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Student Held Misconceptions Regarding Area and Perimeter of Rectangles

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STUDENT-HELD MISCONCEPTIONS

REGARDING AREA AND PERIMETER OF RECTANGLES

A Thesis Presented

by

SUSAN M. CARLE

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Critical and Creative Thinking Program

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ABSTRACT

STUDENT-HELD MISCONCEPTIONS

REGARDING AREA AND PERIMETER OF RECTANGLES

DECEMBER, 1993

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Students enter the classroom with individual schemas,
based on their experiences and ideas, which influence the
reception, interpretation, and recollection of new
information. Effective teachers must understand the
implications of these existing schemas. As an experienced
classroom teacher, the author finds students often
manipulate and apply new information well in class, only to
forget or alter the material a few weeks later. When
misconceptions are woven into schemas, they interfere with
reception of information. This thesis examines specific
student-held misconceptions about the area and perimeter of
rectangles and the process of their identification and
eradication.

Identification of the misconception is the first step
in bringing about change. The process of identification
begins through the analysis of a pre-test which is designed
to highlight specific erroneous ideas that the students
hold. Through this pre-test, the author identifies five
misconceptions. For ease of discussion they have been

named: Fallacy of Multiples, Increase/Decrease Assumption, Conversion Conclusion, Spatial Bias, and Equality Assumptions. Each misconception is defined and explained and the specific pre-test questions used for its identification are included.

There are several learning theories which can aid the teacher in establishing a process of misconception eradication and educational change. The author works within a framework including theoretical components of cognitive psychology, Anderson's theory of memory, and Ennis' definition of critical thinking and taxonomy of critical thinking dispositions and abilities. The mathematical components of this framework are developed utilizing metacognition, transfer, and recent curriculum and professional development standards of the National Council of Teachers of Mathematics.

This multi-faceted framework provides the foundation on which to build lessons targeting the eradication of specific misconceptions. Three lesson plans are presented to illustrate the practical implementation of the theories in the classroom. Each lesson contains four components: Motivation, Activity, Metacognition, and Transfer. The author concludes the thesis with more general classroom teaching suggestions and a review of current innovative educational approaches.

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INTRODUCTION

Overview

The focus of this thesis is the identification and
eradication of misconceptions regarding area and perimeter
of rectangles. I am a high school mathematics teacher
writing for my colleagues (although there are general ideas
which transcend subject areas). I wish to share with them
the importance of identifying misconceptions in order to
provide ideas and tools for them to take into their own
classrooms.

Misconceptions can be found at all levels in all
subjects. I have limited my study to specifically
identified misconceptions held by high school geometry
students. Using an informal pre-test I designed, I analyzed
the students' incorrect answers and identified five discrete
misconceptions. I present and define each of these
misconceptions, along with several multiple choice questions
targeting each, to be used as a tool for identification in
other students. Basing my work on Perkins (1981, 1986),
Gardner (1983, 1985, 1991), Ennis (1987), Schoenfield
(1987a, 1987b), Costa (1991a, 1991b), and others and
integrating the theory with nationally accepted standards
for teaching mathematics, I present lesson plans intended to
lead students to identify their erroneous beliefs and

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encourage them to construct the correct concepts
independently.

Rationale and Purpose

High school mathematics can be both challenging and fun
for students and teachers alike. Teaching students to
approach conventional problems with unconventional methods
can increase motivation and further understanding.
Utilization of these innovative approaches demands that the
teacher develop new perspectives for presenting traditional
subject matter. Quite often, there are obstacles to
overcome when trying to reach this challenging goal. Among
these obstacles are subtle misconceptions that students
themselves often don't recognize. These misconceptions are
generally deeply rooted and hinder the acquisition of
correct information.

Too often, lessons presented to students are seemingly
understood at the moment, but after a short period of time
the new concepts often disappear leaving behind the more
entrenched prior knowledge. Like many classroom teachers, I
look for an explanation behind my students' short lived
success with various concepts and conclude that their
misconceptions are a factor. I find that it is not easy to
uncover their erroneous beliefs because the misconceptions
are not easily recognized through verbal communication. If
a student is asked "Do you understand?", the common response
is "Yes." Even when asked "Are there any questions?", the
misconceptions still go undetected because the typical

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student response to that question is "No." It isn't until a student must write down an answer, whether on a homework problem, a classroom example, or a test, that the misconception becomes evident. The Massachusetts Educational Assessment Program also found that misconceptions about area and perimeter became apparent when students had to perform tasks. On assessment tests given to fourth and eighth graders in 1988, students showed limited understanding of area and perimeter. "Students do not understand the relationship between the properties of figures and their measurement" (Badger 1989, 39). If I were able to specifically identify misconceptions before a lesson was presented and then target the erroneous beliefs during the lesson, I would be better prepared to help the students overcome their misconceptions.

My initial identification came through the use of an informal multiple choice pre-test that I designed. I used my experience as a classroom teacher when writing the questions. I already had some thoughts about the beliefs my students held and used this insight to arrive at appropriate distracters for each question. Through the use of this pre-test, I identified five different misconceptions which my students held about the relationship of area and perimeter of rectangles. Lessons were then developed to target three of the five misconceptions I had identified. The pre-test and lessons presented in this thesis have been revised based on both the success and problems encountered with two of my geometry classes. The purpose of this thesis is to pass on

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to other experienced high school mathematics teachers the insights that I have gained. It is the goal of this thesis to offer a way to identify misconceptions and to suggest strategies that will help to make instruction more effective in terms of correcting students' misconceptions and transferring correct information.

What is a Misconception?

For this thesis I define a misconception as a misunderstanding based on personal biases, false assumptions, and/or incorrect conclusions. When embarking on a new subject, students bring with them an intellectual framework which may contain incorrect ideas and perceptions. Very often these ideas are formulated before the students enter school. Each person interprets the world around him and from that input creates an understanding specific to his own needs. Even if these understandings are false, individuals work with them and use reasoning that makes these interpretations functional. The danger in not identifying and correcting misconceptions is that as new information is received, it is interpreted through this incorrect reasoning and framework already in place.

Misconceptions in Mathematics.

Misconceptions can be seen across the mathematics curriculum. In basic mathematics, students sometimes have difficulties when "borrowing" to subtract fractions. If given the problem 12 3/8 - 6 5/8, students mistakenly place

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the 1 borrowed from the 12 next to the 3 to create the incorrect fraction 11 13/8. Students understand the process of subtraction and the need to borrow, but they haven't correctly calculated how the 1 they have borrowed changes the fraction. In algebra, students have difficulty with the use of variables. When asked to simplify $x + 2 - 2$, students may say it can't be done without knowing what x is. With the simplification of fractions, when asked to simplify 36/56, students may cancel the 6's. When simplifying rational expressions, students may cancel the n's in $(n + 5)/(n - 2)$.

Misconceptions in Geometry.

There are problems in geometry as well. Many students think that right triangles must always be drawn in the same configuration (right angle in the lower left) or that the base of an isosceles triangle is always on the bottom. Students think that lines can be straight or curved and because of this don't always interpret the concept of parallelism correctly; for example, they might say that two concentric circles are parallel because they don't intersect.

I became particularly interested in the misconceptions about area and perimeter after I wrote some lessons using these concepts to encourage my students to be better critical thinkers. For the purpose of this thesis, I have limited this discussion of misconceptions to rectangles.

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<u>Misconceptions in Area and Perimeter of Rectangles</u>.

My students were very capable of calculating area and perimeter, but whenever I asked questions that required them to analyze or synthesize the relationship between the measurements, they ran into problems. For instance, a typical question might ask the following: Given rectangle A, with a base of 3 inches and a height of 5 inches and rectangle B, whose base and height are respectively twice those of rectangle A, how does the area of rectangle B compare to the area the rectangle A? The typical response was that the area had doubled as well. It was through analysis of students' responses to questions like this, that I came to the conclusion that my students had some misconceptions which needed to be addressed.

Overview of Chapters to Follow

Chapter II of my thesis presents the theoretical framework which I utilize. Its major components are cognitive psychology and critical thinking.

Cognitive psychology is a science which helps us understand the workings of the mind. Using Anderson's (1990) model of memory, I discuss how information is encoded within our brains. During this encoding process misconceptions may be developed and embraced. To help explain how this may happen, I review the work of four authors. Alan Schoenfield (1987a) discusses how individuals construct their own interpretive framework and process

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information to fit that framework, despite what he calls "objective" (22) reality. David Perkins and Fay Martin (1986) develop the concept of "fragile knowledge" (216) which gives some insight as to how well people really know something. And finally, Howard Gardner (1983) proposes the idea of multiple intelligences and hypothesizes that the failure to address each of these intelligences during teaching may result in only partial learning. All four of these theorists have ideas that, in conjunction with each other, help explain the difficulties of accurately acquiring information.

I also discuss the use of critical thinking skills as a means to enhance learning. There are several definitions and taxonomies of skills. I focus on Ennis' (1987) definition and his table of abilities and dispositions. I highlight the following dispositions:

> 1) Seek a clear statement of the thesis or question,
> 2) Seek reasons,...
> 8) Look for alternatives,...
> 10) Take a position (and change a position) when the evidence and reasons are sufficient to do so,...
> 11) Seek as much precision as the subject permits. (Ennis 1987, 12)

From his list of critical thinking abilities, I include:

> 1) Focusing on a question,...
> 7) Inducing and judging inductions,...
> 9) Defining terms and judging definitions in three dimensions,...
> 10) Identifying assumptions. (Ennis 1987, 12-14)

Each of these abilities is broken into specific sub-skills and expanded upon in my thesis. (Ennis' complete list of dispositions and abilities is included in Appendix A.) I

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use these dispositions and abilities as a foundation for my lesson plans.

Chapter III includes a discussion of the mathematical framework of my lessons. I state and discuss the specific Standards of the National Council of Teachers of Mathematics (NCTM) (1989,1991) which my lessons target. The complete list of standards are established as a guideline for all current mathematical teaching. (The complete set of standards is included in Appendix B and Appendix C.)

There are two sets of the NCTM guidelines, one for curriculum and evaluation (1989) and one for the professional development of teachers (1991). From the Curriculum Standards (1989) I use "Standard 3: Mathematical Reasoning" (143) and "Standard 7: Geometry from a Synthetic Perspective" (157). The focus of Standard 3 is inductive and deductive reasoning. It highlights the importance of recognizing patterns, making generalized conjectures and then testing one's hypothesis for verification. The focus of Standard 7 is to provide students with experiences that deepen their understanding of shapes and their properties. It also emphasizes the importance of infusing examples with applicability to human activities into the curriculum. From the Teaching Standards (1991), I use "Standard 1: Worthwhile Mathematical Tasks," (25) "Standard 3: Students' Role in Discourse," (45) and "Standard 4: Tools for Enhancing Discourse" (52). Standard 1 emphasizes the teacher's responsibility for providing a comfortable learning environment, complete with an array of learning materials.

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Teachers should promote problem solving and communication in their students. Standard 3 stresses the need for students to be engaged in making conjectures, proposing solutions to problems and presenting valid arguments to particular claims. Finally, Standard 4 instructs the teacher to enhance the discourse in the classroom by allowing and encouraging the use of manipulatives, computers, calculators, diagrams or any tool which encourages the student to more effectively explore and communicate mathematics.

Along with the NCTM guidelines, there are some general concepts that can be used to help eradicate already formed misconceptions. The first of these is metacognition. Loosely defined, metacognition means thinking about one's thinking. Arthur Costa (1991b) and Alan Schoenfeld (1987b) discuss the benefits of engaging students in this process, assisting the students to develop an awareness of how they think and why they think that way, and gaining the power to edit and accurately refine the information they are processing.

Another principle which aids in the correction of misconceptions is transfer. If information is not transferred correctly, a misconception is born. How do we, as teachers, better ensure transfer of what we are teaching? I discuss the theory of "prompts, hints and provides" by Perkins and Martin (1986, 218) as well as their emphasis on providing students with varied experiences during the learning process. I explore Gardner's (1991) idea of

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"Christopherian Encounters" (151) and his beliefs on how its use helps students confront and replace their misconceptions.

Chapter IV defines the five misconceptions that I have identified. For ease of reading and discussion, I have named each misconception. Included with each are several item specific multiple choice questions. These questions give a clearer picture of each defined misconception. Taken collectively, these questions can be used by teachers as a means of identifying the specific misconceptions their students hold. (The complete pre-test is presented in Appendix D.)

Students may hold all or none of the following misconceptions about area and perimeter of rectangles:

1. Fallacy of Multiples - If the base and height of a rectangle are doubled, then the new rectangle's area is twice that of the original rectangle's area, or if the base of a rectangle is tripled, then the perimeter of the new rectangle is triple that of the original. The fallacy is the assumption that the multiple which occurs, whether to base, height or both, will always be the same multiple which describes the change in area or perimeter.

2. Increase/Decrease Assumption - If the height of a rectangle increases by three units then the perimeter also increases by three units. The assumption neglects to take into account that there are two heights in the rectangle.

3. Conversion Conclusion - If there are three feet in a yard, then there are three square feet in a square yard.

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The mistaken conclusion is that two dimensional unit conversions are equivalent to linear unit conversions.

4. Spatial Bias - The number representing area must be greater than the number representing perimeter, since "space is more," as a student once told me. The students fail to understand the importance and value of the units attached to the number.

5. Equality Assumption - If area is conserved, then so is perimeter. For example, if a rectangle has an area of 24 sq. in. and a perimeter of 28 in., then another rectangle having an area of 24 sq. in. must also have a perimeter of 28 in. The students incorrectly assume that area and perimeter are directly correlated and that one specifically determines the other.

Chapter V of my thesis contains lesson plans that target three of the five misconceptions. There are lessons for the Fallacy of Multiples, Conversion Conclusion and Equality Assumption misconceptions. Each lesson plan follows the same four part format: motivation, activity, metacognition and transfer. I also include the rationale behind each lesson based on the theoretical framework presented in Chapters II and III.

Finally, in Chapter VI, I reflect back on the previous chapters. The theories are unraveled and I discuss the impact that they have on the ever-changing face of education. I talk about the implications for the classroom teacher and suggest a few general strategies that can be easily integrated into general classroom practice as a means

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of getting started. For instance, students can be taught to test cases by substituting numbers for variables, to use pictures drawn to scale, or to use and manipulate variables to work through a generalized version of the problem. As a classroom teacher, I recognize the importance of practical information, and hence I hope to provide guidance for my colleagues to use these ideas effectively in their classrooms.

CHAPTER II

THEORETICAL FRAMEWORK

Overview

In order to understand the origins of misconceptions
and to develop strategies to correct them, it is necessary
to understand the thinking and learning process. The
theoretical framework I use is based on two fundamental
branches of study: cognitive psychology and critical
thinking. Cognitive psychology is the science which
explores how the mind works. Critical thinking is "thinking
that displays mastery of intellectual skills and abilities"
(Paul 1992, 643).

In this chapter I give a brief introduction to
cognitive psychology, focusing on Anderson's theory of
memory. The discussion includes long term memory,
propositions and schemas. It is important for teachers to
understand how the brain takes in information, categorizes
it and, most importantly, retrieves the information.

> It is not helpful to plan as though we, as teachers,
> are putting new ideas into otherwise blank minds.
> All learners have to construct meanings from what
> they see and hear, using existing ideas in longterm
> memory. (Osborne and Fryeberg 1985, 144)

Once an understanding of how information becomes
encoded is achieved, the natural question becomes: "How does
a misconception arise?". There is no succinct answer to
that question. Each individual operates within his own

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framework, interpreting new information in a manner which fits his preconceived ideas.

I review three authors who each bring a unique insight to our understanding of the learning process. Alan Schoenfeld (1987a) studies cognitive science and its implications in mathematics. David Perkins is interested in how the mind works and has been involved in studies, with Fay Martin (1986), concerning misconceptions and what they call "fragile knowledge" (216). Furthermore, Howard Gardner (1983) proposes the theory of multiple intelligences, believing that there are many ways in which information is accessed. While some students may fully understand a concept presented in a lecture format, others may benefit from a visual or experimental approach. Together, these theorists offer some explanations and insights as to how a misconception may be formed.

The final part of the chapter focuses on critical thinking. There are several definitions and taxonomies of skills from which to choose. I highlight Robert Ennis' (1987) definition and his table of dispositions and abilities. This table is extensive and I do not attempt to discuss it all. Instead, I focus on the following dispositions:

    1) Seek a clear statement of the thesis or question,
    2) Seek reasons,...
    8) Look for alternatives,...
    10) Take a position (and change a position) when the
        evidence and reasons are sufficient to do so,...
    11) Seek as much precision as the subject permits.
        (Ennis 1987, 12)

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From his list of abilities, I discuss:

1) Focusing on a question,...
7) Inducing and judging inductions,...
9) Defining terms and judging definitions in three dimensions,...
10) Identifying assumptions. (Ennis 1987, 12-14)

I conclude with a discussion of critical thinking as a means of conquering misconceptions.

Cognitive Psychology

Introduction.

How people think and the nature of human intelligence have interested scientists for decades. Although much has been learned about the anatomy and physiology of the brain, the mechanisms for processing information have yet to be completely understood. The newest branch of science to attempt to unravel this mystery is Cognitive Psychology.

In an attempt to explain information processing and problem solving, cognitive psychologists have looked at the cognitive process as a sequence of ordered stages. I am particularly interested in the stages of encoding, "the process by which new incoming information is related to and transformed by preexisting knowledge structures" (Schacter 1989, 689), and retrieval, as I think it is at these stages that misconceptions are born. How information is encoded and retrieved is quite complex, and it is not within the scope of this paper to present a detailed explanation. However, there are two topics, propositions and schemas, which are particularly relevant to this study. It is

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through these two mechanisms that information is stored in and retrieved from memory.

Propositions.

According to Anderson (1990) a "proposition is the smallest unit of knowledge that can stand as a separate assertion; that is, the smallest unit about which it makes sense to make the judgement true or false" (123). For example, **this is a square**, is a proposition. As we process information, whether auditory or visual, we break it down into propositions. This is similar to dissecting a sentence into its most basic components in order to understand its meaning. Just as some sentences can contain another separate sentence within them, so can propositions. They actually enter "into hierarchial relationships where one proposition occurs as a unit within another proposition" (Anderson 1990, 128).

An example of a proposition within a proposition is the following: The angle is obtuse because it has a measure of 120 degrees. The sentence contains both the proposition **the angle is obtuse** and the proposition **the angle has a measure of 120 degrees**. In our organization of the sentence, both propositions would be encoded and they would be linked together by **angle**, the shared subject. This linking of propositions is called "networking" (Anderson 1990, 126). The more links we have between propositions and the stronger the links are, the easier it is to retrieve the encoded information from our memory.

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<u>Schemas</u>.

Schemas are more general than propositions.

> They are ways of encoding regularities into
> categories, whether these regularities are perceptual
> or propositional. They are abstract in the sense
> that they encode what is generally true rather than
> what is true about a specific instance. (Anderson
> 1990, 134)

For example, **squares** would represent a schema. Even without

a specific picture, people are able to think of a figure

having four sides of equal lengths and having four right

angles. Schemas are not concerned about the specifics of

information. A schema would not contain information

describing the actual size of the square or its position or

orientation on the paper.

Much of the organizing we do with information takes

place within schemas. They provide us with the "big

picture."

> The general notion is that new, incoming information
> activates a higher-order body of relevant past
> knowledge and experience - a schema - and is encoded
> in terms of the information already present in the
> schema. (Schacter 1989, 692)

It is within this self defined framework that people

operate. Individual frameworks may differ depending on

culture, upbringing or education. Because people don't all

have the same schemas, it is possible for each to remember

identical information differently.

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At the time of retrieval, when the schema is activated, recall of a prior experience typically consists of some combination of specific information that was stored about that experience and general information that is contained in the relevant schema. This notion allows for, and even predicts the occurrence of, various kinds of distortions and biases in remembering. (Schacter 1989, 692)

Long-Term Memory.

According to Anderson (1990), "memories which have sufficiently strong encodings that they can be reactivated are referred to as long-term memory because they can be recalled at long delays" (150). Both propositions and schemas affect the encoding of information, whether accurate or inaccurate.

Accurate encoding. There are several ways to make an encoding strong. One is elaboration. As information comes in and is encoded, propositional networks are developed. These networks can be enhanced by using visual images and by learning additional facts to expand upon the original information. By the use of elaboration, more links are created between propositions. Then, if one link is weak, another retrieval path can be activated, and the thought is not forgotten. "Thus, elaborations help to the degree that they are an effective means of redundancy and so provide a means for reconstructing the to-be-remembered information" (Anderson 1990, 184).

Creating meaning out of received information is another way to create a stronger encoding. Some people do this naturally. For instance, if you ask someone how they

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remember a new phone number, many people respond with an explanation of how some of the digits add up to another digit or perhaps some of the digits are representative of familiar birthdays. Whatever the case, the individual is creating meaning in order to improve memory. "It has been shown over and over again that meaningful information is better remembered than meaningless information" (Anderson 1990, 121). Like elaboration, creating meaning strengthens encoding of propositions, thus strengthening memory.

Inaccurate encoding. While propositions generally provide accurate encoding, schemas can aid in inaccuracies. Anderson (1990) found that people's belief in their set of schemas is so strong that even when information "does not fit with their own schemas, they will exhibit a powerful tendency to distort it to make it fit" (197). This explains how a teacher could present a student with accurate information and have the student interpret it inaccurately. Schacter (1989) offers this explanation. He states:

> The only aspects of an event that will be represented are those that are relevant to a schema that is activated at the time of encoding. Therefore the stored representation of an event may be incomplete because a schema appropriate to incoming information is not activated, or because schema-irrelevant details are not encoded. (693)

Because schemas are general ideas, they may lead the student to make inferences.

> Activated schematic knowledge provides a basis for making inferences and suppositions about the meaning of an event, and such interpretive activities may become part of an event's representation. (Schacter 1989, 693)

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Unfortunately, these inferences may not be appropriate for the problem a student is working on at a particular moment and instead of providing accurate information may instead offer erroneous information. To avoid this, specifics of the problem at hand must be considered and results based on inferences in the past must be weeded out before the student can reach the correct conclusions.

Origins of Misconceptions.

Although no one has clearly identified the origin of misconceptions, several theorists have offered their ideas. There are two general concepts I discuss. One is that students do not make a connection between the "theories" they created as natural scientists in their early years and the theories presented to them later in textbooks. Because they see school as unconnected to their daily lives, they tend to believe that the information presented to them in school is of no use in the "real world." The other suggests that pre-formed schemas override new information and the new information is skewed to fit into the schemas already acquired. Three theorists highlight these ideas and each offers a unique perspective.

Objective reality. Alan Schoenfeld (1987a) is known for his work in cognitive science and its significance for mathematics. He believes that "objective" (22) reality and one's interpretive framework offer an explanation for misconceptions.

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"Objective" reality refers to what is real. Schoenfeld (1987) points out that what we see "may or may not correspond to 'objective' reality" (22). Because of our backgrounds, experiences and prior knowledge, what we think we see is not always accurate. To illustrate this Schoenfeld uses the example of an optical illusion. The question accompanying the illusion may ask a question like: "Are the two lines parallel?". While the lines are actually parallel, it is difficult for one to see the lines as such, despite measuring to prove it. When we look at the picture we are interpreting what we see, not processing what is actually there.

This leads to the idea of interpretive frameworks. As each of us experiences the world, we create our own frameworks based upon our own interpretations. These interpretations become our personal schemas and we use them to guide us in the future. When this happens, a misconception can occur.

> When a student retains and continues to use his preconception to interpret classroom information, he is likely to give it meaning which differs from or even conflicts with the meaning intended by his teacher. (Nussbaum and Novick 1982, 184)

The new concept the student adopts does not develop without thought. On the contrary, many times he has an elaborate reason to explain the conclusion he has reached.

> It is not a matter of 'not understanding' but of 'understanding differently' from what was intended. In doing so, he (the student) may very well give the new information a rationale and even a sophisticated meaning which is indeed at variance with the accepted meaning. (Nussbaum and Novick 1982, 184)

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From different experiences in his past, the student makes what he thinks is a reasonable abstraction from his framework and uses it to answer the new problem. Unfortunately, what seems to make such obvious sense to him and his framework may conflict with proven fact.

Fragile knowledge. David Perkins and Fay Martin (1986) studied novice programmers to look at the difficulties encountered in mastering programming skills even after the material was well presented. One of their main findings was what they termed "fragile knowledge" (216). This refers to incomplete, hard to access, and misused knowledge.

> Common experience testifies that often a person does not simply 'know' or 'not know' something. Rather, the person sort of knows, has some fragments, can make some move, has a notion, without being able to marshall enough knowledge with sufficient precision to carry a problem through to a clean solution. (Perkins and Martin 1986, 216)

When a person relies on this "fragile knowledge" to reach a conclusion, a misconception may occur.

To expose students' "fragile knowledge", Perkins and Martin (1986) used "prompts, hints, and provides" (218) when answering students' questions . These were very often in the forms of questions. A **prompt** would give no specific information to the problem, but would instead attempt to elicit strategic thinking from the students. A **hint** would ask the students about specific strategies they might know, leading the students to a solution. Finally, a **provide** would give the students an exact solution to the problem.

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Depending on whether the students needed a prompt, hint, or provide, an assessment of the students' level of knowledge could be made.

> The phenomena of fragile knowledge say that students know more than you might think. To be sure, that knowledge is often inert, undifferentiated, undergeneralized, and so on....Moreover, the fragile knowledge phenomena of misplaced and conglomerated knowledge catch students in the midst of seeking to cope with the task in an exploratory way. (Perkins and Martin 1986, 225)

Consequently, as students grope to make meaning out of what they practically know, misconceptions may arise. Prompts, hints, and provides allow the teacher a window into the students' thinking process and allow her the chance to identify and point out the various misconceptions to the students. More about the identification of misconceptions is provided in Chapter IV.

Multiple intelligences. Howard Gardner, in his book Frames of Mind (1983), theorizes that human intelligence is not limited to linguistic and reasoning skills. He instead proposes that there are seven human intelligences: linguistic, musical, logical-mathematical, spatial, bodily-kinesthetic, and personal (which has two parts, inter-personal and intra-personal), and suggests that measuring one does not give a complete picture of how smart someone is. Not everyone can obtain the same mastery level in each of the distinct intelligences. In fact, a single individual may be better equipped to learn in some realms of intelligences than in others.

> For example, students with strengths in the spatial,
> musical, or personal spheres may find school far more
> demanding than students who happen to posses the
> 'text friendly' blend of linguistic and logical
> intelligences. (Gardner 1991, 149)

Because people have these different facets of intelligences,
it is possible that instruction which fails to target the
intelligence where their strength lies will allow them to
fall back on what they believe to be true, rather than seek
to incorporate the new information correctly into their
schemas. Gardner feels that schools encourage the use of
existing schemas and don't correctly teach students to
incorporate new knowledge.

> School is difficult because much of the material
> presented in school strikes students as alien, if not
> pointless, and the kinds of supporting context
> provided for pupils in earlier generations has become
> weakened. (Gardner 1991, 149)

Because examples of theories used during a class are
frequently contrived, students see little, if any, relevance
to their lives. When this happens, the material presented
can only be recalled in these contrived problems that prime
the student to use the new information. If the wording
changes or the situation is presented differently, students
naturally fall back on their previous, more powerful set of
schemas. One must remember that not all misconceptions are

> completely wrong or completely useless. Indeed, they
> develop and endure precisely because they prove
> sufficiently functional in the world of the young
> child and can be drawn upon with some utility even in
> the adult world. (Gardner 1991, 156)

Implications. The implications for teaching are
numerous. The common thread through the three theorists is

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the idea that each student enters the classroom with his own set of schemas. These schemas are quite strong and tend to overshadow any new information presented. This suggests that a teacher must not assume that the information she presents will be identically received by the students. The teacher should ask the students to explain the concepts in their own words. This will stress that the correct answer is not a verbatim repetition of the teacher's words. It is important for the teacher to make sure that the students have made the information "theirs" and that it is correct.

The idea of "fragile knowledge" (Perkins and Martin 1986, 216) is also important for the teacher to understand. Too often, teachers are quick to answer students' questions directly, without allowing or encouraging the students to think about the question and reason an answer for themselves.

Finally, as teachers, we must remember that all students have strengths in different areas of intelligence. It is important for us to identify and target various learning styles. We also must work harder at providing a link between what is learned at school and the "real world." Students need to see how classwork applies to their lives and to the framework that they have already established. Chapter III elaborates on how these ideas can aid in the transfer of new material.

Critical Thinking

General Definition.

Defining critical thinking is a challenge. Some people equate it with logical reasoning. Others feel it means taking the time to examine a subject thoroughly before making a decision. Critical thinking is all of that and more. It includes reasoning carefully, questioning references, examining points of view, looking at decisions and verifying their effects, and many other skills. Because there are so many varying ideas about critical thinking, I elected to choose one theorist's definition and work within its framework. I chose Robert Ennis' definition and have focused on his list of dispositions and abilities.

Ennis' Definition.

"Critical thinking is reasonable reflective thinking that is focused on deciding what to believe or do" (Ennis 1987, 10). This definition, although succinct, includes a host of skills and dispositions. It focuses and elaborates on the upper levels of Bloom's taxonomy: analysis, synthesis, and evaluation. However, Ennis finds that those terms are too vague and each one can be expanded upon. He also feels that his definition leaves room for creativity when solving a problem. Ennis has fourteen dispositions and twelve abilities (with each ability broken into several sub-categories). Although I find them all useful in the classroom, for the purposes of this paper, I limit my

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discussion to only a few selected dispositions and abilities. His complete list can be found in Appendix A.

Dispositions.

There are five of Ennis' (1987) dispositions which I find pertinent to this discussion. Each one is valuable when attempting to solve a problem. If my students are able to embrace these dispositions they become better equipped to explore mathematical ideas.

The first disposition is "1) Seek a clear statement of the thesis or question" (12). When approaching any problem, it is important to clarify exactly what it is that you are being asked to do. Too often students read through a problem and then seek a quick solution. For example, a question might ask: "How many cartons would you need to pack 45 eggs, if each carton holds 12 eggs?". Too often students give an answer such as 3 and 3/4 cartons. While this answer is correct for the simplification of 45/12, it does not answer the question asked. This first disposition, when utilized, puts students on the right path towards finding a solution and eliminates an unnecessary waste of time as they continue through the solution process and eliminates unreasonable answers.

Disposition "2) Seek reasons" (12) can mean to seek a reason for each step one takes when solving a problem or it can mean to seek a reason for the occurrence of a certain phenomenon. To be good critical thinkers, students need to become adept at both aspects. It is important for students

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to understand the reason behind each step they take when solving a problem. They also need to be able to analyze retrospectively a correct answer that may have been achieved by chance.

Too often when students find themselves stuck on a problem they give up. Ennis offers disposition "8) Look for alternatives" (12). This disposition is one of the most important things a good problem solver does. When students are stumped they need to take a different approach to the problem. If they experiment with ideas, and are not singularly focused, they are more likely to find success with the problem.

Disposition "10) Take a position (and change a position) when the evidence and reasons are sufficient to do so" (12) requires the students to use their ability to reason. Students may present two extremes. They may take a position on an answer and not be persuaded to change what they think, even if contrary evidence is presented. Their schemas are overriding the instruction. At the other extreme, students may easily surrender their position without understanding a reason to do so. These students have little confidence in their schemas and are easily dissuaded. This disposition requires students to look at their position, evaluate its accuracy and change that position if, and only if, there exist reasons to do so.

Disposition "11) Seek as much precision as the subject permits" (12) is vital to their ability to focus. Sometimes students get bogged down with problem solving because they

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are concerned with details not necessary to the problem's solution. On the other hand, at times, they go to the opposite extreme and answer a problem in generalities rather than the specifics needed. This disposition requires students to judge the problem and answer it with the appropriate amount of precision, thus neither hindering their progress nor leaving an answer too vague.

Abilities.

Unlike dispositions, which are traits of thinking patterns, abilities represent the skills that a person possesses. Ennis (1987) suggests four abilities which I find relevant to my discussion.

Ability "1) Focusing on a question" (12) is three fold. The ability to identify or formulate a question, to identify or form criteria that would judge possible answers to a question, and to maintain parameters of the given situation are necessary components when focusing on a question. It is not enough to be able to simply identify the question; one needs to be able to examine and focus on all aspects of it, from start to finish.

Ability "7) Inducing and judging inductions" (13) is the one I target most frequently. Ennis breaks this into four components. They include generalizing, inferring explanatory conclusions and hypotheses, investigating, and criteria. This ability includes being able to recognize a pattern and discern whether or not you have enough information to actually claim the pattern exists. With this

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skill the student is able to accurately interpret
information presented. She is also able to set up
experiments to gather her own information. Through these
experiments she is able to make conclusions from the
evidence gathered and is able to reason that the conclusions
she has drawn fit the data and are plausible. This skill
allows and encourages the student to be an active
participant in her own learning.

Ability "9) Defining terms and judging definitions in
three dimensions" (14) requires the student to be able to
judge the validity of results based on the definitions she
is using. By three dimensions, Ennis means the ability to
interpret results based on different forms, definitional
strategies, or context of the same word. It is important
that a student is able to differentiate meaning based on
context of a word.

Ability "10) Identifying assumptions" (14) requires the
student to review the problem, the process, and the answer
for hidden assumptions. Are these assumptions valid? If
not, does that change the results she has obtained?
Sometimes, to progress through a problem it is acceptable to
make an assumption. The student needs to be able to
recognize when this is appropriate and to define the
assumption she makes.

Implications.

Since misconceptions tend to be deeply rooted, it is
necessary for students to be good critical thinkers in order

for them to replace their misconceptions with accurate information. The dispositions and abilities that were highlighted all require the students to question the validity of their beliefs. They also ask students to give reasons for and substantiate claims that they make. Students should not be allowed to say that they believe something just "because." It is necessary for the teacher to make the students accountable for what they believe. It is the teacher's responsibility to foster these skills within her students. She must encourage them to ask questions and not let them be satisfied with unexplained results.

When utilized, these abilities also encourage students to be more active in their learning by experimenting. Of course this only happens if the teacher is willing to step back from lecturing and give the students a chance. Teachers need to devise lessons that encourage and expect the students to think for themselves. Examples of this type of lesson are offered in Chapter V. The teacher should not be too quick to step in with an answer when students are frustrated; rather she should gently lead them, perhaps through questioning, in the right direction. The students should be encouraged to make a connection between what they are learning in school and their outside lives. The more they make these connections, the more likely it is that their interpretations will be accurate.

Chapter Summary

Propositions, long term memory, schemas, objective reality, and fragile knowledge all offer their own lens to look at misconceptions. It is clear that people all operate within their own framework and that this framework greatly influences how they interpret information presented to them. It is also clear that in order to change their frameworks to be correct, the errors need to be pointed out to them, or discovered by them, so that they may adapt their schemas accordingly.

> However, there has been no well-articulated theory explaining or describing the substantive dimensions of the process by which people's central, organizing concepts change from one set of concepts to another set, incompatible with the first. (Posner et al. 1982, 211)

It is known that students tend to investigate when faced with the unknown. They also interpret rather than learn information by seeking patterns and regularities in what the teacher says. That is the good news.

> The bad news is the type of generalizations the students try. . . . the generalizations are based on surface properties rather than meaning . . . (and) unlike good scientists, most students don't consciously test their generalizations. (Maurer 1987, 171)

This is where critical thinking can be helpful. It offers skills and dispositions to help students who are confronted with problems which can't easily be answered with information from their present schemas. These abilities and skills need to be fostered in students to be utilized to the fullest extent. This hope of eradicating misconceptions

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leads to the concept of transfer. It is important to remember that

> learning is concerned with ideas, their structure and the evidence for them. It is not simply the acquisition of a set of correct responses, a verbal repertoire or set of behaviors. We believe it follows that learning, like inquiry, is best viewed as a process of conceptual change. (Posner et al. 1982, 212)

Chapter III provides some possible ways to increase transfer and to encourage this conceptual change through the use of metacognition and by following the guidelines set forth by the National Council of Teachers of Mathematics.

C H A P T E R I I I

MATHEMATICAL FRAMEWORK

Overview

This chapter presents the theoretical framework specifically related to mathematics. Lessons presented in Chapter V are based on this framework which includes three major components: Metacognition, Transfer, and the National Council of Teachers of Mathematics (NCTM) Standards (1989, 1991). Although the first two topics do not solely apply to mathematics, their application in a mathematics classroom is advantageous and useful to students and is strongly encouraged in the Standards.

The goal of my lessons is to get students to confront and correct any misconceptions they have developed about area and perimeter of rectangles. To encourage students to acknowledge the misconceptions that they have, the lesson plans must challenge them to think about what they think and how they think. Metacognition is a term which defines that process. Loosely translated, it means "thinking about one's thinking" (Schoenfeld 1987b, 189). There are two authors whose works I reference in discussing the concept of metacognition: Arthur Costa and Alan Schoenfeld. Each has written on the value of engaging students in the process of metacognition, by assisting the students to develop an awareness of the way they think and why they think that way

and helping them gain the power to edit and accurately refine the information they are processing.

Once information is processed, it is necessary to think about how students transfer that information. An initial form of transfer occurs when the student gains the ability to use particular skills and techniques as needed in given problems. Often referred to as rote learning, it is useful as long as no broader application is expected. The ability to transcend the specific examples and apply the new concept to other areas and varying types of problems is indicative of a higher level of transfer. The teacher's role in this process is limited to the presentation of skills, concepts, and strategies. The teacher can't transfer the information for the student, he has to do that himself.

A student's ability to transfer is important because if information is not transferred correctly, then a misconception could be born or could be allowed to persist. I discuss how teachers can aid students in the process of transfer using the help of several authors. Perkins and Martin (1986) emphasize providing students with varied experiences during instruction. Perkins and Salomon (1991) suggest using the techniques of "bridging" and "hugging" (220). Gardner (1991) helps us understand how better transfer can help students replace misconceptions with correct information.

The chapter concludes with a discussion of the NCTM Standards. These standards have been developed to be used as a unifying guide for the instruction of mathematics. The

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complete document includes the <u>Curriculum and Evaluation</u> <u>Standards for School Mathematics</u> (1989) and the <u>Professional</u> <u>Standards for Teaching Mathematics</u> (1991). Although I do not address either set completely, there are a few select standards which I have highlighted. From the <u>Curriculum and</u> <u>Evaluation Standards for School Mathematics</u>, I have chosen "Standard 3: Mathematical Reasoning" (143) and "Standard 7: Geometry from a Synthetic Perspective" (157). From the <u>Professional Standards for Teaching Mathematics</u>, I discuss "Standard 1: Worthwhile Mathematical Tasks" (25), "Standard 3: Students Role in Discourse" (45), and "Standard 4: Tools for Enhancing Discourse" (52).

Metacognition

A general theme that runs through the <u>Standards</u> is the idea that having students reason, think for themselves, and then communicate their thoughts is important to learning. One way to enhance these processes is to have students examine their own ways of thinking. By having students focus on their own minds, they develop an insight that helps them become better thinkers and, ultimately, better learners. Metacognition defines this process.

<u>Definition</u>.

To say that metacognition is "thinking about one's thinking" Schoenfeld (1987b, 189) only begins to explain what this term represents. He offers a more thorough

definition by describing metacognition as having three distinct categories of intellectual behavior. They are:

1. Your knowledge about your own thought processes.
2. Control or self-regulation.
3. Beliefs and intuitions. (190)

The first category refers to a student's ability to know and respect the limitations of his own thinking. This could refer to knowing how well he performs certain tasks, for example, memorizing a group of facts, or understanding what types of explanations work the best for him, auditory or visual. The second category is the ability to know and use limitations wisely. For example, students should not proceed in solving a problem without knowing where the adopted strategy will take them. Also, when solving problems, it is important to understand what the rationale is at all times during the solution. The student doesn't get caught up in the process of answering just for the sake of giving an answer.

The last category reminds teachers that all students enter situations with prior knowledge. It is important for students to know what their biases are, before they begin to apply those beliefs in a new situation.

Metacognition in Mathematics.

Mathematics is a subject which focuses on problem solving and offers the student a challenge to extend its abstract concepts to the concrete world. Schoenfeld's categories 2 and 3 target some important aspects found in a mathematics classroom.

The notions of control and self-regulation play heavily in problem solving. Very often students face time limitations when solving problems. It may be due to the length of time for the class period or perhaps it is because students are working on a timed assessment test, such as the Standard Achievement Test (SAT). Students must be able to manage their time wisely and make efficient use of what they know.

> Aspects of management include (a) making sure that you understand what a problem is all about before you hastily attempt a solution; (b) planning; (c) monitoring, or keeping track of how well things are going during a solution; and (d) work on the problem. (Schoenfeld 1987b, 190-191)

Too often students skip the initial steps and simply start working on the problem. This is neither efficient nor successful in most cases. By urging students to engage in an organized process, teachers are encouraging them to be better thinkers and to look at their thought patterns as they proceed.

Beliefs and intuitions also influence students' ability to be good problem solvers. It might be a belief about what mathematics is, or it might be a belief formed by using their schemas. It doesn't matter where the belief originated, as long as the erroneous belief is confronted and challenged. The notions that students bring with them to the classroom are very powerful and, if not addressed, can hinder the students' progress. Chapter II discussed how schemas can influence and color new information. Metacognition offers students a strategy for realizing what

is actually right versus what only "seems" right, so that information can be accurately encoded into their schemas.

Strategies for Improving Metacognition.

Understanding what metacognition is and getting students to improve in their use of it are worlds apart. Most students are not aware of how they think or why they think in the manner they do. If instruction in this area is too explicit, it may turn students off. If it is too subtle, they may miss the point altogether. Arthur Costa (1991b) suggests twelve methods which the classroom teacher might use to approach this task:

1. Planning strategy
2. Generating questions
3. Choosing consciously
4. Evaluating with multiple criteria
5. Taking credit
6. Outlawing 'I can't'
7. Paraphrasing or reflecting back students' ideas
8. Labeling students' behaviors
9. Clarifying students' terminology
10. Role playing and simulations
11. Journal keeping
12. Modeling (212-213)

An important part of metacognition is the planning stage. Costa suggests that before an activity begins, the teacher should discuss strategies for solving the problem, rules to be followed, time constraints to be applied, and the purpose of the activity. By focusing on these guidelines early, the teacher helps the students to keep them in mind throughout the problem.

During and after the activity the teacher should invite the students to share their thought processes and assess how

well they did following the guidelines laid out at the beginning of the activity. This is helpful for both student and teacher as it provides the teacher with a window into the students' approach, and the student is able to assess the efficiency of the method he chose to solve the problem.

Getting the student to pose his own questions is another effective strategy. If the student is required to generate questions on his own, he is forced to pause and reflect on what he is doing and comprehension is increased. It is important to encourage the student to become involved at this stage and excuses of "I can't" and "I don't know how" should not be allowed. A student needs to realize that he has a contribution to make and it is important to share it with everyone.

Another strategy that Costa suggests is paraphrasing or reflecting back students' ideas. By hearing a restatement of what he has said, the student is made aware of his own thinking. It may be that what the student conveyed was not what he was trying to say at all. This technique helps him become aware of how to accurately communicate what he is thinking.

Costa stresses the role of teacher as role model. An actively metacogitating teacher challenges and encourages her students to follow her lead. The teacher should talk through the strategies she is using to solve a problem. She should provide the students with a view of her thought processes and share with the students how this ability to understand her own thinking is helpful.

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Using Metacognition as a Means to Correct Misconceptions

Both Costa and Schoenfeld discuss the merits of utilizing metacognition during problem solving. When a student stops to reflect on his thinking, he is more likely to be aware of mistakes in either his logic or procedure.

In my classroom, I observe how my students approach problems and take notice of their tendency to rely on old, incorrect methods even after new, correct methods are presented and successfully employed by them in previous lessons. I ask myself "Why didn't the new strategy take?". I feel that the lack of metacognitive skills is the answer. Students fail to take the time to examine their thought processes and fail to evaluate new methods versus old. It is as if they simply go through the motions but don't incorporate the new ideas into their repertoire.

By making students aware of how they think and by encouraging them to improve this skill of self reflection, teachers help students to face their misconceptions and reconcile them. Many times an incorrect answer, thought to be a misconception, can be explained by the lack of consideration given to the question before answering it, or perhaps once an answer was obtained its validity was never assessed. With improved metacognitive skills, it is hoped that these errors become less frequent. Students become better problem solvers, more likely to take the time to know and understand a question, reason an answer, and reflect back on its correctness.

Transfer

Definition.

There are several types of transfer discussed in the
literature. I am only concerned with the idea of students
incorporating new information into their schemas in such a
way that they readily use the information when appropriate.
"Transfer goes beyond ordinary learning in that the skill or
knowledge in question has to travel to a new context"
(Perkins and Salomon 1991, 215). It is this idea of
transfer that is central to the eradication of
misconceptions. If material is not being transferred, then
it will have little to no effect in future situations. The
question then becomes "How does a teacher teach for
transfer?".

"Hugging" and "Bridging".

Perkins and Salomon (1991) introduced the two terms
"hugging" and "bridging" (220). Each refers to a different
type of transfer. "Hugging" refers to the transfer of facts
and basic skills. It suggests teaching skills in the
specific contexts you would like your students to use them.
"Bridging" refers to transfer of ideas and skills through
abstraction into other subjects and situations differing
from the norm of your particular classroom. It suggests
that the teacher make connections between varying subjects
and problems, rather than leaving it to happen

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spontaneously. Together these two ideas help aid in the development of transfer.

How does a teacher bring these ideas into the classroom? In geometry, if students are asked to draw a triangle, most will draw an equilateral triangle. Even if they are familiar with other triangles - obtuse, right, or scalene - they are unlikely to draw them. To encourage students to consider all types of triangles, the teacher needs to frequently vary the types of triangles used in classroom examples (Perkins and Martin 1986). If the teacher only uses equilateral triangles, it is no wonder that this is the triangle that comes to mind for the students. This is one example of "hugging". Another example might be found during problem solving. If the teacher wants to improve her students' ability to judge the validity of answers, then she would consistently present problems with the goal of getting her students to look back at the answers. She would want to choose problems where they would have absurd answers if they didn't reflect back, and she would want to continually discuss the merits of reflection. An example of this type of task would be a typical age problem. Students should recognize that if they arrived at an answer of 5 and -20, that the -20 couldn't possibly be the age of one of the people referred to in the problem.

"Bridging" is somewhat different. It requires the students to take the knowledge to a level beyond mere memorization of facts. To do this

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teachers can point out explicitly the more general
principles behind particular skills or knowledge or,
better, provoke students to attempt such
generalizations themselves. (Perkins and Salomon
1991, 220)

Questions that ask students to use a new skill in a

different context get the students to think about how what

they have learned can be useful to them in other ways. When

this happens, it is more likely that the student will

remember the skill and recall it at a later date when

appropriate.

When used together by the teacher, "hugging" and

"bridging" can greatly aid in student transfer (Perkins and

Salomon, 1991). It is important to make the students

themselves aware of what transfer is and get them to focus

on what type of strategies work best for their own learning.

Accordingly, a major goal of teaching for transfer
becomes not just teaching particular knowledge and
skills for transfer but teaching students in general
how to learn for transfer. (221)

"Christopherian Encounters".

Howard Gardner (1991), in The Unschooled Mind argues

that misconceptions are most effectively addressed
in 'Christopherian Encounters'; that rigidly applied
algorithms require explorations of the relevant
semantic domains; and that stereotypes and
simplifications call for the adoption of multiple
perspectives. (151)

He looks at the origins of misconceptions based differently

on particular subjects. In the sciences, he sees

misconceptions as naive theories that children develop as

they experience the world around them. In the social

sciences, he feels misconceptions are caused by biases and

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prejudices. In mathematics, he sees students'
misconceptions arising from their failure to understand
theorems and equations and simply using them as answer
generators.

> In mathematics, the difficulties are manifested in
> the rigid application of algorithms: Rather than
> appreciating how formalism captures objects and
> events in a domain, students simply treat the
> expression as a string of symbols into which values
> are to be 'plugged'. (251)

To encourage students to see math as more complete and not
simply a set of equations, the teacher must continually work
with the students to show them a more correct perspective.
Gardner suggests that when working on a new concept the

> teacher needs to work with the students on three
> dimensions: (1) an understanding of what is involved;
> (2) an exploration of the particular semantic domain
> being investigated; (3) how best to relate the formal
> algorithmic rules to the particulars of a given
> semantic world. (164)

In doing this, it is hoped that a more thorough
understanding of the concept is obtained by the students and
that complete transfer is achieved.

Sometimes teachers think that they are providing enough
challenge to the students' erroneous beliefs by providing a
single counterexample. However the use of infrequent,
singular examples is not enough to override the students
existing schema. Instead

> it is important that these prejudicial views be
> regularly and repeatedly recognized as such and that
> the students have ample opportunities to develop
> richer and more rounded views of the subject.
> (Gardner 1991, 236)

This can begin as the teacher models correct and accurate
use of information. The students also need to be active in

the use of new information. The teacher needs to provide

tasks that challenge the students' beliefs and engage them

in "situations where students' earlier models or

misconceptions are brought into sharp focus . . . and

directly challenge the viability of the model they have been

favoring" (Gardner 1991, 157-158).

If the teacher can successfully provide these

situations for her students and she is able to successfully

incorporate "hugging" and "bridging", then the better her

chances of having transfer occur. And if transfer has

occurred, the more likely it is that misconceptions have

been eradicated.

> This state of affairs can come to pass only if the
> students become familiar with the new models,
> understand the reasons for them, perceive why they
> are more appropriate than the older, competing ones,
> which may well have retained their attractiveness,
> and are then able to draw upon them when they
> encounter a new problem, puzzle or phenomenon.
> (Gardner 1991, 157)

The National Council of Teachers of Mathematics Standards

Curriculum Standards.

The NCTM Curriculum Standards (1989) were developed to

guide schools in changing and developing their curriculum to

maximize effectiveness of mathematics education. They

provide guidelines which present several goals for the

teacher to attain. Each one is designed to help her enhance

her teaching and foster a rich learning environment. The

Standards emphasize a shift from the traditional methods of

teaching, such as "rote memorization, instruction by teacher

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exposition, and extended periods of individual seatwork practicing routine tasks" (NCTM 1989, 129), to a more student oriented classroom that emphasizes

> the active involvement of students in constructing and applying mathematical ideas, problem solving as a means as well as a goal of instruction, and the use of a variety of instructional formats, including small group work, whole-class discussions, and peer instruction. (129)

By setting up such an instructional environment, the teacher enables

> the students to approach the learning of mathematics both creatively and independently and thereby strengthens their confidence and skill in doing mathematics. (128)

The standards are clearly written to express how mathematics should be presented and what type of skills should be stressed in the classroom. I find the Standards to be a practical guide to using the theories of cognitive psychology, critical thinking, metacognition, and transfer, as they incorporate many of the ideas already discussed. Although each standard is important, it is only practical to stress one or two within a single lesson. I feel that the few I have chosen are particularly important in geometry and help students become better learners by encouraging them to be good reasoners and by highlighting the applicability of the math learned in the classroom to their daily lives.

Standard 3: Mathematics as Reasoning. This standard focuses on increasing the student's ability to reason. It states:

In grades 9-12, the mathematics curriculum should
include numerous and varied experiences that
reinforce and extend logical reasoning skills so that
all students can
- make and test conjectures;
- formulate counterexamples;
- follow logical arguments;
- judge the validity of arguments;
- construct simple valid arguments;
and so that, in addition, college-intending students
can
- construct proofs for mathematical assertions,
  including indirect proofs and proofs by
  mathematical induction. (NCTM 1989, 143)

In traditional methods of instruction, the rationale
behind the answer was not stressed and the need for giving
valid arguments was not addressed. Teachers who used answer
sheets, rather than asking to see the work, were interested
in refining students' computational skills and considered a
correct answer to be a sign of success. In doing this, the
skills of reasoning and metacogitating were neglected. Yet
in daily life, students reason and present logical arguments
with very little difficulty without being cognizant of the
process. If Sally reports a piece of social information to
Bob, and Bob responds with the question "Why?" or "How do
you know that?", Sally is generally able to answer and give
a satisfactory reason for her statement.

This ability to draw conclusions and present logical
arguments from given information is not new to students in
their "out of school" life. I find too often, however, that
this ability seems to vanish when it comes to the
mathematics classroom. One of the ways to get the students
to realize that they already possess this ability is to make
them aware of the situations in which they use it. In doing

that, not only do the students realize that they already are capable of doing what is asked of them, but also are able to readily transfer the information presented. Because the students are able to take ownership of the material, it will replace any misinformation in their schemas. As this occurs, it becomes easier for the teacher to engage them in tasks that ask them to reason mathematically.

These tasks can be provided in lessons that ask the students to find patterns and then extrapolate ideas from them.

> A mathematician or a student who is doing mathematics often makes a conjecture by generalizing from a pattern of observations made in particular cases (inductive reasoning) and then tests the conjecture by constructing either logical verification or a counterexample (deductive reasoning). (NCTM 1989, 143)

Activities that promote pattern recognition and generalizing allow the student to see the usefulness of logical reasoning and foster the use of their own reasoning skills. I provide examples of these activities in the lessons presented in Chapter V.

Standard 7: Geometry from a Synthetic Perspective. This standard's focus is on providing instruction that deepens students' understanding of geometric shapes and their properties. Specifically it stresses the importance of incorporating practical applications of the geometry presented.

The curriculum should be infused with examples of how geometry is used in recreations (as in billiards or sailing); in practical tasks (as in purchasing paint for a room); in the sciences (as in the description and analysis of mineral crystals); and in the arts (as in perspective drawing). (NCTM 1989, 157)

When teaching the topics of area and perimeter, I am able to show several ways in which their applicability can be shown to students. I quite often use examples of fences when discussing perimeter and buying paint or carpeting when discussing area as a means of linking the concepts presented in class to the students' life outside of school. For example, in my lesson targeting the Conversion Conclusion misconception (see Chapter IV for definition), I present a problem to the students that incorporates converting units of area when buying carpeting.

These real life applications help students to see that the knowledge they attain in school is connected to their lives. When this happens, the information is more readily transferred by the student, and misconceptions held in place by old schemas may be broken.

Professional Standards for Teaching Mathematics.

These Professional Standards (NCTM, 1991) highlight the teacher's role in implementing the curriculum standards. They speak specifically to the types of activities the teacher should use and to the role the teacher should take in the classroom. Again, there are several standards presented. However, I am limiting my discussion to three that I find particularly helpful as a means of getting

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students to confront misconceptions. In Chapter V, the implementation of these standards can be seen in the lesson plans presented.

Standard 1: Worthwhile Mathematical Tasks. This standard states:

> The teacher of mathematics should pose tasks that are based on
> - sound and significant mathematics;
> - knowledge of students' understandings, interests, and experiences;
>
> and that
> - engage students' intellect;
> - develop students' mathematical understandings and skills;
> - stimulate students to make connections and develop a coherent framework for mathematical ideas;
> - call for problem formulation, problem solving, and mathematical reasoning;
> - promote communication about mathematics;
> - display sensitivity to, and draw on, students' diverse background experiences and dispositions; promote the development of all students' dispositions to do mathematics. (NCTM 1991, 25)

The responsibility for finding tasks that meet these requirements belongs to the teacher. It is her job to understand these guidelines and to either develop or identify appropriate tasks. An experienced teacher will know how to incorporate the use of cooperative learning, group work and manipulatives as a means of targeting various learning styles and to provide the students with a chance to have a dialogue about the math they are using. The value of a chosen task can't be overlooked. The teacher needs to ask herself: Does the task simply allow a student to master a skill through repetition and simple calculation, or does it engage the student in an intellectual process that

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challenges the student to think and make sense of the
concept presented so as to reach a higher level of transfer?

> Good tasks are ones that do not separate
> mathematical thinking from mathematical concepts or
> skills, that capture the students' curiosity, and
> that invite them to speculate and to pursue their
> hunches. (NCTM 1991, 25)

This type of task often encourages lively discussions about
mathematical concepts which readily engage the student in
his own learning. The more involved a student is in his
learning, the more likely that incorrect schemas will be
broken and correct information transferred.

When designing a task, the students' mathematical
backgrounds must be considered. A teacher must understand
what her students believe and she must continually examine
their work to make note of where their difficulties lie.

> An awareness of common student confusions or
> misconceptions around a certain mathematical topic
> would help a teacher select tasks that engage
> students in exploring critical ideas that often
> underlie those confusions. (NCTM 1991, 27)

A task has much more value if it targets specific problems
the students are having and if it can provide a window of
insight for the teacher into the students' method of
learning and thinking about mathematics.

Lessons, including a variety of tasks, should be
designed around the needs of the students and should promote
their ability to understand concepts, solve problems, and
communicate mathematics.

Standard 3: Students' Role in Discourse. This standard
encourages the teacher to be a classroom facilitator by

highlighting the importance of the students' role in the
classroom and in their own learning. It states:

> The teacher of mathematics should promote classroom
> discourse in which students
> - listen to, respond to, and question the teacher
>   and one another;
> - use a variety of tools to reason, make
>   conjectures, solve problems, and communicate;
> - initiate problems and questions;
> - make conjectures and present solutions;
> - try to convince themselves and one another of
>   the validity of particular representations,
>   solutions, conjectures, and answers;
> - rely on mathematical evidence and argument to
>   determine validity. (NCTM 1991, 45)

What students talk about in a mathematics classroom and
what they hear other students discuss heavily influences
what they learn about mathematics. Since students are more
likely to listen to their peers for an explanation, a good
mathematics class should not be a lecture, but rather a
discussion that engages all students in the learning
process. A teacher should not be too quick to dominate the
discussion and should encourage and seek explanations for
critical ideas from the students. This method of providing
explanations aids and enhances the students' understanding
of a concept. If students are able to communicate a concept
presented in class, it strengthens and reaffirms their own
knowledge.

> Students should engage in making conjectures,
> proposing approaches and solutions to problems, and
> arguing about the validity of particular claims.
> They should learn to verify, revise, and discard
> claims on the basis of mathematical evidence and use
> a variety of mathematical tools... Above all, the
> discourse should be focused on making sense of
> mathematical ideas sensibly in setting up and solving
> problems. (NCTM 1991, 45)

This notion of students learning to make sense of mathematical ideas on their own is very powerful. The ability to be an independent learner is a skill that remains valuable from subject to subject and year to year. More importantly, however, is the idea that this independence will foster metacognitive skills and increase transfer.

Standard 4: Tools For Enhancing Discourse. This standard emphasizes the importance of making available all resources to students as a means of improving discourse. It states;

> The teacher of mathematics, in order to enhance discourse, should encourage and accept the use of
> - computers, calculators, and other technology;
> - concrete materials used as models;
> - pictures, diagrams, tables, and graphs;
> - invented and conventional terms and symbols;
> - metaphors, analogies, and stories;
> - written hypotheses, explanations, and arguments;
> -oral presentations and dramatizations. (NCTM 1991, 52)

Certainly, availability will have an effect on the teacher's ability to utilize all of the tools mentioned here. However, I think that the atmosphere of the classroom and the teacher's approachability greatly influence the success of using tools in the classroom. People have all known teachers who want math done "their" way. Although they may provide the students with various tools, they may show little interest in allowing the students to experiment and find which method is comfortable for them. In contrast, a good teacher not only provides the tools for the students,

but encourages them to try different methods and draws upon the creativity and individuality of each student.

> Given the range of mathematical tools available, teachers should often allow and encourage students to select the means they find most useful for working on or discussing a particular mathematical problem. (NCTM 1991, 52)

The teacher who allows a problem to be solved with a variety of methods and shares an appreciation of the diversity of these approaches with her students, rather than imposing on them a single solution method, is ultimately successful.

By encouraging this variety, the teacher is able to reiterate the importance of correct reasoning and explanation, without succumbing to an explanation of rote methodology. This atmosphere has a tremendous effect on the spontaneity and joy in the discourse in the classroom. Ultimately, students feel freer to express themselves if they know that their medium of expression will be accepted and respected.

Chapter Summary

This chapter presented some very powerful ideas for the teaching of mathematics. Included were some methods that would help a teacher get her students to confront and correct any misconceptions that they had. The teacher's role was shown to be extremely important as an influential partner of the classroom. No longer should the teacher be the sole voice in the classroom. It is necessary for her to fully engage the students in their own learning. Students must be brought from passive learners to active learners.

They need to work their minds, to better understand their own thinking, and to become an integral part of the learning process.

The Standards (1989, 1991) presented provide the teacher with a picture of how to accomplish this task. It is not an easy one, as most students have preconceived notions about what a mathematics classroom should be. The Standards emphasize the importance of fostering in the students an ability to metacogitate, the need for accurate transfer, and techniques that would encourage both.

Used with Anderson's (1990) theory of memory and Paul's (1992) theory of critical thinking, metacognition, transfer and the NCTM Standards (1989, 1991) provide the teacher with a set of tools to take into her classroom that will help her begin to "see" what her students are thinking, identify the notion of misconceptions they may have, and develop and present lesson plans to help correct those misconceptions.

C H A P T E R   I V

FIVE IDENTIFIED MISCONCEPTIONS

Overview

Students do not enter the classroom as a "tabula rosa".
Their years as independent learners have given each of them
an individual framework within which to work.
Unfortunately, not only is each framework different,
creating one of many challenges for the teacher, but it is
also likely to contain misconceptions, thereby furthering
the challenge. The identification of these misconceptions
by the teacher and their recognition by the individual
student play a vital role in the effectiveness of the
eradication of these misconceptions.

This chapter discusses the teacher's role in this
process of identification. I share with the reader five
misconceptions regarding area and perimeter of rectangles
which I have identified in my students. For ease and
clarity of discussion, I have given each misconception a
name. They are as follows:

      1.) Fallacy of Multiples

      2.) Increase/Decrease Assumption

      3.) Conversion Conclusion

      4.) Spatial Bias

      5.) Equality Assumptions

The chapter includes a discussion of each of the
misconceptions. Some of them are complex and contain
several variations. I discuss how I identified each

misconception and present a rationale for each of the names, in hopes that a better understanding by the reader will make the names more descriptive and memorable.

Accompanying each definition are several multiple choice questions. Each of these questions was designed to allow the teacher, upon analysis of the student's answers, to recognize the misconceptions that a student may hold. Taken together, the questions provide an appropriate pretest that could be used to identify the various types of misconceptions before teaching the unit. The complete pretest can be found in Appendix D.

Identification of Student Misconceptions

An experienced classroom teacher knows that a good source of information about students' understanding is the questions they ask. This is one of the most effective and accurate ways a teacher has to recognize that students are struggling with or have not yet understood a concept. It was the questions my students were asking that made me cognizant of the fact that they held various misconceptions. The following quotes are from students who answered my questions about area and perimeter of rectangles.

> If they (two rectangles) both have an area of 100, how can they be different in terms of perimeter? If they have the same area then they should be equal. This means that the perimeter (for both) should be the same. (The student thinks that if area is conserved then perimeter must be as well.)

> The square yard is three times the square foot because there are three feet in a yard. (The student fails to account for the two dimensions.)

> If the base of the rectangle decreases by 2, then
> the perimeter will decrease by 2. (The student
> fails to account for both bases of the rectangle.)

The idea of misconceptions intrigued me. I noticed that not only did my students have mistaken beliefs before their study of area and perimeter, they held onto these beliefs even after participating in many lessons that presented ideas contrary to what they thought to be true.

Once aware of this phenomenon, I began to examine more closely the array of ideas my students held. To my surprise, there were several different types of misconceptions and not all were held by every student. It was necessary for me to try to identify, specifically, what each student thought. I decided that one possible avenue of identification would be the use of a pre-test.

When compiling a test, its format must be given consideration. There are various types of questions that can be used to generate a test. Among them are "fill in the blank" questions, open-ended questions, multiple choice questions and computational questions. In order to address different learning and testing styles of students and to provide myself with a more flexible window into their understanding, I chose both multiple choice and open-ended questions. The majority of the questions were multiple choice and by writing several questions in this form with appropriate distracters, I was able to isolate various misconceptions because as Oppenheim (1987) suggests, "by examining the responses to distracters in multiple choice items, certain patterns of misconceptions may emerge" (361).

The open-ended questions in my original pre-test reinforced my identification of the misconceptions seen in the answers to the multiple choice questions.

As I was writing the questions, I found that I had to evaluate each one several times. I wanted the questions to highlight students' misconceptions, but I found it difficult not to introduce other variables that might contribute to a wrong answer. For instance, was a student's incorrect answer caused by a question's awkward or misleading wording or the lack of an accompanying diagram, or did it truly represent the students' flawed thinking? Choosing distracters which reflected common mistakes or change in operations also required revisions. I had to change my reference from that of teacher to that of student. It was important to consider not only the types of mistakes that students might make, but also what concepts they might bring with them that would influence their choices.

Despite my awareness of these potential problems, analysis of the pre-test suggested that it contained some flaws. Some of the questions were poorly worded and some actually could have promoted misconceptions. Although the pre-test was in a preliminary stage, the insights that it gave me were helpful. I was able to identify five distinct misconceptions. I have since rewritten the test questions with my new insight and it is my hope that the questions are improved enough to be able to provide teachers with a useful assessment of students' knowledge and biases.

Definition.

Fallacy of Multiples is the misconception that changing the base or height of a rectangle to a multiple of the original, changes the perimeter and/or area of that rectangle by the same multiple. For example, students assume that doubling both the base and height of a rectangle will double its area. Students think that tripling the height of a rectangle will triple the perimeter of that rectangle. The fallacy is the assumption that the multiple which occurs, whether to base, height, or both, will be the same multiple which always describes the change in area or perimeter.

Richard Paul (1992) defines a fallacy as "an error in reasoning" (640). This definition accurately describes what the students are doing. In algebra, students have learned the multiplication postulate which states that if $a = b$, then $ca = cb$. I have found that they use this reasoning to make similar conclusions in the cases involving the rectangles. The students fail to recognize that the relationship between height, base, area, and perimeter are not simple equalities. To reflect this flawed reasoning, I named this misconception Fallacy of Multiples.

This misconception has four cases to be defined, although only two of them reflect a misconception. Since a rectangle is two dimensional, students are asked to consider what happens to the area and perimeter if one dimension is

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multiplied by a factor, and what happens to the area and perimeter if both dimensions are multiplied by a factor. Because of these variations, it is possible for a student to get both correct and incorrect answers using the same logic.

Change in One Dimension - Effect on Area.

Definition. The first case assesses what happens to the area of a rectangle if either the base or the height is changed to a multiple of the original.

Many students will correctly assume that the area has changed by the same multiple as the base or height. Although this is not a common misconception, it does highlight the student's thinking on multiples.

Background. Questions addressing this facet (see Fig. 4.1) of the Fallacy of Multiples are useful because they allow the teacher to assess the students' basic understanding. In my original pre-test, I included problems of this nature to reinforce my feeling that students assume that a multiple in one part of the problem leads to the same multiple in the answer.

Rationale. Each question contains the correct answer with two distracters. One of the distracters contains a mistake students may make by distributing the multiple to both the height and base when computing area. The other distracter is twice or half the multiple and represents a guess.

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Sample Pre-Test Questions.

1. If the base of a rectangle is tripled and the height
   is unchanged, then the new rectangle's area will:

   a) triple.   b) be 6 times as big.
   c) be 9 times as big.

2. If the height of a rectangle is doubled and the base
   is unchanged, then the new rectangle's area will:

   a) quadruple.   b) double.
   c) be 1/2 as big.

3. If the height of a rectangle is increased n times and
   the base is unchanged, then the new rectangle's area
   will:

   a) be n times as big.   b) be n^2 times as big.
   c) be 2n times as big.

Fig. 4.1 Change in one dimension - effect on area pre-test questions.

Change in One Dimension - Effect On Perimeter.

Definition. This case assesses the changes to the perimeter of a rectangle if the base or height is changed to a multiple of itself.

There are two common mistakes that students make. First, many incorrectly assume that the perimeter will change by the same multiple as the base or height. In the second case, students don't make the multiple assumption, but they neglect to consider both heights or bases when figuring the new perimeter. This second case can be more generalized and I have classified it as a separate misconception, the Increase/Decrease Assumption.

Background. Of the four cases of Fallacy of Multiples, this one presents the most difficulty for students. An explanation for what is occurring is not easily written in a concise general formula that the students can memorize. Since the multiple can only be factored out of one of the terms in the perimeter equation, there is no common change to the perimeter of any rectangle. For example, since $P = 2b + 2h$ (P represents perimeter, b represents base, and h represents height) is the formula for perimeter, then a formula representing a multiple of the base only would look like this, $N = 2(bn) + 2h$ (N represents new perimeter and n represents the multiple affecting the base). The difference between the two perimeters is $N - P$ or $[2(bn) + 2h] - [2b + 2h]$. That equation simplifies to $2(bn) - 2b$ or $2b(n-1)$, which represents the exact modification to the perimeter.

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The increase by a multiple will always depend on the specific dimensions of the individual rectangle because it is dependent on b (base). This same rationale also works for a multiple of the height. Each problem presents the same challenge, but each one must be addressed and analyzed separately. My findings indicated that even when students understood the explanation for a particular answer, they were not able to consistently transfer the concept to other problems.

Rationale. Each question (see Fig. 4.2) has the same type of distracters. One answer, the same multiple as presented in the question, indicates that the student has adopted the Fallacy of Multiples. The other distracter assumes that the student doesn't adopt the multiple fallacy, but when she calculates the perimeter, she neglects to account for the increase on two heights or two bases (see Increase/Decrease Assumption).

Change in Two Dimensions - Effect on Area.

Definition. This case of the Fallacy of Multiples considers what happens to the area of a rectangle if both its height and base are multiplied by the same positive integer. Students incorrectly assume that the area will change by the same multiple. They forget to account for the squaring of the multiple which occurs when the base and height are multiplied together to calculate area. Algebraically, the change is easily seen. The formula for

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Sample Pre-Test Questions.

4. If the base of rectangle A below is tripled and the height remains unchanged (see rectangle B below), then the perimeter will:

a) triple.
c) increase by 32.

b) increase by 16.

4 | A | 4 | B |

8   24

5. If the height of rectangle C below is quadrupled and the base is unchanged (see rectangle D below), then the perimeter of the rectangle will:

a) increase by 9.
c) increase by 18.

b) quadruple.

3 | C |

7

12 | D |

7

6. If the height of rectangle E below is increased n times and the base is unchanged (see rectangle F below), then the perimeter will:

a) be n times as big.
c) increase by $8(n-1)$.

b) increase by $4(n-1)$.

4 | E |

6

$4n$ | F |

6

Fig. 4.2. Change in one dimension - effect on perimeter pre-test questions.

area is A = bh (A stands for area, b stands for base, and h stands for height). If n represents the multiple used to change both base and height, then the new formulas would be N = (nb)(nh), or n^2bh (N represents new area). Clearly the effect on area is n^2 rather than n.

Background. This is the case that I first identified. It was the one concept where I saw my students consistently making errors. On my initial pre-test, 80% of the students who answered these questions incorrectly chose the answer indicating a Fallacy of Multiples misconception.

Rationale. Each of these questions (see Fig. 4.3) has two distracters. One represents the Fallacy of Multiples misconception. The other shows the student's attempt to account for the change in both base and height to the area, but the student has made an operational error, adding rather than multiplying.

Change in Two Dimensions - Effect on Perimeter.

Definition. The final case is similar to the first case, since it is not actually a misconception. Students correctly assume that if the base and the height of a rectangle are multiplied by the same integer then the perimeter is also changed by the same multiple.

Background. Although most students will answer these questions correctly, I find them valuable as a means to confirm the notion that students assume a multiple to both

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Fig. 4.3. Change in two dimensions - effect on area
pre-test questions.

the base and height yields the same multiple to the

perimeter. However, their ability to answer correctly

does not confirm their understanding of factoring and balancing equations. Most students are unable to explain that the multiple can be factored out of each term of the perimeter equation, thus showing how the perimeter is affected by the same multiple. Since $P = 2b + 2h$, if both b and h are multiplied by n, the new perimeter is represented by the equation $N = 2bn + 2hn$ (P represents perimeter, b represents base, h represents height and N represents new perimeter). Because n is a common factor, it can be factored out to yield the equation $N = n(2b + 2h)$ or $N = nP$ by substitution.

Rationale. Each question (see Fig. 4.4) contains two distracters. One represents the correct answer if the question were asking about the change in area. The other represents a student's answer if he doubles the multiple, indicating that he is guessing at the answer.

Increase/Decrease Assumption

Definition.

Increase/Decrease Assumption is the misconception that if the base or the height of a rectangle is increased or decreased by a rational number, then the perimeter increases or decreases by that same number. Students incorrectly assume, for example, that if the height of a rectangle is increased by 5 inches, then the perimeter also increases by 5 inches. The students neglect to consider that the

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Sample Pre-Test Questions.

10. If the base and height of a rectangle are tripled, then the perimeter will:

   a) be 6 times as big.   b) triple.
   c) be 9 times as big.

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2│              │              │                          │
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        7                      └──────────────────────────┘
                                            21
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11. If the base and height of a rectangle are both multiplied by 5, then the perimeter will:

   a) be 10 times as big.   b) be 25 times as big.
   c) be 5 times as big.

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 └────────┘           5 │                            │
     2                  │                            │
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                                    10
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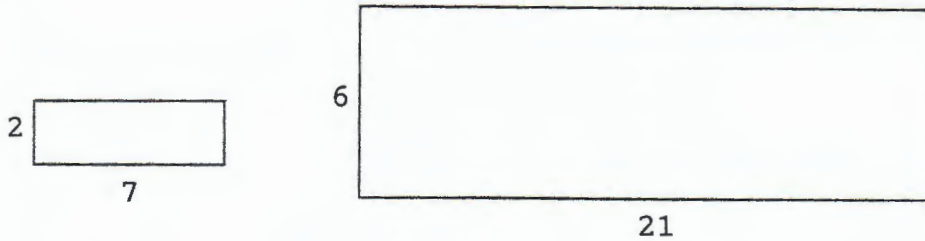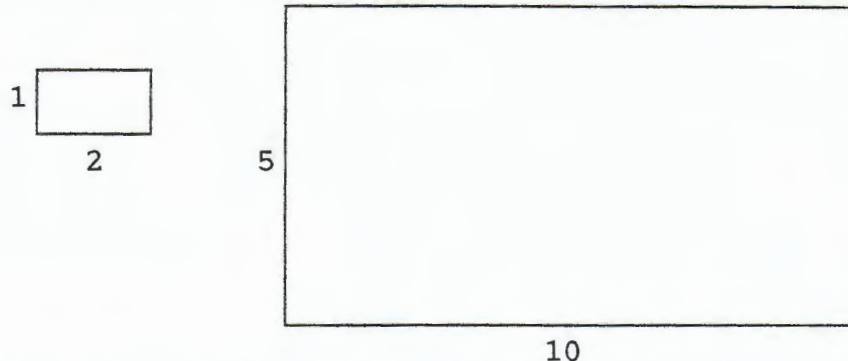
12. If the base and height of a rectangle are both multiplied by n, then the perimeter will:

   a) be 2n times as big.   b) be n times as big.
   b) be n^2 times as big.

Fig. 4.4. Change in two dimensions - effect on perimeter pre-test questions.

rectangle has two heights and therefore the increase occurs twice.

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Paul (1992) defines an assumption as a "statement accepted or supposed to be true without proof or demonstration" (640). Because "we are typically unaware of what we assume and therefore rarely question our assumptions," (Paul 1992, 640) the students are making a generalization without checking to see if what they assumed was correct. Since this misconception involves an increase or decrease to the perimeter and is based on a false assumption, I named it the Increase/Decrease Assumption.

Background.

After identifying Fallacy of Multiples, I began to think about what other misconceptions my students might have. Since they had made an assumption when dealing with multiples of height and base, I tried some questions using different operations, and the students' responses were similar. On my original pre-test, 83% of the students who answered these questions incorrectly chose the answer indicating an Increase/Decrease Assumption. They failed to recognize that the increase would occur on each height, thus increasing the perimeter by a factor of two. This is easily calculated using algebra and the perimeter formula. Since $P = 2b + 2h$, an increase or decrease to the base alone can be shown as $N = 2(b+n) + 2h$ (where P stands for perimeter, N stands for new perimeter, b represents the base, n stands for the increase or decrease and h represents the height). Distributing the 2 yields the equation $N = 2b + 2n + 2h$. By regrouping and making a substitution the equation for the

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new perimeter becomes N = P + 2n. From this it is clear that the increase or decrease to perimeter is twice the increase or decrease to the base. The same rationale also holds true for an increase or decrease to the height. It is my feeling that the students did not use their algebra skills and simply jumped to a faulty assumption.

Rationale.

Each question (see Fig. 4.5) has two distracters. One of the distracters would indicate the Increase/Decrease Assumption by allowing the students to choose the number that was used as the increase or decrease in the question. The other distracter would show the student trying to account for the change to two sides of the rectangle, but rather than doubling the number, the student squares it instead, perhaps showing some confusion between calculating area and perimeter.

Conversion Conclusion

Definition.

This misconception defines the incorrect conclusion students make when asked to convert from one unit of area to another. They fail to delineate between linear unit conversions and square unit conversions. Their incorrect conclusion is that whatever the conversion factor is for the linear units it must apply to square units. For example, if asked to convert 3 square yards to square feet, many

Sample Pre-Test Questions.

13. If the base of a rectangle is increased by 3 inches and the height remains the same, then the perimeter will:

a) increase by 6 inches.   b) increase by 3 inches.
c) increase by 9 inches.

3 in. [ ] 3 in. [ ]
     4 in.   7 in.

14. If the height of a rectangle is increased by 4 inches and the base is unchanged, then the perimeter will:

a) increase by 4 inches.   b) increase by 16 inches.
c) increase by 8 inches.

4 in. [ ]
     16 in.

8 in. [ ]
     16 in.

15. If both the base and height of a rectangle are increased by 5 inches, then the perimeter will:

a) increase by 20 inches.   b) increase by 25 inches.
c) increase by 5 inches.

3 in. [ ]
     8 in.

8 in. [ ]
     13 in.

16. If the base of a rectangle is decreased by 7 inches and the height is unchanged, then the perimeter will:

a) decrease by 14 inches.   b) decrease by 7 inches.
c) decrease by 49 inches.

17. If the height of a rectangle is decreased by n inches and the base remains the same, then the perimeter will:

a) decrease by n inches.   b) decrease by n^2 inches.
c) decrease by 2n inches.

Fig. 4.5. Increase/Decrease Assumption pre-test questions.

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students reply with an answer of 9 square feet. When they are asked for an explanation, their response is that there are three feet in a yard. The units have been neglected and thus ignored during the conversion.

When deciding on a name for this misconception, I considered Paul's (1992) comments about conclusions that people make. "We rarely monitor our thought processes, we don't critically assess the conclusions we come to, to determine whether we have sufficient grounds or reasons for accepting them" (641). My students failed to judge their conclusions for validity; instead, they simply accepted them. For this reason, I named this misconception Conversion Conclusion.

Background.

In my original pre-test, I didn't have any questions to identify this misconception. It was while I was searching for practical applications for area, that these problems arose. What would happen, for instance, if one were to measure a room for carpet in square feet and find out later that carpeting was sold in square yards? This happened to me, and I am embarrassed to admit that when I converted from square feet to square yards, I jumped to the wrong conclusion and used the linear conversion factor. I quickly realized my mistake when the saleswoman gave me the correct conversion, but my experience planted the seed for consideration of this misconception.

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Rationale.

Each question (see Fig. 4.6) has three distracters. One distracter identifies the Conversion Conclusion misconception by presenting an answer obtained using the linear conversion factor. Another reflects a basic fault in the students' thinking. Without considering the relative size of the various units, students mistakenly use the inverse operation. A combination of the utilization of the linear conversion factor and the inverse operation would lead to the fourth answer.

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Sample Pre-Test Questions.

18. How many square feet are there in 18 square yards?

    a) 162 sq.ft.              b) 54 sq.ft.
    c) 2 sq.ft.                d) 6 sq.ft.


19. How many square inches are there in 12 square feet?

    a) 144 sq.in.              b) 1 sq.in.
    c) 1728 sq.in.             d) 1/12 sq.in.


20. How many square yards are there in 27 square feet?

    a) 243 sq.yd.              b) 3 sq.yd.
    c) 9 sq.yd.                d) 81 sq.yd.


21. How many square feet are there in n square yards?

    a) 3n sq.ft.               b) 9n sq.ft.
    b) n/3 sq.ft.              d) n/9 sq.ft.
```

Fig. 4.6. Conversion Conclusion pre-test questions.

<center>Spatial Bias</center>

Definition.

This misconception represents the students' idea that "space is more." Many students feel that a rectangle could not have a perimeter of 4 inches and an area of 1 square inch. Their understanding is that the space occupied by area should be "more" than the length of the perimeter. From this bias they conclude that the number for area must be larger than the number for perimeter. This interpretation completely ignores the significance of the units attached to the measurements.

I have used the word bias in a neutral sense to mean "a mental leaning or inclination" (Paul 1992, 640). Too often in the students' earlier mathematics education, units of measurement have been ignored or have been a lower priority than the perfection of computational skills. Because of this, students lack the understanding of the meaning and importance of units of measurement and are inclined to see answers in terms of numbers only. They fail to realize that one can't compare the "size" (space) of area with the "size" (length) of perimeter. Because of this, I have named this misconception Spatial Bias.

Background.

These questions are the result of several questions asked by students as we studied area and perimeter. I did not originally test for this misconception, but became

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increasingly aware of its presence when students asked how it was possible for a rectangle to have a perimeter "larger" than its area. I realized that many students had a bias that the number representing area had to be greater than the number representing perimeter. These questions are designed to test that bias.

Rationale.

Each question (see Fig. 4.7) contains two correct answers with one distracter. The distracter adheres to the idea that area must be larger than perimeter. I find that identifying this misconception is easier through open-ended questions or by questioning students directly. I have included these questions to keep the multiple choice format set by the rest of the pre-test.

Equality Assumptions

Definition.

This last misconception identifies the idea that many students hold about the correlation between area and perimeter. I have found that students assume all rectangles with equal areas must have equal perimeters. They also assume that the converse is true. I look at both cases in making this assumption. Students neglect the fact that most numbers have several integral factors which would allow rectangles with the same area to have varying perimeters. Conversely they "forget" that a sum could be created with

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Sample Pre-Test Questions.

Students may choose more than one answer.

22. If the perimeter of a rectangle is 16 inches, which
    of the following are possible values for the
    rectangle's area?

    a) 7 sq.in.                    b) 15 sq.in.
    c) 24 sq.in.


23. If the area of a rectangle is 24 square inches, which
    of the following represent possible values for its
    perimeter?

    a) 20 in.                      b) 50 in.
    c) 30 in.


24. If the area of a rectangle is 1 square yard, which of
    the following represent possible values for its
    perimeter?

    a) 4 yds.                      b) 1/2 yd.
    c) 5 yds.
```

Fig. 4.7. Spatial Bias pre-test questions.

several different addends, thus allowing rectangles with

equal perimeters to have different areas. Because the

misconception represents the students' acceptance of flawed

reasoning concerning the equality of area and perimeter in

certain rectangles, I have named it Equality Assumption.

Area Conservation.

Definition. Given two rectangles with equal area, many

students think that they must also have the same perimeter.

The mistaken thought is if area is conserved, then so is

perimeter. For example, if rectangle A has an area of 24 square inches and a perimeter of 20 inches and rectangle B has an area of 24 square inches, it is falsely assumed that rectangle B also must have a perimeter of 20 inches.

Background. On my original pre-test, two questions targeted this misconception. On the question without a diagram, 100% of the students who answered incorrectly chose the answer indicating the Equality Assumption. On a similar question with a diagram, of those who answered incorrectly, only 90% chose the answer indicating the Equality Assumption. Along with the results from the pre-test, comments and questions from students assured me that there were misunderstandings about the concept.

Rationale. Each question (see Fig. 4.8) contains two distracters. One indicates the Equality Assumption misconception by allowing the student to choose the same perimeter as the given perimeter in the question. The other distracter is the given area and may represent either a guess or a failure to understand concepts of area and perimeter.

Perimeter Conservation.

Definition. In this case of Equality Assumption, it is incorrectly assumed that if the perimeters of two rectangles are equal, then the areas must be equal as well. For example, if two rectangles both have a perimeter of 20

Sample Pre-Test Questions.

25. Both rectangle A and B have an area of 36 square inches and rectangle A has a perimeter of 24 inches. What is the perimeter for rectangle B?

   a) 26 in.   b) 24 in.
   c) 36 in.

   6 in. | A | (6 in.)   4 in. | B | 9 in.

26. Both rectangle C and D have an area of 12 square feet and rectangle D has a perimeter of 16 feet. What is the perimeter for rectangle C?

   a) 16 ft.   b) 12 ft.
   c) 14 ft.

   3 ft. | C | 4 ft.   2 ft. | D | 6 ft.

27. Both rectangle E and F have an area of 20 square yards and rectangle E has a perimeter of 24 yards. What is the perimeter for rectangle F?

   a) 20 yds.   b) 18 yds.
   c) 24 yds.

   2 yds. | E | 10 yds.   5 yds. | F | 4 yds.

Fig. 4.8. Area conservation pre-test questions.

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inches and one of them has an area of 9 square inches, students assume that the area of the other rectangle is also 9 square inches.

Background. Having identified the misconception when area was conserved, it was important to see if the misconception held when perimeter was conserved. Although this was not tested originally, during classroom discussions it became obvious that many students held the misconception in this case as well.

Rationale. Each question (see Fig. 4.9) has two distracters. One of the distracters indicates the Equality Assumption. It allows the student to choose the same area as presented in the problem. The other distracter is the rectangle's perimeter and represents a guess.

Chapter Summary

The identification process of misconceptions is not an easy one. There are no guarantees that a student's incorrect answers on a pre-test identify true misconceptions. The answer may be a guess or, perhaps, the question was misunderstood. However, with the use of a pre-test, observation of student participation in groups or activities and analysis of questions during class, it is possible for the teacher to arrive at some solid conclusions about her students' prior knowledge and understanding about a subject. This increased student baseline information is

Sample Pre-Test Questions.

28. Both rectangle A and B have a perimeter of 16 inches
    and rectangle A has an area of 12 square inches.
    What is the area of rectangle B?

    a) 16 sq.in.   b) 15 sq.in.
    c) 12 sq. in.

2 in. | A | (width 6 in.)   3 in. | B | (width 5 in.)

    6 in.   5 in.

29. Both rectangle C and D have a perimeter of 28 yards
    and rectangle D has an area of 48 square yards. What
    is the area of rectangle C?

    a) 40 sq.yds.   b) 48 sq.yds.
    c) 28 sq.yds.

4 yds. | C | 6 yds. | D |

    10 yds.   8 yds.

30. Both rectangle E and F have a perimeter of 8 feet and
    rectangle E has an area of 4 square feet. What is
    the area of rectangle F?

    a) 4 sq.ft.   b) 8 sq.ft.
    c) 3 sq.ft.

2 ft. | E | 3 ft. | F |
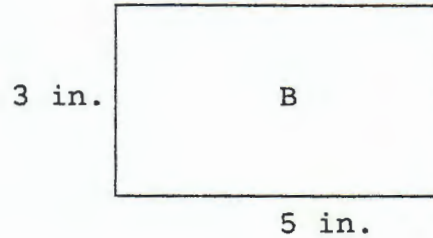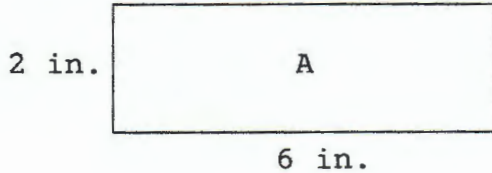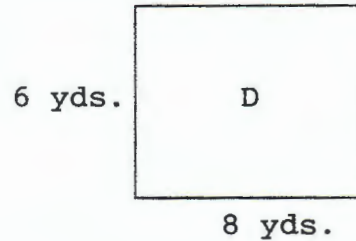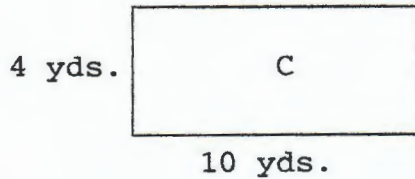
    2 ft.   1 ft.
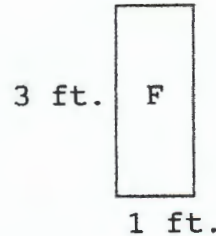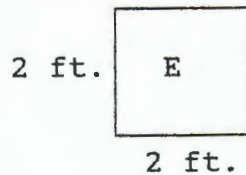
Fig. 4.9. Perimeter conservation pre-test questions.

beneficial and can aid the teacher in preparing upcoming

lessons to convey the correct concepts to her students.

C H A P T E R   V

THREE SAMPLE LESSONS

<u>Overview</u>

The previous chapters have presented various theories and methodologies to assist the classroom teacher in her daily job of educating students and helping them replace any misconceptions that they may hold. Learning theories and their utilization in the identification of misconceptions must be put into the context of classroom lessons. This chapter presents examples of activities and lessons that implement the theories in an appropriate and realistic manner, as they are geared toward the eradication of students' misconceptions about area and perimeter of rectangles. Specifically, the lessons target the Fallacy of Multiples, Conversion Conclusion, and Equality Assumption misconceptions.

These lessons are designed for an experienced geometry teacher to use, and although there are some suggestions for questions to raise during class, it is hoped and assumed that the teacher would modify these lessons to meet the needs of her class. It is also hoped that the teacher would elaborate on any points where she found her students to be struggling and also reduce or delete exercises in which her students didn't need work. It is assumed that both similar figures and how to calculate area and perimeter have been

covered. The terms "half", "double", "triple", and "quadruple" will need to be defined if students do not know what they mean. I have not included time limitations as I find that the time needed for adequate completion of each lesson varies from class to class.

Each lesson follows the same format of <u>Motivation</u>, <u>Activity</u>, <u>Metacognition</u>, and <u>Transfer</u> and presents different exercises to target each aspect of the learning process. <u>Motivation</u> refers to an opening exercise that will engage the students in the lesson of the day. It is designed to capture the students' attention and interest and, in these lessons, it confronts the students' erroneous beliefs about area and perimeter of rectangles.

<u>Activity</u> contains the bulk of the lesson and is the place where the correct concepts are presented. All of the activities presented in this chapter involve the students in pattern recognition, hypothesis formulating, hypothesis testing, and the use of reasoning to reach and substantiate conclusions.

<u>Metacognition</u> is the part of the lesson that allows and encourages the students to reflect on their thought processes. During this time, students should "look" at what they believed prior to the lesson and compare it with what was just presented. The teacher needs to ensure that students take this time to understand why the new material is correct and to identify why they held an erroneous belief.

Transfer is the part of the lesson designed specifically to help the student "own" the new concept. Certainly, the Activity will play a role in the transfer of new material, but the exercises presented at this stage focus on maximizing that transfer.

Accompanying each lesson is a rationale for the activities presented and their link to the theories presented in Chapters II and III.

Fallacy of Multiples

This lesson is presented as a sample lesson to help students recognize and correct their Multiple Fallacy misconception. It challenges the students to assess their ideas about how multiples of height and base of a rectangle influence its perimeter and area. This misconception was broken into four parts, two of which yielded correct answers by the students. The exercises presented here focus on the two parts that highlighted the misconception.

Motivation.

   - Show students the following rectangle and have them calculate its area and perimeter.

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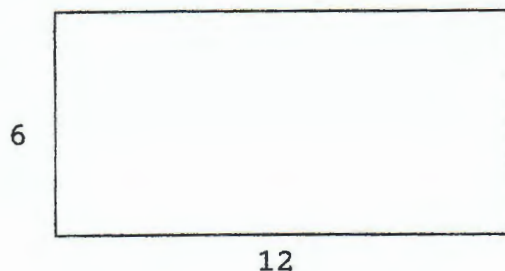
- Ask the students the following: What if you wanted a rectangle with a height only half as big; how could you change this rectangle to accomplish that? Let the students direct you in marking the diagram.

- Ask the students: What do you think has happened to the area? to the perimeter?

- Have the students calculate the area and perimeter of each of the new rectangles. Ask: Do your calculations agree with your predictions?

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    ┌─────────────────┐
  3 │                 │
6   ├─────────────────┤
  3 │                 │
    └─────────────────┘
           12
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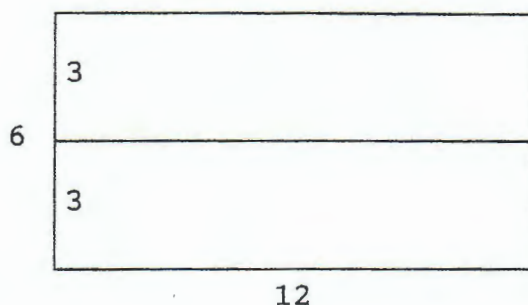
- Now suppose that the length in each new rectangle is half as big. Again calculate area and perimeter of each new rectangle. Compare them with the original area and perimeter.

```
        6         6
    ┌────────┬────────┐
  3 │        │        │
6   ├────────┼────────┤
  3 │        │        │
    └────────┴────────┘
        12
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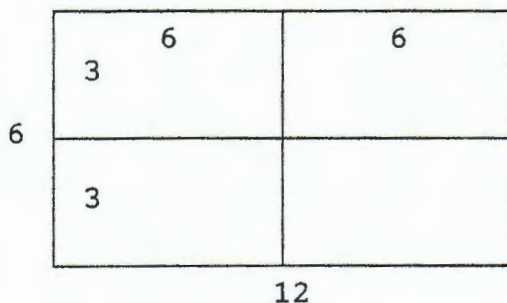
- Ask the students to explain to you what they have observed. Can they give you an explanation for what

has occurred? Are they willing or able to make any initial generalizations about what they have seen? It is important to stress to the students that any generalizations made here will need to be tested and verified before being accepted as valid.

Activity.

- Give each student a piece of dot paper with various rectangles already drawn on it. (see Fig. 5.1) Each horizontal or vertical segment connecting two dots represents a single unit of length.
- Have the students group all the related rectangles (those in which bases or heights are multiples of each other) together. To help the students with this task, it may be advantageous to have the students cut out the rectangles.
- Looking within these groups of rectangles, ask the students to pair rectangles together based on a multiple. For example, if rectangle A has a height of 10 units and rectangle B has a height of 20 units they would be paired together. Some rectangles may have more than one match. Students should look for pairings based on height, as well base.
- Have the students calculate the area and perimeter of each rectangle.
- Ask students to observe and make note of any relationship between the multiple of the height and base from the old rectangle to new rectangle and the

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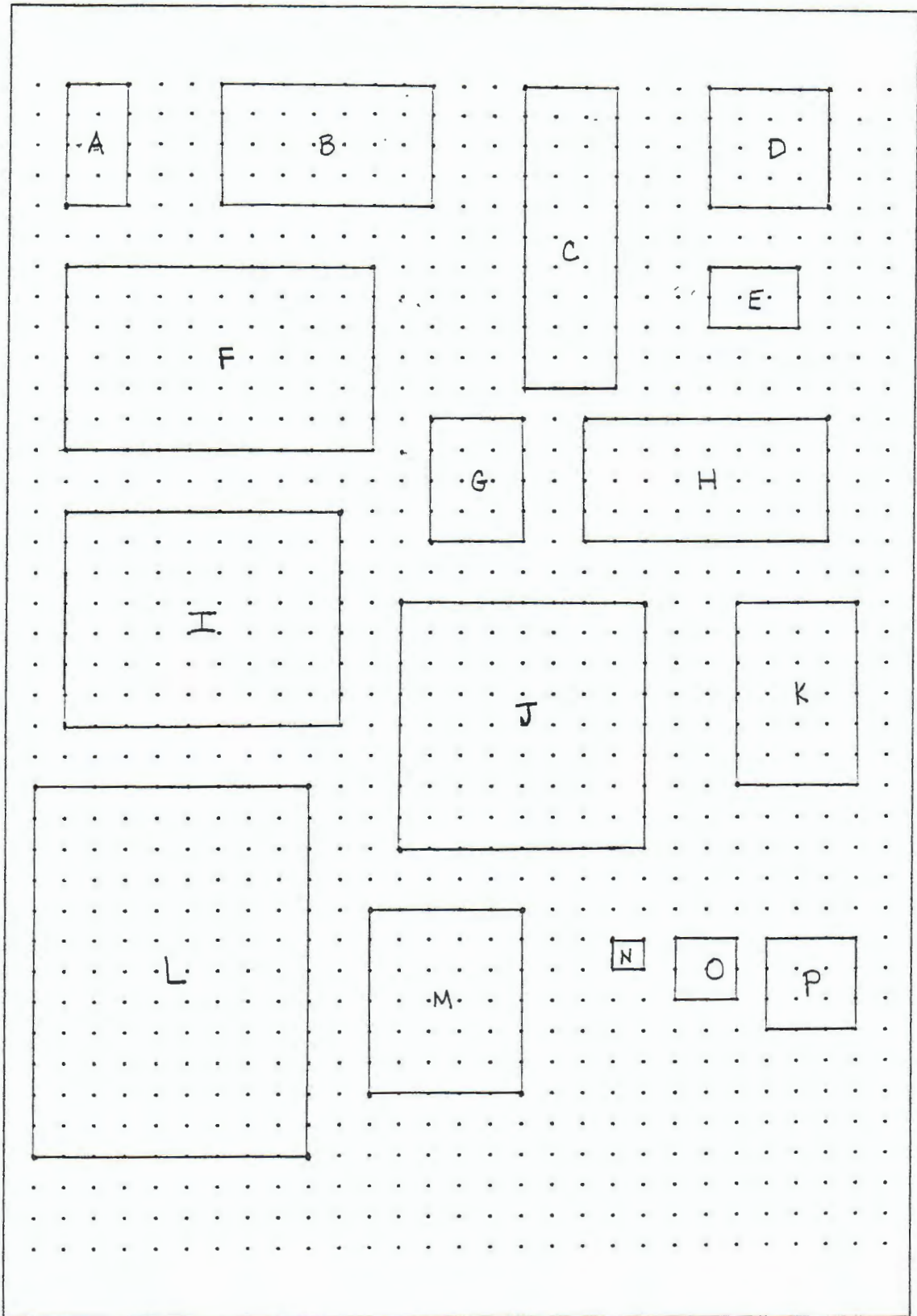
Fig. 5.1. Dot paper worksheet for Fallacy of Multiples Lesson

change in the perimeter or area. One way to help students see the relationship clearly is to have them set up a chart. The headings could be base, height, area, and perimeter.

- Ask: If the height or base doubled (tripled, quadrupled...), what were the changes in perimeter and area?

- Ask: If both the height and base doubled (tripled, quadrupled...), what were the changes in perimeter and area?

- Have students work in groups to make and verify conjectures about how doubling (tripling, quadrupling...) the height and/or base of a rectangle will change the perimeter and area. Remind students how to calculate area and encourage the use of an algebraic solution. Provide paper and scissors and encourage students to test their theories with other rectangles.

- Each group is responsible for presenting its findings to the class. If a group finishes early, present them with this challenge: Why is it that not all pairs of rectangles with perimeters in a 2 to 1 ratio have areas in a 4 to 1 ratio?

- After each group has presented its findings, the teacher should summarize all information and make sure that the final presentation is correct.

Metacognition.

  - Have students write answers to the following
questions:

    - Before we began today, what did you think would
    happen to the perimeter of a rectangle if its
    length or base was changed to a multiple of the
    original (for example doubled or tripled)? What
    did you think would happen to its area?
    - What were the conclusions we reached today?
    - How were they similar or different from what you
    had originally thought? Were you surprised by the
    results?
    - Explain in your own words why you got the
    results you did.
    - How are you going to remember what happens to
    area and perimeter when you change a side of a
    rectangle to one of its multiples?
    - Explain why this is different from other effects
    of multiples. For example, if you receive $3.00
    an hour for babysitting and you usually work 5
    hours, what happens to the amount you make if you
    double the number of hours to 10?

Transfer.

- Give all students a blank piece of dot paper.
    - Ask them to draw a rectangle.
    - Ask them to calculate its area and perimeter.

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- Ask them to draw a second rectangle, with the same height and a base three times as long as the original rectangle's base. Without recalculating area and perimeter, have students state what the area and perimeter of the new rectangle would be? Once they have written down an answer, they may verify it by calculating area and perimeter. (This process can be repeated as many times as is necessary.)

Rationale.

In this lesson students are looking for patterns, using various materials to assist them, making conjectures, proving their conjectures, and communicating their reasoning to the class. In doing this, the following dispositions and abilities, as discussed in Chapter II, are targeted: seeking a clear statement of the problem, seeking reasons, looking for alternatives, taking a position, and inducing and judging inductions. From the NCTM Standards (1989,1991) the following standards have been utilized: mathematics as reasoning, worthwhile mathematical tasks, students' role in discourse, and tools for enhancing discourse. By encouraging the students to go through a process of discovery, the students find their beliefs directly confronted, as suggested by Gardner (1991), and they are then forced to resolve any discrepancies.

While the students are working in groups, the teacher is free to move from group to group to observe the types of

processes the students are using. It is at this time that the teacher can reinforce critical thinking skills (dispositions and abilities).

When students have questions, rather than immediately answering them, the teacher can redirect the students by asking them to explain to her the task at hand (seeking a clear statement of the question) or she can use the techniques of "prompts, hints, and provides" (Perkins and Martin 1986, 218) to clarify their fragile knowledge. If students have come up with conclusions (taken a position), the teacher is given the perfect opportunity to get them to strengthen their reasoning skills by asking for the verification of their results (inducing and judging inductions). The teacher is also afforded the opportunity to encourage the students to try various strategies and to encourage persistence (look for alternatives).

The exercises presented promote lively discussion in the class (students' role in discourse) and it is important that the teacher not be the sole person carrying on the discussion. Students ask many questions, and if asked clarifying questions in return (which helps improve their metacognitive skills), rather than given the answers, they gain a better understanding about the effects of multiples on area and perimeter of rectangles. Costa (1991b) agrees, saying: "Clarifying helps students to reexamine their own problem-solving processes, to identify their errors, and to self-correct" (213).

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The methods of transfer employed here are varying, and include examples of "hugging" (Perkins and Salomon 1991, 220) and the general idea of confronting students' beliefs and then allowing them the time to explore and resolve the differences.

Equality Assumptions

This lesson is presented as an example which would help students come to see their flawed reasoning when assuming that rectangles with the same perimeter will have the same area and when assuming that rectangles with the same area will have the same perimeter.

Motivation.

- Show students the following two rectangles.

- Ask the following questions:
    - Find the perimeter of each of the rectangles. What do you notice? (They are the same.)
    - Without calculating the area of the two rectangles, do you know anything about their areas? (Are they the same also?)
    - What do you discover if you calculate the area of each rectangle?

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- How can it be that if two rectangles have the same perimeter they don't have the same area?

- What if we took two rectangles that had the same area?
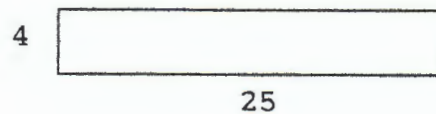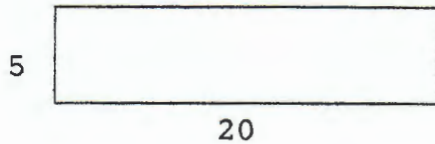
- Does each rectangle have the same perimeter?

- How can this be?

Activity.

Looking at rectangles with fixed perimeters.

- Have students work in pairs and give each pair some graph paper.

- Have each group draw as many rectangles as possible, using the graph paper as a guide, that have a perimeter of 24 units. Calculate the area of each rectangle. A limitation to using the graph paper is that students tend to fix all the values of the heights and bases as whole numbers to simplify their calculations. For a review of mixed number calculations, the teacher could encourage the students to use fractional values as the heights and bases.

- Compare the various rectangles from each group. Have the students discuss how it could be that all the

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rectangles have a perimeter of 24 inches but varying areas.

- You can use a looped string of 24 inches to demonstrate to the students how perimeter can stay fixed while area can be changed.

- Have the students make a table displaying the various integral heights (h) and bases (b) that yielded their area (A) and a perimeter (P) of 24. Have the students graph the relationships (h,b), (h,P), and (h,A). Compare the three graphs and have the students discuss any relationships that they find (Biggs and MacLean 1969).

- Have a contest to see which group can make the rectangle with the largest area and which one can make the rectangle with the smallest area. This can lead to an interesting discussion about the possibilities of making rectangles with an infinitely small area. This would also lead to a discussion about the possibilities of fractional values being used for the height and base if it has not already been discussed.

Looking at rectangles with fixed area.

- Give each pair of students 36 square tiles. Each tile represents 1 square unit of area.

- Have each group make a rectangle and find its area and perimeter.

- Have groups compare perimeters. You may have to explain to the students that counting the lengths of

the exposed sides of the squares will allow them to figure perimeter.

- Have the students discuss how it is possible for all the rectangles to have an area of 36 square units and still have different perimeters.

- Have the students make a table displaying the various integral heights (h) and bases (b) that yielded their various perimeters (P) and an area (A) of 36. Have the students graph the relationships (h,b), (h,P), and (h,A). Compare the three graphs and have the students discuss any relationships that they find (Biggs and MacLean 1969).

- Which group can make the rectangle with the largest perimeter? smallest perimeter?

- Are there any limitations to this because of the rigidity of the tiles? Explain. (No fractional lengths are used.)

Metacognition.

- Have the students write answers for the following questions:

    - What did you learn today about rectangles with the same perimeter? area?

    - Were you surprised by the results? Were they different from what you had previously believed?

    - Write down, in your own words, how you would explain the results obtained today to another

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geometry student who had not yet studied area and perimeter.

Transfer.

- Have students answer the following questions:

    - Will changing the perimeter of a rectangle necessarily change its area?

    - If you are told to construct a figure that has a perimeter of 100 inches, can you do it so the area will be as small as you want? Would you want such a small area? Give an example.

    - If you are working for a landscaping company, and your customer wants to maximize an area to be fenced off for a garden, what type of rectangle would you recommend?

    - If two figures have different areas, must they have different perimeters?

    - Consider the following scenario:

> Recently a friend of mine had her living room floor scraped. The cost was to be reckoned by the square foot. She measured the floor to be 19 1/2 feet long and 12 1/2 feet wide. In order to know how much the job was going to cost, she had to know how many square feet were going to be scraped. Thus she had to calculate the area of the floor.... So that she didn't have to multiply numbers involving halves, she decided to calculate 19 x 13 instead of 19 1/2 x 12 1/2. She realized that the calculation would have been still simpler if she had rounded off the numbers the other way and calculated 20 x 12. What a surprise she got when the two answers did not come out to be the same (Walter 1970, 286-287).

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- Draw on graph paper the three different rectangles discussed in the problem. Calculate the area of each one. Explain what assumption the friend made and why it was incorrect.

Rationale.

This lesson emphasizes the importance of students being involved in their own learning. The students are actively engaged in an experimental process and are drawn into the discussion of mathematics by experiencing it first hand. From Ennis'(1987) list of dispositions and abilities, this lesson targets: seeking a clear statement of the question, seeking as much precision as the subject permits, and identifying assumptions. From the NCTM Standards (1989, 1991) the following standards are highlighted: mathematics as reasoning, geometry from a synthetic perspective, worthwhile mathematical tasks, students' role in discourse, and tools for enhancing discourse.

This lesson asks the students to use previously learned algebraic skills during the graphing exercise and shows the students how interconnected the various aspects of math are. The questions asked during transfer are intended to show the students how the concepts they are learning apply to their lives outside of school (geometry from a synthetic perspective). One of the questions asks the students to identify where a person went wrong in solving a problem by finding the mistaken assumption (identifying assumptions). It is hoped that this task will not only help the students

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with the transfer of the concepts, but will also draw the student into thinking about any assumptions they may have made and how their strategies may have been originally flawed (refining metacognitive skills).

The teacher's role throughout this lesson is to be a facilitator and to prod the students in the correct direction of a solution. Because of the pairing of students, the teacher is free to circulate from group to group and observe the thoughts of her students. She is also available to model different strategies and provide assistance as needed. She should actively encourage students to define the problem (seek a clear statement), use all materials available (tools for enhancing discourse), and to accurately measure and graph all information (seek as much precision as the subject permits).

During the activity and transfer parts of the lesson, the technique of "bridging" (Perkins and Salomon 1991, 220) was used. Students were shown how the concept they were studying was connected to another branch of mathematics and their everyday lives.

Conversion Conclusion

This last lesson presents some exercises which have students actively looking at how to convert various square units, square yards to square feet, square inches to square feet, etc. Most students incorrectly assume that the conversion factors for square units are the same as those

for linear measurement. The lesson starts by letting them test their theory in a "real life" situation.

Motivation.

- Present the students with the following problem:
You have decided to re-carpet your living room. You know that carpet is sold using some unit of measurement for area, but you can't remember which one. You decide to measure your living room in square feet. After careful measurement, you conclude that your living room is 144 sq.ft. When you get to the carpet dealer you see that you made a mistake. They sell carpet by the square yard, not the square foot. Your budget will allow you to spend $350.00. If the carpet you want costs $20.00 per square yard, can you afford it?

- Allow students time to try to solve the problem. You may want to allow them to work in groups. Two approaches may be presented by your students; divide 144 by 3 or divide 144 by 9. Ask for their rationale.

Activity.

- Have students work in pairs.
- Give each pair a different rectangle. All rectangles should have an area easily measured in square feet.

- Have the students first calculate the area in square inches, using rulers to find the length of the height and the base.
- Have the students use manipulatives (cardboard square feet) to find the area in square feet.
- Compare the area in square feet to the area in square inches.
- Have the students answer the following questions:
    - Was the first area in square inches divided by 12 or 144 to arrive at the area in square feet?
    - Why would you divide by 144 if there are only 12 inches in a foot?
    - What is the correct answer to the opening problem?
    - How would you convert from square inches to square feet? How about square yards to square feet?

Metacognition.

- Have students write answers to the following questions:
    - What is the importance of being able to convert back and forth between units?
    - Explain how converting from square yards to square feet differs from converting from yards to feet.
    - What strategies are you going to use so that you don't confuse conversion in the future?

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- Give each pair of students the task of finding the area of various walls in the room (or school). Require students to express the area in two different units, but only allow them to make one initial measurement.
- Can you ever foresee a time when you might have to do this (other than buying carpet)? If so, give an example.
- Have students complete the attached worksheet (see Fig. 5.2).

Rationale.

This lesson presents some concrete examples of why it is important to understand conversion factors. Highlighted were the following dispositions and abilities: looking for alternatives, seeking as much precision as the subject permits, and defining terms and judging definitions in three dimensions. From the NCTM Standards (1989, 1991), the following standards are utilized: geometry from a synthetic perspective, worthwhile mathematical tasks, and tools for enhancing discourse.

The students are asked to take some measurements using rulers and square units (tools for enhancing discourse). When taking these measurements, students must realize what types of measurements would be erroneous (seek as much precision as the subject permits). They are also asked to reason why the conversions they know for linear measurements

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Worksheet

Complete each conversion.

1. How many square feet are there in 108 square yards?

2. How many square inches are there in 5 square feet?

3. How many square yards are there in 90 square feet?

4. How many square feet are there in 288 square inches?

5. How many square yards are there in 6480 square inches?
```

Fig. 5.2. Worksheet for Conversion Conclusion lesson.

don't work for two dimensional measurements (defining terms and judging definitions).

The teacher's role is to guide the students to the correct understanding. She should be inquisitive about students' ideas and should continually have students explain their reasoning.

Transfer is fostered through the use of realistic examples and through the use of the worksheet. The students are asked to complete several types of conversions, to assure their familiarity with the varying ways in which it might be encountered later. Both "hugging" (Perkins and Salomon 1991, 220) and "bridging" (Perkins and Salomon 1991,

220) can be seen in the lesson in the variety of conversions and in the problems that pose scenarios outside of the classroom.

Chapter Summary

The lessons presented in this chapter are different than many lessons presented in mathematics classrooms. They have been revised based on individual student suggestions and classroom trials. These lessons are focused on students and utilize their natural curiosity and their ability to explore various aspects of a concept. Students particularly liked the challenges presented to them and the use of manipulatives to test their own theories. They also enjoyed the visual parts of the lessons, including the graphing and the opening motivators. One student commented: "Being able to see what was happening, really helped me understand better." By asking students to be directly involved in their learning, the teacher's role changes to that of facilitator and moderator.

It is difficult to have lessons like this every day. Both constraints on the teacher's planning time and limitations based on the need to adhere to a curriculum have an impact on the frequency of using lessons of this nature. It is important to realize that devoting an entire lesson to this style of presentation is not necessary to achieve optimum learning. The teacher, instead, needs to understand the various aspects of learning presented in the past chapters.

An experienced teacher is able to incorporate the ideas presented in such a way as to complement her usual manner of teaching. It can be subtle, such as the use of more questioning to get students to elaborate and clarify their thoughts, or it can be more comprehensive, like redoing entire lessons. Whichever method a teacher chooses, it is important to better understand how students learn and how teachers can make a more positive impact during that process.

C H A P T E R   V I

REFLECTIONS

A Look Back

Misconceptions arise in all facets of people's lives.
They occur across the disciplines taught in school and
across social and political realms outside of school. This
thesis has focused on misconceptions in the mathematics
classroom, specifically misconceptions about area and
perimeter of rectangles.

Five distinct misconceptions are defined: Fallacy of
Multiples, Increase/Decrease Assumption, Conversion
Conclusion, Spatial Bias, and Equality Assumptions. Each
represents a different misconception held by students. Like
many misconceptions, they are born out of flawed reasoning
yet make sense to the students and become part of their
schemas. Because the students believe in the validity of
their assumptions and because these beliefs influence how
all new material is interpreted, it is important to be able
to identify the misconceptions students hold and to work
with students as they try to eradicate them.

The pre-test questions presented in Chapter IV are
sample problems which teachers could use to identify which
misconceptions their students hold about area and perimeter
of rectangles. These multiple choice questions provide
specifically chosen distracters which, through analysis,

enable the teacher to identify the various misconceptions. Although it is not always practical for the teacher to approach identification through a pre-test, there are some signals from students which a teacher should be able to recognize. "Well chosen tasks afford teachers opportunities to learn about their students' understandings even as the tasks also press the students forward" (NCTM 1991, 27).

An experienced teacher will know how to interpret the questions students ask and will also know how to analyze the reasoning behind errors that her students make. An insightful teacher is able to use that information and assess the misconceptions her students hold. Based on this insight, the teacher can then verbally test the students to confirm her beliefs. Although this method may not be as thorough or formal as a written exam, it does provide the necessary identification of misconceptions and gives the teacher the opportunity to address them with her students.

The importance of identification before instruction can not be overemphasized. Chapter II points out that students arrive in a classroom with prior knowledge and set schemas. In order for new material to be learned effectively and to replace incorrect schemas, it must be presented in such a manner that the students are willing to examine their previous beliefs, understand why they are not correct, and realize why the new concepts are better. If the teacher can "'get inside students' heads' and locate the causes of their difficulties . . . (this) . . . makes it much easier to remedy them" (Schoenfeld 1987a, 29). If the teacher fails

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to acknowledge students' previous beliefs, the effects of the instruction may be short lived. A temporary understanding may be attained by students, but the correct concepts are not likely to be incorporated into their schemas and thus the misconceptions may prevail.

Several suggestions are made for eradicating misconceptions and sample lessons are presented in Chapter V. An understanding of the learning process is crucial for the teacher as she guides her students through assigned subject matter. The lessons included in this thesis are based on the theories and <u>Standards</u> (NCTM 1989, 1991) presented in Chapters II and III. Their structure and format, as well as the defined role of the teacher, are more important than the specific information presented in each lesson.

Unraveling the Theories

This thesis reviews and highlights some of the theories and tools available to the mathematics teacher: critical thinking skills, metacognition, transfer and the NCTM <u>Standards</u> (1989, 1991). The underlying message is that the successful classroom of the 90's must evolve from the traditional teacher-centered classroom where the student is seen as a "retainer of, rather than a processor of, experience and information" (Strike and Posner 1985, 211).

No longer is it enough for the teacher to stand at the front of the classroom, lecturing to her students, and testing them periodically to see what information they have

absorbed. Instead, students must be active participants in their own learning. Society has changed and factual knowledge is not sufficient for productivity. Instead, being able to think critically and use information in a variety of ways is necessary.

> In a rapidly changing technological environment, it is difficult to predict what knowledge students will need or what problems they will have to solve 20 years from now. What they really need to know, it seems, is how to learn the new information and skills that they will require throughout their lives... However, skill in learning, reasoning, and general problem solving - including the more sophisticated aspects of reading and elementary mathematics are neglected by the schools. (Chipman and Segal 1985, 1)

In order for educators to meet this need, they have to involve the students while they are in school. These various skills can't be assumed in students; instead they have to be fostered and practiced. In perfecting these skills, students become more confident and better independent learners and thinkers. Once students are better able to monitor their own learning, their schemas become stronger and more accurate. Teachers can not "fix" students' misconceptions, they can only lead the students to a recognition of the faulty reasoning. Students must correct their erroneous beliefs themselves.

Metacognition and critical thinking skills provide the students with the tools they need to eradicate their misconceptions. Both of these enable students to become better at monitoring and maximizing their thinking. Critical thinking encompasses a variety of abilities and dispositions that, when used, enable the student to better

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understand, clarify and interpret information. They also make the student a better problem solver, by giving the student several techniques to employ while working on a problem's solution. Metacognition teaches students to recognize and interpret their own thought processes. If a student is able to do that, the student is able to make his learning more effective. The student will recognize and then utilize strategies that are effective for him. He will also be able to police himself during problem solving and reduce the numbers of errors he makes.

Transfer and the NCTM Standards (1989, 1991) direct the teacher's role during the learning process. Theories about transfer provide the teacher with tools to aid her in guiding students to receive and retain information. The Standards (NCTM 1989, 1991) offer her direction for improving curriculum as well as instruction and evaluation. The theme running through the Standards (NCTM 1989, 1991) is one that suggests that the students' role is no longer one of a passive receiver of information. It is up to the teacher to design lessons that engage the students in the act of learning. "When students engage in activities that require them to use new learning, both their knowledge of content and skills and their ability to use them develop productively together" (Applebee, Langer and Mullis 1991, 17).

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Getting Started

It is overwhelming to think about adequately incorporating all of these ideas into any one class. It seems almost impossible to implement each one fully every day. Because that is such an extraordinary task, it is more advantageous to weave pieces of the theories into daily lessons. The teacher's understanding of the concepts and the ramifications of them that are most important.

There may be some days when the entire class period will be devoted to a particular thinking skill and all activities used will be a tool to help students refine that particular skill. There may be other days when students will explore and validate various mathematical concepts using different methods and tools. Students should be routinely engaged in a dialogue which encourages them to question, reason, and communicate their ideas to others.

Basic strategies can be taught quickly and are helpful as students begin to monitor their own learning. Students should be encouraged to "test" their answers using each strategy before seeking the correct answer and accompanying explanation from the teacher.

The simplest strategy is to draw a "picture". Students will readily admit that a diagram makes the problem easier and quickly provides them with information in a concrete form. For example, if a question asks what happens to the area of a rectangle if the base and height are doubled in length, have the students actually draw to scale two

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appropriate rectangles. With a visual to guide them, most students will not adopt the Fallacy of Multiples misconception and will realize that it would take four of the original rectangles to "fill in" the area of the second rectangle. This strategy is advantageous regardless of the concept being presented.

Use of algebraic proofs is another way to have students confirm the reasoning behind correct ideas. In problems involving area and perimeter, two formulas are used to show students what is taking place as the length of heights and bases of rectangles are changed. Not only do these proofs remove all doubt and speculation about what is happening, they also remind students that material learned in a previous mathematics course is still valid and useful as they progress through the mathematics curriculum. This connection helps students to see mathematics as a subject that continually builds upon itself and that concepts presented daily will have applications in the future.

Another strategy students can use to check for mistakes is numerical substitution. There are two uses of this strategy. One is for the student to check an answer that he has arrived at **after** solving an equation by going back and substituting his answer for the variable. The other allows the student to check his hypothesis by plugging in values to see if his idea is correct **before** making any attempts to solve the problem algebraically. In algebra this idea is usually emphasized as a means for validating an answer obtained after solving an equation. Many of the sample

pre-test questions in Chapter IV present students with diagrams which designate the various lengths of the rectangles. Many students never use these values as a means of checking their hypotheses. If they did, they would begin to recognize and question their misconceptions. The strategy of numerical substitution works in other branches of mathematics as well. While numerical substitution is a method students can use independently to "prove" certain ideas, the teacher needs to caution students against the use of the numbers 0, 1, and 2. Zero and 1 are numbers to avoid as each represents an identity element. Zero is the identity element for addition and one is the identity element for multiplication. As identity elements, they will leave some equations unchanged. The number 2 is a bad choice because $2 + 2 = 4$ and $2 \times 2 = 4$. This can give a false sense of correctness if the problem centers on the operations of addition or multiplication.

Stressing these more general strategies with students encourages them to be more accountable for their thinking processes and their answers. It also gives them a chance to witness their own reasoning and confront any misconceptions. Since students can use these strategies independently, it is more likely that they will be convinced of the validity of the outcomes and change their schemas accordingly.

Where to Go from Here

Certainly identifying and eradicating misconceptions in mathematics should become a standard practice in schools.

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Misconceptions need to be addressed in other disciplines as well as mathematics. The various means of instruction suggested here, if routinely presented throughout the students' day rather than during a single class, would be much more effective. It is not enough, however, to address misconceptions at the high school level. Students arrive at school for the first time with naive theories and preconceived ideas that are incorrect. In order for this process of identification and eradication to be the most effective, it needs to begin at the earliest stage of organized instruction.

It is at these early stages, when students are most eager to learn and are naturally curious, that the process of identification should begin. The methodology presented in this paper to empower students and to help them be accountable for their education should not be overlooked during these beginning years. If students were exposed early to the skills of pattern recognition, making deductions, questioning hypotheses, reasoning, and communication of ideas the easier it would be in secondary education to help students correct their misconceptions and to continue to refine these skills before the students graduate. Unfortunately, not enough time is given to these methods in earlier grades and time is lost in the high school classroom trying to acclimate the students to this approach to education. Too often high school students are passive and simply want to memorize information. They have lost their curiosity and natural love for learning.

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Research and testing of theories continues. The
Teacher Education Research Center (TERC) located in
Cambridge, Massachusetts has developed a teacher training
program called "Talking Mathematics". This program adopts a
whole language approach to math and "stresses math as a
language and means of communicating; it concentrates on
understanding concepts by hands-on experiences and through
questioning and investigation" (Berger 1993a, B42). This
program has received notice from the NCTM and Virginia
Williams, field coordinator for NCTM commented that the
program "adheres to the Council's list of 'Professional
Standards for Teaching Mathematics'" (Berger 1993a, B42).

TERC has also sponsored other training sessions for
teachers and has enlisted the help of Eleanor Duckworth, a
professor at Harvard. She believes that "students must be
given room to explore. Teachers, in her view, should avoid
prodding children toward correct answers and instead, should
invite them to discover and be confident learners" (Duffy
1993, 31). She feels that teachers need to be trained to
let students experience new concepts first hand. Too many
teachers don't give students a chance and have little faith
that the students could, on their own, arrive at the same
conclusions the teacher was seeking. Duckworth says it
". . . is always a big revelation to teachers: to watch kids
work and not tell them what to think and to see that they
could come up with stuff on their own" (Duffy 1993, 34).

There is also a new push in education to readdress the
needs of students in middle schools. There are several

programs currently utilized, including The Coalition of
Essential Schools Network, founded by Theodore Sizer, The
School-College Partnership Project, and Performance
Assessment Collaboratives for Education. These programs
emphasize the need for more student oriented curricula and
the value that these types of curricula have on education.
Dennie Palmer Wolf, executive director of Performance
Assessment Collaboratives for Education says

> we want a kind of curriculum that is based on serious
> understanding of the subject matter but also enables
> the students to think broadly. We want kids who can
> be tough critics of their own work and teachers who
> understand what the standards for good work are.
> (Berger 1993b, 33)

There are some schools that are in the process of
testing different theories about learning. One example, is
an elementary school in Maryland that has been testing
Howard Gardner's theory of multiple intelligences for the
past three years. Rather than use a traditional test to
identify "gifted" students, they have been using a testing
method that incorporates Gardner's ideas about various types
of intelligences. The test is not strictly a written exam,
but includes problems that allow the student to demonstrate,
among others, their spatial skills.

The fact that such programs are in existence gives
credibility to the power of the ideas presented in this
thesis. Clearly misconceptions are not unique to area and
perimeter of rectangles and can occur in any subject.
Because of the theories presented, it has become evident
that one of the most effective ways to eradicate

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misconceptions is through a change in teaching instruction. NCTM leads the way for change with its Professional Standards (1991). These Professional Standards encourage teachers to confront their own misconceptions about the role of student and teacher in the classroom and to adopt the use of more non-traditional methods. Although each teacher or program may have a slightly different approach, all should embody the notion that the student needs to be active in his own learning and that traditional methods of instruction and assessment do not fully accomplish the goals of the twenty-first century.

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Ennis' Taxonomy of Critical Thinking Dispositions and Abilities

A. Dispositions
1. Seek a clear statement of the thesis or question
2. Seek reasons
3. Try to be well informed
4. Take into account the total situation
5. Try to remain relevant to the main point
6. Keep in mind the original and/or basic concern
7. Look for alternatives
9. Be open minded
    a) Consider seriously other points of view than one's own (dialogical thinking)
    b) Reason from premises with which one disagrees-without letting the disagreement interfere with one's reasoning (suppositional thinking)
    c) Withhold judgement when the evidence and reasons are insufficient
10. Take a position (and change a position) when the evidence and reasons are sufficient to do so
11. Seek as much precision as the subject permits
12. Deal in an orderly manner with the parts of a complex whole
13. Use one's critical thinking abilities
14. Be sensitive to the feelings, level of knowledge, and degree of sophistication of others

B. Abilities
1. Focusing on a question
    a) Identifying or formulating a question
    b) Identifying or formulating criteria for judging possible answers
    c) Keeping the situation in mind
2. Analyzing arguments
    a) Identifying conclusions
    b) Identifying stated reasons
    c) Identifying unstated reasons
    d) Seeing similarities and differences
    e) Identifying and handling irrelevance
    f) Seeing the structure of an argument
    g) Summarizing
3. Asking and answering questions of clarification and/or challenge, for example:
    a) Why?
    b) What is your main point?
    c) What do you mean by "_____"?
    d) What would be an example?
    e) What would not be an example (though close to being one)?

f) How does that apply to this case (describe a counterexample)?
g) What difference does it make?
h) What are the facts?
i) Is this what you are saying: "_____"?
j) Would you say some more about that?
4. Judging the credibility of a source
   a) Expertise
   b) Lack of conflict of interest
   c) Agreement among sources
   d) Reputation
   e) Use of established procedure
   f) Known to risk reputation
   g) Ability to give reasons
   h) Careful habits
5. Observing and judging observation reports; criteria:
   a) Minimal inferring involved
   b) Short time interval between observation and report
   c) Report by observer, rather than someone else (i.e., not hearsay)
   d) Records are generally desirable; if report is based on a record, it is generally best that
      1) The record was close in time to the observation
      2) The record was made by the observer
      3) The record was made by the reporter
      4) The statement was believed by the reporter, either because of a prior belief in its correctness or because of a belief that the observer was habitually correct
   e) Corroboration
   f) Possibility of corroboration
   g) Conditions of good access
   h) Competent employment of technology, if technology is useful
   i) Satisfaction by observer (and reporter, if a different person) of credibility criteria (item B4)
6. Deducing and judging deductions
   a) Class logic
   b) Conditional logic
   c) Interpretation of statements
      1) Double negative
      2) Necessary and sufficient conditions
      3) Other logical words and phrases: only, if and only if, or some, unless, not, not both, etc.
7. Inducing and judging inductions
   a) Generalizing
      1) Typicality of data
      2) Limitation of coverage
      3) Sampling

b) Inferring explanatory conclusions and
       hypotheses
       1) Types of explanatory conclusions and
          hypotheses
             a) Causal claims
             b) Claims about the beliefs and
                attitudes of people
             c) Interpretations of authors' intended
                meaning
             d) Historical claims that certain things
                happened
             e) Reported definitions
             f) Claims that something is an unstated
                reason or unstated conclusion
       2) Investigating
             a) Designing experiments, including
                planning to control variables
             b) Seeking evidence and counterevidence
             c) Seeking other possible explanations
       3) Criteria: Given reasonable assumptions
             a) The proposed conclusion would explain
                the evidence (essential)
             b) The proposed conclusion is consistent
                with known facts (essential)
             c) Competitive alternative conclusions
                are inconsistent with known facts
                (essential)
             d) The proposed conclusion seems
                plausible (desirable)
8. Making value judgments
    a) Background facts
    b) Consequences
    c) Prima facie application of acceptable
       principles
    d) Considering alternatives
    e) Balancing, weighing, and deciding
9. Defining terms, and judging definitions in three
   dimensions
    a) Form
       1) Synonym
       2) Classification
       3) Range
       4) Equivalent expression
       5) Operational
       6) Example-nonexample
    b) Definitional strategy
       1) Acts
             a) Report a meaning (reported
                definition)
             b) Stipulate a meaning (stipulative
                definition)
             c) Express a position on an issue
                (positional, including programmatic
                and persuasive definition)

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2) Identifying and handling equivocation
                        a) Attention to context
                        b) Possible types of response
                            i) The simplest response: "The
                               definition is just wrong."
                           ii) Reduction to absurdity:
                               "According to that definition,
                               there is an outlandish result."
                          iii) Considering alternative
                               interpretations: "On this
                               interpretation, there is this
                               problem; on that interpretation
                               there is that problem."
                           iv) Establishing that there are two
                               meanings of key term and a shift
                               in meaning from one to the other
                            v) Swallowing the idiosyncratic
                               definition
                        c) Content
        10. Identifying assumptions
            a) Unstated reasons
            b) Needed assumptions; argument reconstruction
        11. Deciding on an action
            a) Define the problem
            b) Select criteria to judge possible solutions
            c) Formulate alternative solutions
            d) Tentatively decide what to do
            e) Review, taking into account the total
               situation, and decide
            f) Monitor the implementation
        12. Interacting with others
            a) Employing and reacting to fallacy labels
               including
                    1) Circularity
                    2) Appeal to authority
                    3) Bandwagon
                    4) Glittering term
                    5) Name calling
                    6) Slippery slope
                    7) Post hoc
                    8) Non sequitur
                    9) Ad hominem
                   10) Affirming the consequent
                   11) Denying the antecedent
                   12) Conversation
                   13) Begging the question
                   14) Either-or
                   15) Vagueness
                   16) Equivocation
                   17) Straw person
                   18) Appeal to tradition
                   19) Argument from analogy
                   20) Hypothetical question
                   21) Oversimplification
                   22) Irrelevance

b) Logical strategies
c) Rhetorical strategies
d) Argumentation; Presenting a position, oral or written
   1) Aiming at a particular audience and keeping it in mind
   2) Organizing (common type; main point; clarification; reasons; alternatives; attempt to rebut prospective challenges; summary, including repeat of main point)

(Ennis 1987, 9-26)

APPENDIX B

National Council of Teachers of Mathematics Curriculum
Standards for Grades 9-12

Standard 1: Mathematics as Problem Solving

In grades 9-12, the mathematics curriculum should
include the refinement and extension of methods of
mathematical problem solving so that all students can

- use, with increasing confidence, problem-solving
  approaches to investigate and understand
  mathematical content;
- apply integrated mathematical problem-solving
  strategies to solve problems from within and
  outside of mathematics;
- recognize and formulate problems from situations
  within and outside of mathematics;
- apply the process of mathematical modeling to
  real-world problem situations.

Standard 2: Mathematics as Communication

In grades 9-12, the mathematics curriculum should
include the continued development of language and
symbolism to communicate mathematical ideas so that all
students can

- reflect upon and clarify their thinking about
  mathematical ideas and relationships;
- formulate mathematical definitions and express
  generalizations (potential theorems) discovered
  through investigations;
- express mathematical ideas orally and in writing;
- read written presentations of mathematics with
  understanding;
- ask clarifying and extending questions related to
  mathematics they have read or heard about;
- appreciate the power, elegance, and economy of
  mathematical notation and its role in the
  development of mathematical ideas.

Standard 3: Mathematics as Reasoning

In grades 9-12, the mathematics curriculum should include numerous and varied experiences that reinforce and extend logical reasoning skills so that all students can

- make and test conjectures;
- formulate counterexamples;
- follow logical arguments;
- judge the validity of arguments;
- construct simple valid arguments;

and so that, in addition, college-intending students can

- construct proofs for mathematical assertions, including indirect proofs and proofs by mathematical induction.

Standard 4: Mathematical Connections

In grades 9-12, the mathematics curriculum should include investigation of the connections and interplay among various mathematical topics and their application so that all students can

- recognize equivalent representations of the same concept;
- utilize and value the connections among mathematical topics;
- utilize and value the connections between mathematics and other disciplines.

Standard 5: Algebra

In grades 9-12, the mathematics curriculum should include the continued study of algebraic concepts and methods so that all students can

- represent situations that involve variable quantities with expressions, equations, inequalities, and matrices;
- use tables and graphs as tools to interpret expressions, equations, and inequalities;
- operate on expressions and matrices, and solve equations and inequalities;
- appreciate the power of mathematical abstraction and symbolism;

and so that, in addition, college-intending students
can

- use matrices to solve linear systems;
- demonstrate technical facility with algebraic
  transformations, including techniques based on
  the theory of equations.

Standard 6: Functions

In grades 9-12, the mathematics curriculum should
include the continued study of functions so that all
students can

- model real-world phenomena with a variety of
  functions;
- represent and analyze relationships using tables,
  rules, and graphs;
- translate among tabular, symbolic, and graphical
  representations of functions;
- recognize that a variety of problem situations
  can be modeled by the same type of function;
- analyze the effects of parameter changes on the
  graphs of the functions;

and so that, in addition, college-intending students
can

- understand operations on, and the general
  properties and behavior of, classes of functions.

Standard 7: Geometry from a Synthetic Perspective

In grades 9-12, the mathematics curriculum should
include the continued study of the geometry of two and
three dimensions so that all students can

- interpret and draw three-dimensional objects;
- represent problem situations with geometric
  models and apply properties of figures;
- classify figures in terms of congruence and
  similarity and apply these relationships;
- deduce properties of, and relationships between,
  figures from given assumptions;

and so that, in addition, college-intending students
can

- develop an understanding of an axiomatic system
  through investigating and comparing various
  geometries.

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Standard 8: Geometry from an Algebraic Perspective

In grades 9-12, the mathematics curriculum should include the study of geometry of two and three dimensions from an algebraic point of view so that all students can

- translate between synthetic and coordinate representations;
- deduce properties of figures using transformations and using coordinates;
- identify congruent and similar figures using transformations;
- analyze properties of Euclidean transformations and relate translations to vectors;

and so that, in addition, college-intending students can

- deduce properties of figures using vectors;
- apply transformations, coordinates, and vectors in problem solving.

Standard 9: Trigonometry

In grades 9-12 the mathematics curriculum should include the study of trigonometry so that all students can

- apply trigonometry to problem situations involving triangles;
- explore periodic real-world phenomena using the sine and cosine functions;

and so that, in addition, college students can

- understand the connection between trigonometric and circular functions;
- use circular functions to model periodic real-world phenomena;
- apply general graphing techniques to trigonometric functions;
- solve trigonometric equations and verify trigonometric identities;
- understand the connections between trigonometric functions and polar coordinates, complex numbers, and series.

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Standard 10: Statistics

In grades 9-12, the mathematics curriculum should include the continued study of data analysis and statistics so that all students can

- construct and draw inferences from charts, tables and graphs that summarize data from real-world situations;
- use curve fitting to predict from data;
- understand and apply measures of central tendency, variability, and correlation;
- understand sampling and recognize its role in statistical claims;
- design a statistical experiment to study a problem, conduct the experiment, and interpret and communicate the outcomes;
- analyze the effects of data transformations on measures of central tendency and variability;

and so that, in addition, college students can

- transform data to aid in data interpretation and prediction;
- test hypotheses using appropriate statistics.

Standard 11: Probability

In grades 9-12, the mathematics curriculum should include the continued study of probability so that all students can

- use experimental or theoretical probability, as appropriate, to represent and solve problems involving uncertainty;
- use simulations to estimate probabilities;
- understand the concept of a random variable;
- describe, in general terms, the normal curve and use its properties to answer questions about sets of data that are assumed to be normally distributed;

and so that, in addition, college students can

- apply the concept of a random variable to generate and interpret probability distributions including binomial, uniform, normal, and chi square.

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Standard 12: Discrete Mathematics

In grades 9-12, the mathematics curriculum should include topics from discrete mathematics so that all students can

- represent problem situations using discrete structures such as finite graphs, matrices, sequences, and recurrence relations;
- represent and analyze finite graphs using matrices;
- develop and analyze algorithms;
- solve enumeration and finite probability problems;

and so that, in addition, college students can

- represent and solve problems using linear programming and difference equations;
- investigate problem situations that arise in connection with computer validation and the application of algorithms.

Standard 13: Conceptual Underpinnings of Calculus

In grades 9-12, the mathematics curriculum should include the informal exploration of calculus concepts from both a graphical and a numerical perspective so that all students can

- determine maximum and minimum points of a graph and interpret the results in problem situations;
- investigate limiting processes by examining infinite sequences and series under curves;

and so that, in addition, college students can

- understand the conceptual foundations of limit, the area under a curve, the rate of change, and the slope of a tangent line, and their applications in other disciplines;
- analyze the graphs of polynomial, rational, radical, and transcendental functions.

Standard 14: Mathematical Structure

In grades 9-12, the mathematics curriculum
should include the study of mathematical structure so
that all students can

- compare and contrast the real number system and
  its various subsystems with regard to their
  structural characteristics;
- understand the logic of algebraic procedures;
- appreciate that seemingly different mathematical
  systems may be essentially the same;

and so that, in addition, college students can

- develop the complex number system and demonstrate
  facility with its operations;
- prove elementary theorems within various
  mathematical structures, such as groups and
  fields;
- develop an understanding of the nature and
  purpose of axiomatic systems.

(NCTM 1989, 137-184)

APPENDIX C

National Council of Teachers of Mathematics Professional
Standards for Teaching Mathematics

Standard 1: Worthwhile Mathematical Tasks

The teacher of mathematics should pose tasks that
are based on

- sound and significant mathematics;
- knowledge of students' understandings, interests,
  and experiences;

and that

- engage students' intellect;
- develop students' mathematical understandings and
  skills;
- stimulate students to make connections and
  develop a coherent framework for mathematical
  ideas;
- call for problem formulation, problem solving,
  and mathematical reasoning;
- promote communication about mathematics;
- display sensitivity to, and draw on, students'
  diverse background experiences and dispositions;
  promote the development of all students'
  dispositions to do mathematics.

Standard 2: The Teacher's Role in Discourse

The teacher of mathematics should orchestrate discourse
by

- posing questions and tasks that elicit, engage,
  and challenge each student's thinking;
- listening carefully to students' ideas;
- asking students to clarify and justify their
  ideas orally and in writing;
- deciding what to pursue in depth from among the
  ideas that students bring up during a
  discussion;
- deciding when and how to attach mathematical
  notation and language to students' ideas;
- deciding when to provide information, when to
  clarify an issue, when to model, when to lead,
  and when to let a student struggle with a
  difficulty;
- monitoring students' participation in discussions
  and deciding when and how to encourage each
  student to participate.

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Standard 3: Students' Role in Discourse

The teacher of mathematics should promote classroom discourse in which students

- listen to, respond to, and question the teacher and one another;
- use a variety of tools to reason, make conjectures, solve problems, and communicate;
- initiate problems and questions;
- make conjectures and present solutions;
- try to convince themselves and one another of the validity of particular representations, solutions, conjectures, and answers;
- rely on mathematical evidence and argument to determine validity.

Standard 4: Tools for Enhancing Discourse

The teacher of mathematics, in order to enhance discourse, should encourage and accept the use of

- computers, calculators, and other technology;
- concrete materials used as models;
- pictures, diagrams, tables, and graphs;
- invented and conventional terms and symbols;
- metaphors, analogies, and stories;
- written hypotheses, explanations, and arguments;
- oral presentations and dramatizations.

Standard 5: Learning Environment

The teacher of mathematics should create a learning environment that fosters the development of each student's mathematical power by

- providing and structuring the time necessary to explore sound mathematics and grapple with significant ideas and problems;
- using the physical space and materials in ways that facilitate students' learning of mathematics;
- providing a context that encourages the development of mathematical skill and proficiency;
- respecting and valuing students' ideas, ways of thinking, and mathematical dispositions;

and by consistently expecting and encouraging students to

- work independently or collaboratively to make sense of mathematics;
- take intellectual risks by raising questions and formulating conjectures;
- display a sense of mathematical competence by validating and supporting ideas with mathematical argument.

Standard 6: Analysis of Teaching and Learning

The teacher of mathematics should engage in ongoing analysis of teaching and learning by

- observing, listening to, and gathering other information about students to assess what they are learning;
- examining effects of the tasks, discourse, and learning environment on students' mathematical knowledge, skills, and dispositions;

in order to

- ensure that every students is learning sound and significant mathematics and is developing a positive disposition toward mathematics;
- challenge and extend students' ideas;
- adapt or change activities while teaching;
- make plans, both short and long-range;
- describe and comment on each student's learning to parents and administrators, as well as to the students themselves.

(NCTM 1991, 25-75)

APPENDIX D

Sample Pre-test

1. If the base of a rectangle is tripled and the height
   is unchanged, then the new rectangle's area will:

   a) triple.   b) be 6 times as big
   c) be 9 times as big.

2. If the height of a rectangle is doubled and the base
   is unchanged, then the new rectangle's area will:

   a) quadruple.   b) double.
   c) be 1/2 as big.

3. If the height of a rectangle is increased n times and
   the base is unchanged, then the new rectangle's area
   will:

   a) be n times as big.   b) be n^2 times as big.
   c) be 2n times as big.

4. If the base of rectangle A below is tripled and the height remains unchanged (see rectangle B below), then the perimeter will:

   a) triple.   b) increase by 16.
   c) increase by 32.

5. If the height of rectangle C below is quadrupled and the base is unchanged (see rectangle D below), then the perimeter of the rectangle will:

   a) increase by 9.   b) quadruple.
   c) increase by 18.

6. If the height of rectangle E below is increased n times and the base is unchanged (see rectangle F below), then the perimeter will:

   a) be n times as big.   b) increase by 4(n-1).
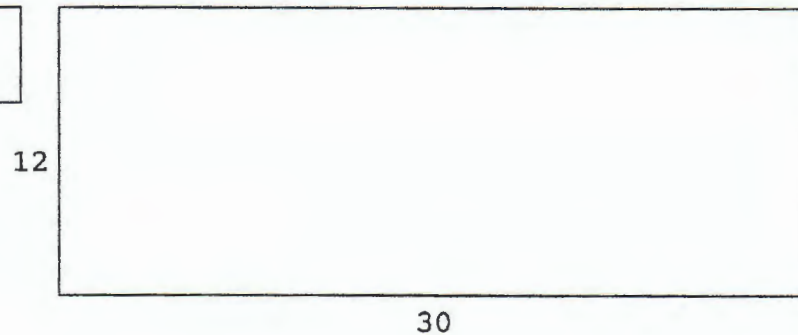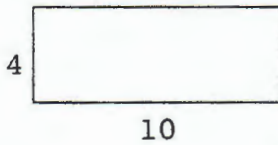   c) increase by 8(n-1).

7. If the base and height of a rectangle are both doubled, then the area will:

   a) double.   b) quadruple.
   c) increase by 14.
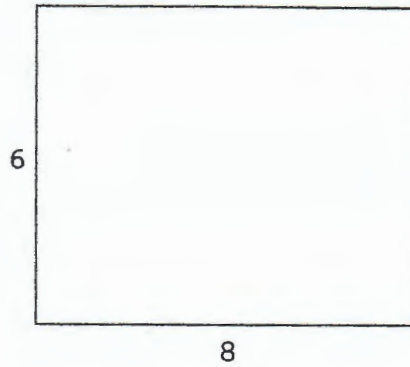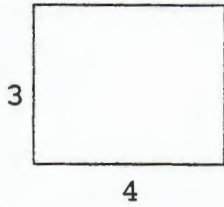
8. If the base and the height of a rectangle are both tripled, then the area will:

   a) be 6 times as big.   b) triple.
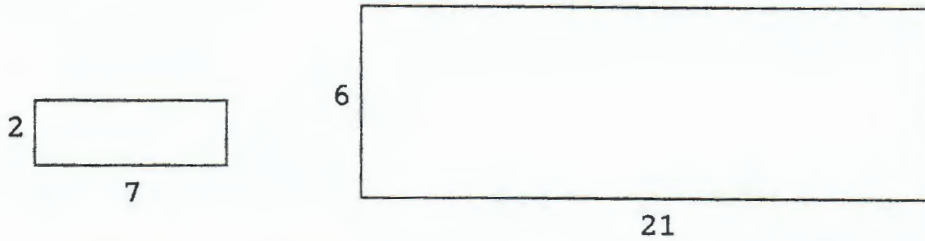   c) be 9 times as big.

9. If the base and height of a rectangle are both multiplied by n, then the area will:

   a) be n^2 times as big.   b) be 2n times as big.
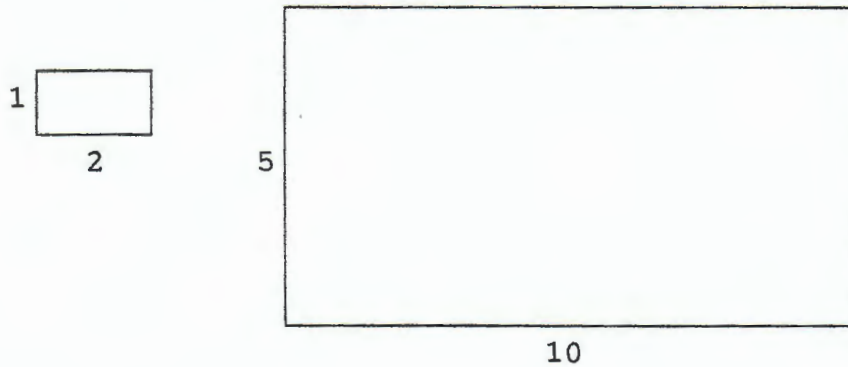   c) be n times as big.

10. If the base and height of a rectangle are tripled, then the perimeter will:

    a) be 6 times as big.   b) triple.
    c) be 9 times as big.

11. If the base and height of a rectangle are both multiplied by 5, then the perimeter will:

    a) be 10 times as big.   b) be 25 times as big.
    c) be 5 times as big.
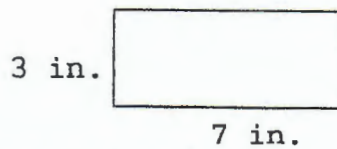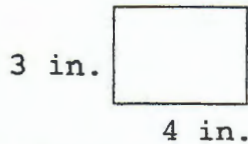

12. If the base and height of a rectangle are both multiplied by n, then the perimeter will:

    a) be 2n times as big.   b) be n times as big.
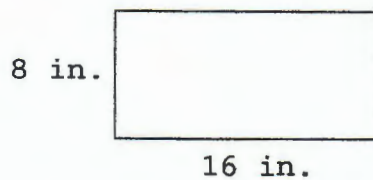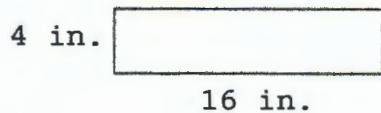   b) be n^2 times as big.

13. If the base of a rectangle is increased by 3 inches and the height remains the same, then the perimeter will:

    a) increase by 6 inches.   b) increase by 3 inches.
    c) increase by 9 inches.

3 in. ⬚   3 in. ⬚
    4 in.   7 in.

14. If the height of a rectangle is increased by 4 inches and the base is unchanged, then the perimeter will:

    a) increase by 4 inches.   b) increase by 16 inches.
    c) increase by 8 inches.

4 in. ⬚
    16 in.   8 in. ⬚
                 16 in.

15. If both the base and height of a rectangle are increased by 5 inches, then the perimeter will:

    a) increase by 20 inches.   b) increase by 25 inches.
    c) increase by 5 inches.

3 in. ⬚
   8 in.   8 in. ⬚
             13 in.

16. If the base of a rectangle is decreased by 7 inches and the height is unchanged, then the perimeter will:

    a) decrease by 14 inches.   b) decrease by 7 inches.
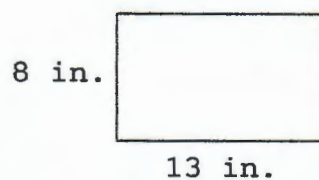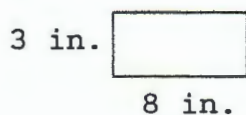   c) decrease by 49 inches.

17. If the height of a rectangle is decreased by n inches and the base remains the same, then the perimeter will:

    a) decrease by n inches.   b) decrease by n^2 inches.
    c) decrease by 2n inches.

18. How many square feet are there in 18 square yards?

   a) 162 sq.ft.   b) 54 sq.ft.
   c) 2 sq.ft.   d) 6 sq.ft.

19. How many square inches are there in 12 square feet?

   a) 144 sq.in.   b) 1 sq.in.
   c) 1728 sq.in.   d) 1/12 sq.in.

20. How many square yards are there in 27 square feet?

   a) 243 sq.yd.   b) 3 sq.yd.
   c) 9 sq.yd.   d) 81 sq.yd.

21. How many square feet are there in n square yards?

   a) 3n sq.ft.   b) 9n sq.ft.
   b) n/3 sq.ft.   d) n/9 sq.ft.

22-24 Students may choose more than one answer.

22. If the perimeter of a rectangle is 16 inches, which
    of the following are possible values for the
    rectangle's area?

   a) 7 sq.in.   b) 15 sq.in.
   c) 24 sq.in.

23. If the area of a rectangle is 24 square inches, which
    of the following represent possible values for its
    perimeter?

   a) 20 in.   b) 50 in.
   c) 30 in.

24. If the area of a rectangle is 1 square yard, which of
    the following represent possible values for its
    perimeter?

   a) 4 yds.   b) 1/2 yd.
   c) 5 yds.

25. Both rectangle A and B have an area of 36 square inches and rectangle A has a perimeter of 24 inches. What is the perimeter for rectangle B?

    a) 26 in.   b) 24 in.
    c) 36 in.

6 in. | A | 6 in.

4 in. | B | 9 in.

26. Both rectangle C and D have an area of 12 square feet and rectangle D has a perimeter of 16 feet. What is the perimeter for rectangle C?

    a) 16 ft.   b) 12 ft.
    c) 14 ft.

3 ft. | C | 4 ft.

2 ft. | D | 6 ft.

27. Both rectangle E and F have an area of 20 square yards and rectangle E has a perimeter of 24 yards. What is the perimeter for rectangle F?

    a) 20 yds.   b) 18 yds.
    c) 24 yds.

2 yds. | E | 10 yds.

5 yds. | F | 4 yds.

28. Both rectangle A and B have a perimeter of 16 inches and rectangle A has an area of 12 square inches. What is the area of rectangle B?

    a) 16 sq.in.   b) 15 sq.in.
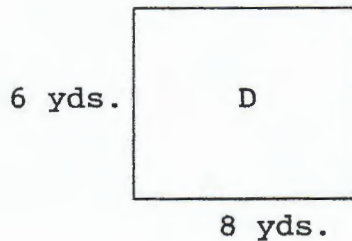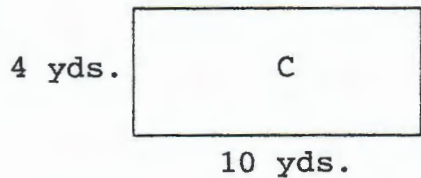   c) 12 sq. in.

2 in. | A |
6 in.

3 in. | B |
5 in.

29. Both rectangle C and D have a perimeter of 28 yards and rectangle D has an area of 48 square yards. What is the area of rectangle C?

    a) 40 sq.yds.   b) 48 sq.yds.
    c) 28 sq.yds.

4 yds. | C |
10 yds.

6 yds. | D |
8 yds.

30. Both rectangle E and F have a perimeter of 8 feet and rectangle E has an area of 4 square feet. What is the area of rectangle F?

    a) 4 sq.ft.   b) 8 sq.ft.
    c) 3 sq.ft.

2 ft. | E |
2 ft.

3 ft. | F |
1 ft.

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