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MODIFIED ABOODH HOMOTOPY PERTUBATION METHOD FOR SOLVING NONLINEAR GAS DYNAMIC EQUATION

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ABSTRACT

In this paper, we present a reliable combination of Aboodh Transform and Modified Homotopy perturbation method to solve nonlinear gas dynamic equations. Some problems were solved to demonstrate the capability and reliability of the mixture of Aboodh Transform and Modified Homotopy perturbation method. We have compared the result obtained with the available Laplace Transform New Homotopy Perturbation Method solution and homotopy perturbation method of solution which is found to be exactly the same. The result revealed that the combination of the Aboodh Transform and Modified homotopy perturbation method is practically well appropriate for use in such problems.

Keywords: Aboodh Transform, Homotopy Perturbation Method, Modified Homotopy Perturbation Method, Nonlinear Gas Dynamic Equation, Partial Differential Equation.

1. INTRODUCTION

Nonlinear models has been very useful in describing various phenomena that appear in many areas of scientific fields. We know that except in a limited number of these problems, there are difficulties in obtaining their exact solutions. Thus, many researchers had made several attempts to develop various analytical methods for obtaining solutions which reasonably approximate the exact solutions see Bellman (1964), Cole (1968). In recent years, most nonlinear problems were solved by numerical methods and their improvement led to improvement in analytical methods. Several scientists believe that the combination of numerical and analytical methods will also produce meaningful results. Many new techniques have been widely used to solve nonlinear problems. Example of such is homotopy perturbation method established by He to obtain series solution of nonlinear differential equations (He 1999, 2000a, 2000b, 2003, 2004, 2005, 2006, 2008). The method has the advantages of simplicity and easy execution.

Gas dynamics equation is a mathematical model which is based on the physical laws of conservation, namely: the laws of conservation of mass, conservation of momentum, conservation of energy and many more. The nonlinear equations of ideal gas dynamics are applicable for the three classes of nonlinear waves like shock fronts, rare factions and contact discontinuities. Several methods have been used to solve different classes of gas dynamics equations see Evans and Bulut (2002), Polyanin and Zaitsev (2004), Elizarova (2009), Jafari, Chun, Seifi, and Saeidy (2009), Ames (1965), Rasulov and Karaguler (2003), Aminikhah and Jamalian (2013), Jafari, Zabihi and Saidy (2008). Recently, Aboodh introduced a new set of integral transform called Aboodh transform and it has been applied to obtain solution of several classes of both linear and nonlinear partial differential equations (Aboodh 2013, 2014, 2015, 2016).

In this paper, we construct the solution of gas dynamic equation by using Modified Aboodh Homotopy Pertubation Method. Analytical approximation to the solution of the nonlinear gas dynamic equation is obtained by using the mixture of Modified Homotopy Pertubation Method and Aboodh Transform. The gas dynamic equation as a nonlinear partial differential model is as follows (Evans and Bulut 2002, Jafari *et al.* 2009, Aminikhah and Jamalian 2013):

$$u_t + uu_x - u(1-u) = 0 \qquad 1$$

where $0 \le x \le 1$ *and* $t > 0$
The result obtained by Modified Aboodh Homotopy Pertubation Method confirm the applicability and effectiveness of the proposed method in solving the nonlinear gas dynamic equation.

2. MODIFIED ABOODH HOMOTOPY PERTURBATION METHOD (MAHPM)

In this section, we propose MAHPM, which is the combination of Aboodh Transform and modified homotopy perturbation methods (MHPM) for solving nonlinear gas dynamic equation. This method is simple and obtain the exact solution of the equations analytically using the initial condition only. This method provides the solution in a closed form. To illustrate the basic idea of this method, we consider the following nonlinear differential equation (Aminikhah and Jamalian, 2013):

$$A(u) - f(r) = 0, r \in \Omega \quad 2$$

with the following initial conditions

$$u(0) = \alpha_0, \acute{u}(0) = \alpha_1, \cdots, u^{(n-1)}(0) = \alpha_{n-1} \quad 3$$

where A is a general differential operator and $f(r)$ is a known analytical function. The operator A can be divided into two parts, L and N, where L is a linear and N is a nonlinear operator. Thus, equation (2) can be rewritten as

$$L(u) + N(u) - f(r) = 0 \quad 4$$

From MHPM [23], we construct a homotopy $U(r,p): \Omega X[0,1] \to \mathbb{R}$, which satisfies

$$H(U,p) = (1-p)[L(U) - u_0] + p[A(U) - f(r)] = 0,\ p \in [0,1],\ r \in \Omega \quad 5$$

$$H(U,p) = L(U) - u_0 + pu_0 + p[N(u) - f(r)] = 0,\ p \in [0,1],\ r \in \Omega \quad 6$$

Where , $p \in [0,1]$ is an embedded parameter and u_0 is an initial approximation for the solution of equation (2)

In the case of $p = 0$, then equation (4) becomes

$$H(U,0) = L(U) - u_0 = 0 \quad 7$$

In the case of $p = 1$, then equation (5) becomes

$$H(U,1) = A(U) - f(r) = 0 \quad 8$$

Now, applying the Aboodh transform to both sides of equation (6), we have

$$A\{L(U) - u_0 + pu_0 + p[N(u) - f(r)]\} = 0 \quad 9$$

Using the differential property of the aboodh transform we obtain

$$v^n U(x,v) - \frac{U(x,0)}{v^{2-n}} - \frac{U^1(x,0)}{v^{3-n}} - \cdots - \frac{U^{(n-1)}(x,0)}{v} = A\{u_0 - pu_0 + p[N(u) - f(r)]\} \quad 10$$

$$U(x,v) = \frac{1}{v^n}\left\{\frac{U(x,0)}{v^{2-n}} + \frac{U^1(x,0)}{v^{3-n}} + \cdots + \frac{U^{(n-1)}(x,0)}{v} + A\{u_0 - pu_0 + p[N(u) - f(r)]\}\right\} \quad 11$$

Taking the Inverse Aboodh transform on both sides of equation (11), we have

$$U(x,t) = A^{-1}\left\{\frac{1}{v^n}\left\{\frac{U(x,0)}{v^{2-n}} + \frac{U^1(x,0)}{v^{3-n}} + \cdots + \frac{U^{(n-1)}(x,0)}{v} + A\{u_0 - pu_0 + p[N(u) - f(r)]\}\right\}\right\} \quad 12$$

From Homotopy Perturbation Method we considered the embedding parameter p as a small parameter and assume that the solution of equation (12) can be written as a power series of p as:

$$U(x,t) = \sum_{n=0}^{\infty} p^n U_n(x,t) \quad 13$$

Substituting the equation (12) into (13), we have

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = A^{-1}\left\{\frac{1}{v^n}\left\{\frac{U(x,0)}{v^{2-n}} + \frac{U^1(x,0)}{v^{3-n}} + \cdots + \frac{U^{(n-1)}(x,0)}{v} + A\{u_0 - pu_0 + p[N(u) - f(r)]\}\right\}\right\} \quad 14$$

This is the coupling of the Aboodh Transform and the modified homotopy perturbation method. Equating the coefficients of the corresponding powers of p, we have

p^0: $U_0(x,t) = A^{-1}\left\{\frac{1}{v^n}\left\{\frac{U(x,0)}{v^{2-n}} + \frac{U^1(x,0)}{v^{3-n}} + \cdots + \frac{U^{(n-1)}(x,0)}{v} + A\{u_0\}\right\}\right\}$

p^1: $U_1(x,t) = A^{-1}\left\{\frac{1}{v^n}\{A\{[N(U_0) - u_0 - f(r)]\}\}\right\}$

p^2: $U_2(x,t) = A^{-1}\left\{\frac{1}{v^n}\{A\{N(U_0, U_1)\}\}\right\}$

p^3: $U_3(x,t) = A^{-1}\left\{\frac{1}{v^n}\{A\{N(U_0, U_1, U_2)\}\}\right\}$ and so on

In general, the recursive relation is given by: 15

p^m: $U_m(x,t) = A^{-1}\left\{\frac{1}{v^n}\{A\{N(U_0, U_1, U_2, \cdots, U_{m-1})\}\}\right\}$

Assuming that the initial approximate solution of equation (2) has the form $U(0) = u_0 = \propto_0, U^1(0) = \propto_1, \cdots,, U^{(n-1)}(0) = \propto_{n-1}$. Thus, the exact solution may be obtained as follows:

$$u = \lim_{p\to 1} U(x,t) = U_0(x,t) + U_1(x,t) + U_2(x,t) + \cdots \qquad 16$$

3. APPLICATION

In this section, in order to demonstrate the effectiveness and applicability of the Modified Aboodh Homotopy perturbation method for solving gas dynamics equations, we will solve the following examples.

Example1

Consider the nonlinear gas dynamic equation (Aminikhah and Jamalian 2013)

$$u_t + uu_x - u(1-u) = 0 \qquad 17$$

subject to the constant initial conditions

$$u(x,0) = ae^{-x} \qquad 18$$

By applying the modified homotopy perturbation method, we construct the following homotopy:

$$H(U,p) = U_t - u_0 + p[u_0 + UU_x - U(1-U)] = 0 \qquad 19$$

where $p \in [0,1]$ is an embedding parameter, u_0 represent the initial approximation that satisfies

the solution of the equation

In the case of $p = 0$, then equation (19) becomes

$$H(U,0) = U_t - u_0 = 0 \qquad 20$$

In the case of $p = 1$, then equation (19) becomes

$$H(U,1) = U_t + UU_x - U(1-U) = 0 \qquad 21$$

Now taking the Aboodh Transform on both sides of equation (19) we have

$$A[H(U,p)] = A\left[U_t - u_0 + p[u_0 + UU_x - U(1-U)]\right] = 0 \qquad 22$$

Using the initial property of Aboodh transform, we obtain

$$vU(x,v) - \frac{1}{v}U(x,0) = A\{u_0 - p[u_0 + UU_x - U + U^2]\} \qquad 23$$

$$U(x,v) = \frac{1}{v^2}U(x,0) + \frac{1}{v}A\{u_0 - p[u_0 + UU_x - U + U^2]\} \qquad 24$$

By applying the Inverse Aboodh transform on both sides of equation (24), we have

$$U(x,t) = U(x,0) + A^{-1}\left\{\frac{1}{v}A\{u_0 - p[u_0 + UU_x - U + U^2]\}\right\} \quad 25$$

From the homotopy perturbation method, we use the embedding parameter p as a small parameter and assume that the solution of equation (25) can be written as power series of p as

$$U(x,t) = \sum_{n=0}^{\infty} p^n\, U_n(x,t) \quad 26$$

Substituting equation (26) into (25)

$$\sum_{n=0}^{\infty} p^n\, U_n(x,t) = U(x,0) + A^{-1}\left\{\frac{1}{v}A\left\{u_0 - p\left[u_0 + \sum_{n=0}^{\infty} p^n \left\{U_n U_{nx} - U_n + {U_n}^2\right\}\right]\right\}\right\} \quad 27$$

Equating the coefficients of the corresponding powers of p in equation (27)

$$p^0: \quad U_0(x,t) = U(x,0) + A^{-1}\left\{\frac{1}{v}A\{u_0\}\right\}$$

$$U_0(x,t) = ae^{-x} + A^{-1}\left\{\frac{1}{v}A\{ ae^{-x}\}\right\}$$

$$U_0(x,t) = ae^{-x} + ae^{-x}t$$

$$U_0(x,t) = ae^{-x}(1+t)$$

$$p^1: \quad U_1(x,t) = -A^{-1}\left\{\frac{1}{v}A\left\{u_0 + U_0(U_0)_x - U_0 + {U_0}^2\right\}\right\}$$

$$U_1(x,t) = -A^{-1}\left\{\frac{1}{v}A\left\{ ae^{-x} + ae^{-x}(1+t)\left(-ae^{-x}(1+t)\right) - ae^{-x}(1+t) + \left(ae^{-x}(1+t)\right)^2\right\}\right\}$$

$$U_1(x,t) = \frac{1}{2!}ae^{-x}t^2$$

$$p^2: \quad U_2(x,t) = -A^{-1}\left\{\frac{1}{v}A\{U_0(U_1)_x + U_1(U_0)_x - U_1 + 2U_0U_1\}\right\}$$

$$U_2(x,t) = -A^{-1}\left\{\frac{1}{v}A\left\{ae^{-x}(1+t)\left(-\frac{1}{2!}ae^{-x}t^2\right) + \frac{1}{2!}ae^{-x}t^2\left(-ae^{-x}(1+t)\right) - \frac{1}{2!}ae^{-x}t^2 + 2\left(ae^{-x}(1+t)\right)\left(\frac{1}{2!}ae^{-x}t^2\right)\right\}\right\}$$

$$U_2(x,t) = \frac{1}{3!}\, a\, e^{-x}t^3$$

$$p^3: \quad U_3(x,t) = -A^{-1}\left\{\frac{1}{v}A\{U_0(U_2)_x + U_1(U_1)_x + U_2(U_0)_x - U_2 + U_0U_1 + U_0U_2 + U_1U_1 + U_2U_0\}\right\}$$

$$U_3(x,t) = -A^{-1}\left\{\frac{1}{v}A\left\{ae^{-x}(1+t)\left(-\frac{1}{3!}\ ae^{-x}t^3\right) + \frac{1}{2!}ae^{-x}t^2\left(-\frac{1}{2!}ae^{-x}t^2\right) + \frac{1}{3!}\ ae^{-x}t^3\left(-ae^{-x}(1+t)\right) - \frac{1}{3!}\ ae^{-x}t^3 + ae^{-x}(1+t)\frac{1}{3!}\ ae^{-x}t^3 + \frac{1}{2!}ae^{-x}t^2\left(\frac{1}{2!}ae^{-x}t^2\right) + \frac{1}{3!}\ e^{-x}t^3\left(ae^{-x}(1+t)\right)\right\}\right\}$$

$$U_3(x,t) = \frac{1}{4!}\, a\, e^{-x}t^4$$

$$p^4: \quad U_4(x,t) = -A^{-1}\left\{\frac{1}{v}A\{U_0(U_3)_x + U_1(U_2)_x + U_2(U_1)_x + U_3(U_0)_x - U_3 + U_0U_3 + U_1U_2 + U_2U_1 + U_3U_0\}\right\}$$

$$U_4(x,t) = -A^{-1}\left\{\frac{1}{v}A\left\{ae^{-x}(1+t)\left(\frac{1}{4!}\ a\,e^{-x}t^4\right)+\frac{1}{2!}ae^{-x}t^2\left(-\frac{1}{3!}\ a\,e^{-x}t^3\right)\right.\right.$$
$$+\frac{1}{3!}\ a\,e^{-x}t^3\left(-\frac{1}{2!}ae^{-x}t^2\right)+\frac{1}{4!}\ a\,e^{-x}t^4\left(-ae^{-x}(1+t)\right)-\frac{1}{4!}\ a\,e^{-x}t^4\Big\}$$
$$+\ ae^{-x}(1+t)\left(\frac{1}{4!}\ a\,e^{-x}t^4\right)+\ ae^{-x}(1+t)\left(\frac{1}{3!}\ a\,e^{-x}t^3\right)+\frac{1}{3!}\ a\,e^{-x}t^3\left(\frac{1}{2!}ae^{-x}t^2\right)$$
$$\left.+\frac{1}{4!}\ a\,e^{-x}t^4\left(ae^{-x}(1+t)\right)\ \}\quad\right\}$$

$$U_4(x,t) = \frac{1}{5!}\ a\,e^{-x}t^5$$

$$\vdots$$

$$p^m:\quad U_m(x,t) = A^{-1}\left\{-\tfrac{1}{v}A\{\textstyle\sum_{k=0}^{m-1}U_k(U_{m-k-1})_x - U_{m-1} + \sum_{k=0}^{m-1}U_kU_{m-k-1}\}\right\} \qquad 28$$

The rest of the components of iteration formula can be obtained by following the same procedure.

Suppose the initial approximation has the form

$U(x,0) = u_0(x,t) = ae^{-x}$

Thus, the exact solution can be obtained as follows:

$u(x,t) = \lim_{p\to 1} U(x,t)$

$u(x,t) = U_0(x,t) + U_1(x,t) + U_2(x,t) + \cdots$

$$u(x,t) = ae^{-x}(1+t) + \tfrac{1}{2!}ae^{-x}t^2 + \tfrac{1}{3!}\ a\,e^{-x}t^3 + \tfrac{1}{4!}\ a\,e^{-x}t^4 + \tfrac{1}{5!}\ a\,e^{-x}t^5 + \cdots \qquad 29$$

$$u(x,t) = ae^{-x}(1 + t + t^2 + t^3 + t^4 + t^5 + \cdots) \qquad 30$$

Hence, the solution can be written in the closed form as:

$$u(x,t) = ae^{t-x} \qquad 31$$

Equation (31) is the exact solution for equation (17) which is the same as the solution obtained in (Aminikhah and Jamalian 2013).

Example 2

Consider the nonlinear homogenous partial differential equation (Jafari, Zabihi and Saidy 2008)

$$u_t + \tfrac{1}{2}(u^2)_x - u(1-u) = 0,\ 0 \le x \le 1,\ t > 0 \qquad 32$$

with specified conditions

$$u(x,0) = e^{-x} \qquad 33$$

By applying the modified homotopy perturbation method, we construct the following homotopy:

$$H(U,p) = U_t - u_0 + p\left[u_0 + \tfrac{1}{2}U^2{}_x - U(1-U)\right] = 0 \qquad 34$$

where $p \in [0,1]$ is an embedding parameter, u_0 represent the initial approximation that satisfies

the solution of the equation

In the case of $p = 0$, then equation (34) becomes

$$H(U,0) = U_t - u_0 = 0 \qquad 35$$

In the case of $p = 1$, then equation (34) becomes

$$H(U,1) = U_t + \frac{1}{2}U^2{}_x - U(1-U) = 0 \qquad 36$$

Now taking the Aboodh Transform on both sides of equation (34) we have,

$$A[H(U,p)] = A\left[U_t - u_0 + p\left[u_0 + \frac{1}{2}U^2{}_x - U(1-U)\right]\right] = 0 \qquad 37$$

Using the initial property of Aboodh transform, we obtain

$$vU(x,v) - \frac{1}{v}U(x,0) = A\left\{u_0 - p\left[u_0 + \frac{1}{2}U^2{}_x - U + U^2\right]\right\} \qquad 38$$

$$U(x,v) = \frac{1}{v^2}U(x,0) + \frac{1}{v}A\left\{u_0 - p\left[u_0 + \frac{1}{2}U^2{}_x - U + U^2\right]\right\} \qquad 39$$

By taking the Inverse Aboodh transform on both sides of equation (39), we have

$$U(x,t) = A^{-1}\left\{\frac{1}{v^2}U(x,0)\right\} + A^{-1}\left\{\frac{1}{v}A\left\{u_0 - p\left[u_0 + \frac{1}{2}U^2{}_x - U + U^2\right]\right\}\right\} \qquad 40$$

$$U(x,t) = U(x,0) + A^{-1}\left\{\frac{1}{v}A\left\{u_0 - p\left[u_0 + \frac{1}{2}U^2{}_x - U + U^2\right]\right\}\right\} \qquad 41$$

From the homotopy perturbation method, we use the embedding parameter p as a small parameter and assume that the solution of equation (41) can be written as power series of p as:

$$U(x,t) = \sum_{n=0}^{\infty} p^n U_n(x,t) \qquad 42$$

Substituting equation (42) into (41)

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = U(x,0) + A^{-1}\left\{\frac{1}{v}A\left\{u_0 - p\left[u_0 + \sum_{n=0}^{\infty} p^n\left\{\frac{1}{2}U^2{}_{nx} - U_n + U_n{}^2\right\}\right]\right\}\right\} \qquad 43$$

Equating the coefficients of the corresponding powers of p in equation (43)

$$p^0: \quad U_0(x,t) = U(x,0) + A^{-1}\left\{\frac{1}{v}A\{u_0\}\right\}$$

$$U_0(x,t) = e^{-x} + A^{-1}\left\{\frac{1}{v}A\{e^{-x}\}\right\}$$

$$U_0(x,t) = e^{-x} + e^{-x}t$$

$$U_0(x,t) = e^{-x}(1+t)$$

$$p^1: \quad U_1(x,t) = -A^{-1}\left\{\frac{1}{v}A\left\{u_0 + \frac{1}{2}\left(U_0{}^2\right)_x - U_0 + U_0{}^2\right\}\right\}$$

$$U_1(x,t) = -A^{-1}\left\{\frac{1}{v}A\left\{e^{-x} + \frac{1}{2}(-2\,ae^{-2x}(1+t)^2) - e^{-x}(1+t) + \left(e^{-2x}(1+t)\right)^2\right\}\right\}$$

$$U_1(x,t) = \frac{1}{2!}e^{-x}t^2$$

$$p^2: \quad U_2(x,t) = -A^{-1}\left\{\frac{1}{v}A\{U_0(U_1)_x + U_1(U_0)_x - U_1 + 2U_0U_1\}\right\}$$

$$U_2(x,t) = -A^{-1}\left\{\frac{1}{v}A\left\{e^{-x}(1+t)\left(-\frac{1}{2!}e^{-x}t^2\right) + \frac{1}{2!}e^{-x}t^2\left(-e^{-x}(1+t)\right) - \frac{1}{2!}e^{-x}t^2 + 2\left(e^{-x}(1+t)\right)\left(\frac{1}{2!}e^{-x}t^2\right)\right\}\right\}$$

$$U_2(x,t) = \frac{1}{3!}e^{-x}t^3$$

$$p^3: \quad U_3(x,t) = -A^{-1}\left\{\frac{1}{v}A\left\{\frac{1}{2}(U_1^2)_x + U_0(U_2)_x + U_2(U_0)_x - U_2 + U_0U_2 + U_2U_0 + U_1{}^2\right\}\right\}$$

$$U_3(x,t) = -A^{-1}\left\{\frac{1}{v}A\left\{\frac{1}{2}\left(-\frac{1}{2}\ e^{-2x}t^4\right) + e^{-x}(1+t)\left(-\frac{1}{3!}e^{-x}t^3\right) + \frac{1}{3!}\ e^{-x}t^3\left(-e^{-x}(1+t)\right) - \frac{1}{3!}\ e^{-x}t^3 + e^{-x}(1+t)\frac{1}{3!}\ e^{-x}t^3 + \frac{1}{3!}e^{-x}t^3\left(e^{-x}(1+t)\right) + \frac{1}{4!}\ e^{-2x}t^4\right\}\right\}$$

$$U_3(x,t) = \frac{1}{4!}\ e^{-x}t^4$$

$$\vdots$$

$$p^m: \quad U_m(x,t) = A^{-1}\left\{-\frac{1}{v}A\left\{\sum_{k=0}^{m-1}U_k(U_{m-k-1})_x - U_{m-1} + \sum_{k=0}^{m-1}U_kU_{m-k-1}\right\}\right\} \qquad 44$$

The rest of the components of iteration formula can be obtained by following the same procedure.

Suppose the initial approximation has the form

$U(x,0) = u_0(x,t) = e^{-x}$

Thus, the exact solution can be obtained as follows:

$$u(x,t) = \lim_{p\to 1} U(x,t)$$

$$u(x,t) = U_0(x,t) + U_1(x,t) + U_2(x,t) + \cdots$$

$$u(x,t) = e^{-x}(1+t) + \frac{1}{2!}e^{-x}t^2 + \frac{1}{3!}e^{-x}t^3 + \frac{1}{4!}\ e^{-x}t^4 + \cdots \qquad 45$$

$$u(x,t) = e^{-x}(1 + t + t^2 + t^3 + t^4 + \cdots) \qquad 46$$

Hence, the solution can be written in the closed form as:

$$u(x,t) = e^{t-x} \qquad 47$$

Equation (47) is the exact solution for equation (32) which is the same as the solution obtained in (Jafari, Zabihi and Saidy 2008).

Example 3

Consider the following non homogeneous nonlinear gas dynamic equation (Aminikhah and Jamalian 2013)

$$u_t + uu_x - u(1-u) = -e^{t-x} \qquad 48$$

subject to initial condition

$$u(x,0) = 1 - e^{-x} \qquad 49$$

By applying the modified homotopy perturbation method, we construct the following homotopy:

$$H(U,p) = U_t - u_0 + p[u_0 + UU_x - U(1-U) + e^{t-x}] = 0 \qquad 50$$

where $p \in [0,1]$ is an embedding parameter, u_0 represent the initial approximation that satisfies

the solution of the equation

In the case of $p = 0$, then equation (50) becomes

$$H(U,0) = U_t - u_0 = 0 \qquad 51$$

In the case of $p = 1$, then equation (50) becomes

$$H(U,1) = U_t + UU_x - U(1-U) + e^{t-x} = 0 \qquad 52$$

Now taking the Aboodh Transform on both sides of equation (50) we have

$$A[H(U,p)] = A\big[U_t - u_0 + p[u_0 + UU_x - U(1-U) + e^{t-x}]\big] = 0 \quad 53$$

Using the initial property of Aboodh transform,we obtain

$$vU(x,v) - \frac{1}{v}U(x,0) = A\{u_0 - p[u_0 + UU_x - U + U^2 + e^{t-x}]\} \quad 54$$

$$U(x,v) = \frac{1}{v^2}U(x,0) + \frac{1}{v}A\{u_0 - p[u_0 + UU_x - U + U^2 + e^{t-x}]\} \quad 55$$

By taking the Inverse Aboodh transform on both sides of equation (55), we have

$$U(x,t) = A^{-1}\left\{\frac{1}{v^2}U(x,0)\right\} + A^{-1}\left\{\frac{1}{v}A\{u_0 - p[u_0 + UU_x - U + U^2 + e^{t-x}]\}\right\} \quad 56$$

$$U(x,t) = U(x,0) + A^{-1}\left\{\frac{1}{v}A\{u_0 - p[u_0 + UU_x - U + U^2 + e^{t-x}]\}\right\} \quad 57$$

From the homotopy perturbation method, we use the embedding parameter p as a small parameter and assume that the solution of equation (57) can be written as power series of p as

$$U(x,t) = \sum_{n=0}^{\infty} p^n\, U_n(x,t) \quad 58$$

Substituting equation (58) into (57)

$$\sum_{n=0}^{\infty} p^n\, U_n(x,t) = U(x,0) + A^{-1}\left\{\frac{1}{v}A\Big\{u_0 - p\Big[u_0 + \sum_{n=0}^{\infty} p^n\left\{U_nU_{nx} - U_n + {U_n}^2\right\} + e^{t-x}\Big]\Big\}\right\} \quad 59$$

Equating the coefficients of the corresponding powers of p in equation (59)

$$p^0: \quad U_0(x,t) = U(x,0) + A^{-1}\left\{\frac{1}{v}A\{u_0\}\right\}$$

$$U_0(x,t) = 1 - e^{-x} + A^{-1}\left\{\frac{1}{v}A\{1 - e^{-x}\}\right\}$$

$$U_0(x,t) = 1 - e^{-x} + t - e^{-x}t$$

$$U_0(x,t) = 1 + t - e^{-x}(1+t)$$

$$p^1: \quad U_1(x,t) = -A^{-1}\left\{\frac{1}{v}A\Big\{u_0(x,t) + U_0(U_0)_x - U_0 + {U_0}^2 + e^{t-x}\Big\}\right\}$$

$$U_1(x,t) = -A^{-1}\left\{\frac{1}{v}A\Big\{(1-e^{-x}) + \big(1+t-e^{-x}(1+t)\big)\big(e^{-x}(1+t)\big) - \big(1+t-e^{-x}(1+t)\big) + \big(1+t-e^{-x}(1+t)\big)^2 + e^{t-x}\Big\}\right\}$$

$$U_1(x,t) = -t - \frac{t^2}{2} - \frac{t^3}{3} - e^{t-x} + \frac{1}{6}e^{-x}(6 + 6t + 3t^2 + 2t^3)$$

$$p^2: \quad U_2(x,t) = -A^{-1}\left\{\frac{1}{v}A\{U_0(U_1)_x + U_1(U_0)_x - U_1 + 2U_0U_1\}\right\}$$

$$U_2(x,t) = -A^{-1}\left\{\frac{1}{v}A\left\{\big(1+t-e^{-x}(1+t)\big)\left(e^{t-x}-\frac{1}{6}e^{-x}(6+6t+3t^2+2t^3)\right) + \left(-t-\frac{t^2}{2}-\frac{t^3}{3}-e^{t-x}+\frac{1}{6}e^{-x}(6+6t+3t^2+2t^3)\right)\big(e^{-x}(1+t)\big) - \left(-t-\frac{t^2}{2}-\frac{t^3}{3}-e^{t-x}+\frac{1}{6}e^{-x}(6+6t+3t^2+2t^3)\right) + 2\big(1+t-e^{-x}(1+t)\big)\left(-t-\frac{t^2}{2}-\frac{t^3}{3}-e^{t-x}+\frac{1}{6}e^{-x}(6+6t+3t^2+2t^3)\right)\right\}\right\}$$

$$U_2(x,t) = \frac{t^2}{2}+\frac{5}{6}t^3+\frac{1}{3}t^4+\frac{2}{15}t^5-\frac{1}{30}e^{-x}(-30+30t^2+25t^3+10t^4+4t^5)+e^{t-x}\ (t-1)$$

$$p^3:\quad U_3(x,t) = -A^{-1}\left\{\frac{1}{v}A\left\{U_0(U_2)_x+U_1(U_1)_x+U_2(U_0)_x-U_2\ +U_0U_1+U_0U_2+{U_1}^2+U_2U_0\right\}\right\}$$

$$U_3(x,t) = -A^{-1}\left\{\frac{1}{v}A\left\{\big(1+t-e^{-x}(1+t)\big)\left(\frac{1}{30}e^{-x}(-30+30t^2+25t^3+10t^4+4t^5)-e^{t-x}\ (t-1)\right) + \left(-t-\frac{t^2}{2}-\frac{t^3}{3}-e^{t-x}+\frac{1}{6}e^{-x}(6+6t+3t^2+2t^3)\right)\left(e^{t-x}-\frac{1}{6}e^{-x}(6+6t+3t^2+2t^3)\right) + \left(\frac{t^2}{2}+\frac{5}{6}t^3+\frac{1}{3}t^4+\frac{2}{15}t^5-\frac{1}{30}e^{-x}(-30+30t^2+25t^3+10t^4+4t^5)+e^{t-x}\ (t-1)\right)\big(e^{-x}(1+t)\big) - \left(\frac{t^2}{2}+\frac{5}{6}t^3+\frac{1}{3}t^4+\frac{2}{15}t^5-\frac{1}{30}e^{-x}(-30+30t^2+25t^3+10t^4+4t^5)+e^{t-x}\ (t-1)\right) + \big(1+t-e^{-x}(1+t)\big)\left(-t-\frac{t^2}{2}-\frac{t^3}{3}-e^{t-x}+\frac{1}{6}e^{-x}(6+6t+3t^2+2t^3)\right) + \big(1+t-e^{-x}(1+t)\big)\left(\frac{t^2}{2}+\frac{5}{6}t^3+\frac{1}{3}t^4+\frac{2}{15}t^5-\frac{1}{30}e^{-x}(-30+30t^2+25t^3+10t^4+4t^5)+e^{t-x}\ (t-1)\right) + \left(-t-\frac{t^2}{2}-\frac{t^3}{3}-e^{t-x}+\frac{1}{6}e^{-x}(6+6t+3t^2+2t^3)\right)^2 + \left(\frac{t^2}{2}+\frac{5}{6}t^3+\frac{1}{3}t^4+\frac{2}{15}t^5-\frac{1}{30}e^{-x}(-30+30t^2+25t^3+10t^4+4t^5)+e^{t-x}\ (t-1)\right)\big(1+t-e^{-x}(1+t)\big)\right\}\right\}$$

$$U_3(x,t) = -\frac{1}{2}t^3-\frac{17}{24}t^4-\frac{7}{12}t^5-\frac{17}{90}t^6-\frac{17}{315}t^7-\frac{1}{6}e^{t-x}(6-6t+3t^2+2t^3) + \frac{1}{1260}e^{-x}\ (1260+840t^3+1155t^4+735t^5+238t^6+68t^7)$$

$$p^4: \quad U_4(x,t) = -A^{-1}\left\{\frac{1}{v}A\{U_0(U_3)_x + U_1(U_2)_x + U_2(U_1)_x + U_3(U_0)_x - U_3 + U_0U_3 + 2U_1U_2 + U_3U_0\}\right\}$$

$$\begin{aligned}
U_4(x,t) = -A^{-1}\Bigg\{\frac{1}{v}A \quad \Bigg\{&\left(1+t-e^{-x}(1+t)\right)\Bigg(\frac{1}{6}e^{t-x}(6-6t+3t^2+2t^3) \\
&-\frac{1}{1260}e^{-x}\ (1260+840t^3+1155t^4+735t^5+238t^6+68t^7)\Bigg) \\
&+\Bigg(-t-\frac{t^2}{2}-\frac{t^3}{3}-e^{t-x} \\
&+\frac{1}{6}e^{-x}(6+6t+3t^2+2t^3)\Bigg)\Bigg(\frac{1}{30}e^{-x}(-30+30t^2+25t^3+10t^4+4t^5) \\
&-e^{t-x}\ (t-1)\Bigg) \\
&+\Bigg(\frac{t^2}{2}+\frac{5}{6}t^3+\frac{1}{3}t^4+\frac{2}{15}t^5-\frac{1}{30}e^{-x}(-30+30t^2+25t^3+10t^4+4t^5) \\
&+e^{t-x}\ (t-1)\Bigg)\Bigg(e^{t-x}-\frac{1}{6}e^{-x}(6+6t+3t^2+2t^3)\Bigg) \\
&+\Bigg(-\frac{1}{2}t^3-\frac{17}{24}t^4-\frac{7}{12}t^5-\frac{17}{90}t^6-\frac{17}{315}t^7-\frac{1}{6}e^{t-x}(6-6t+3t^2+2t^3) \\
&+\frac{1}{1260}e^{-x}\ (1260+840t^3+1155t^4+735t^5+238t^6+68t^7)\Bigg)\left(e^{-x}(1+t)\right) \\
&-\Bigg(-\frac{1}{2}t^3-\frac{17}{24}t^4-\frac{7}{12}t^5-\frac{17}{90}t^6-\frac{17}{315}t^7-\frac{1}{6}e^{t-x}(6-6t+3t^2+2t^3) \\
&+\frac{1}{1260}e^{-x}(1260+840t^3+1155t^4+735t^5+238t^6+68t^7)\Bigg) \\
&+\left(1+t-e^{-x}(1+t)\right)\Bigg(-\frac{1}{2}t^3-\frac{17}{24}t^4-\frac{7}{12}t^5-\frac{17}{90}t^6-\frac{17}{315}t^7 \\
&-\frac{1}{6}e^{t-x}(6-6t+3t^2+2t^3) \\
&+\frac{1}{1260}e^{-x}\ (1260+840t^3+1155t^4+735t^5+238t^6+68t^7)\Bigg) \\
&+2\Bigg(-t-\frac{t^2}{2}-\frac{t^3}{3}-e^{t-x}+\frac{1}{6}e^{-x}(6+6t+3t^2+2t^3)\Bigg)\Bigg(\frac{t^2}{2}+\frac{5}{6}t^3+\frac{1}{3}t^4+\frac{2}{15}t^5 \\
&-\frac{1}{30}e^{-x}(-30+30t^2+25t^3+10t^4+4t^5)+e^{t-x}\ (t-1)\Bigg) \\
&+\Bigg(-\frac{1}{2}t^3-\frac{17}{24}t^4-\frac{7}{12}t^5-\frac{17}{90}t^6-\frac{17}{315}t^7-\frac{1}{6}e^{t-x}(6-6t+3t^2+2t^3) \\
&+\frac{1}{1260}e^{-x}(1260+840t^3+1155t^4+735t^5+238t^6+68t^7)\Bigg)\Big(1+t \\
&-e^{-x}(1+t)\Big)\Bigg\}\Bigg\}
\end{aligned}$$

$$\begin{aligned}
U_4(x,t) = &\frac{3}{8}t^4+\frac{31}{40}t^5+\frac{23}{36}t^6+\frac{113}{315}t^7+\frac{31}{315}t^8+\frac{62}{2835}t^8 \\
&+\frac{1}{45360}e^{-x}\ (-45360+28350t^4+41958t^5+32823t^6+16272t^7+4464t^8+992t^9) \\
&+\frac{1}{30}e^{t-x}(-30+30t-15t^2+5t^3+10t^4+4t^5)
\end{aligned}$$

⋮

$$p^m: \quad U_m(x,t) = A^{-1}\left\{-\frac{1}{v}A\{\sum_{k=0}^{m-1} U_k(U_{m-k-1})_x - U_{m-1} + \sum_{k=0}^{m-1} U_k U_{m-k-1}\}\right\} \tag{60}$$

The rest of the components of iteration formula can be obtained by following the same procedure.

Suppose the initial approximation has the form

$U(x,0) = u_0(x,t) = 1 - e^{-x}$

Thus, the exact solution can be obtained as follows:

$$u(x,t) = \lim_{p\to 1} U(x,t)$$

$$u(x,t) = U_0(x,t) + U_1(x,t) + U_2(x,t) + U_3(x,t)\cdots$$

$$u(x,t) = \left(1 + t - e^{-x}(1+t)\right) + \left(-t - \frac{t^2}{2} - \frac{t^3}{3} - e^{t-x} + \frac{1}{6}e^{-x}(6 + 6t + 3t^2 + 2t^3)\right) + \left(\frac{t^2}{2} + \frac{5}{6}t^3 + \frac{1}{3}t^4 + \frac{2}{15}t^5 - \frac{1}{30}e^{-x}(-30 + 30t^2 + 25t^3 + 10t^4 + 4t^5) + e^{t-x}\ (t-1)\right) + \left(-\frac{1}{2}t^3 - \frac{17}{24}t^4 - \frac{7}{12}t^5 - \frac{17}{90}t^6 - \frac{17}{315}t^7 - \frac{1}{6}e^{t-x}(6 - 6t + 3t^2 + 2t^3) + \frac{1}{1260}e^{-x}\ (1260 + 840t^3 + 1155t^4 + 735t^5 + 238t^6 + 68t^7)\right) + \left(\frac{3}{8}t^4 + \frac{31}{40}t^5 + \frac{23}{36}t^6 + \frac{113}{315}t^7 + \frac{31}{315}t^8 + \frac{62}{2835}t^8 + \frac{1}{45360}e^{-x}\ (-45360 + 28350t^4 + 41958t^5 + 32823t^6 + 16272t^7 + 4464t^8 + 992t^9) + \frac{1}{30}e^{t-x}(-30 + 30t - 15t^2 + 5t^3 + 10t^4 + 4t^5)\right) + \cdots \tag{61}$$

$$u(x,t) \cong 1 + 4e^{-x} - 5e^{t-x} - \frac{13}{35}t^8 - \frac{172}{945}t^9 - \frac{11}{60}e^{t-x}t^5 + \frac{7}{24}e^{t-x}t^4 + \frac{16361}{40320}e^{-x}t^8 + \frac{172}{945}e^{-x}t^9 - \frac{17}{90}e^{t-x}t^6 - \frac{17}{315}e^{t-x}t^7 + \frac{1382}{155925}e^{-x}t^{11} + \frac{691}{14175}e^{-x}t^{10} - \frac{21}{80}t^6 - \frac{79}{168}t^7 - \frac{3}{2}e^{t-x}t^2 + 4e^{t-x}t + \frac{407}{720}e^{-x}t^7 - \frac{1}{24}e^{-x}t^4 + \frac{33}{80}e^{-x}t^6 + \frac{1}{120}e^{-x}t^5 + \frac{1}{6}e^{-x}t^3 - \frac{691}{14175}t^{10} - \frac{1382}{155925}t^{11} - \frac{1}{2}e^{-x}t^2 \tag{62}$$

Equation (62) is the exact solution for equation (48) which is the same as the solution obtained in (Aminikhah and Jamalian 2013).

4. CONCLUSION

In this paper, we introduced the Modified Aboodh Homotopy Perturbation Method which is the combination of Aboodh transform and Modified homotopy perturbation method to solve nonlinear gas dynamic equations. The main advantage of this method is that, it provides the user an analytical approximation to the solution in series of rapidly convergent sequence with elegantly computed terms. The results obtained show that the method is trustworthy and introduces a significant advancement in solving nonlinear partial differential equations over existing methods.

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