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Geometry in Action: A Curriculum Unit Utilizing Dynamic Geometry Software to Enhance Students' Comprehension

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GEOMETRY IN ACTION
A curriculum unit utilizing dynamic geometry software to enhance students'
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A Synthesis Project Presented

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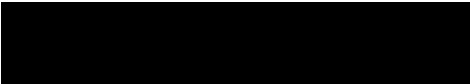
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ABSTRACT

GEOMETRY IN ACTION
A curriculum unit utilizing dynamic geometry software to enhance students'
comprehension

May 2001

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The paper identifies two critical obstacles to student success in a traditional geometry classroom and examines the role dynamic geometry software can play in overcoming these obstacles. The two concerns are that students complete geometry classes with minimal understanding due to low teacher and parental expectations and that students lack visualization skills. Student investigations in a dynamic geometry environment address both of these issues in that they raise expectations and provide a visualization aide. Additionally, dynamic geometry software produces precise, manipulable sketches that represent a continuum of related geometric figures on which students can base conjectures.

The focus of the project is a curriculum unit that was implemented in a high school geometry elective entitled Geometry in Action. A total of seven students were enrolled in the semester long elective that focused on student investigations in a dynamic geometry environment. The curriculum unit investigated the medians, altitudes, perpendicular bisectors, and angle bisectors of a triangle and the points of concurrency. The unit also emphasized critical thinking skills to maximize student comprehension.

The integrated thinking skills included predicting results of manipulations, identifying characteristics, making decisions, forming and justifying conjectures, and seeing relationships.

The discussion of each of the nine lessons in the unit contains the conceptual emphasis of the lesson, the key thinking processes incorporated in the lesson, a detailed description of the lesson, and a wrap-up activity. Actual student work and feedback are included within the paper. Also included are the author's reflections on the execution of the lessons and suggestions for improvement and adaptations in future implementations.

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CHAPTER ONE

INTRODUCTION

"So, what do you teach?"

"High school geometry."

"Oh, I *hated* geometry when I was in school. I never could understand it."

Like many geometry teachers, I have experienced this scenario all too often throughout my teaching career. The person with whom I am conversing typically delivers the last statement with a look of sheer terror. Unfortunately, many students, both past and present, merely "survive" geometry in contrast to actually understanding the concepts.

An educator's role is to strive for comprehension on the part of his/her students. If a particular approach or activity does not result in a student's understanding of a particular concept, an effective teacher will alter the instruction. In my experience, the traditional approaches utilized in the geometry classroom are frequently unsuccessful in promoting student understanding of geometric concepts. For this reason, I feel there is a strong need to investigate alternative approaches to geometry education. Such an alternative, the development of a curriculum unit on the points of concurrency, is the focus of this synthesis project. The curriculum unit was designed to be incorporated in a geometry elective, Geometry in Action, offered at a local suburban high school.

Obstacles to Students' Understanding of Geometry

To ensure an improvement in instruction as opposed to a change for the sake of change, it is necessary to understand why the current approach is failing. There are two main reasons for the previously described attitude towards geometry and the lack of geometric learning.

The first obstacle involves expectations of students. Geometry students are often not required, or even encouraged, to fully comprehend the material being presented. It is more time-efficient to suffice with a class' shallow understanding and rote memorization of teacher-presented concepts. "Instead of understanding what they are doing, students parrot what they have seen and heard" (Battista 1999, 427). Students are also told by parents and friends, "I never understood geometry. Just get through the course and you won't have to deal with it anymore."

The second hindrance involves student limitations. It is common for students to lack the visualization skills necessary to grasp the abstract topics central to the field of geometry. Without the ability to visualize an abstract concept, understanding of the concept is not possible.

How, then, do we improve the state of geometry education? If the first social constraint were removed and the second mental obstacle were circumvented, students would have a greater chance for success in geometry. Student-based investigations involving the use of computers are the tools with which we can achieve this goal. Student-based investigations remove the socially imposed barrier, making students responsible for their own understanding of the concepts. Computers equipped with

dynamic geometry software can provide the visualization skills necessary to transcend the levels of geometric thought.

Background Information

Newly implemented this year, the high school at which I teach is offering an alternative geometry elective. As the developer of the course, I have focused the course around computer-based constructions utilizing the Geometer's Sketchpad (Jackiw, 1995) and student explorations that require students to form conjectures and investigate their hypotheses.

The goal of the course is to promote comprehension of geometric concepts through the development of thinking skills and habits of mind that are important in the study of geometry: accurate conjecture forming, a questioning attitude, and the ability to provide support for ideas. To fulfill these goals, students are encouraged to develop their own individual understanding of geometric concepts through the manipulation and investigation of computer constructions. By observing the manner in which the computer constructions behave when manipulated, students form conjectures regarding the properties and characteristics of the computer constructions, which are generalized to apply to all related geometric shapes. Students are repeatedly asked to defend their conjectures and to explain *why*. For example, it is not sufficient to state that the centroid of a triangular region is the center of gravity. Students must also discuss the validity of the statement. Ultimately, students should provide justification for their conclusions.

The Role of Computers in Teaching Understanding

The rationale behind the intensive use of computers in Geometry in Action is discussed in chapter two. The subject is approached from two perspectives: pedagogical research and first hand experience.

This chapter references the works of Judah L. Schwartz (1992, 1997[?]), co-creator of the Geometric Supposer, Paul Goldenberg, Al Cuoco, and Orit Hazzan (Goldenberg and Cuoco 1996, Hazzan and Goldenberg 1996), researchers and curriculum developers for Education Development Center, and Colette Laborde (1993), a well-published mathematics professor. Seven themes of dynamic geometry software which are found woven through the articles authored by these researchers are discussed within the chapter: precision, manipulability, explicitness, visualization, an improved learning process, directed attention, and motivation.

Although most of the research in the field of dynamic geometry software is favorable, two drawbacks to the use of computers can be found in the writing of Laborde (1993) and Enrique Galindo (1997). Laborde focuses on the difficulty students have in learning how to use the programs. Recent versions of dynamic geometry software, however, have become more user-friendly and minimize the complexity of interacting with the computer. Galindo is uncomfortable with students' eagerness to accept computer-generated sketches and their manipulations as proof. This shortcoming can be overcome by a classroom and a teacher that use the sketches as a springboard for further discussion instead of an end product.

The chapter also cites the experience of Judah Schwartz and mathematics professor Claudi Giamati (1995). Each of these individuals describes how dynamic

of traditional programs. Geometry in Action is one possible alternative gateway.

Computer-based student investigations are the keys that unlock the gate.

CHAPTER TWO

THE BENEFITS OF COMPUTER USAGE IN THE GEOMETRY CLASSROOM

Computer software designed for the geometry classroom has been improving and gathering support at a tremendous rate over the past ten to fifteen years. The advent of dynamic geometry software in the eighties revolutionized the manner in which students could learn geometric concepts. "Learning geometry could ... become an open-ended exploration of relationships in geometric figures rather than a rerun of proofs of theorems" (Twomey n.d.). Computers now provide the opportunity for students to produce accurate, manipulable geometric constructions. Like a snowball in rapid descent, dynamic geometry software has continued to expand with each passing year, in terms of both volume (the multitude of software programs available) and covered ground (the growing number of wired classrooms that offer computer use).

This chapter provides support from research and classroom experience for the inclusion of computer activities in a geometry classroom. Judah L. Schwartz (1992, 1997[?]), co-creator of the Geometric Supposer, provides insights into the strengths of computer programs in the geometry classroom. Paul Goldenberg, Al Cuoco, and Orit Hazzan (Goldenberg and Cuoco 1996, Hazzan and Goldenberg 1996), researchers and curriculum developers for Education Development Center, distinguish between Euclidean geometry and Dynamic geometry, clarifying the differences and strengths of each version of geometry. This chapter includes a look at the work of Colette Laborde (1993), a French mathematics professor and researcher in the field of dynamic geometry software, who discusses the visualization obstacle in geometry and the ability of computers to

surmount that obstacle. Support is also found in the writings of Douglas Clements and

Michael T. Battista (1994, 1995), researchers and advocates in the field of mathematics

education reform who itemize the numerous benefits and educational implications of

dynamic geometry software. Finally, experiential support is found in the classroom of

Claudia Giamati (1995), a mathematics professor at Northern Arizona University.

<u>What is Dynamic Geometry Software?</u>

In 1983, Judah Schwartz and Michal Yerushalmy started a revolution in geometry

education. Geometry's version of the shot heard around the "classroom" was a computer

program they created called the Geometric Supposer that was to become the precursor to

today's dynamic geometry programs. In reflecting on their work, Schwartz (1992),

whose thinking can be extended to apply to all subsequent dynamic geometry programs,

describes the program as follows:

> The Geometric Supposer is what I call an intellectual mirror
> environment. It is a series of computer programs that deal with
> geometry and that allow – indeed invite – the user to explore and
> experiment. Just as the child's blocks do not by themselves pose
> problems, neither does the Supposer. Just as the blocks, in
> conjunction with interesting challenges, provide an opportunity for
> invention, exploration, and problem posing and solving, so does the
> Supposer.

Dynamic geometry software is comprised of computer programs that allow the

user to construct and manipulate geometric drawings. As implied by the adjective

"dynamic" the software does not produce static diagrams but rather generates diagrams

that can be altered to represent a continuum of related examples. With the aid of a

mouse, the software user can alter the size and the shape of a sketch by "dragging" a

point, a line, or a set of points. The manipulation of the computer sketch allows for some

explicitly defined properties of the diagram to remain unchanged. For example, the user may define point M to be the midpoint of segment $\overline{AB}$. Although the length and location of segment $\overline{AB}$ may be altered, point M will continue to be located at the midpoint of the segment. In this manner, dynamic geometry software allows the user to form conjectures and conclusions by observing what is or is not altered by the manipulation of the sketch.

It is important to note that although dynamic geometry software appears to depict a continuum of Euclidean geometry, it is in actuality a closely related branch of geometry (Goldenberg and Cuoco 1996). The dynamic characteristic of the computer program introduces objects that do not exist in Euclidean geometry and requires additional postulates and definitions.

> There's plenty of evidence that Euclid and others have thought in terms of dynamic images…but things like line segments, or triangles, though they could be *various* (lengths, shapes), weren't really treated as being vari*able*, and therefore the consequences of varying them – the effect that would have on other features of a figure – were not considered. (Goldenberg, e-mail message to author, Dec. 27, 2000)

Goldenberg illustrates this distinction with two examples – a monster-producing quadrilateral (Goldenberg and Cuoco, 1996) and the center of a circle through three points (Goldenberg, e-mail message to author, Dec. 27, 2000). If an arbitrary quadrilateral *ABCD* is constructed and one or more of the vertices dragged, the quadrilateral produces some "monsters" (Goldenberg and Cuoco 1996), cases which occur in the dynamic geometry environment that do not adhere to Euclidean definitions. These monsters include the case where three of the points are collinear, producing a triangle, the case where all points are collinear, producing a line segment, and the case where the quadrilateral crosses over itself (figure 1). These monster cases force an examination and a clarification of the definition of a quadrilateral.

Goldenberg's second example constructs a circle through three points, A, B, and C and locates the center of the circle. What happens to the center when the three points are collinear and no triangle can be constructed? "[T]he answer cannot be given at all without introducing a new postulate – a postulate of dynamic (not Euclidean) geometry.... What *does* happen is a decision not of geometry but of some new systems, and so it is a new postulate" (Goldenberg, e-mail message to author, Dec. 27, 2000).

Dynamic geometry software is available in many different looks and formats. Although there are many programs currently marketed, two programs – Geometer's Sketchpad (Jackiw, 1991; 1995) and Cabri (Baulac, Bellemain, and J.M. Laborde, 1994) – are the most commonly used for geometry education. Geometer's Sketchpad is the program that was utilized in the development and implementation of this synthesis project.

| Quad. *ABCD* | Monster #1: points A, D, and C are collinear | Monster #2: All four vertices are collinear | Monster #3: The quadrilateral crosses over itself |

Figure 1. Dynamic geometry programs can produce "monster" cases of a quadrilateral
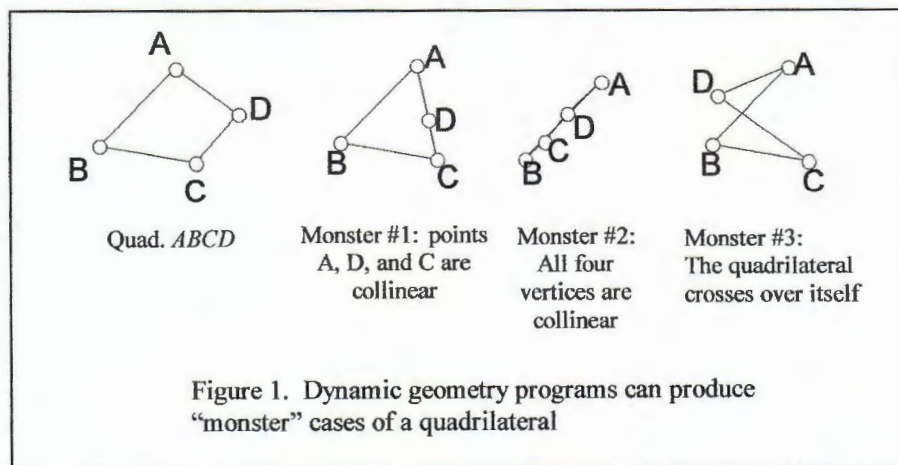
Students' Conceptual and Visualization Difficulties

> The amazing visual possibilities afforded by computers could lead to
> a belief that students could easily understand and conceptualize
> complex objects, in particular the notion of geometrical figure.
> (Laborde 1993, 60)

Colette Laborde, a respected French professor of mathematics, has been vocal in

the area of dynamic geometry software, presenting lectures and writing several articles.

Laborde believes that computers can enhance student conceptualization of geometric

objects. In "The computer as part of the learning environment: the case of geometry,"

Laborde (1993) clearly articulates a central conceptual difficulty in geometry and the

manner in which dynamic geometry software can aid in conceptual understanding by

distinguishing between *figures* and *drawings*.

Geometry is a study of abstract concepts. These concepts are understood to have

characteristics that are not representable in concrete models – planes continue without

end, lines have no thickness, circles are perfectly round, etc. Laborde refers to these

idealized concepts as "figures". The physical representations of these concepts do not

possess the aforementioned qualities. For example, planes must be drawn with edges,

lines are seen with thickness, and circles are sketched in a skewed manner. Laborde

labels these representations, which are typically paper sketches or compass and

straightedge constructions, as "material drawings" (1993, 52). Laborde notes that

"several difficulties arise because students actually work on *material drawings* while they

are expected to work on *figures* or on descriptions of figures" (1993, 52*)*. Students are

often unable to see the plane through the points, so to speak; they have difficulty

separating the limitations of the drawing from the ideal characteristics of the figure.

A second drawback to the use of material drawings is illustrated in the following example. A student is asked to investigate a point, labeled point C, which lies on a circle as is illustrated in figure 2. It is to be noted that in terms of the *figure*, point C may lie anywhere on the circle. However, in the *material drawing* above, point C is confined to one and only one
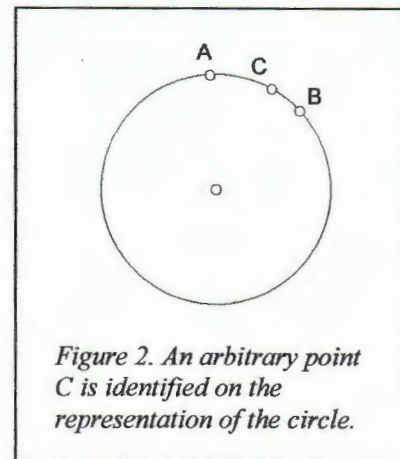
Figure 2. An arbitrary point C is identified on the representation of the circle.

location. The student, therefore, may easily be misled by details pertaining to this particular location (e.g., point C lies on minor arc AB) whereas the assignment requires that generalizations independent of the exact location of point C be formed. As Laborde states:

> The ambiguity of the drawing comes here from the impossibility of accounting through graphical means for the fact that C is *anywhere* on the circle. Usual geometric drawings do not express the variability of the elements of the figure and even less the range of the values of variable elements: there is only one position for C in the drawing, C must belong on the drawing to one of the circle segments. (1993, 51)

Once again, the physical limitations of the *material drawing* are creating a barrier to conceptual understanding of the *figure*.

Computer generated constructions, while still prone to some of the setbacks of material drawings, significantly close the gap between *material drawings* and *figures*. Dynamic geometry programs are able to minimize both of the previously mentioned drawbacks. First, computer generated constructions are able to depict many of the idealized qualities of a *figure*, although some qualities still cannot be represented. For example, although the circle is still represented on the computer screen as having width, the points on a computer-generated circle are all equidistant from a fixed center point.

Second, the dynamic nature of the computer program eliminates the erroneous fixation of points. Locations can be generalized. Laborde acknowledges that the precision and manipulability of computer sketches minimizes the shortcomings of material drawings, giving them a more figure-like quality. Dynamic geometry software, according to Laborde, creates an environment in which students are "often enabl[ed] to visualize geometric properties" (1993, 48), and encouraged to "conceive theoretical objects or relations" (1993, 60). "The amazing visual possibilities afforded by computers could lead to a belief that students could easily understand and conceptualize complex objects, in particular the notion of geometrical figure" (1993, 60).

Benefits and Implications of Computer Programs

A substantial amount of research has been completed in the area of dynamic geometry. In particular, Douglas Clements and Michael Battista (1994) have outlined several key benefits and educational implications of the use of dynamic geometry programs in the classroom. It is important to point out that these themes are not unique to Clements and Battista, but are echoed throughout the publications of the other cited authors. Each of these themes – precision, manipulability of sketches, explicitness in student thinking, visualization aid, an improved learning process, and motivation – is discussed below.

Precision: Computers generate accurate sketches. A computer creates straight lines, perfect circles, and exact midpoints. This is a major improvement over constructions done by hand. For example, the three angle bisectors of a triangle always meet in exactly one point. This is obvious in a sketch constructed with Geometer's

Figure 3. A computer-generated sketch clearly shows the one point of intersection of the angle bisectors of a triangle.

Figure 4. A pencil-and-paper construction does not accurately depict a single point of intersection.
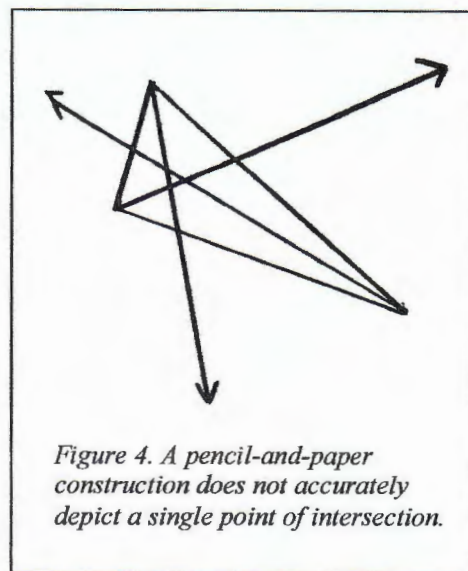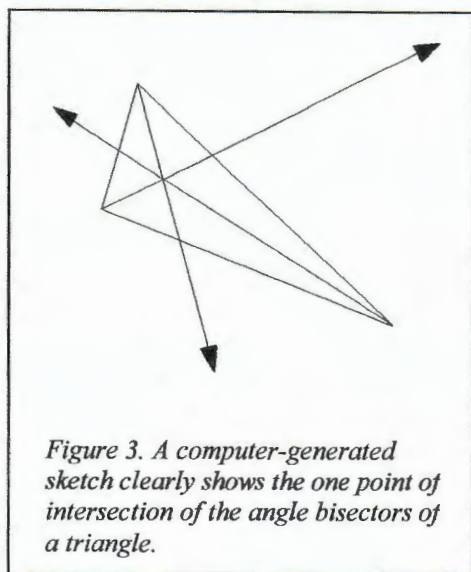
Sketchpad (figure 3). However, due to inaccuracies, a student's pencil and paper sketch

often shows three points of intersection as illustrated in figure 4. As Douglas Clements

and Michael Battista state, "Computer environments demand and thus facilitate precision

and exactness in geometric thinking. In contrast, when working with paper and pencil,

there is imprecision and students are distracted by the actual effort of the drawing"

(Clements and Battista 1994, 189).

Manipulability: Computer programs allow students to drag elements of the

sketches. In this manner, students can investigate multiple examples in order to establish

generalities applicable to a figure as opposed to characteristics specific to one drawing

(Clements and Battista 1994, Hazzan and Goldenberg 1996, Laborde 1993, Schwartz

1997[?]). Dynamic geometry programs "allow the manipulations of specific screen

objects in ways that assist students in viewing them as geometric (rather than visual)

objects and as representatives of a class of geometric objects. Such activities develop

students' ability to reflect on the properties of the class of objects and to think in a more

general and abstract manner" (Clements and Battista 1994, 188). For example, a student

could alter the shape of the triangle in figure 3, forming an acute triangle or a right triangle, and observe the effects of the manipulation on the point of intersection. Judah Schwartz (1997[?]) feels strongly that the manipulation of geometric sketches increases students' comprehension of geometric figures and concepts:

> By virtue of the fact that software environments can make the intangible interactively manipulable, and do so in several different representations simultaneously, the depth and breadth of understanding that can be reached by students (and teachers) is substantially enlarged. This expanded understanding comes about by virtue of the ability of the user of the technology to manipulate an abstraction in one representation and see the consequences of his or her actions simultaneously in several different representations. (online)

Students are provided the opportunity to form and test conjectures and to experiment with geometric concepts by manipulating the computer construction.

Explicitness: In order to construct a sketch with the aid of a computer program, students must provide an explicit description of the sketch. As Laborde observes, "A drawing on the screen is the result of a process performed by the user in which s/he makes explicit the definition of the figure" (Laborde 1993, 54). The computer cannot interpret vague descriptions or assume what the student is intending. Thus, students must be clear in their interaction with the computer. For this to be possible, students must also be clear in their personal understanding. Douglas Clements and Michael Battista view dynamic geometry environments as reflections of students' understanding. "The computer environments mirror students' geometric thinking. Researchers and teachers consistently report that in such context students cannot 'hide' what they do not understand" (Clements and Battista 1994, 189). By making descriptions explicit, students' interactions with the computer can help the student and the teacher identify areas of confusion or lack of understanding. For example, in order to construct an

altitude of a triangle, a student must first know the definition of the term and then be able to apply that definition. An altitude is a segment from a vertex of a triangle that is perpendicular to the line containing the opposite side of the triangle. In order to apply this definition, it is important that the student recognize that the altitude intersects the line containing the opposite side of the triangle, and not necessarily the side itself. If a student cannot construct an altitude, it is most likely due to the fact that the student does not fully comprehend the definition. It is commonly believed that if a person explains a concept to another individual, then that person will better understand and internalize the material himself. The computer plays the role of a second person by responding directly to the student's instructions.

Visualization: As discussed earlier in this chapter, dynamic geometry software allows students to see objects on the screen that they may not be able to visualize in their minds. If a student cannot visualize a figure, then it is extremely difficult, if not impossible, for that student to pose questions relating to the figure and to search for answers. Computers produce precise sketches of two- or three-dimensional objects with which the user can interact and which the user can investigate.

Learning process: Dynamic geometry software changes the manner in which geometry concepts are presented, shifting emphasis from the memorization of facts to the understanding of underlying ideas (Clements and Battista 1994, Schwartz 1997[?]). Dynamic geometry involves the student as an active, critical thinker, replacing the passive student that is often present in the traditional classroom. "Students can make conjectures, evaluate visual manifestations of those conjectures, and reformulate their thought. This seems to be essential for developing reasoning skills in geometry"

(Clements and Battista 1994, 187 – 188). The dynamic geometry classroom is an educational setting that is unfamiliar to most students who are more familiar with the traditional lecture and practice classroom. In a class that utilizes dynamic geometry software, students are expected to ask questions and search for answers. By providing opportunities for investigation and exploration, computer programs hold students accountable for their own learning and understanding, and thus encourage students to become life-long learners.

Hazzan and Goldenberg (1996) have also been vocal in their support of the use of dynamic geometry software in the classroom. "…[G]eometry software with or without the dynamic feature (used prudently) can help students focus their mathematical thinking on bigger and often more abstract mathematical ideas than is common in paper-and-pencil work" (Hazzan and Goldenberg 1996, 2) Two additional benefits, not discussed by Clements and Battista, are found in the works of Hazzan and Goldenberg (1996). These benefits – directed attention and motivation – are as important as the previously mentioned strengths.

Directed attention: When presented with a diagram, a student must decide which properties to explicitly recognize and which properties to ignore or overlook. This decision may be either a conscious or an unconscious one. The selection of properties affects which conclusions are drawn and the degree to which the conclusion can be generalized. Hazzan and Goldenberg state:

> …[C]onstruction on paper virtually forces one's attentions (at least initially) on the actual lengths and positions of objects. Geometry software designed for experimentation renders these features changeable, and leaves one to focus on more abstract relationships, and invariant (often qualitative) properties of geometric entities – properties that are

independent of the (often quantitative) particularities that one cannot easily avoid in a paper work. (1996, 2 - 3)

Geometry software, therefore, emphasizes qualities that can be applied to a general case while de-emphasizing characteristics specific to a given object. For this reason, conjectures are more easily formed, investigated, and supported by the student while working in a dynamic geometry environment.

Motivation: The use of computer programs and the opportunity to investigate geometric figures motivate students. "DGEs [Dynamic Geometry Environments] evoke some of the curiosity people have about how things 'work'" (Hazzan and Goldenberg 1996, 2–3). For many students, dynamic geometry is a motivation tool that inspires them to want to learn and to understand.

Drawbacks to Dynamic Geometry

Throughout the articles and books published in the field of dynamic geometry research, seldom does one find a drawback to the use of dynamic geometry software in the classroom. It is important to identify the specific context referenced, as there are many works published that discuss the negative aspects of general computer use in the classroom. Typically, these shortcomings refer to Internet use or education programs that drill students in a worksheet-like format. These comments are not relevant to dynamic geometry software which focuses on investigation.

Laborde does hesitate in her complete support of the use of computers in the geometry classroom by commenting that such programs are difficult to use *(1993, 59)*. The complex syntax required to communicate with the computer frustrates many students. As a result, these students are not motivated to investigate and do not benefit

from the computer experience. This disadvantage, however, has been greatly reduced in the eight years that have passed between the publication of the article and the present. By adding toolbars and straightforward commands to the computer programs, software programs have become increasingly easy to use to the point that this remark is no longer relevant.

A second drawback to the use of dynamic geometry is the students' tendencies to accept their work in the computer environment as sufficient proof. Enrique Galindo, a mathematics professor at Indiana University Bloomington, found that "students often view as proof the multiple examples that are generated easily and quickly in an interactive geometry environment and do not see a need to rely on formal reasoning in constructing proofs" (Galindo 1997, 228). In the absence of a computer-generated counterexample, students will often assume that a conjecture *must* be true and therefore does not necessitate any further justification. This, however, is not a shortcoming of the software programs but of the classes in which the programs are utilized. The teacher in a dynamic geometry classroom must recognize that the computer is a means to inspire, not replace, students' thinking. If the teacher insists on using the computer-generated examples as merely a foundation for further reasoning, the students will learn to seek additional justification for their conjectures.

Dynamic Geometry Software in Practice

Claudia Giamati is a mathematics professor at Northern Arizona University, where she utilizes the Geometer's Sketchpad in her classroom. Giamati believes that dynamic geometry software is a crucial element in the learning process. She describes a

lesson in which students were asked to determine a method for locating the center and the angle of a rotation and concludes:

> The use of the Geometer's Sketchpad in this type of exploration was invaluable. The students gained a deeper understanding of the problem by using their scripts to explore it and make conjectures than they would have if the results had merely been explained to them. (Giamati 1995, 457 – 458)

The computer program aided the students in comprehending the material as opposed to simply memorizing the information.

In addition to strengthening understanding, dynamic geometry programs provide students with the experience of seeing mathematics as a vibrant field to be created and expanded, not a static field to be merely studied and applied. "Explorations that give students the opportunity to make reasonable conjectures deepen the students' understanding of what it is to *do* mathematics" (Giamati 1995, 458). By "*doing* mathematics", Giamati is emphasizing a distinction between learning mathematics and exploring mathematics. A typical learning experience involves studying mathematical concepts that have previously been discovered, invented, or proven. While "*doing*" mathematics, students are invited to discover, invent, or prove original mathematical ideas. Judah Schwartz echoes this view:

> It has been found over and over again, and in many parts of the world, that if students are given tools that make it easy for them to explore conjectures, they will both make and explore conjectures. This means that they will have, in some measure, something of the experience of making mathematics. (Schwartz 1997[?])

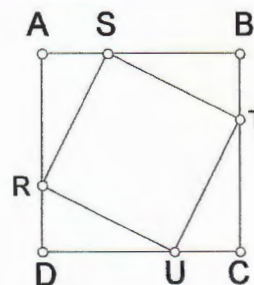This is exactly what happened when some high school sophomores made an original contribution

Figure 5. The sides of square ABCD have been divided such that AR = 2(RD), SB = 2(AS), TC = 2(BT), and DU = 2(UC). Quadrilateral RSTU is a square with area $\frac{5}{9}$ the area of ABCD.

to geometry with the aid of dynamic geometry software. "[S]ome 10th-grade students working with the Supposer discovered something that had not hitherto appeared in any geometry book.... Allowed to wander intellectually with the Supposer, our sophomore students devised a new piece of mathematics" (Schwartz 1992). The students began with a theorem that is proved in many geometry classrooms. This theorem states that if the sides of a square are divided in a 2:1 ratio as shown in figure 5, these four points can be joined to form a square with an area that is $\frac{5}{9}$ the area of the original square. The students used the Geometric Supposer to explore non-square quadrilaterals with sides divided in a 2:1 ratio. It was discovered that any quadrilateral with sides divided in a 2:1 ratio would produce a region with an area that is $\frac{5}{9}$ the area of the original quadrilateral. Although their work in the dynamic geometry environment did not constitute a proof, it did provide the students with an idea that they could then attempt to prove in a more deductive manner (Schwartz 1992). Dynamic geometry programs provide students with the opportunity to explore the role of mathematicians in addition to the opportunity to explore and deepen geometric knowledge.

NCTM Principles

The use of dynamic geometry software in the classroom is in accordance with two of the overarching principles of mathematics education recently published by the National Council of Teachers of Mathematics (NCTM). These principles are:

The learning principle: Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge. (NCTM 2000, 20)

The technology principle: Technology is essential in teaching and learning

mathematics; it influences the mathematics that is taught and enhances students'

learning. (NCTM 2000, 24)

As was discussed in the preceding section, dynamic computer programs allow

students to experience both the creation and the application of geometric concepts.

Through investigation and exploration, students can more fully comprehend concepts.

Dynamic geometry programs have the potential to enhance students' learning.

The use of such programs is an appropriate use as outlined by NCTM: "Students can

learn more mathematics more deeply with the appropriate use of technology.

Technology should not be used as a replacement for basic understandings and intuitions;

rather, it can and should be used to foster those understandings and intuitions" (NCTM

2000, 25). Dynamic geometry programs do not explicitly provide knowledge, but allows

students to explore and to investigate their own conjectures.

Applications of Research to Concurrency Project

The precision, explicitness, manipulability, and directed attention features of

dynamic software have the potential to enhance students' understanding of the points of

concurrency, whereas the visualization, motivation, and learning process benefits provide

students with tools to investigate the lessons. Precision is crucial in determining that the

special segments of a triangle do in fact intersect in exactly one point (refer to the earlier

example involving angle bisectors, figures 3 and 4 on page 14). Precision is also

necessary to establish the unique quality that each point of intersection possesses (for

example, the intersection point of the angle bisectors is the center of a circle inscribed in

the triangle). Manipulability allows students to generalize information pertaining to the intersection points by investigating all classifications of triangles – acute, obtuse, right, scalene, isosceles, and equilateral. Explicit descriptions require that students comprehend the definitions of the special segments and, in turn, the definitions of the intersection points. Directed attention de-emphasizes the consideration of specific lengths and types of triangles when the student forms generalized conjectures. Attention is instead focused on characteristics that remain constant when the sketch is altered, such as the ratio between the lengths of the segments formed on the median by the centroid. As a result of these features, students can potentially achieve a more complete understanding of the points of concurrency.

Conclusion

Dynamic geometry programs provide students with an opportunity for a deeper understanding of geometric concepts. Through their design, the programs offer precise representations of geometric concepts to aid in students' visualization. By allowing the manipulation of the computer-generated sketches, dynamic geometry programs permit the investigation of countless related examples and thereby encourage the formation of conjectures.

Computer programs enhance students' understanding by creating an environment that encourages learning. This is a motivating environment where the student is an active participant in his or her own education. It is also an environment that requires explicitness on the part of a student's commands, which in turn can clarify a student's understanding or expose a student's misunderstanding of a concept. As a result of their

design and the learning environment they generate, dynamic geometry programs are invaluable tools that can potentially augment students' understanding.

CHAPTER THREE

THE POINTS OF CONCURRENCY CURRICULUM UNIT

> A lesson that exploits the use of technology enables students to
> participate in constructing their own knowledge. Students might be
> involved in running experiments, testing conjectures, solving and
> posing problems, and exchanging ideas.... During the activity,
> questions should direct students to analyze and synthesize their
> observations. The atmosphere should encourage communication
> between students – questioning, suggesting, challenging, listening,
> and striving to develop convincing arguments for beliefs and
> conjectures. (Schoaff 1993, 21)

As the focal point of my synthesis project, I have designed and taught a

curriculum unit on the points of concurrency of a triangle. The lessons are structured

around the use of dynamic geometry software, specifically the Geometer's Sketchpad

(Jackiw 1995) This chapter provides an outline of the lessons that comprise the unit.

The curriculum unit covers the special segments of a triangle and their points of

concurrency. Every triangle has four types of special segments: medians, altitudes,

perpendicular bisectors, and angle bisectors. Each type of segment has three

representations in any given triangle. In each case these three representations, or in the

case of the altitudes, the lines containing the three representations, are concurrent. For

example, any triangle will have three medians and these three medians will intersect in

exactly one point. Each of these four intersection points, known as the points of

concurrency, has unique characteristics that distinguish it from the other three points of

concurrency.

The unit has been designed for implementation in a geometry elective at a

suburban high school. The elective, Geometry in Action, is structured around active

student learning through computer exploration of geometric concepts. Critical and creative thinking skills are emphasized throughout the course, as students are encouraged to explore, form conjectures, and support their conjectures using the computer program and their peers for assistance.

Geometry in Action is designed to be an informal look at geometric concepts. It is intended to supplement the standard geometry course that is taught at the sophomore level in the mathematics curriculum. The elective does not provide the geometry credit required of students. Consequently, although students are expected to provide justifications of their conjectures, Geometry in Action is primarily an intuitive course. The rigor of formal proof is reserved for the standard geometry class.

Geometry in Action is a newly introduced course that was offered to students for the first time in the fall 2000 semester. The elective is open to all students enrolled at the high school, regardless of previous mathematics experience. As a result, this first class is a cross section of the school student body. The class is composed of seven students: three freshmen, two sophomores, one junior, and one senior. One of the seven students is also enrolled in accelerated math courses and two others are honor level mathematics students. Five of the seven students have not previously studied geometry at the high school level, while the remaining two students are currently enrolled in geometry classes. When asked what their primary motivation was for selecting the class, five students voiced an interest in mathematics and a desire to study geometry. The remaining two students admitted that they needed a credit and enrolled in the class by default.

<u>An overview of the unit</u>

By focusing on student investigations, the unit increases the potential for student understanding of the segments, the points of concurrency, and their properties as opposed to rote memorization. It is the role of the computer software to provide the tools necessary for students to form accurate conjectures in the course of their investigations. All lessons in the unit require the use of dynamic geometry software, such as Geometer's Sketchpad. Additional materials needed are listed under the appropriate lesson. Built into this curriculum are the following aspects:

Conceptual Emphases:

- ☞ Students will be able to define and construct the four types of special segments of a triangle.
- ☞ Students will recognize that the three representations of the median, the angle bisector, and the perpendicular bisector of a triangle are concurrent, and that the lines containing the three altitudes of a triangle are concurrent.
- ☞ Students will be able to define and construct the four points of concurrency.
- ☞ Students will apply the unique characteristics of the points of concurrency.

NCTM Standards: This unit is also aligned with the following National Council of Teachers of Mathematics (NCTM) standards:

Geometry Standard

Instructional programs should enable students to:

- ☞ Analyze properties and determine attributes of two-dimensional objects

- Explore relationships among classes of two-dimensional objects, make and test conjectures about them, and solve problems involving them
- Establish the validity of geometric conjectures using deduction, prove theorems, and critique arguments made by others
- Draw and construct representations of two-dimensional geometric objects using a variety of tools (NCTM 2000, 308)

Reasoning and Proof Standard

Instructional programs should enable students to:

- Make and investigate mathematical conjectures
- Develop and evaluate mathematical arguments and proofs (NCTM 2000, 342)

Communication Standard

Instructional programs should enable students to:

- Organize and consolidate their mathematical thinking through communication
- Communicate their mathematical thinking coherently and clearly to peers and teachers
- Analyze and evaluate the mathematical thinking and strategies of others
- Use the language of mathematics to express mathematical ideas precisely (NCTM 2000, 348)

Connections Standard

Instructional programs should enable students to:

- Recognize and use connections among mathematical ideas

- Understand how mathematical ideas interconnect and build on one another to produce a coherent whole

- Recognize and apply mathematics in contexts outside of mathematics (NCTM 2000, 354)

Lesson one: Defining "median" and "altitude" of triangles

Conceptual Emphasis: Students will formulate definitions for "median" and "altitude" as the terms apply to triangles.

Key Thinking Processes: Brainstorming, analogizing, identifying characteristics, decision-making

Activity:

1. Brainstorming & Analogizing: Working in small groups, the class brainstorms possible definitions for the terms "median of a triangle" and "altitude of a triangle". The teacher encourages students to draw upon prior knowledge of medians and altitudes in non-geometry settings and to make analogies between previous uses of the terms and the current applications. The teacher might ask, "Where, outside of this class, have you used the terms 'median' and 'altitude'? Which aspects of these non-geometry definitions could apply to the median or altitude of a triangle?" When presented with this activity, F.R. wrote the following response regarding medians:

    "1. Median of a triangle

    2. Median of a number line

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3. Median of a road

Each of these definitions describes something in the middle which evenly

separates something else."

After several minutes, groups "jigsaw". Using this cooperative learning strategy,

one or two members of each group stays at their assigned table while the

remaining students in the group move to other tables, forming new groups. Each

newly formed group compares and builds upon brainstorm lists.

2. Identify Characteristics: Once groups have completed brainstorming, students

move to the computers, where the teacher has constructed and saved two sketches.

The first sketch contains a triangle and one median (figure 6). The second sketch

contains a triangle and one altitude (figure 7). Students use these sketches as

tools in the formation of the definitions of each term. Students investigate the

Segment AD is a median
of triangle ABC.
Determine the definition of
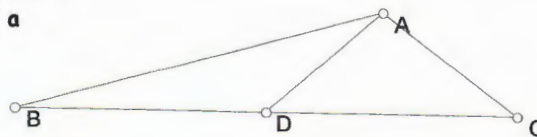the term "median of a
triangle".

Figure 6

Segment LO is an
altitude of triangle
LMN. Determine the
definition of the term
"altitude of a
triangle".

Figure 7

segments by dragging the vertices of each triangle and observing the results on the sketches. These manipulations allow students to identify the defining characteristics of the median and altitude. For example, although J.Z. initially believed that the altitude was a segment "from the vertex to the other side of the triangle", as she altered the sketch represented by figure 7 she soon observed that $\overline{LO}$ does not necessarily intersect $\overline{MN}$. However, J.Z. did soon recognize that the segment $\overline{LO}$ continually intersected the line containing $\overline{MN}$. She then identified "The altitude intersects the line containing the opposite side" as a key characteristic of the altitude of a triangle.

3. <u>Define and Decide</u>: Based on their observations, groups refine their brainstorming and articulate a clear definition for each of the terms. Groups decide which of their previous ideas are best supported by the characteristics identified in working with the sketchpad and how these ideas can be enhanced in response to their observations.

4. <u>Share & Refine:</u> Each group shares its definition with the class. Students then engage in a discussion focused on synthesizing the various definitions and refining the language of the definitions. The teacher plays a minor role at this point, allowing the discussion and the ideas to be student-generated. Once the class as a whole agrees on a definition for each term, the definitions are to be included in each student's notebook.

Wrap-up activity: Students write a letter to Noah Webster in an attempt to convince him to accept their definitions for altitude and median of a triangle. The letters must be

persuasive, explaining why the student believes Webster should include his/her definition in Webster's dictionary.

Reflections on the implementation of the lesson:

Students seemed to have a clearer understanding of the terms "median" and "altitude" after this lesson, compared to students in a traditional class who are given the definitions by the teacher or the textbook. I base that observation on my experiences over the last nine years using traditional methods of teaching these concepts. On average, the students in Geometry in Action could more readily apply the definitions of median and altitude and correctly use the vocabulary in their classroom conversations. In the past, many students have needed time to stop and refer to definitions in their notes before applying definitions for median and altitude. Former students also have had difficulty understanding that an altitude intersects the line containing the opposite side, and not necessarily the side itself. This misunderstanding causes problems when trying to construct the altitudes of an obtuse triangle, as is seen in figure 8. All seven of the Geometry in Action students recognized this concept. Particularly significant was that the class quickly refined its first definition, "The altitude of a triangle is a segment from a vertex of a triangle, perpendicular to the opposite side" to include the phrase "perpendicular to *the line containing* the opposite side" without any prompting from the teacher.

Figure 8. Two of the altitudes in an obtuse triangle fall outside of the triangle, intersecting the line but not the side itself.
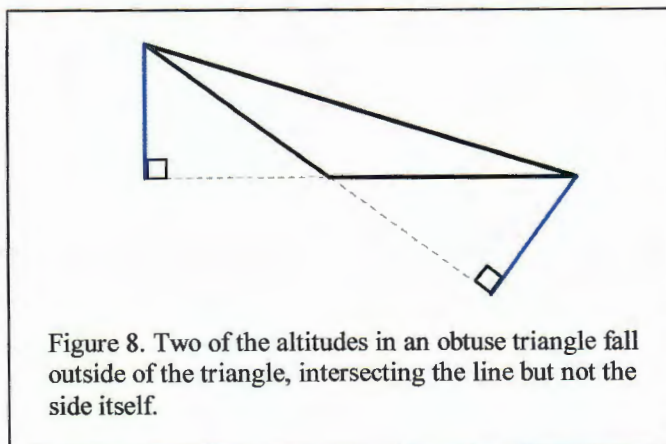
The Geometry in Action students were able to accurately define the median of a triangle without much difficulty. The class did, however, struggle with the definition of an altitude of a triangle. The obstacle was caused by the formation of an inappropriate analogy with a previous definition. The students recognized altitude as the height of an object. The students then found it difficult to define the altitude in terms of a segment and not in terms of a distance. This was apparent both in the many wordings and rewordings of their definition and in a later activity that asked the students to construct the altitudes of a triangle. Students did not know where to begin the construction since they believed they were constructing a distance as opposed to a segment. After constructing the altitudes, students were able to recognize their misunderstanding and understand that the altitude referred to a segment.

Lesson two: An investigation of medians

Conceptual Emphases: Students will determine the following properties of medians:

- ☞ Every triangle has three medians.

- ☞ The medians of a triangle are concurrent.

- ☞ The point of concurrency lies in the interior of the triangle.

- ☞ The distance between the vertex and the point of concurrency is twice the distance between the point of concurrency and the opposite midpoint.

Key Thinking Processes: Making observations, identifying characteristics, forming of conjectures

Activity: Either by sharing a computer or using neighboring computers, students work in pairs to investigate medians of triangles.

Figure 9. A triangle and its three medians
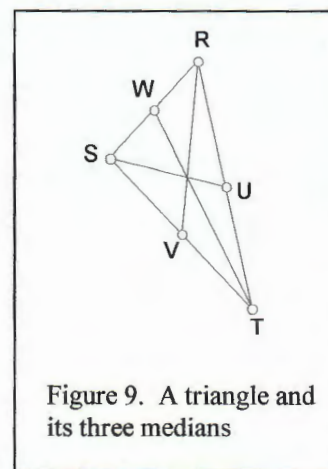
1. Construct, Observe & Identify: Each pair of students constructs an arbitrary triangle on the computer and determines the number of medians belonging to the triangle. Each group constructs all possible median(s) of their triangle (figure 9) and explores their sketch, making observations regarding the medians of the triangle. The teacher encourages students to change the size and shape of the triangle by dragging one or more vertices, and to measure any segments the pair feels is appropriate. Students observe the effect of each change on their sketch. Through these observations, they identify the important characteristics of the medians of a triangle

2. Forming Conjectures: Students use the characteristics identified in the previous step to formulate conjectures regarding the median(s) of a triangle. Conjectures are expressed in general terms, referring to all triangles or to specific types of triangles. For example, within his sketch (figure 10), O.F. stated that "the segment from the vertex to the intersection of the medians is always twice the length of the segment on that line from the intersection to the opposite line." This is an acceptable conjecture, whereas "AG = 2 GD" would not have been acceptable since it refers to the labels given in one particular diagram and it does not articulate the underlying concept. Some students have difficulty with the proper wording of a conjecture. The teacher needs to observe each group closely

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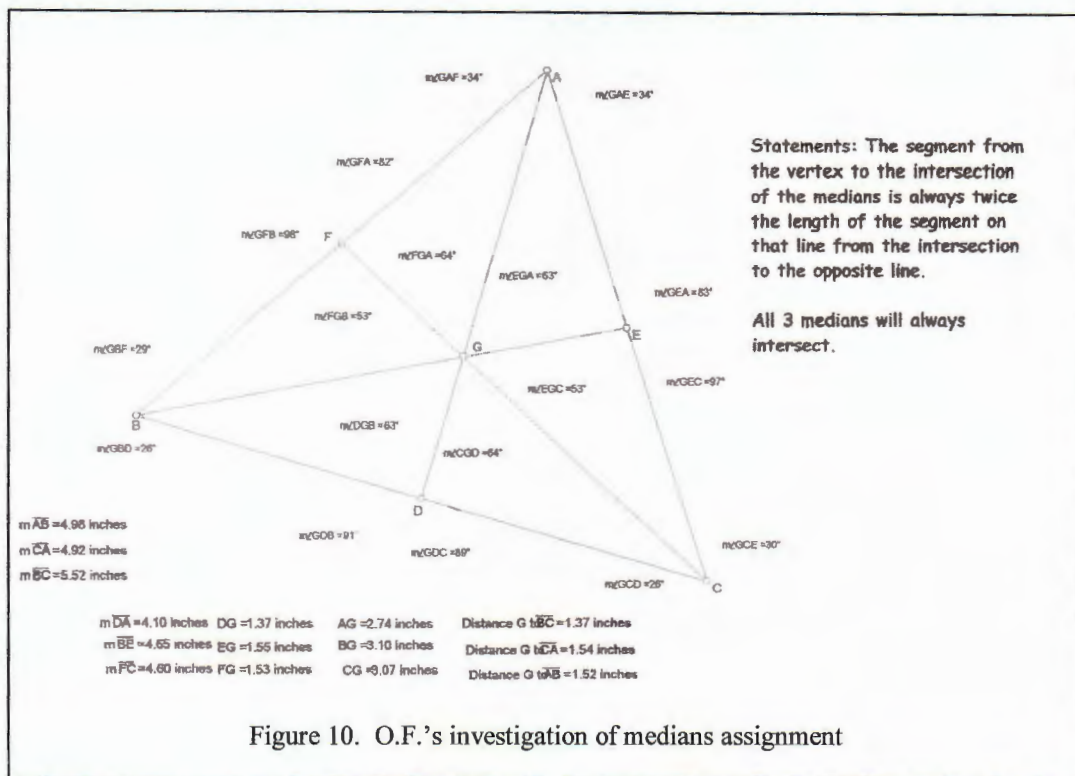
Figure 10. O.F.'s investigation of medians assignment

and offer assistance where necessary. The teacher may ask questions such as "What is the underlying concept you are trying to convey?" or "In general terms, what does this mean?" to encourage students to articulate conjectures as opposed to reporting observations. The teacher may also move groups to new computers where they must align their conjecture with another group's sketch. This will encourage students to write conjectures in more general terms.

3. Share & Compare: Each pair in turn records one of their conjectures on the board and elicits feedback from the class by asking if other groups observed anything similar or anything contradictory. The pair leads a discussion in which the conjecture is discussed, refined, and ultimately approved or rejected by the class. If the conjecture is approved, the statement is added to students' notebooks. The remaining pairs continue in this manner until all conjectures have been presented.

Once again, the teacher's role in this activity should be minimal. The discussion, the ideas, and the questions are to be student-generated. The teacher acts more like a guide and a safety net than an active participant or a leader. The teacher joins the dialogue only to provide guiding questions or comments if the discussion begins to stall or wander or to encourage students to more fully address a topic or thought process. The teacher may, for example, ask how the group arrived at a particular conjecture or what motivated them to measure specific segments. Ultimately, however, it is the teacher's responsibility to ensure that information copied into notebooks is, in fact, correct and that students have effectively justified the approval or rejection of the conjecture.

Wrap-up activity: Students are asked to answer the following questions:

What strategies did you use to investigate the medians?

What surprises did you encounter?

Did anything not work out the way you expected? If so, explain what occurred and why.

Responses can be written or discussed orally. If time permits, the latter forum would allow students the opportunity to listen to, learn from, and build upon each other's comments.

Reflections on the implementation of the lesson:

The lesson successfully provided the students with the intended concepts. The students easily identified the key properties of the median with little or no assistance. By

having the students construct the medians of a triangle, the activity also reinforced the definition of a median from the previous lesson.

The students easily recognized the four properties that were identified as the conceptual emphases of the lesson. When J. Z. noted that the three medians "came together", the other students quickly echoed the concept, refining the language. Once the students had agreed that "the three medians of a triangle intersect in one point" I provided the term "concurrent" and illustrated how the term is applied to this statement. O. A. elaborated on the preceding statement by observing that "the point of intersection is always inside the triangle". The other students agreed that they had also observed the property. O. F. remarked that each median was divided into two segments with lengths in a ratio of 1:2. This surprised and intrigued the other students. F. R. became interested in the six non-overlapping triangles formed by the medians. He felt they were the same somehow, but was unable to be more specific. This discussion provided an introduction to the following day's lesson.

One drawback of the lesson was the lack of computer time available after the students shared their observations and conjectures. Students were surprised by O. F.'s conjecture relating the lengths of the segments on the medians. Follow-up computer time would allow each of the students to investigate this property. The time could also have been used to investigate F. R.'s comments concerning the six triangles formed by the medians. This oversight will be corrected in subsequent classes.

Lesson three: Balancing on the centroid

Conceptual Emphasis: Students will identify the point of concurrency of the medians as the centroid, the center of gravity on which the triangular region can be balanced.

Key Thinking Processes: Making observations, brainstorming, justifying; seeing relationships

Additional Materials: card stock or a similar heavy weight paper, scissors, compasses, straightedges

Activity:

1. Construct (paper) and Observe: With the aid of compasses and straightedges, students construct a triangle and its three medians on a piece of card stock. The triangles are then cut out of the paper. The teacher directs the students to place the tip of their pens or pencils directly under the point of concurrency and to balance the triangular region. If the medians were constructed accurately, the triangular regions will balance.

2. Name & Brainstorm: The teacher names the point of intersection of the medians the "centroid", noting that the centroid is the center of gravity of the triangular region. The teacher asks why the point of concurrency is the center of gravity; what allows the triangular region to balance on this specific point? The class brainstorms ideas.

3. Construct (computer), Observe, & Justify: Using a computer program, students construct an arbitrary triangle and its three medians. The teacher directs students' attention to the six smaller non-overlapping triangular regions formed by the

medians (figure 11). They measure the area of each triangular region and make an observation(s) regarding the areas of the triangles. Students should notice that the areas of the regions are equal and provide a justification for this observation. Within a class discussion aimed at identifying why the areas are equal,

Area I = 0.3 in^2 Area IV = 0.3 in^2
Area II = 0.3 in^2 Area V = 0.3 in^2
Area III = 0.3 in^2 Area VI = 0.3 in^2

Figure 11. The areas of the six triangles formed by the medians are equal.
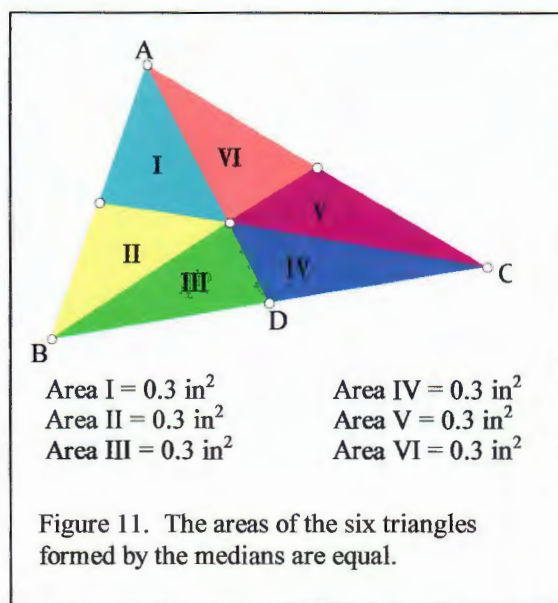
the teacher elicits from the students the formula for the area of a triangle (Area = ½ base times height) and the definitions of the terms "base" and "height" as they are used in the formula. It may be beneficial for the students to construct the altitude from the centroid in each of the six triangles and to calculate the area of each triangle using the length of the altitude and the length of the side to which it is perpendicular. Students at this point should be able to show that ΔABD and ΔADC have the same area since their bases are equal and they share an altitude. For the same reason, the areas of ΔIII and ΔIV are equal, as are the areas of ΔV and ΔVI and the areas of ΔI and ΔII. By way of substitution and subtraction, the class can justify the equality of the six areas.
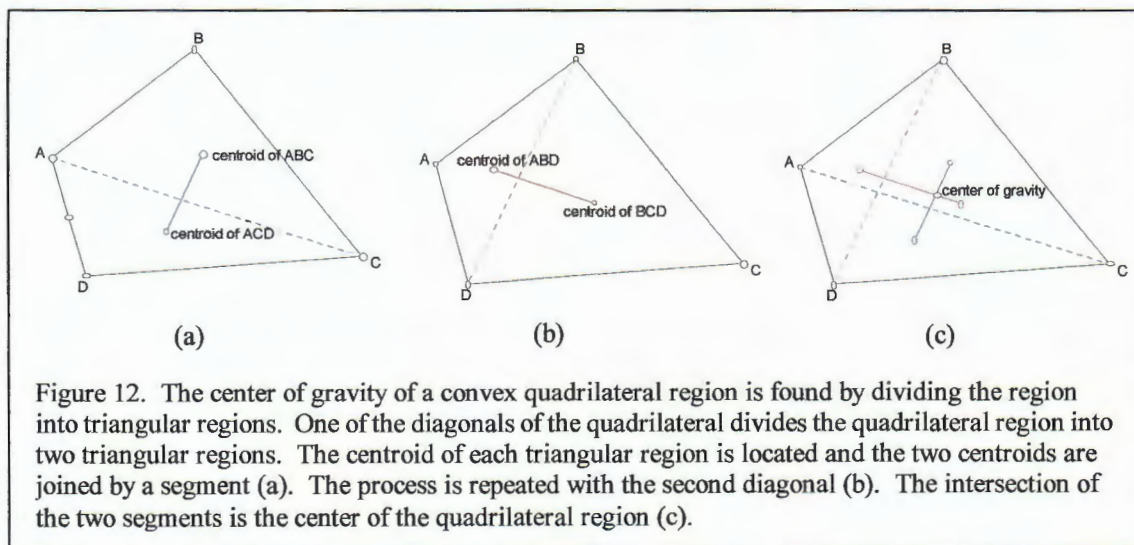
4. <u>Relating Area to Balance:</u> The teacher asks students to explain the relationship between the equal areas and the point of concurrency being the centroid.

Wrap-up activity: Students apply the concepts from this lesson to quadrilaterals by developing a method for locating the center of gravity of a convex quadrilateral region.

The activity can be completed by compass and straight edge on card stock or on the computer depending upon students' preferences and/or materials available. Through investigations, students should recognize that the center of gravity of a convex quadrilateral region is found by dividing the region into triangular regions. One of the diagonals of the quadrilateral divides the quadrilateral region into two triangular regions. The centroid of each triangular region is located and the two centroids are joined by a segment (figure 12a). The process is repeated with the second diagonal (figure 12b). The intersection of the two segments is the center of the quadrilateral region (figure 12c) (Shilgalis and Benson 2001, 303-304).

Reflections on the implementation of the lesson:

The lesson satisfied the learning styles of both the visual learners and the kinesthetic learners in the class. The lesson allowed the students to see the equal areas on the Sketchpad and to experience the center of gravity while physically balancing the triangle. The balancing activity by itself is effective in conveying to students the defining characteristic of the centroid. The sketchpad project, however, supplements this activity

Figure 12. The center of gravity of a convex quadrilateral region is found by dividing the region into triangular regions. One of the diagonals of the quadrilateral divides the quadrilateral region into two triangular regions. The centroid of each triangular region is located and the two centroids are joined by a segment (a). The process is repeated with the second diagonal (b). The intersection of the two segments is the center of the quadrilateral region (c).

by providing the foundations of a justification for the property. The computer can more accurately calculate the area of the triangles than the student working with a ruler, emphasizing the fact that the areas are equal.

It is important for students to recognize that the triangles formed by the medians are not necessarily congruent. I did not explicitly discuss this with the class, believing that the observation was obvious. However, O. F. did not initially recognize this and, believing the triangles were congruent, he spent a considerable amount of time trying to prove the triangles congruent in order to demonstrate that the areas are equal. I would amend this lesson by asking students to determine which case(s) produce congruent triangles.

Lesson four: An investigation of altitudes

Conceptual Emphases: Students will determine the following properties of altitudes:

- ➡ Every triangle has three altitudes. In an acute triangle, the three altitudes lie in the interior of the triangle. In a right triangle, two of the altitudes lie on the triangle. In an obtuse triangle, two of the altitudes lie in the exterior of the triangle.
- ➡ The lines containing the three altitudes are concurrent.
- ➡ The point of concurrency lies in the interior of the triangle if the triangle is acute, on the triangle if the triangle right, or in the exterior of the triangle if the triangle is obtuse.

Key Thinking Skills: Making observations, identifying characteristics, forming conjectures

Activity: In a manner similar to that outlined in activity two, students work in pairs to observe altitudes of triangles and to form conjectures relating to altitudes. Conjectures are presented to the class and discussed. The teacher refers to the point of concurrency of the lines containing the altitudes as the "orthocenter." Students also answer the following questions:

1. When, if ever, is the median of a triangle also the altitude?
2. Is it possible to have exactly three medians also be altitudes? Exactly two? Exactly one?

Students should recognize and be able to prove that the median will also be the altitude if it is drawn from the vertex angle to the base of an isosceles triangle. They should also be able to prove that exactly one median coincides with an altitude if the triangle is isosceles, but not equilateral, whereas all three medians coincide with altitudes if the triangle is equilateral. It is not possible to have exactly two medians in a triangle that are also altitudes.

Wrap-up activity: Students are asked to answer the following question (adapted from Serra 1997, 164):

A large triangular plot of land was left to a brother and sister by their father. His will states that the property is to be divided in two plots of land by the altitude from the northernmost point of the property for the heirs. However, that end of the property is covered with quicksand at the northern vertex. The will stipulates that the heir who figures out how to draw the altitude without using the northern vertex gets to choose his or her parcel first. Can you determine a method for locating the altitude? Do you think

this is a fair way to divide the land? Why or why not? (The answer to the question is provided at the end of the chapter.)

Reflections on the implementation of the lesson:

The activity supplied a needed reinforcement of the definition of altitudes. The sketchpad project made apparent students' misconceptions concerning altitudes. As previously stated in the reflections to activity one, the majority of the students mistakenly identified the height of a triangle as the altitude. The influence of this inappropriate definition, illustrated in F.R.'s comments on his assignment (figure 13), resulted in two obstacles. First, the students found the sketch difficult to construct since they were attempting to construct a distance instead of a segment. Second, it was believed that only one altitude existed in a triangle. Students were surprised to learn that three altitudes

F.R.'s observations:
The altitude of a triangle can be either inside or in the outer region of the triangle. It will show the shortest distance from the segment to the opposite angle. All three of the altitudes will intersect at the same point which wouldn't nessasarily be inside the triangle.

Figure 13. F. R.'s comments on his altitude sketch reflect the student's previous definition of the altitude as a distance.

exist in any triangle. By resolving the difference in definitions, students were able to surmount each of the obstacles. Similar difficulties did not arise in subsequent activities. Students were then able to recognize the altitude as a segment.

When sharing conjectures with the class, J. Z. observed that one altitude always remained in the interior of the triangle, regardless of the shape of the triangle (figure 14).

J.Z.'s observations:

1. All the altitudes intersect at one point, and they will always intersect because of the way they are constructed.
2. There is always one altitude left inside the triangle.

Figure 14. J.Z.'s altitude sketch and comments

O.F.'s observations:

If an angle is obtuse, then the altitudes not connected to it will be outside the triangle. However, because there can only be 1 obtuse angle, 1 must always be inside at least.

They will not always connect within the triangle. Opposite triangles not necessarily be congruent.

If any, only 2 altitudes can be outside the triangle.

Figure 15. O. F.'s alitude sketch and comments

She continued by erroneously stating that the altitudes always intersected in exactly one point, but that point was not necessarily inside the triangle. O. F. elaborated upon this by outlining the locations of the altitudes in acute, right, and obtuse triangles (figure 15).

He, too, agreed that the altitudes were concurrent and that the point of concurrency was sometimes in the interior of the triangle, sometimes on the triangle, and sometimes outside the triangle. None of the students questioned the concurrency of the altitudes. By asking students to sketch an obtuse triangle and its three altitudes, I was able to elicit from the students the observation that the altitudes themselves did not necessarily intersect. The previous comments were then clarified, stating that the lines containing the altitudes were concurrent.

Lesson five: Anticipating properties of the perpendicular bisectors and the angle bisectors

Conceptal Emphases: Students will determine the following properties of the perpendicular bisectors of the sides of a triangle and the bisectors of the angles of a triangle:

- ☞ Every triangle has three perpendicular bisectors.
- ☞ The perpendicular bisectors are concurrent. The point of concurrency lies in the interior of an acute triangle, on the triangle itself if the triangle is right, and in the exterior of an obtuse triangle.
- ☞ Every triangle has three angle bisectors.
- ☞ The angle bisectors are concurrent. The point of concurrency lies in the interior of the triangle.

Key Thinking Processes: Predicting; identifying characteristics

Prerequisite Knowledge: A perpendicular bisector of a side of a segment is a line that is perpendicular to the segment at its midpoint. Although the perpendicular bisector is a line, it is grouped under the heading "special segments of a triangle" for convenience of terminology. An angle bisector of a triangle is a segment contained within the ray that divides the angle into two congruent halves. The segment has one endpoint at the vertex of the angle being bisected and the second endpoint on the opposite side of the triangle.

Activity:

1. Predicting: The teacher organizes the class into groups of 3 or 4 students. Half of the groups in the class are directed to discuss angle bisectors; the remaining groups are assigned perpendicular bisectors. Each group forms a preliminary list of conjectures they predict will be observed when they construct their sketches of either the bisectors of the angles of a triangle or the perpendicular bisectors of the sides of a triangle. Groups should consider earlier investigations of medians and altitudes and prior studies of angle bisectors and perpendicular bisectors when forming these conjectures.

2. Share: Each group lists their conjectures on the board. Students are required to justify why they expect to observe the predicted results.

3. Identify characteristics: With the aid of a computer, each student, working individually or in pairs, constructs a triangle and its angle bisectors and a second triangle and its perpendicular bisectors. Students use the sketches to identify the

characteristics of the bisectors. In doing so, students verify or refute the predicted conjectures.

Wrap-up Activity: Students report individual findings back to the class. The teacher asks students to explain which conjectures did or did not work. The teacher should also ask groups to discuss the thought process that lead to the conjectures and to describe their investigations. It must be made clear to the students that the process is as important as the product. If any conjectures are disproved, the class explains why the rejected conjecture(s) is/are not valid and investigate where an error in reasoning was made.

Reflections on the implementation of the lesson:

The class had previously studied angle bisectors and perpendicular bisectors independently of triangles. Since the concepts are familiar to the students, the angle bisector and perpendicular bisectors were combined into one lesson. Students appeared to be comfortable discussing both concepts in relation to triangles within the same lesson.

Students have become accustomed to the structure of the course. When divided into groups to predict the characteristics of the bisectors, members of both groups searched for construction aides without hesitation. J. Z. and O. A. logged onto a computer accessing the Geometer's Sketchpad. O. F. found a compass and straightedge and began sketching on paper. The students appear to recognize a need for accurate sketches in the conjecture-forming process.

The free-hand sketches of perpendicular bisectors produced by K. N. during the first part of this lesson showed an understanding that is often lacking in geometry students. Geometry students frequently assume that the perpendicular bisector of a side
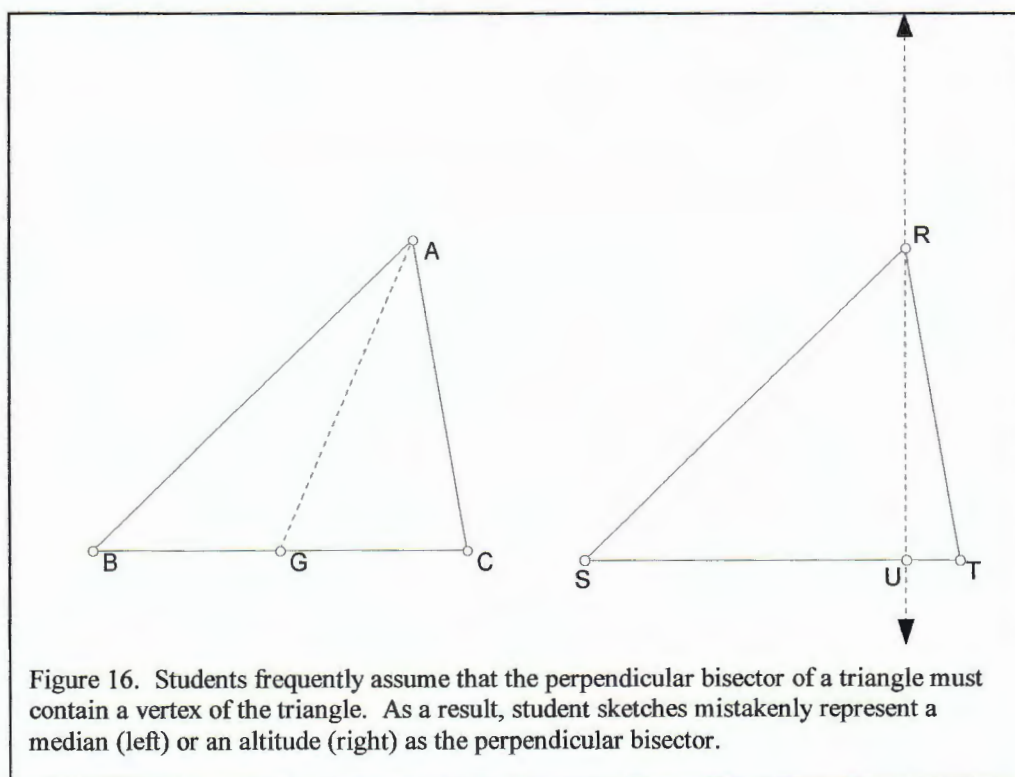
of a triangle must contain the opposite vertex. Although this situation does occur in an isosceles triangle, this is a special case and should not be generalized to all triangles. As a result, their sketches of perpendicular bisectors mistakenly represent a segment that is not perpendicular to a side of the triangle or a segment that does not bisect the side (figure 16). On a consistent basis, however, K.N. accurately depicted a perpendicular bisector that did not contain a vertex reflecting a comprehension of the definition of the segment. The remaining class members followed K.N.'s lead, also exhibiting an understanding that the perpendicular bisector does not necessarily contain a vertex. This understanding was also displayed in the class discussion:

> Mrs. Mignini: "What do you have to say about perpendicular bisectors?"
>
> O. F.: "They all intersect at one point. And, it's kinda like the altitudes."
>
> Mrs. Mignini: "What do you mean by that?"

Figure 16. Students frequently assume that the perpendicular bisector of a triangle must contain a vertex of the triangle. As a result, student sketches mistakenly represent a median (left) or an altitude (right) as the perpendicular bisector.

O. F. : "In the sense that they're basically...basically the altitudes are from a vertex and they must be perpendicular to a line except these have to be perpendicular to a line and it doesn't matter where they go."

Mrs. Mignini: "What do you mean?"

O. F.: "It doesn't matter where their endpoint is. They're kinda the same in that they're perpendicular to a side."

I concluded the discussion with a question: Is it possible for the perpendicular bisector to contain a vertex? The students quickly recognized that a vertex would lie on the perpendicular bisector if the triangle were isosceles.

Lesson six: Naming the circumcenter

Conceptual Emphasis: Students will identify the point of concurrency of the perpendicular bisectors of a triangle as the circumcenter, the center of the circumscribed triangle.

Key Thinking Processes: Seeing relationships

Prerequisite Knowledge: Perpendicular Bisector Theorem (Any point on the perpendicular bisector of a segment is equidistant from the endpoints of the segment.)

Activity:

1. Seeing relationships in definitions: The instructor names the point of intersection of the perpendicular bisectors of a triangle the "circumcenter." Small groups of students determine the defining property of the circumcenter by relating the term

to the meaning and usage of the prefix and the root of the word. To aid in their definition, students should answer the following questions:

What does the prefix "circum-" mean?

When is the term "center" used in geometry?

2. <u>Verify:</u> Students use the computer software to verify their proposed definition. Completed sketches contain an arbitrary triangle, its three perpendicular bisectors, and the circumscribed circle (figure 17).

3. <u>Seeing relationships in properties:</u> Once groups recognize that the circumcenter is the center of the circumscribed circle (also referred to as a circumcircle), the students determine why the point of concurrency of the perpendicular bisectors and the circumcenter are in fact the same point. They are expected to refer back to previous work completed with the perpendicular bisectors, specifically the Perpendicular Bisector Theorem that states any point on the perpendicular bisector of a segment is equidistant from the endpoints of the segment, and relate the theorem to the current situation.
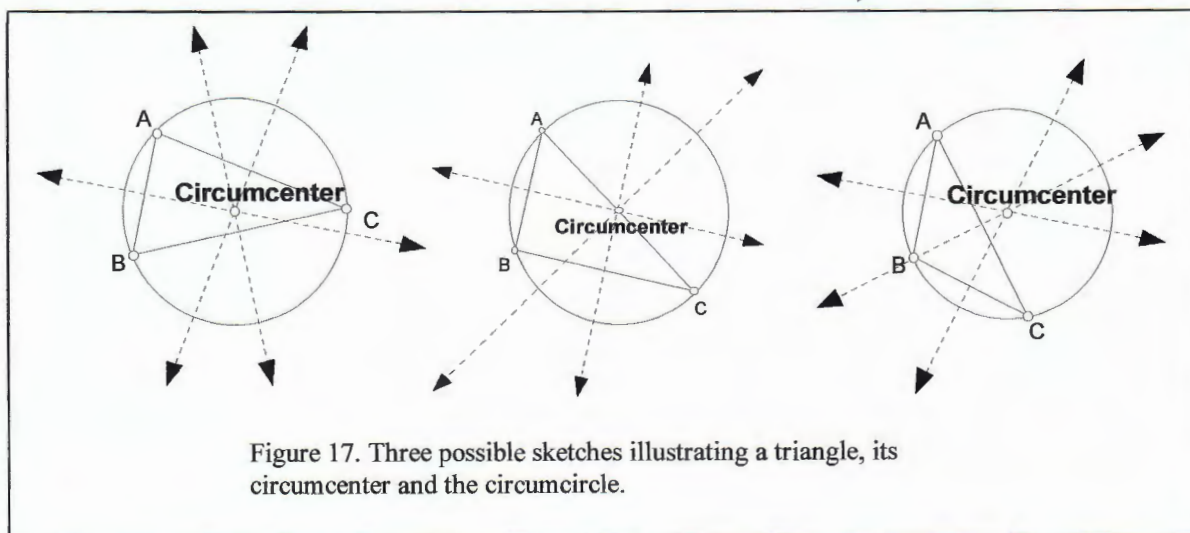

Figure 17. Three possible sketches illustrating a triangle, its circumcenter and the circumcircle.

Wrap-up activity: Students submit a written explanation of why the point of concurrency of the perpendicular bisectors and the circumcenter are the same point. Students describe how such a property can be useful by citing examples where the circumcenter would be necessary. Examples are to be both construction-related and "real-world" applications.

Reflections on the implementation of the lesson:

The use of the computer program in this lesson allows students to visualize the defining property of the circumcenter. The dynamic geometry program provides the student with an accurate representation of the circumcircle of the triangle. It has been my experience that students who construct the circumcircle using a compass, straightedge, and/or other construction tools do not take the care and effort to produce an accurate construction. As a result, the constructed circumcircle does not contain the three vertices of the triangle, although it may be close. Such an experience is not as convincing and reinforcing as the computer experience.

The last activity in the lesson was difficult for the students. They did not recall the Perpendicular Bisector Theorem. Consequently, considerable class time was needed to review the theorem. Immediately following the review, without any prompting, O. F. recognized the relationship between the Perpendicular Bisector Theorem and the defining property of the circumcenter. He expressed this relationship by saying, "so, $[\overrightarrow{RO}]$ is equidistant

Figure 18. The circumcenter of a triangle
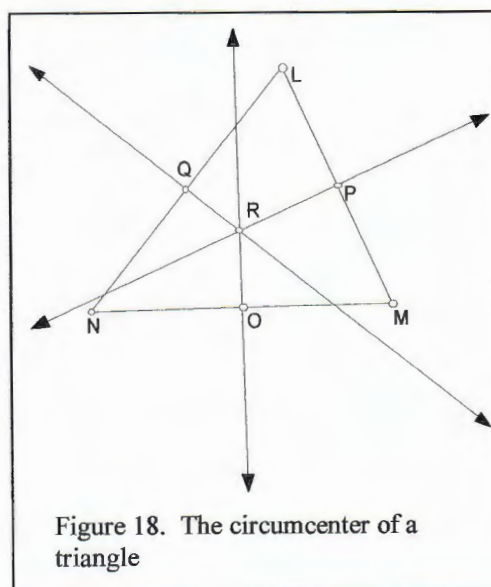
from [N] to [M]. So that means…this point, [R] is too. [R] is equidistant from [M] and [L], and [R] is equidistant from [L] and [N]. Which would be equidistant from all three"(figure 18). The remaining students needed some time to process this explanation, but soon they also understood the relationship.

The class period concluded with an "ah-ha" experience as O. F. continued his train of thought.

> *O. F.: Wait a sec. That means, if this works, that if you change it around a bit, the point of concurrency of the angle bisectors will be equidistant from the sides…The point of concurrency of the angle bisectors will be equidistant."*
>
> *Mrs. Mignini: Why?*
>
> *O. F.: Because each one of those points is equidistant from two sides. The circle we make [with that point as a center] will touch the nearest point on the triangle.*

O. F. understood the relationship between the theorem and the circumcenter well enough to form a conjecture regarding the point of concurrency of the angle bisectors. This provided an excellent preview of the next lesson.

Lesson seven: Identifying the incenter and applying the points of concurrency

Conceptual Emphasis: Students will identify the point of concurrency of the angle bisectors of a triangle as the incenter, the center of the inscribed triangle.

Key Thinking Skills: Formulating conjectures, comparing and contrasting, seeing relationships, thinking flexibly

Prerequisite Knowledge: Angle Bisector Theorem (Any point on the bisector of an angle is equidistant from the sides of the angle.) and the distance between a line and a point not on the line is the length of the perpendicular segment

Additional Materials: compasses, straightedges, local maps (optional)

Activity:

1. Formulate conjectures through comparison and contrast: Working in small groups, students formulate conjectures regarding the point of concurrency of the angle bisectors of a triangle and suggest possible names for the point. The teacher encourages students to refer back to the previous lesson and to make a comparison between the point of concurrency of the perpendicular bisectors of a triangle and the point of concurrency of the angle bisectors of a triangle. If necessary, the teacher also directs students' attention to the Angle Bisector Theorem. The teacher may ask "In what ways is the Angle Bisector Theorem and the Perpendicular Bisector Theorem similar? In what ways are they different? How would their differences affect the point of concurrency?"

2. Share: The teacher leads a class discussion focusing on groups' conjectures. Students present both their conjectures and their reasoning. All conjectures discussed must be justified by the group. The teacher concludes the discussion by naming the point of concurrency of the angle bisectors of the triangle the "incenter."

3. Construct: Using the computer software, students construct an arbitrary triangle, its three angle bisectors, and the incenter using the computer program. In order to construct the inscribed circle (also referred to as the incircle), students must locate a point on the circle. The teacher asks students how to determine the location of a point of the circle. Students must

Figure 19. The incenter is the center of the circle inscribed in the triangle.
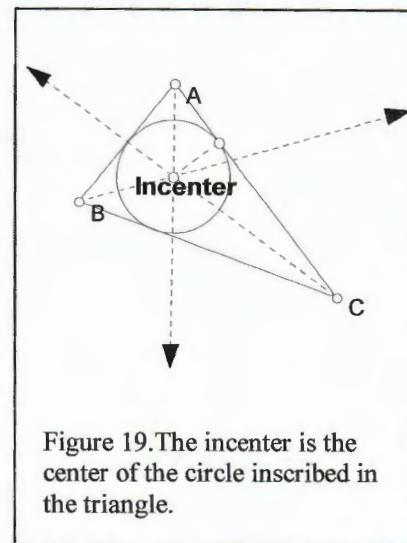
recognize that the circle will intersect each side of the triangle at the point closest to the incenter. The closest point is found by constructing a segment from the incenter perpendicular to one of the sides of the triangle. The distance from the incenter to this point is the length of the radius of the incircle. Using this method, students complete the sketch of the inscribed circle (figure 19).

4. Seeing relationships in properties: Once groups recognize that the incenter is the center of the inscribed circle (also referred to as a incircle), the students determine why the point of concurrency of the angle bisectors and the incenter are in fact the same point. Students are expected to refer back to previous work completed with the angle bisectors, specifically the Angle Bisector Theorem that states any point on the angle bisector of an angle is equidistant from the sides of the angle, and relate the theorem to the current situation. If the class is struggling with the connection, the teacher may need to direct them by asking, "How does the Angle Bisector Theorem apply to the bisector of ∠A? How does the theorem apply to the bisector of ∠B or ∠C? What can be said about the point of intersection

between the angle bisectors?" The teacher then guides the class to recognize that the point of intersection is the center of the incircle (figure 19).

4. <u>Seeing relationships in applications:</u> The teacher will present the following two problems to the class (Answers are provided at the end of the chapter.):

> *Example 1: The town of Westborough is considering constructing a new playground. The town feels that the children of Westborough would be best served by a playground that is equidistant from the Fales School, Hastings School, and Armstrong School. Where should the town look for available land?*
>
> *Example 2: A large corporation is considering opening an office complex in central Massachusetts. Since it expects that employees will commute from various parts of the state, the corporation believes that the ideal location for the complex would be equidistant from I-495, I-290, and the Mass Pike. Where should the complex be built? (Hint: Approximate the routes with straight segments.)*

The class discusses strategies for solving each problem. The teacher directs the students to apply the points of concurrency if the class does not recognize the connection. If maps are available, groups will determine the requested locations with the use of compasses and straightedges. Alternatively, maps can be scanned and inserted into Geometer's Sketchpad files.

5. <u>Flexibility:</u> Each group writes two original word problems involving the points of concurrency, similar to the problems stated in part 5. The group is responsible for both the word problem and the solution. The problems are shared with the class.

This can be accomplished in two ways: 1) the problems can be written on transparencies and displayed with the use of an overhead projector, or 2) the problems can be written on a piece of paper and taped to the desk, forming a "station"; groups of students circulate from one station to the next answering the questions they find at each desk.

Wrap-up activity: Students are asked to explain the strategies they invoked both in the creating of the problems and in the solution process.

Reflections on the implementation of the lesson:

As stated with the circumcenter, the use of the computer software provides students with an accurate visual experience involving the incircle. Constructions of the incircle completed with the aid of a compass, straightedge, and/or other construction tools are typically inaccurate, producing a circle that does not intersect each side of the triangle or intersects the sides in two points instead of the appropriate one point. Students appear to be more accepting of the concept of an incircle after experiencing the accurate representation provided by the computer.

To encourage students to explore the relationship between the Angle Bisector Theorem and the incenter, I asked the class to explain why the point of concurrency of the angle bisectors and the incenter were the same point. With the exception of O. F., who had recognized the relationship during the previous class, students struggled with the connection. The students could not break out of circular reasoning:

O. A.: *"The intersection is equidistant because the inner circle that touches all 3 sides uses the intersection point as the center of the circle."*

> *Mrs. Mignini:* *"The circle is tangent to the triangle <u>because</u> the point is equidistant, not the other way around. Why is the point equidistant from the sides?"*
>
> *O. A.:* *"The point is equidistant because of the angle bisectors and because the point they intersect in is equidistant."*

It was only through very leading questions were the students able to recognize the relationship between the Angle Bisector Theorem and the incenter.

Some of the constructed word problems illustrated the group's creativity and comprehension of the points of concurrency. For example, J.Z., O.A., and C.N. presented a problem involving three computers in a room. The three computers are to be connected to a network server. The group asked where in the room should the server be placed if it is to be equidistant from all three computers. The group also wrote a problem that asked how a triangular platform could be balanced in a sculpture. The remaining two groups, however, produced problems with contexts and wordings that very closely mirrored the original problems. Unfortunately, a lack of time due to snow days prevented a further development of word problems. If there had been time, it would have been interesting to push the groups to fully explore where they could use the points of concurrency in the world around them.

Lesson eight: The nine-point circle, and the Euler line

Conceptual Emphasis: Students will reinforce their understanding of the special segments of a triangle and the points of concurrency.

Key thinking skills: Making observations

Activity:

Students work individually on the following computer projects applying knowledge

of the points of concurrency to nine-point circles and to Euler lines (All answers are

included at the end of the chapter; activities 2 and 3 are adapted from O'Daffer and

Clemens 1976, 66.)

1. The teacher defines a nine point circle of a triangle as a circle containing the

   midpoint of each of the three sides of the triangle, the midpoint of each segment

   between the orthocenter and each vertex, and the feet of the three altitudes (figure

   20). A foot of an altitude is the point of intersection of the altitude and the side of

   the triangle to which it is perpendicular or the line containing the side if the

   triangle is obtuse. The center of the nine-point of the circle is located by finding

   the midpoint between the orthocenter and the circumcenter of the triangle.

   Students construct a triangle and its nine-point circle using the dynamic geometry

   software and answer the following questions:
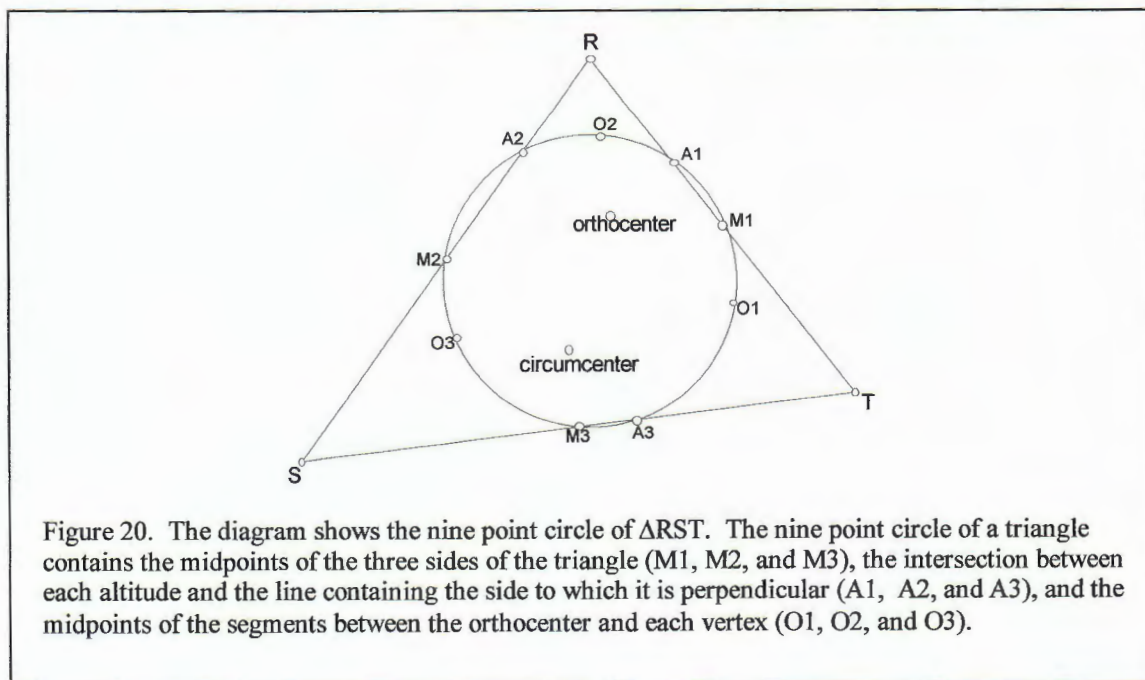

Figure 20. The diagram shows the nine point circle of ∆RST. The nine point circle of a triangle contains the midpoints of the three sides of the triangle (M1, M2, and M3), the intersection between each altitude and the line containing the side to which it is perpendicular (A1, A2, and A3), and the midpoints of the segments between the orthocenter and each vertex (O1, O2, and O3).

a. What is the relationship between the nine-point circle of a triangle and the incircle?

b. Can you determine a second method of locating the center of the nine-point circle? (Hint: choose any three of the nine points and use the three points as vertices of a triangle.)

c. Based on your answer to part b, what can you conclude regarding the circumcircle of each of the triangles formed with vertices at three of the nine points?

2. If given a non-right triangle ABC, three triangles can be determined by the orthocenter of △ABC and two of the vertices of △ABC (see figure 21). Students investigate the relationship between the nine-point circle of an arbitrary non-right triangle and the three triangles with vertex at the orthocenter. This can be done in the following manner:

a. Students construct a non-right triangle, △ABC, its orthocenter, H, and its

Figure 21. If △ABC is not a right triangle, then three triangles can be determined by the orthocenter of △ABC and two of the vertices of △ABC. The three triangles are △ABH, △ACH, and △BCH.
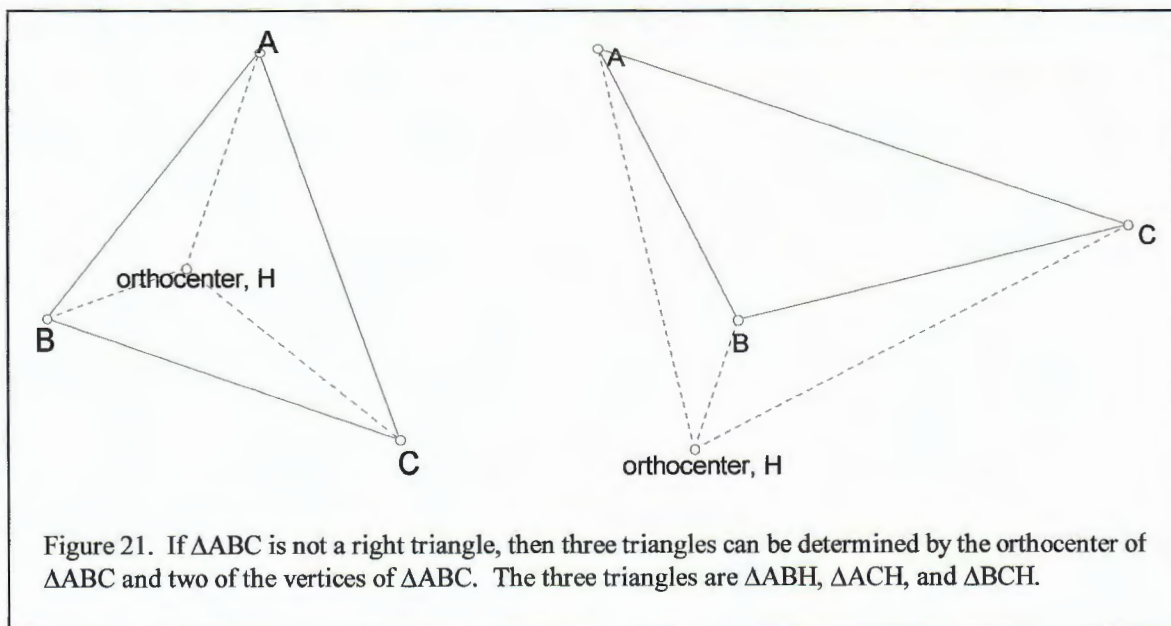
nine-point circle. Students "hide" the nine-point circle so that it is not displayed on the monitor.

    b.   Students construct $\triangle ABH$ and its nine-point circle. They display the nine-point circle of $\triangle ABC$ and determine the relationship between the two circles. This works particularly well if the nine-point circles are two different colors.

    c.   Students repeat the previous step for $\triangle ACH$ and $\triangle BCH$.

    d.   Students write a conjecture regarding the relationship between the investigated nine-point circles.

3.   In any given triangle, three of the four points are collinear. The line determined by the three points is called the Euler Line. Students construct a triangle and its orthocenter, circumcenter, incenter, and centroid and answer the following questions:

    a.   Which of the three points are collinear?

    b.   What happens to the Euler Line when the triangle is isosceles?

    c.   What happens to the Euler Line when the triangle is equilateral?

    d.   The Euler Segment refers to the segment that contains all three points such that two points are endpoints and the third point is between the other two. Determine which point of concurrency is between the other and the ratio of the lengths of the segments formed on the Euler Segment by that point.
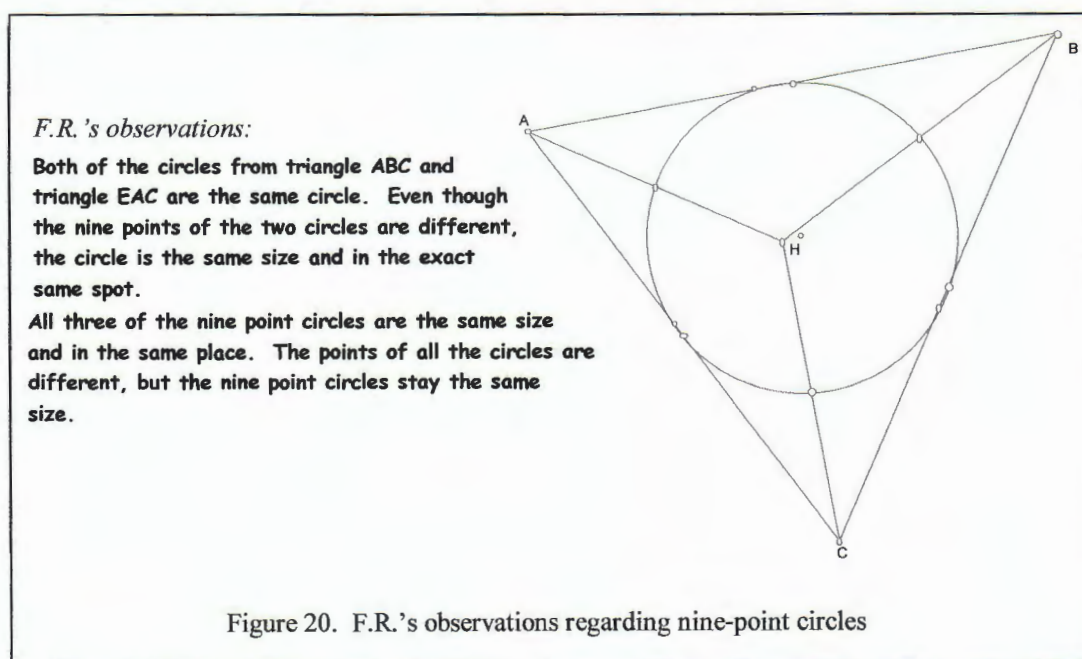
Reflections on the implementation of the lesson:

This is an activity that I do not include in a non-computer classroom. The constructions, if completed by hand, are time consuming and require a tremendous

amount of effort and attention to detail. The paper constructions also cannot be manipulated. Therefore, students can make observations regarding a specific triangle, but they cannot easily generalize their observations to apply to all triangles. With the aid of a computer program, however, constructions are simplified and are manipulable.

The students were motivated by the lesson. They eagerly worked on the computers, beginning before I arrived to class and staying through lunch. At the completion of the concurrency unit, the students almost unanimously voted this activity as their favorite. "It helps to mess around, to see when things don't work," stated O. F., "It's a challenge."

As students worked through the sketches, it became apparent that students understood the definitions of the four special segments and their points of concurrency. With relative ease, the students were able to construct the required points, including the orthocenter that had been a source of misunderstanding in earlier lessons. Student comments, both oral and written, were well thought out. For example, in response to the

F.R.'s observations:

Both of the circles from triangle ABC and triangle EAC are the same circle. Even though the nine points of the two circles are different, the circle is the same size and in the exact same spot.

All three of the nine point circles are the same size and in the same place. The points of all the circles are different, but the nine point circles stay the same size.

Figure 20. F.R.'s observations regarding nine-point circles

K.N.'s observations:

The circles of any triangle are the same no matter what. The nine point circle of ABH and ABC are congruent and form to look like one. The same applies for ABC and BCH. The orthocenter for ACH will lie on point C. If we look at BHC, the orthocenter will be on H. The nine point circle will stay in the same place. The nine points will also stay in the same place.
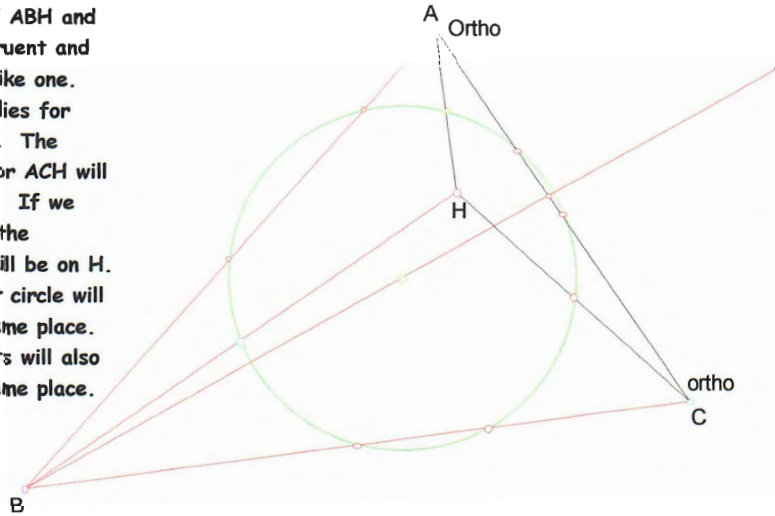
Figure 21. K. N. discusses the location of the orthocenters in his nine-point circle observations.

second activity, F. R. noted that although the nine-point circles coincided, the nine points themselves differed (figure 20). K. N. discussed the location of the orthocenters in relation to the original triangle (figure 21), echoing comments made orally by J. Z. and O. A. The students were motivated to push beyond the questions being asked.

Lesson nine: Applying concurrency points to miniature golf

Conceptual Emphasis: Students will reinforce their understanding of the properties of the points of concurrency.

Key thinking skills: Originality, imagery

Materials: foam board, cutting utensils, glue, string

Activity: Students enrolled in Geometry in Action have been designing a miniature golf course throughout the semester. At the completion of each unit, the design is amended to include concepts from the unit.

1. Design: Individually, students design an original obstacle for a hole in a miniature golf course. The obstacle has two criteria: it must include the use of at least one of the following points of concurrency: incenter, circumcenter, or centroid, and it must be buildable using available materials. Students must be able to visualize a three-dimensional obstacle in order to incorporate the point(s) of concurrency.

2. Construct: Students construct a scale model of their obstacles out of foam board and other needed materials.

3. Present: Students present their models to the class, explaining how the point(s) of concurrency were utilized. Students must provide the scale factor and the anticipated measurements of the actual obstacle if the golf course were completed.

Reflections on the implementation of the lesson:

Once again, students were motivated by the lesson. This was especially apparent when L. J., a student who has consistently shown little interest in the class in general, exhibiting only the bare minimum effort level, began working through lunch and after the bell each day. As a result, the activity motivated L. J.
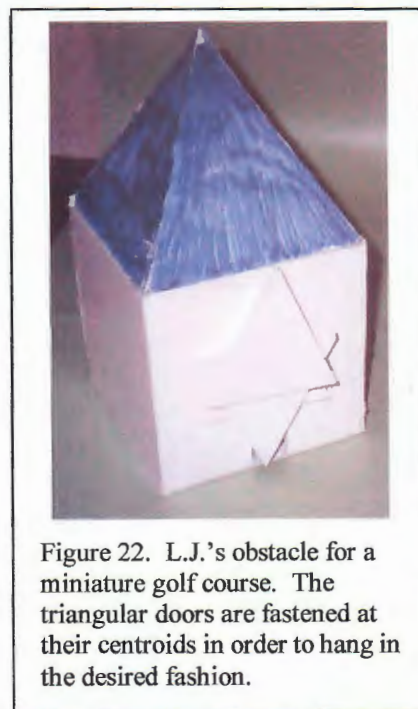

Figure 22. L.J.'s obstacle for a miniature golf course. The triangular doors are fastened at their centroids in order to hang in the desired fashion.

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to learn the special segments and their points of concurrency. Although his project did

not maximize the potential of the centroid (figure 22), L. J. did learn from the activity.

The students took differing approaches to the project. This is apparent in the

design of the final project (figures 22 through 27) and in the design process. Several of

the students utilized the centroid (figures 22, 23, 24, 25 and 27). The incenter was also

Figure 23. K. N.'s miniature golf obstacle. The symmetric pendulum is hung from the centroid of the triangle. As a result, the obstacle hangs almost perfectly horizontally.

Figure 24. C. N.'s miniature golf obstacle. The circle is congruent to the incircle of the triangle hanging below it. The triangle was designed to be hung from its centroid so that it hung horizontally. Unfortunately, the project was damaged before the picture could be taken.
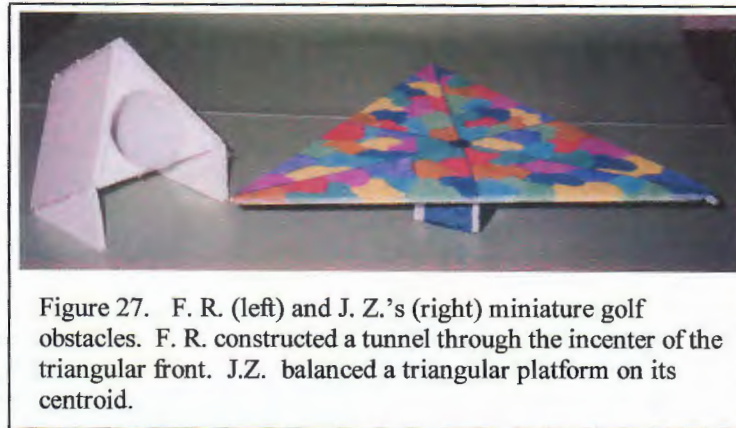

Figure 25. O. F.'s miniature golf obstacle. The bases of the pendulum are triangles. The pendulum hangs from the centroid of the top base. Also, the golf hole found beneath the pendulum is congruent to the incircle of the bottom base.

Figure 26. O. A.'s miniature golf obstacle. The golf hole is the incircle to the triangular obstacle.

Figure 27.   F. R. (left) and J. Z.'s (right) miniature golf obstacles. F. R. constructed a tunnel through the incenter of the triangular front. J.Z. balanced a triangular platform on its centroid.

incorporated in several projects (figures 24, 25 20, and 27). Within each of the designs,

comprehension of the point(s) of concurrency is exhibited. This comprehension was

reinforced and strengthened throughout the course of this project. The method students

employed varied greatly as well. It was interesting to see individual learning styles and

personalities displayed through the design process. Some students, such as L. J. (figure

22), K. N. (figure 23) and F. R. (figure 27), could visualize the obstacle. These students

made few, if any, rough sketches, working instead from a mental image.   The sketches

that were drawn were typically two-dimensional sketches of specific faces of the project.

These students appeared to work part-to-whole, obtaining a product through a focus on

the individual faces of the obstacle.   Other students, such as O. F. (figure 25) O. A.

(figure 26), and J. Z. (figure 27), relied on the Geometer's Sketchpad to provide them

with accurate representations of the obstacles. These sketches tended to be three-

dimensional representations. The students appeared to work whole-to-part, focusing on

the overall project and then paying attention to details as needed. The remaining student,

C. N., chose to sketch free hand like the former group. Her sketches, however, were

more detailed and were three-dimensional, similar to the latter group. She appeared to

have the most difficulty with the visualization process. She did not exhibit a clear-cut part-to-whole or whole-to-part preference. Instead, she appeared to latch on to any technique that seemed promising at the time. Of the seven students in the class, C. N. struggled most with the project. The project provided an opportunity for application which benefited students of various learning modalities.

Summary

The points of concurrency curriculum unit represents a typical unit in the Geometry in Action elective. Each unit presented in the course is designed around student-based investigations that are completed within the dynamic geometry environment. It is important to note that the computer programs themselves do not produce understanding of geometric concepts, but rather they facilitate the learning of those topics. The software can only react to a student's commands, and thus it can provide answers to questions only after those questions are asked. The student must know which questions to ask and how to synthesize the information received from the software if he is to use the software to its fullest potential. Therefore, it is crucial that students understand how to investigate geometric conjectures in a critical and creative manner. This is accomplished through frequent questioning by a student's teacher and classmates.

Each student responded to Geometry in Action and to the concurrency unit in his or her own way. O.F., a freshmen enrolled in accelerated mathematics courses, used the sketchpad to compensate for his poor visualization skills. With the use of his sketches, O.F. successfully reached for concepts beyond the class expectations. J.Z., a sophomore

who is also studying geometry on an honors level, used the sketchpad investigations to complement the foundation of geometric understanding that she is attaining in her honors class. J.Z. stated that she liked working through the computer projects since it made her think and it motivated her. Motivation was also a key factor for O.A., a sophomore geometry student, F.R., a freshman, and K.N, an honors level freshman. I believe O.A. also benefited greatly from the visualization aide that the computer provided. With the assistance of the sketchpad, O.A. appeared to understand material with which she otherwise struggled. For example, the concept of the incenter as the center of an inscribed circle had eluded O.A. until she was able to construct the sketch on the Geometer's Sketchpad. C.N., a junior who transferred into the class late in the term, had difficulty with the structure of the course due to a language barrier. English is not her first language. As a result, C.N. struggled with the discussions that are central to many of the lessons in the unit. The Geometer's Sketchpad was useful for C.N. to identify areas of misunderstanding. For example, when attempting to construct a circumcircle, it quickly became apparent that C.N. could not distinguish between an altitude and a perpendicular bisector. Once this was discovered, I took the time to sit with her and communicate the concepts to her through constructions of altitudes and constructions of perpendicular bisectors. The use of the computer neither helped nor hurt L.J., a senior with a history of low achievement and effort in mathematics classes. He enrolled in Geometry in Action as a result of having no other scheduling options. L.J. quickly learned the computer program, but was reluctant to explore and form conjectures.

For the students in Geometry in Action, the dynamic geometry projects were necessary components of the class but not sufficient. To varying degrees, the students

were motivated by the projects and benefited from the visualization aide. However, each of the students needed the guidance of myself and their fellow students to maximize their thinking and their understanding.

Answers

Lesson four, wrap-up activity: The altitude can be located by first constructing the altitudes from the two known vertices. The lines containing the two altitudes will intersect at the orthocenter. Since the lines containing all three altitudes intersect at the orthocenter, this point is contained in the third altitude. Constructing a line through the orthocenter that is perpendicular to the line containing the third side of the triangle will result in a line that contains the third altitude.

Lesson seven, example 1: The town should look for land at the circumcenter of the triangle formed between the three schools. The circumcenter is located by constructing the perpendicular bisector of each side of the triangle.

Lesson seven, example 2: The complex should be built at the incenter of the triangle determined by the three roads. The incenter is located by constructing the angle bisectors of the triangle.

Lesson eight, computer activity 1:

    a. The incircle and the nine-point circle are internally tangent circles.

    b. The circumcenter of any of the triangles determined by three of the nine-points will coincide with the center of the nine-point circle.

c. The circumcircle of each triangle determined by three of the nine-points will coincide with the nine-point circle.

Lesson eight, computer activity 2: The circles coincide.

Lesson eight, computer activity 3:

a. The othocenter, circumcenter, and the centroid are collinear.

b. The Euler Line also contains the incenter in an isosceles triangle.

c. The four points coincide in an equilateral triangle. Therefore, there are infinitely many lines that contain the four points.

d. The centroid is between the orthocenter and the circumcenter. The centroid divides the Euler Segment in a 2:1 ratio.

CHAPTER FOUR

CONCLUDING REMARKS

As evidenced by students' attitudes and performance, the needs of many geometry students are not currently being met by the traditional geometry classroom. The utilization of available resources and the implementation of new classroom structures may strengthen the potential and the achievement of these students. It is my belief that structuring activities around student investigations that are conducted in a dynamic geometry environment can attain this goal. These investigations assign to the students a more active role in the learning process, making them more responsible for their own education.

As mentioned in the previous chapter, the role played by the computer software varies from student to student. Some students benefit primarily from the computer as a visualization aide, whereas other students rely on the program to clarify their understanding and to identify areas of misunderstanding. Still other students use the computer as a motivator. All students, however, benefit from the precision and the manipulability of the computer-generated sketches. The dynamic geometry software allows the user to investigate a continuum of related examples as opposed to one static drawing. In this manner, the dynamic geometry software provides an environment conducive to the formation of conjectures.

It is important to note that the software provides only the environment and the potential for student learning. Dynamic geometry software provides a tool with which students can visualize and explore geometric concepts. The programs augment, but do

not replace, effective teaching in the geometry classroom. Effective teaching in a dynamic classroom should focus on relevant thinking skills – predicting results of manipulations, identifying characteristics, making decisions, forming and justifying conjectures, and seeing relationships. The software only reacts to commands made by the students. In order for a student to be successful in a dynamic geometry environment, he must recognize what to look for, how to search for the information, and how to synthesize the results.

The topics of dynamic geometry and the points of concurrency, specifically the centroid, are timely issues. Shortly before the completion of this paper, three relevant articles were published in issues of *Mathematics Teacher* and *Scientific American Explorations*. In "Making Better Use of Computer Tools in Geometry", Brad Glass and Walter Deckert ask the question, "Are [students'] interactions [with the computing tool] helping them learn geometrical concepts?" (2001, 224) Their answer is yes. Their research is consistent with the main premise of this paper:

> A major goal in the geometry classroom is having students develop and validate conjectures. High school students are able to develop conjectures by generalizing the patterns that unfold during explorations in interactive geometry environments. Although interactive geometry, static construction, and Logo-based environments do not directly help students generalize the patterns, they offer mathematically accurate environments in which patterns can be observed. Pattern recognition is stimulated by the ability to manipulate diagrams and to observe the resulting changes. (Glass and Deckert 2001, 227)

The authors also emphasize the role of dynamic geometry software as a "springboard" (228) to further investigation and justification. Likewise, Thomas W. Shilgalis and Carol T. Benson found it beneficial to use the Geometer's Sketchpad "for exploration and clarification" (2001, 307) in their search for the center of gravity of a polygonal region.

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The authors build upon the methods of locating the centroid of a triangle and the center of gravity of a quadrilateral region in order to locate the center of gravity of any polygonal region. Lastly, Krista West (2001) outlined physical experiments that investigate an individual's center of gravity. The experiments, which involve bending over or shifting one's weight while standing against a wall, provide practical activities with which to convey to students a clearer understanding of what is meant by the phrase "center of gravity". These articles give support to the importance of these topics in the current geometry classroom.

A Vision of the Future of Geometry in Action

As the end of the school year approaches, the future of Geometry in Action is being discussed. My principal, my advisor and my colleagues have asked several questions concerning the curriculum unit, the course in general and what I envision the future of the course to be. Some of these questions and my responses are outlined in the following pages.

1. *Would the class be successful with a larger student enrollment?* Although the small class size had both its advantages and disadvantages, I believe the size of seven students was more of a shortcoming than a strength. The greatest advantage of the size was the individual attention I could provide to each student. The minimal number of viewpoints that could be offered, however, outweighed this attention. Within the scope of average class sizes (typically 20 - 30 students per class), the potential of a course that is structured around student investigations and the

discussion of observations and conjectures is increased as the number of voices in the class increases. One student may recognize a pattern that the other students do not see and share such insights with the class, leading to further channels for investigation.. A larger class would be conducted on two levels – group discussions of individual student observations and class discussions outlining the groups' conclusions.

2. *Should the class remain heterogeneous or should the enrollment become more focused?* ·Students of all ability levels and math backgrounds can benefit from working in a dynamic geometry environment. A heterogeneous group, however, produces less focused objectives within the class and the investigations. I believe that classes consisting of students with similar backgrounds would increase the potential benefit for all students. Ideally, I would like to see Geometry in Action divided into two courses, an introductory course for students who have not yet formally studied geometry and an elaboration course for those who have completed a year of geometry. This would allow the sketchpad investigations of each course to concentrate on more clearly defined goals. The introductory course would create a foundation of geometric understanding upon which a subsequent geometry course could build. Such a foundation would improve the chances for success in the required geometry class, especially for students who are identified as "at-risk" due to poor spatial abilities. The elaboration course, on the other hand, would provide an elective for those students who enjoy geometry. Only algebraic math electives are

currently offered. A dynamic geometry elective would allow these students to deepen their understanding of geometry and to explore topics that interest them.

3. *How could the unit be integrated into a standard geometry class?* Although the geometry concepts involved are appropriate for any geometry class, the class structure and the learning process detailed in this curriculum unit may need to be modified when integrated into a standard geometry class. The points of concurrency unit was implemented approximately halfway through the Geometry in Action course. Throughout the first few months of the course, the students received instruction on strategies for successfully experimenting in the sketchpad environment. They learned what is expected of them and what to expect from the teacher and the class in a course structured around student investigations. Although the students in my standard geometry classes, as well as the students in the classes of my colleagues, utilize the Geometer's Sketchpad throughout the year to introduce or reinforce topics, the students do not receive this extensive instruction. As a result, a teacher in a standard geometry class may need to provide more structure and guidance in the sketchpad activities. It may become necessary to focus students' attention. The teacher may need to incorporate the teaching of critical thinking skills listed previously in this chapter. I believe that if given a supportive classroom and the guidance of a teacher interested in a critical thinking process, a standard geometry class can successfully complete the points of concurrency curriculum unit.

Summary

There are too many students who complete a geometry course with a minimal level of geometric knowledge and comprehension. Since there are a variety of reasons why students do poorly in geometry classes, there cannot be one blanket "cure". Dynamic geometry software, however, can overcome many of the obstacles that hinder student success when augmented by instruction in the areas of thinking skills. Although I had become an advocate of dynamic geometry software after observing the results in my standard geometry classes, I didn't fully recognize the potential benefits until I taught Geometry in Action. The structure of the elective values and emphasizes these thinking skills, resulting in more successful investigations. When used as an aide to learning, dynamic geometry software can raise expectations of student learning and provide a tool for visualization. As a result, students can potentially deepen their understanding of geometric concepts.

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