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Solutions of Second Order Nonlinear Singular Initial Value Problems by Modified Laplace Decomposition Method

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ABSTRACT

In this paper, we presented a reliable modification of the Laplace Decomposition Method (LDM) for solving non linear singular second order initial value problems. The method is used to obtain the exact solutions of the nonlinear singular initial value problems. The validity of the method is verified by using nonlinear Lane-Emden type differential equations as an example to show the effectiveness of the method.

Keywords: Singular, Initial value Problems, Adomian Polynomial, Laplace Transform, Lane-Emden type differential equations, Modified Laplace Decomposition Method.

1. INTRODUCTION

Most of the physical phenomena are nonlinear in nature and many analytical and numerical methods have been developed by various scientists to cope with the nonlinearity of such problems. In reality, the Laplace transform is one of the few methods that can be useful to linear systems with periodic or discontinuous driving inputs. In spite of its great usefulness in solving linear problems however, the Laplace is totally incapable of handling nonlinear equations because of the difficulties that are caused by the nonlinear terms.

Lane-Emden type IVPs has many applications in Mathematics and astrophysics [1], [2]. The nonlinear Lane-Emden type problem of the form

$$y'' + \frac{2}{x}y' + f(x,y) = g(x) \; 0 \le x \le 1 \qquad (1)$$

Subject to the conditions

$$y(0) = A, y^{'}(0) = B \qquad (2)$$

Where $f(x,y)$ is a continuous real valued function.

The solution of Lane-Emden equation is numerically challenging because of the singularity behavior at origin. The approximate solutions of the Lane-Emden equation is given by Adomian decomposition, homotopy perturbation, Variational iteration, Differential transform [4]-[12] and so on. Laplace Adomian Decomposition Method (LADM) proposed in [13] and [14] is successfully used to find solutions of differential equations [15]-[19]. We use this method to obtain the exact solutions for nonlinear equations but it generates noise term for inhomogeneous differential equations [20]. Hussain [21] found a modified method which increases the convergence of solution when compared with Laplace Adomian Decomposition Method (LADM). In this paper, we will use Modified Laplace Decomposition Method (MLDM) obtain the exact or approximate analytic solutions of the nonlinear Lane-Emden type equations.

2. Analysis of the Method

Modified Laplace Decomposition Method

Here, we will briefly discuss the procedure of the Modified Laplace Decomposition Method (MLDM) for solution of nonlinear Lane-Emden equation (1)

$$y'' + \frac{2}{x}y' + f(x, y) = g(x), \qquad 0 \le x \prec 1$$

Multiplying through by x and then taking the Laplace transform on both sides gives:

$$L\{xy''\} + 2L\{y'\} + L\{xf(x, y) - xg(x)\} = 0 \tag{3}$$

$$-s^2L'\{y\} - y(0) + L\{xf(x, y) - xg(x)\} = 0 \tag{4}$$

Where L is the operator of the Laplace transform operator and it is define as

$$L\{x\} = \int_0^\infty e^{-sx} f(x)dx$$

$$L'\{y\} = \frac{dL\{y\}}{ds}$$

We decompose $f(x, y)$ into two parts.

$$f(x, y) = M[y(x)] + N[y(x)]$$

Where $M[y(x)]$ *and* $N[y(x)]$ denote the linear term and the nonlinear term respectively.

The MLDM gives a solution $y(x)$ an infinite series as:

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \tag{5}$$

and the nonlinear term takes the form of infinite series of the Adomian polynomials A_n of the form:

$$N\{y(x)\} = \sum_{n=0}^{\infty} A_n(x) \tag{6}$$

Where A_n are the Adomian polynomials and it can be obtained by the formula

$$A_n = \frac{1}{n!} = \left[\frac{d^n}{d\lambda^n} N\left(\sum_{n=0}^{\infty} \lambda^n u_n\right)\right] \qquad n = 0, 1, 2, \ldots \tag{7}$$

Therefore,

$$A_n = N[u_0]$$

$$A_1 = u_1 N'[u_0]$$

$$A_2 = u_2 N'[u_0] + \frac{1}{2!}u_1^{\,2} N''[u_0] \tag{8}$$

$$A_3 = u_3 N'[u_0] + u_1 u_2 N''[u_0] + \frac{1}{3!}u_1^{\,3} N'''[u_0]$$

$$\vdots$$

Substituting (7), (8) into (4) and applying the linearity property of Laplace transform, we have

$$-s^2 L'\left\{\sum_{n=0}^{\infty} y_n(x)\right\} - y(0) - L\{xg(x)\} + \sum_{n=0}^{\infty} L\{xR[y_n(x)] + xA_n(x)\} = 0 \tag{9}$$

In general, the recursive relation is given by:

$$\begin{aligned} L'\{y_0(x)\} &= -s^{-2}y(0) - s^{-2}L\{xg(x)\}, \\ L'\{y_{n+1}(x)\} &= s^{-2}L\{xM[y_n(x)] + xA_n(x)\}, \end{aligned} \tag{10}$$

Integrating (10) from 0 to s, we have

$$\begin{aligned} L\{y_0(x)\} &= \int\left[-s^{-2}y(0) - s^{-2}L\{xg(x)\}\right]ds, \\ L\{y_{n+1}(x)\} &= \int s^{-2}L\{xM[y_n(x)] + xA_n(x)\}ds \end{aligned} \tag{11}$$

Taking the inverse Laplace transform of (11), we have

$$\begin{aligned} y_0(x) &= L^{-1}\left\{\int\left[-s^{-2}y(0) - s^{-2}L\{xg(x)\}\right]ds\right\} = H(x), \\ y_{n+1}(x) &= L^{-1}\left\{\int s^{-2}L\{xM[y_n(x)] + xA_n(x)\}ds\right\}, \end{aligned} \tag{12}$$

Where $H(x)$ arises from the source term and the given initial condition. The choice of (12) as the initial solution produces noise oscillation in iteration process. To overcome this problem, we assume that $H(x)$ can be decomposed as

$$H(x)=H_0(x)+H_1(x) \tag{13}$$

Instead of (12), we suggest the following modification:

$$\begin{aligned} y_0(x)&=H_0(x),\\ y_1(x)&=H_1(x)+L^{-1}\{s^{-2}L\{xM[y_n(x)]+xA_n(x)\}ds\},\\ y_{n+1}(x)&=L^{-1}\{\int s^{-2}L\{xM[y_n(x)]+xA_n(x)\}ds\}, \end{aligned} \tag{14}$$

The above solution by MLDM depends on $H_0(x)$ and $H_1(x)$.

3. Numerical examples

Modified Laplace Decomposition Method is illustrated by the following examples.

Example 1

Consider a nonlinear Lane-Emden fowler equation

$$y''+\frac{2}{x}y'+\alpha(6y+4y\ln y)=0$$

S
ubject to the initial conditions

$$y(0)=1,\quad y'(0)=0 \tag{16}$$

Applying the MLDM and initial conditions (16), we have

$$-s^2L'\{y\}-1=-\alpha L\{6xy+4xy\ln y\},$$

and then, we obtain the recursive relations as

$$\begin{aligned} L'\{y_0\}&=-s^{-2}\\ L'\{y_n\}&=\alpha s^{-2}L\{6xy_{n-1}+4xA_{n-1}\} \end{aligned} \tag{17}$$

Where the nonlinear operator $N[y]=y\ln y$ is decomposed as in (8) in terms of the Adomian polynomials. $N[y]=y\ln y$ are computed as follow:

$$\begin{aligned} A_0&=N[y_0]=y_0\ln y_0\\ A_1&=y_1N'[y_0]=y_1(1+\ln y_0)\\ A_2&=y_2N'[y_0]+\frac{1}{2}y_1^2N''[y_0]=y_2(1+\ln y_0)+\frac{y_1^2}{2!y_0}\\ A_3&=y_3N'[y_0]+y_1y_2N''[y_0]+\frac{1}{3!}y_1^3N'''[y_0]=y_3(1+\ln y_0)-\frac{y_1^3}{3!y_0^2} \end{aligned} \tag{18}$$

Integrating both sides of (17) and then taking the inverse Laplace transform we have

$$\begin{aligned} y_0(x)&=L^{-1}\{\int -s^{-2}ds\}\\ y_n(x)&=\alpha L^{-1}\{\int s^{-2}L\{6xy_{n-1}+4xA_{n-1}\}ds\} \end{aligned} \tag{19}$$

By substituting (16) into (19) we obtain the series solution as

$$\begin{aligned} y_0&=1\\ y_1&=\alpha L^{-1}\{\int s^{-2}L\{6xy_0+4xA_0\}ds\},\quad y_1=-\alpha x^2\\ y_2&=\alpha L^{-1}\{\int s^{-2}L\{6xy_1+4xA_1\}ds\},\quad y_2=\frac{1}{2!}\alpha^2x^4\\ y_3&=\alpha L^{-1}\{\int s^{-2}L\{6xy_2+4xA_2\}ds\},\quad y_3=-\frac{1}{3!}\alpha^3x^6\\ y_4&=\alpha L^{-1}\{\int s^{-2}L\{6xy_3+4xA_3\}ds\},\quad y_4=\frac{1}{4!}\alpha^4x^8 \end{aligned} \tag{20}$$

$$y(x)=1-\alpha x^2+\frac{1}{2!}\alpha^2x^4-\frac{1}{3!}\alpha^3x^6+\frac{1}{4!}\alpha^4x^8-....$$

Hence the solution is given below in the series form

(21)

The exact solution is $y(t)=e^{-\alpha x^2}$

(22)

Example 2

Consider a nonlinear Lane-Emden differential equation

$$y'' + \frac{2}{x}y' + 4\left(2e^{y} + e^{y/2}\right) = 0, \quad 0 \le x \le 1 \tag{23}$$

Subject to initial conditions

$$y(0) = 0 \quad y'(0) = 0 \tag{24}$$

The exact solution is $y(x) = -2\ln(1 + x^2)$

Applying the MLDM and initial conditions (24), we have

$$-s^2 L'\{y\} + L\left\{4x\left(2e^{y} + e^{y/2}\right)\right\} = 0 \tag{25}$$

and then, we obtain the recursive relation as

$$\begin{aligned} &L'\{y_0\} = 0, \\ &L'\{y_n\} = s^{-2}L\{4xA_{n-1}\} \end{aligned} \tag{26}$$

Where the nonlinear operator $N[y] = 2e^{y} + e^{y/2}$ is decomposed as in (8) in terms of the Adomian polynomials. $N[y] = 2e^{y} + e^{y/2}$ are computed as follow:

$$\begin{aligned} &A_0 = 2e^{y_0} + e^{y_0/2} \\ &A_1 = y_1\left(2e^{y_0} + \frac{1}{2}e^{y_0/2}\right), \\ &A_2 = y_2\left(2e^{y_0} + \frac{1}{2}e^{y_0/2}\right) + \frac{y_1^2}{2!}\left(2e^{y_0} + \frac{1}{4}e^{y_0/2}\right), \\ &A_3 = y_3\left(2e^{y_0} + \frac{1}{2}e^{y_0/2}\right) + y_1 y_2\left(2e^{y_0} + \frac{1}{4}e^{y_0/2}\right) + \frac{y_1^3}{3!}\left(2e^{y_0} + \frac{1}{8}e^{y_0/2}\right), \\ &\vdots \end{aligned} \tag{27}$$

Integrating both sides of (26) and then taking the inverse Laplace transform, we have

$$\begin{aligned} &L'\{y_0\} = 0, \quad y_0 = 0 \\ &y_n(x) = L^{-1}\left[\int \left\{s^{-2}L(4xA_{n-1})\right\}ds\right] \end{aligned} \tag{28}$$

Using the initial conditions and (27), we have these results

$$\begin{aligned} y_0 &= 0 \\ y_1 &= -2x^2 \\ y_2 &= x^4 \\ y_3 &= -\frac{2}{3}x^6 \\ y_4 &= \frac{1}{2}x^8 \end{aligned} \tag{29}$$

Hence the solution is given below in the series form

$$y(x) = -2\left(x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8 + \dots\right) \tag{30}$$

Hence the exact equation has the form

$$y(x) = -2\ln(1 + x^2) \tag{31}$$

Example 3

Consider nonlinear Lane-Emden equation

$$y'' + \frac{2}{x}y' - \left(6 + x^4\right) = 0, \quad t \ge 0 \tag{32}$$

Subject to initial conditions

$$y(0) = 0 \quad y'(0) = 0 \tag{33}$$

Applying the MLDM and initial conditions (33), we have

$$-s^2 L'\{y\} + L\{xy^3\} - L\{6x + x^5\} = 0 \tag{34}$$

The recursive relation is obtained as

$$L'\{y_0\} = -s^{-2}L\{6x\},$$
$$L'\{y_1\} = s^{-2}L\{xy^3{}_{n-1}\} - L\{x^5\}, \qquad (35)$$
$$L'\{y_n\} = s^{-2}L\{xy^3{}_{n-1}\},$$

Integrating both sides of (35) and then taking the inverse Laplace transform, we have

$$L\{y_0\} = \frac{2}{s^3}, \quad y_0 = x^2,$$
$$L\{y_1\} = 0, \quad y_1 = 0 \qquad (36)$$
$$L\{y_n\} = 0, \quad y_n = 0, \quad n \succ 1$$

Example 4.

Richardson's theory of thermionic currents

Consider the nonlinear differential equation of Richardson's theory of thermionic currents

$$y'' + \frac{2}{x}y' + e^{-y} = 0 \qquad (37)$$

Subjects to the initial conditions

$$y(0) = 0 \quad y'(0) = 0 \qquad (38)$$

Applying the MLDM and initial conditions (38), we have

$$-s^2L\{y\} + L\{xA_n\} = 0 \qquad (39)$$

The recursive relation is obtained as

$$L'\{y_0\} = -s^{-2}y(0)$$
$$L'\{y_n\} = s^{-2}L\{xA_{n-1}\} \qquad (40)$$

Where the nonlinear operator $N[y] = e^{-y}$ is decomposed as in (8) in terms of the Adomian polynomials. $N[y] = e^{-y}$ are computed as follow:

$$A_0 = N[y_0] = e^{-y_0}$$
$$A_1 = y_1N'[y_0] = -y_1e^{-y_0}$$
$$A_2 = y_2N'[y_0] + \frac{1}{2}y_1^2N''[y_0] = -y_2e^{-y_0} + \frac{y_1^2}{2!}e^{-y_0}$$
$$A_3 = y_3N'[y_0] + y_1y_2N''[y_0] + \frac{1}{3!}y_1^3N'''[y_0] = y_3e^{-y_0} + y_1y_2e^{-y_0} - \frac{y_1^3}{3!}e^{-y_0}$$

(41)

Integrating both sides of (40) and then taking the inverse Laplace transform, we have

$$y_0 = L^{-1}\left\{\int -s^2y(0)\right\} = 0,$$
$$y_n = L^{-1}\left\{\int s^{-2}L\{xA_{n-1}\}ds\right\} \qquad (42)$$

Using the initial conditions and (41), we have

$$y_0 = 0,$$
$$y_1 = L^{-1}\left\{\int s^{-2}L\{xA_0\}ds\right\}, \quad y_1 = -\frac{1}{6}x^2 \qquad (43)$$
$$y_2 = L^{-1}\left\{\int s^{-2}L\{xA_1\}ds\right\}, \quad y_2 = -\frac{1}{120}x^4$$
$$y_3 = L^{-1}\left\{\int s^{-2}L\{xA_2\}ds\right\}, \quad y_3 = -\frac{1}{1890}x^6$$

Hence we obtain series solution as

$$y(x) = -\frac{1}{6}x^2 - \frac{1}{120}x - \frac{1}{1890}x^6 \ldots \qquad (44)$$

Example 5

Isothermal gas spheres equation

Isothermal gas equation are modeled by

$$y'' + \frac{2}{x}y' + e^{y} = 0 \qquad (45)$$

Subjects to the initial conditions

$$y(0) = 0 \quad y'(0) = 0 \qquad (46)$$

Applying the MLDM and initial conditions (38), we have

$$-s^2L\{y\} + L\{xA_n\} = 0 \qquad (47)$$

The recursive relation is obtained as

$$L'\{y_0\} = -s^{-2}y(0)$$
$$L'\{y_n\} = s^{-2}L\{xA_{n-1}\} \quad (48)$$

Where the nonlinear operator $N[y]=e^y$ is decomposed as in (8) in terms of the Adomian polynomials. $N[y]=e^y$ are computed as follow:

$$A_0 = N[y_0] = e^{y_0}$$
$$A_1 = y_1 N'[y_0] = y_1 e^{y_0}$$
$$A_2 = y_2 N'[y_0] + \frac{1}{2}y_1^2 N''[y_0] = y_2 e^{y_0} + \frac{y_1^2}{2!}e^{y_0}$$
$$A_3 = y_3 N'[y_0] + y_1 y_2 N''[y_0] + \frac{1}{3!}y_1^3 N'''[y_0] = y_3 e^{y_0} + y_1 y_2 e^{y_0} + \frac{y_1^3}{3!}e^{y_0}$$

(49)

Integrating both sides of (48) and then taking the inverse Laplace transform, we have

$$y_0 = L^{-1}\left\{\int -s^2 y(0)\right\} = 0,$$
$$y_n = L^{-1}\left\{\int s^{-2}L\{xA_{n-1}\}ds\right\} \quad (50)$$

Using the initial conditions and (41), we have

$$y_0 = 0,$$
$$y_1 = L^{-1}\left\{\int s^{-2}L\{xA_0\}ds\right\}, \quad y_1 = -\frac{1}{6}x^2$$
$$y_2 = L^{-1}\left\{\int s^{-2}L\{xA_1\}ds\right\}, \quad y_2 = \frac{1}{120}x^4 \quad (51)$$
$$y_3 = L^{-1}\left\{\int s^{-2}L\{xA_2\}ds\right\}, \quad y_3 = -\frac{1}{1890}x^6$$

Hence we obtain series solution as

$$y(x) = -\frac{1}{6}x^2 + \frac{1}{120}x - \frac{1}{1890}x^6 + \ldots \quad (52)$$

4. Conclusion

In this paper, we have successfully applied the modified Laplace Decomposition Method (MLDM) to obtain the exact solutions for nonlinear singular second order initial value problems. It is demonstrated that the presented approach can accelerate the rapid convergence of series solution when compared with other methods. It is shown that MLDM is simple and easy to use and produce reliable result.

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