+h_3]$ and $\tau_5 = [a, a+h_1, a+h_2, a+h_3]$ denote the five 3-tetrahedra forming ∂T. By the previous lemma,

$$\sum_{l=1}^{4} \int_{\tau_l} dF \wedge \delta h = \int_{\tau_5} fdh. \tag{4}$$

At this point, we notice that

$$\begin{aligned}(F(a+h_2)-F(o))\mathfrak{A}(a-o,a+h_1-o)+(F(a)-F(o))\mathfrak{A}(a+h_1-o,a+h_2-o)+\\ (F(a+h_1)-F(o))\mathfrak{A}(a+h_2-o,a-o)+\\ (F(a+h_3)-F(o))\mathfrak{A}(a+h_2-o,a+h_1-o)+\\ (F(a+h_2)-F(o))\mathfrak{A}(a+h_1-o,a+h_3-o)+\\ (F(a+h_1)-F(o))\mathfrak{A}(a+h_3-o,a+h_2-o)+\\ (F(a)-F(o))\mathfrak{A}(a+h_3-o,a+h_1-o)+(F(a+h_3)-F(o))\mathfrak{A}(a+h_1-o,a-o)+\\ (F(a+h_1)-F(o))\mathfrak{A}(a-o,a+h_3-o)+(F(a+h_3)-F(o))\mathfrak{A}(a-o,a+h_2-o)+\\ (F(a)-F(o))\mathfrak{A}(a+h_2-o,a+h_3-o)+(F(a+h_2)-F(o))\mathfrak{A}(a+h_3-o,a-o)=\\ (F(a+h_3)-F(a))\mathfrak{A}(h_1,h_2)+(F(a+h_1)-F(a))\mathfrak{A}(h_2,h_3)+\\ (F(a+h_2)-F(a))\mathfrak{A}(h_3,h_1).\end{aligned}$$

Thus if we divide by $\mathfrak{V}(h_1, h_2, h_3)$ and let $(h_1, h_2, h_3) \to (0, 0, 0)$, equation (4) and the mean value theorem imply that $F'(a) = f(a)$. This holds for all $a \in U$. That is, F is a primitive of f in U and $\int_M f(h)\, dh = 0$. □

Corollary 3.4. *Let $U \subset \mathbb{H}$ be a star domain with center o. Let also $f, g : U \to \mathbb{H}$ be continuous in U. If f is right-holomorphic in $U \setminus \{o\}$ and g is left-holomorphic in $U \setminus \{o\}$, then $fdhg$ is integrable in U.*

4 THE INTEGRAL FORMULA AND ITS CONSEQUENCES

The volume element of a 3-sphere can be revisited by using quaternionic differential forms. By Example 2.4, the complex function $(z - \mathsf{c})^{-1}$ yields the (right- and left-) holomorphic function $(h-c)^{-1}_{\mathfrak{F}} := -4(h-c)^{-1}/|h-c|^2$ in $\mathbb{H} \setminus \{c\}$. Now, the quaternionic volume element of a 3-sphere ∂B with center c is $dh = |dh|(h-c)/|h-c|$. Thus if $\rho = |h-c|$ is the radius of this sphere, the noteworthy 3-form

$$dh\, \frac{(h-c)^{-1}}{|h-c|^2} = \frac{1}{\rho^3}|dh|$$

on ∂B is real and gives the real volume element of the unit 3-sphere. We recall that the total volume of ∂B is $\int_{\partial B} |dh| = 2\pi^2\rho^3$.

Theorem 4.1 (Cauchy-Fueter Integral Formula)**.** *Let f be right-holomorphic in a domain U and let B be an open ball which together with its boundary ∂B lies wholly in U. Then,*

$$f(h) = \frac{1}{8\pi^2} \int_{\partial B} f(\eta)\ d\eta\ (\eta - h)_{\mathfrak{F}}^{-1}$$

for all $h \in B$.

Proof. Let $h \in B$ be fixed and consider the differential form $(f(\eta) + f(h))\ d\eta$ $(\eta - h)_{\mathfrak{F}}^{-1}$. By Corollary 3.4, it is integrable in U. Since $\overline{B}$ lies inside a bigger ball in U, $0 = \int_{\partial B}(f(\eta) + f(h))\ d\eta\ (\eta - h)_{\mathfrak{F}}^{-1} = \int_{\partial B} f(\eta)\ d\eta\ (\eta - h)_{\mathfrak{F}}^{-1} - 8\pi^2 f(h)$. □

Corollary 4.2. *Under the assumptions of the theorem, for $n \in \mathbb{Z}$,*

$$\frac{1}{8\pi^2} \int_{\partial B} d\eta\ (\eta - h)_{\mathfrak{F}}^{n} = \begin{cases} 8\pi^2 & \text{if } n = -1 \\ 0 & \text{if } n \neq -1. \end{cases}$$

Additionally, the continuity of f' need not be assumed.

Corollary 4.3 (Fueter)**.** *The components p, q, r and s of a right holomorphic function $f : U \to \mathbb{H}$, $f = p + iq + jr + ks$, are infinitely differentiable with respect to t, s, y and z in U.*

Due to Proposition 2.2, the maximum principle gives an analogue of Liouville's theorem, *cf.* (Deavours, 1973). However, this result can also be regarded as a consequence of the integral formula.

Theorem 4.4 (Liouville-Fueter)**.** *If $f : \mathbb{H} \to \mathbb{H}$ is a right-holomorphic bounded function, then $f(h)$ is a constant.*

Proof. By hypothesis there is a constant $\mathfrak{m}$ such that $|f(\eta)| \leq \mathfrak{m}$, for all $\eta \in \mathbb{H}$. Let B be an open ball of radius ρ centered at $h = t + ix + jy + kz \in \mathbb{H}$. From $|\partial(\eta - h)_{\mathfrak{F}}^{-1}/\partial t| \leq 20/\rho^4$, we get

$$\left|\frac{\partial f}{\partial t}(h)\right| \leq \frac{40\mathfrak{m}}{\rho}\ \pi^2.$$

The remaining first partial derivatives of f are similarly bounded. As ρ is arbitrary, $f(h)$ is a constant. □

In the proof of theorem 3.3 the only use made of the holomorphy is to show that f is continuous and that the integral over a 4-pentahedron vanishes. Thus the same argument produces the following helpful statement.

Theorem 4.5. *If $f : U \to \mathbb{H}$ is a continuous function in a star domain U and $\int_{\partial T} f(h)dh = 0$ for every closed 4-pentahedron lying in U, then f has a primitive in U.*

In the end, we establish the following Morera-like condition for holomorphy. This condition is the point of departure in (Deavours, 1973).

Theorem 4.6. *Let $f : U \to \mathbb{H}$ be continuous in a domain U. If $\int_{\partial T} f(h)dh = 0$ for every closed 4-pentahedron T lying in U, then f is right-holomorphic in U.*

Proof. Let $a \in U$ be fixed and let $\rho > 0$ be so small that the ball $B = \{h \in \mathbb{H} : |h - a| < \rho\} \subset U$. Then, B is a star domain. By the previous theorem, there is a right-holomorphic function F such that $F' = f$ in B. From (2), we have that $\mathfrak{R}(f) = -\mathfrak{I}(F) \cdot \nabla = -\partial\mathfrak{R}(F)/\partial t$ and $\mathfrak{I}(f) = -\partial\mathfrak{I}(F)/\partial t = \mathfrak{R}(F)\nabla + \mathfrak{I}(F) \times \nabla$. By Fueter's remark (corollary 4.3), f is is $\mathbb{R}^4$-differentiable and the partial derivatives (of all orders) of its components are continuous. Furthermore, f satisfies Cauchy-Fueter equations (1):

$$\begin{aligned} \partial\mathfrak{R}(f)/\partial t &= -\partial\mathfrak{I}(F)/\partial t \cdot \nabla = \mathfrak{I}(f) \cdot \nabla, \\ \mathfrak{R}(f)\nabla &= -\partial\mathfrak{R}(F)\nabla/\partial t = -\partial\mathfrak{I}(f)/\partial t - \mathfrak{I}(f) \times \nabla. \end{aligned}$$

That is to say, f is right-holomorphic in U. □

5 CONCLUDING REMARKS

The correct notion of integral for the construction of calculus in $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ involves always a limiting process in a submanifold of codimension 1. In the quaternion case, it is a 3-submanifold; in the complex case, it is a curve; in the real case, a pair of points. The definition of derivative should be inherent in the notion of differentiability. It is hard to believe the quaternionic derivative has been so often disregarded. The distinct structure of real quaternions can be used to establish most of the key results of the quaternion calculus, in particular, Cauchy-Fueter theorem. No remorse should be felt when the vector space structure is employed, since it is a part of such a structure.

Many interesting topics already examined by Fueter's school have been left behind. Power series, singularities and analytic continuation can be certainly studied by the methods proposed here.

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