

GroupConsensus of Heterogenous Continuous-time Multi-agent

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Abstract:** This paper considers the group consensus problem for continuous-time linear heterogeneous mu **Examples of Science**. North China University of Technology, Beijing 100144, P. R. China
 Abstract: This paper considers the group consensus problem for continuous-time linear heterogeneous multi-agent

systems with undi **Zhang Yanxin**

College of Science, North China University of Technology, Be
 Abstract: This paper considers the group conse

systems with undirected and directed fixed to

coefficients to divide all second-order agents **Abstract:** This paper considers the group consensus problem for continuous-time linear heterogeneous multi-agent systems with undirected and directed fixed topology. In order to obtain group consensus, we use two partitio By constructing the Lyapunov function, a sufficient condition for group consensus under undirected topology are
proved. Based on a system transformation method, the group consensus for heterogeneous multi-agent systems is

Introduction

proved. Based on a system transformation method, the group consensus for heterogeneous multi-agent systems is
transformed into a group consensus for homogeneous multi-agent systems. We also find the convergence points of t **Examplem of multi-agent system of multi-agent systems.** We also find the convergence points of the
two groups, it has great significance. Finally, numerical examples are provided to demonstrate the effectiveness of the
t two groups, it has great significance. Finally, numerical examples are provided to demonstrate the effectiveness of the
theoretical results.
Keywords: Group consensus; heterogeneous multi-agent systems; undirected topolo theoretical results.
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 Introduction

In recent years, distributed multi-agent cooperative control system has been wid **Keywords:** Group consensus; heterogeneous multi-agent systems; undirected topology; directed topology
 Introduction

In recent years, distributed multi-agent cooperative control system has been widely applied in the fie **Keywords:** Group consensus; heterogeneous multi-agent systems; undirected topology; directed topology
 Introduction

In trecent years, distributed multi-agent cooperative control system has been widely applied in the f **Introduction**
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spacecraft cooperative control, control of formation flying satellites, mobile robot dist Introduction
In recent years, distributed multi-agent cooperative control system has been widely applied in the fields of ummanned
spacecraft cooperative control, control of formation flying satellites, mobile robot distri In recent years, distributed multi-agent cooperative control system has been widely applied in the fields of ummanned spacecraft cooperative control, control of formation flying satellites, mobile robot distributed optimi spacecraft cooperative control, control of formation 1
a basic problem of multi-agent system cooperative (
researchers attention^{[1]-[5]}. Using graph theory, Jad
consensus behavior of the Vicsek model. Bases on the *i*
fo Exam or mant-ugant system cooptionary contains proof and a theori-state provided a meta-group of the ansensus behavior of the Visesk model. Bases on the analysis in^[6]. Sholer et al.^[7] investigated the consensus probl messame shearing and the Viesek model. Bases on the analysis in^[6], Subarch can also an concentration for the Circle of the Viesek model. Bases on the analysis in^[6], Suber et al. ^{17]} investigated the consensus probl consensus ochiants of the Vecsel moderation (and suddering topologies by discussing three cases: directed networks with fixed topology, directed networks with switching topology, undirected networks with communication time

real and imaginary parts of the eigenvalues of th

Consensus problem is the design of a suitable co

through a certain amount.

In real life, group collaboration is a very com

military applications in surveillance, reconn Consensus problem is the design of through a certain amount.

In real life, group collaboration is

military applications in surveillance,

consensus not only helps better unde
 $\overline{\text{Copyright © 2017 Zhang YX}}$.

doi: 10.18686/esta.v4 Consensus problem is the design of a suitable control protocol so that all the multi-agent achieve to the same value
through a certain amount.
In real life, group collaboration is a very common phenomenon, and it is very i through a certain amount.

In real life, group collaboration is a very common phenomenon, and it is very important. It has many civil and

military applications in surveillance, reconnaissance, battle field assessment, etc In real life, group collaboration is a very commilitary applications in surveillance, reconnaissan
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This is an Open Acces *Electronic Science Technology and Application*
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useful ideas for distributed cooperative control. In the group consensus problem, the whole network is divided into
multiple sub-networks with information exchanges between them, and the aim is to design appropriate protoc useful ideas for distributed cooperative control. In the group consensus problem, the whole network is divided into
multiple sub-networks with information exchanges between them, and the aim is to design appropriate proto useful ideas for distributed cooperative control. In the group consensus problem, the whole network is divided into multiple sub-networks with information exchanges between them, and the aim is to design appropriate proto useful ideas for distributed cooperative control. In the group consensus problem, the whole network is divided into
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multiple sub-networks with information exchanges between them, and the aim is to design appropriate proto work is divided into
ppriate protocol such
us problem in $[9]$ - $[11]$ is
e are more groups in
stems with switching
mics, they introduce
duced-order systems.
 $[16]$ studied the group
bound of time delay useful ideas for distributed cooperative control. In the group consensus problem, the whole network is divided into
multiple sub-networks with information exchanges between them, and the aim is to design appropriate protoc useful ideas for distributed cooperative control. In the group consensus problem, the w
multiple sub-networks with information exchanges between them, and the aim is to desi
that agents in the same sub-networks reach the s Experience of existing research results about consensus problem of multi-agent systems in the above of sconting above that consensus problem in ^[9]411] is received a for existing reproducts a more proups in practical exa mare agents in the same standard content steam the substitute states. It is casy to see that constrains protocol in the system construents in one group. However, there are more groups in the space in case of group consensu A procedured external impacts. The dynamics of each coupling agents are described by single-integrator dynamics, they introduce the procedure trees from transformation delays, where the agents are described by single-inte

restrictions of exchanging. Therefore, it has very important practice and condensity in the proportation delay
topologies and communication delays, where the agents are described by sigle-integrator dynamics, they introduc ropologies and communication dearys, where the agents each described by singur-integration dynamics, they inducated
Group consensus control for second-order dynamic equations of agents are transformed into reduced-order sy Group consensus control of escond-order dynamic multi-agent systems was investigated in^{[14]-[16]} [^{16]} studied the group consensus problem of second-order dynamic multi-agent systems was investigated in^{[14]-[16]}, ^{[1} control for a class of heterogeneous multi-agent systems, were found the upper bound of time delay
consensus problem of second-order multi-agent systems with time delays, where found the upper bound of time delay
such tha systems with the multi-agent systems with the directed multi-agent systems with the directy, where found the upper bound of this deady
such that the multi-agent systems can achieve group consensus.
From the above, most of stern that the mini-agent systems can achieve group consensus.

From the above, most of existing research results about consensus problem of multi-agent systems are established

mainly based on the homogeneous multi-agent From the above, most of existing research results about consensus problem of multi-agent systems are established mainly based on the homogeneous multi-agent systems, that is, all the agents have the same dynamics behaviou mainly based on the homogeneous multi-agent systems, that is, all the agents have the same dynamics behaviours.
However, the dynamics of each coupling agents may be different in practice work, because of the external impac We continue of exchanged a may be different in practic work, occass of the content in the proposes relations of exchange, Therefore, it has very important practical significance that study group consensus problem heterogen Extractions of xactinguity. Interview, it uses they important practical significant systems profolems of heterogeneous multi-agent systems mainly consider the mix of first-order and second-order systems 1^{17} H¹⁸¹ cons partiti-agent systems mainly consider the mix of first-order and second-order systems. ^[17]-118] considered consensus multi-agent systems with fixed and switching directed communication topology to reach consensus by us multi-agent systems manny constact are mix of inse-ofted and second-ofted systems.²⁰² control for a class of heterogeneous multi-agent systems, some conditions were presented for hetern systems with fixed and switching tems with fixed and switching directed communication topology to reach consensus by using tools from graph
ory and matrix theory. In^[19], Zheng and Wang presented a novel protocol according to the feature of heterogeneou

theory and matrix theory. In^[19], Zheng and Wang presented a novel protocol according to the feature of heterogeneous multi-agent, studied the group consensus in continuous-time system $[20]$ studied consensus problem of multi-agent, studied the group consensus in continuous-time system.^[20] studied consensus problem of heterogeneous
multi-agent in continuous-time system and discrete-time system by a novel consensus algorithm, respective multi-agent in continuous-time system and discrete-time system by a novel consensus algorithm, rest Inspired by the aforementioned work, in this paper, we will investigate the group consens multi-agent systems under undire theory and matrix theory. In^[19], Zheng and Wang presented multi-agent, studied the group consensus in continuous-tim multi-agent in continuous-time system and discrete-time systems under undirected and directed fixed t Inspired by the aforementioned work, in this paper, v
multi-agent systems under undirected and directed fixed to
paper are two folds: (1) Two partition coefficients are in
multi-agent system; (2) The concrete convergence multi-agent system; (2) The concrete convergence points are given.

This paper is organized as follows. Section 2 introduces some basic concepts and notations in graph theory and

presents the models of the problems. Sect

14| Zhang YX. 12 $\forall x, y \in V$, and $\forall y, y \in V$, and $\forall y, y \in V$ and $\forall x, y \in V$ and $\forall y, y, y \in V$, where an edge is an ordered pair of distinct vertices of V, and the non symmetric weighted adjacency matrix $A = [a_{ij}]$, where This paper is organized as follows. Section 2 introduces some basic concepts and notations in graph theory and
presents the models of the problems. Section 3 gives the main results of heterogeneous multi-agent systems und given in Section 4. Finally, the conclusion is made in section 5.
 1. Preliminaries
 1.1 Graph Theory

Graph theory^[21] is an effective tool to study the coupling topology of the communication configuration of the a **1. Preliminaries**
 1.1 Graph Theory

Graph theory^[21] is an effective tool to study the coupling topology of the communication configuration of the agents. In

this section, we briefly review some basic notations an

A graph is called complete if every pair of vertices are adjacent. A path of length *r* from v_i to v_j in a graph is a sequence of $r+1$ distinct vertices starting with viand ending with v_j such that consecutive vert A graph is called complete if every pair of vertices are adjacent. A path of length *r* from v_i to v_j in a graph is a sequence of $r+1$ distinct vertices starting with viand ending with v_j such that consecutive vert A graph is called complete if every pair of vertices are adjacent. A path of length r from v_i to v_j in a graph is a sequence of $r+1$ distinct vertices starting with viand ending with v_j such that consecutive vertic A graph is called complete if every pair of vertices are adjacent. A path of length r from v_i to v_j in a graph is a sequence of $r+1$ distinct vertices starting with viand ending with v_j such that consecutive vertic *Let us* are adjacent. A path of length *r* from v_i to v_j in a graph is a sequence ig with v_j such that consecutive vertices are adjacent. If there is a nnected. The degree matrix D(G) of G is a diagonal matrix with A graph is called complete if every pair of vertices are adjacent. A path of length r from v_i to v_j in a graph is a sequence of $r+1$ distinct vertices starting with viand ending with v_j such that consecutive vertic A graph is called complete if every pair of vertices are adjacent. A path of length r from v_l to v_j in a graph is of $r+1$ distinct vertices starting with viand ending with v_j such that consecutive vertices are adja graph is called complete if every pair of vertices $t+1$ distinct vertices starting with viand ending
th between any two vertices of G, then G is connected
d columns indexed by V, in which the (v_i, v_j) entry
 $L($
the Lapl

$$
L(G) = D(G) - A(G)
$$

A graph is called complete it every pair of vertices are adjacent. A pain of length P from V_i to V_j in a graph is a sequence of $r+1$ distinct vertices strating with viand ending with V_j such that consecutive vert of *i*+1 distinct vertices starting with viand ending with *v*_{*y*} such that consecutive vertices are adjacent. If there is a path between any two vertices of G, then G is connected. The degree matrix D(G) of G is a dia and columns indexed by v, in which the (v_i, v_j) entry is the degree of vertex v_i . The symmetric matrix definies as.
 $L(G) = D(G) - A(G)$

is the Laplacian of G. The Laplacian is always symmetric and positive semi-definite, an In reality, the whole multi-agent system can be divided into some practical significance. Without loss of generality, all a generality, the whole multiplicity of its solid to the number of connected components in the grap

is the Laplacian of G. The Laplacian is always symmetric and positive semi-definite, and the algebraic multiplicity of its
zero eigenvalue is equal to the number of connected components in the graph.
Definition 1^[12] zero eigenvalue is equal to the number of connected components in the graph.
 Definition 1^[12] (Subgraph)

A network with topology G_i=(V₁, E₁, A₁) is said to be a sub-network of a network with topology G=(V, E E, A) if (i)V₁
graph of G.
, we say that
h of G.
ssters, which
enerality, all
¹) consisting
agents with
a example of **Definition 1**¹¹² (Subgraph)

A network with topology $G_1 = (V_1, E_1, A_1)$ is said to be a sub-network of a network with
 $\subseteq V$, (ii)E₁ \subseteq E and (iii) the weighted adjacency matrix A_1 inherits A . Correspondingly a sub-network of a network with topology $G=(V, E, A)$ if (i) V_1
A₁ inherits A . Correspondingly, we call G_1 a subgraph of G .
rict, and $E1 = \{(v_i, v_j) : i, j \in V_1, (v_i, v_j) : i, j \in E\}$, we say that
e. Correspondingly, we cal $x^1 = (x_1, x_2, \ldots, x_m)^T$ and $x^2 = (x_{m+1}, x_{m+2}, \ldots, x_n)^T$, and the agents in each subgroup can establish a sub network. An example of A network with topology $G_1=(V_1, E_1, A_1)$ is
17*z*, (ii)E₁ \subseteq E and (iii) the weighted adjacen
²*x*urthermore, if the inclusion relations in (i) and
the first network is a proper sub-network of th
In reality, the *2*^{*n*} different in the second one *x*^{*n*} different in the second one *x*^{*n*} different in the second one *x* correspondingly, roper sub-network of the second one. Correspondingly, e multi-agent system can be divide (i) is said to be a sub-network of a network with topology G=(V, E, A) if (i)V₁ enercy matrix A₁ inherits A . Correspondingly, we call G₁ a subgraph of G. and (ii) are strict, and E1=((v_i, v_j) : i, $j \in V_1$, (v_i, v_j) EV, (ii)E₁ \subseteq E and (iii) the weighted adjacency matrix A₁ inherits A . Correspondingly, we call G Furthermore, if the inclusion relations in (i) and (ii) are strict, and E1=((v_h, v_j) : i, $j \in V_1$, (v_h, v_j) : i, th *A* and the particular significance. Without loss of generality, all

gents divided into two different subgroups (G_1, x^1) consisting

consisting of $n-m$ first-order (single) integrator agents with

in each subgroup can ded into some complex subgroups or intelligent clusters, which
ave a more practical significance. Without loss of generality, all
1) agents divided into two different subgroups (G₁, x¹) consisting
 x^2) consisting of

$$
A = \begin{pmatrix} A_s & A_{sf} \\ A_{fs} & A_f \end{pmatrix}
$$

Where A_s , A_f , A_{sf} , and A_f correspond to, respectively, the indices of second-order integrator agents, first-order integrator agents, from second-order integrator agents to first-order integrator agents to first-o gents divided into two different subgroups (G_1, x^1) consisting
consisting of $n-m$ first-order (single) integrator agents with
in each subgroup can establish a sub network. An example of
ig. 1.
 $\frac{Group 2}{8}$
A A_{*A*} $\$ The group consensus problem is studied under first-order and second-order integrator agents obeying the neighbor-based law. The position adjacency matrix of the network can be partitioned as $A = \begin{pmatrix} A_s & A_{sf} \\ A_{\beta} & A_{f} \end{$ Fig. 1: A undirected topology.

Fig. 1: A undirected topology.

meighbor-based law. The position adjacency matrix of the network can be partitioned as
 $A = \begin{pmatrix} A_x & A_y \\ A_\beta & A_y \end{pmatrix}$

Where A_s , A_β , $A_{\beta\beta}$, and A_β Fig. 1: A undirected topology.

The group consensus problem is studied under first-order and second-order integrator agents obeying the

neighbor-based law. The position adjacency matrix of the network can be partitioned The group consensus problem is studied under first-order and second-order integrator agents obeying the neighbor-based law. The position adjacency matrix of the network can be partitioned as $A = \begin{pmatrix} A_s & A_{sf} \\ A_{\beta} & A_{f} \end{$ The group consensals problem is statical
neighbor-based law. The position adjacency matri:
Where A_s , A_f , A_{sf} , and A_{fs} correspond to, respectiv-
agents, from second-order integrator agents to 1
second-order integ

$$
L = \begin{pmatrix} \overline{L}_s & -A_{sf} \\ -A_{fs} & \overline{L}_f \end{pmatrix}
$$

 $\begin{pmatrix} -A_{sf} \ A_{fs} & \overline{L}_f \end{pmatrix}$
atrix of corresponding to velocity adjacency ma $\begin{pmatrix} -A_{sf} \\ \overline{L}_f \end{pmatrix}$
trix of corresponding to velocity adjacency matrix \overline{L}_s $-A_{sf}$
 \overline{L}_f \over \overline{L}_s \overline{L}_f \overline{L}_f
A_{fs} \overline{L}_f $\overline{$ $\begin{pmatrix} \overline{L}_s & -A_{sf} \\ -A_{fs} & \overline{L}_f \end{pmatrix}$ matrix of corresponding to velocity adjacency matrix of the $=\begin{pmatrix} \overline{L}_s & -A_s \\ -A_{fs} & \overline{L}_f \end{pmatrix}$
an matrix of corresponding to velocity adjacency matrix of the $L = \begin{pmatrix} \overline{L}_x & -A_y \\ -A_\beta & \overline{L}_f \end{pmatrix}$
where \overline{L}_s $L_s + D_{g}$, \overline{L}_f $L_f + D_{\beta}$. The Laplacian matrix of corresponding to velocity adjacency matrix of the
network is denoted by \hat{L}
1.2 System model
In this $L = \begin{pmatrix} L & L_s + D_{sf} & \bar{L}_f & L_f + D_{fs} \end{pmatrix}$. The Laplacian m
network is denoted by \hat{L}
1.2 System model
In this subsection, we propose the heterogeneous multi-ag $L = \begin{pmatrix} \overline{L}_s & \overline{L}_s + D_{sf} & \overline{L}_f + D_{fs} \end{pmatrix}$. The Laplacian matr
network is denoted by \hat{L}
1.2 System model
In this subsection, we propose the heterogeneous multi-agent
 $\begin{cases} x_i(t) = v_i(t), v_i(t) = u_i(t) \end{cases}$ $L = \begin{pmatrix} \overline{L}_s & -A_{sf} \\ -A_{\beta} & \overline{L}_f \end{pmatrix}$
where \overline{L}_s $L_s + D_{sf}$, \overline{L}_f $L_f + D_{\beta}$. The Laplacian matrix of corresponding to velocity adjacency matrix of the
network is denoted by \hat{L}
1.2 System model
In t $L = \begin{pmatrix} \overline{L}_s & -A_g \\ -A_{fs} & \overline{L}_f \end{pmatrix}$

The Laplacian matrix of corresponding to velocity adjacency matrix of the

geneous multi-agent system which is composed of a CT system
 $(t) = v_i(t), v_i(t) = u_i(t), \quad i \in I_m,$
 $(t) = u_i(t), \quad i \in I$ $L = \begin{pmatrix} \overline{L}_s & -A_g \\ -A_g & \overline{L}_f \end{pmatrix}$

The Laplacian matrix of corresponding to velocity adjacency matrix of the

geneous multi-agent system which is composed of a CT system
 $(t) = v_i(t), v_j(t) = u_j(t), \quad i \in I_n$,
 $(t) = u_j(t), \quad i \in I_n /$ $L = \begin{pmatrix} \overline{L}_s & -A_{sf} \\ -A_{fs} & \overline{L}_f \end{pmatrix}$
The Laplacian matrix of corresponding to velocity adjacency may
geneous multi-agent system which is composed of a CT system
 $i(t) = v_i(t), v_i(t) = u_i(t), \quad i \in I_m,$
 $i(t) = u_i(t), \quad i \in I_n / I_m,$
Tab $L = \begin{pmatrix} \overline{L}_s & -A_{g'} \\ -A_{fb} & \overline{L}_f \end{pmatrix}$

The Laplacian matrix of corresponding to velocity adjacency matrix of the

ogeneous multi-agent system which is composed of a CT system
 $x_i(t) = v_i(t), v_i(t) = u_i(t), \quad i \in \mathbb{I}_m,$
 \over $L = \begin{pmatrix} \overline{t}_x & -A_y \\ -A_{jk} & \overline{L}_f \end{pmatrix}$

The Laplacian matrix of corresponding to velocity adjacency matrix of the

ogeneous multi-agent system which is composed of a CT system
 $x_i(t) = v_i(t), v_i(t) = u_i(t), \quad i \in I_n,$
 $x_i(t) = u_i(t), \$ $L = \begin{pmatrix} \overline{L}_s & -A_{s'} \\ -A_{js} & \overline{L}_j \end{pmatrix}$

The Laplacian matrix of corresponding to velocity adjacency matrix of the

erogeneous multi-agent system which is composed of a CT system
 $\begin{cases} x_i(t) = v_i(t), v_i(t) = u_i(t), & i \in I_m, \\ x_i(t) =$ orresponding to velocity adjacency matrix of the

which is composed of a CT system
 I_m ,
 I_n / I_m ,

-like and control input, respectively, of agent *i*. The 1 matrix of corresponding to velocity adjacency matrix of the

i-agent system which is composed of a CT system
 $\hat{i} = u_i(t), \quad i \in I_m,$
 $i \in I_n / I_m,$

Table (1-1)

-like, velocity-like and control input, respectively, of agen by velocity adjacency matrix of the

by a consection of a CT system

input, respectively, of agent *i*. The
 $\{1, 2\}$,

communication of agent *i* and *j*, B= $-\frac{A_{\varphi}}{L_f}$
 \overline{L}_f

intive of corresponding to velocity adjacency matrix of the

t system which is composed of a CT system

(*i*), $i \in \mathbb{I}_m$,
 $i \in \mathbb{I}_n / \mathbb{I}_m$,

(1-1)

velocity-like and control input, resp

$$
\begin{cases} x_i(t) = v_i(t), v_i(t) = u_i(t), & i \in \mathbb{I}_m, \\ x_i(t) = u_i(t), & i \in \mathbb{I}_n / \mathbb{I}_m, \end{cases}
$$

network is denoted by \hat{L}
 1.2 System model

In this subsection, we propose the heterogeneous multi-agent system which is composed of a CT system
 $\begin{cases} x_i(t) = v_i(t), v_i(t) = u_i(t), & i \in I_m, \\ x_i(t) = u_i(t), & i \in I_n / I_m, \end{cases}$

Table (1 **1.2 System model**

In this subsection, we propose the heterogeneous multi-agent system which is composed of a CT system $\{x_i(t) = v_i(t), v_i(t) = u_i(t), \quad i \in I_m, \ x_i(t) = u_i(t), \quad i \in I_m, \}$
 $x_i(t) = u_i(t), \quad i \in I_n / I_m,$

Table (1-1)

where $x_i \in$

$$
L = \begin{pmatrix} \overline{L}_x & -A_y \\ -A_y & \overline{L}_f \end{pmatrix}
$$

\nwhere \overline{L}_x , $L_x + D_y$, \overline{L}_y , $L_y + D_y$. The Laplacian matrix of corresponding to velocity adjacency matrix of the
\nnetwork is denoted by \hat{L} .
\n**1.2 System model**
\nIn this subsection, we propose the heterogeneous multi-agent system which is composed of a CT system
\n
$$
\begin{cases} x_0(t) = v_1(t), y_1(t) = u_x(t), & i \in I_m, \\ x_2(t) = u_x(t), & i \in I_m \end{cases}
$$

\nwhere $x_i \in \mathbb{R}$, $v_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ and are the position-like, velocity-like and control input, respectively, of agent *i*. The
\ninteractions among agents are realized through the following protocol,
\n
$$
u_x(t) = \begin{cases} \sum_{j=1}^{n} a_{ij} (h_i x_j - h_i x_j) + k_i h_i \sum_{j=1}^{m} b_{ij} (v_j - v_j), i \in I_m, i \in \{1, 2\}, \\ k_2 \sum_{j=1}^{m} a_{ij} (h_i x_j - h_i x_i), i \in I_n \end{cases}, i \in \{1, 2\}.
$$

\nTable(1-2)
\n
$$
[b_y]
$$
 is about velocity communication of agent *i* and *j*, B=
\n**Remark 1** h_i , h_j are constants being used to partition the sub-groups consisting of first-order integrator agents and
\nsecond-order integration agents, respectively.
\n**Definition 2** The heterogeneous multi-agent continuous-time system (1) is said to reach group consensus if for any initial conditions x_0 and v_0 , it follows that
\n
$$
\begin{cases} \lim_{x \to 0} |x_0(t) - x_1(t)| = 0, i, j \in I_m, \\ \lim_{x \to 0} |x_0(t) - x_1(t)| = 0, i, j \in I_m, \\ \lim_{x \to 0} |x_0(t) - x_1(t)| = 0, i, j \in I_m, \end{cases}
$$

*h*₁*x_i*), *i* \in I_n / I_m , *l* \in {1, 2}.

Table(1-2)
 *i*₂>0, A= $[a_{ij}]$, is about position communication
 nd j.

antition the sub-groups consisting of first-order in

ontinuous-time system (1) is said to

$$
\begin{cases}\n\lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0, i, j \in I_n, \\
\lim_{t \to \infty} \|v_i(t) - v_j(t)\| = 0, i, j \in I_m.\n\end{cases}
$$

 $i \in I_{\pi}/I_{\pi}$,

Table (1-1)

On-like, velocity-like and control input, respectively, of agent *i*. The

Dilowing protocol,
 $(x_i) + k_i h_2 \sum_{j=1}^{m} b_{ij} (v_j - v_i), i \in I_{\pi}, l \in \{1, 2\},$
 $h_i x_i, j_i \in I_{\pi}/I_{\pi}, I \in \{1, 2\}.$

Table(1-Table (1-1)

on-like, velocity-like and control input, respectively, of agent *i*. The

following protocol,
 $x_i + k_i h_i \sum_{j=1}^{m} b_{ij} (v_j - v_j), i \in I_m, l \in \{1, 2\}$,
 $h_i x_i, j_i \in I_n / I_m, l \in \{1, 2\}$.

Table(1-2)

Table(1-2)

Table(1 *tt*, *ti*, $i \in I_n / I_m$,

Table (1-1)

position-like, velocity-like and control input, respectively, of agent *i*. The

the following protocol,
 $t_j - h_2x_j + k_jh_2\sum_{j=1}^n b_{ij}(v_j - v_j), i \in I_m, l \in \{1, 2\},$
 $h_jx_j - h_ix_j), i \in I_n / I_m, I \in$ Table (1-1)
 viton-like, velocity-like and control input, respectively, of agent <i>i. The
 v_{is}x₁ + *k₁ h_i* $\sum_{j=1}^{n} b_{ij} (v_j - v_j)$, *i* $\in I_m$, *l* $\in \{1, 2\}$,
 rh_ix₁, *i* $\in I_n / I_m$, *l* $\in \{1, 2\}$. *i* e $\mathbb{I}_n / \mathbb{I}_m$,

Table (1-1)

The filter welocity-like and control input, respectively, of agent *i*. The

Illowing protocol,
 $x, y, h, \epsilon \int_{y=1}^{\pi} b_y (v_j - v_i), i \in \mathbb{I}_m, l \in \{1, 2\},$
 $x^x, h, \epsilon \in \mathbb{I}_n / \mathbb{I}_m, l \in \$ $\nu_z(t), \nu_z(t) = u_z(t), \quad i \in \mathbb{I}_n,$
 $i \in \mathbb{I}_s / \mathbb{I}_m$,
 $\tau(t), \quad i \in \mathbb{I}_s / \mathbb{I}_m$,

Table (1-1)

position-like, velocity-like and control input, respectively, of agent *i*. The

the following protocol,
 $\tau = h_x x, j + k, h_2 \sum_{$ **Remark 1** *h_{i, h₂* are constants being used to partition the sub-groups consisting of first-order integrator agents and ond-order integrator agents, respectively.
 Definition 2 The heterogeneous multi-agent continuo} **Example 2** The heterogeneous multi-agent continuous-time system (1) is said to reach group consensus if for any initial conditions x_0 and v_0 , it follows that
initial conditions x_0 and v_0 , it follows that
 $\lim_{$ **Definition 2** The heterogeneous multi-agent continuou
initial conditions x_0 and v_0 , it follows that
 $\begin{cases} \lim_{t\to\infty} ||x_i(t) - x_j||_{L^{\infty}} \end{cases}$
 $\begin{cases} \lim_{t\to\infty} ||x_i(t) - x_j||_{L^{\infty}} \end{cases}$
Remark 2 In the following analysis, Example $\lim_{t\to\infty} ||v_i(t) - v_j(t)|| = 0, i, j \in I_m$.

Remark 2 In the following analysis, for notational simplicity, we only consider the case, i.e., all age

in one-dimensional space. However, all results we have obtained can be e

2. Group consensus of heterogeneous multi-agent system

In this section, we consider the group consensus of the heterogeneous multi-agent systems composed of a CT system (1)

under undirected and directed fixed topology. Lb_y I is about velocity communication of agent *i* and *j*.
 **2. Group constants being used to partition the sub-groups consisting of first-order integrator agents and

2. Group** consensus if for any
 Definition 2 Th **If** $\lim_{t\to\infty} ||x_i(t) - x_j(t)|| = 0, i, j \in I_n$ **.**

Remark 2 In the following analysis, for notational simplicity, we only consider the case, i.e., all agents are assumed

in one-dimensional space. However, all results we have obtai Etheraire $\lim_{t \to \infty} ||x_i(t) - x_j(t)|| = 0, i, j \in I_n$.

Remark 2 In the following analysis, for notational simplicity, we only consider the case, i.e., all agents are assumed

in one-dimensional space. However, all results we have

 $T = \nabla$ *.*

 $\mathbf{x}_s \quad [x_1, x_2, \dots, x_m], \mathbf{v}_s \quad [v_1, v_2, \dots, v_m], \mathbf{x}_f \quad [x_{m+1}, x_{m+2}, \dots, x_n].$
The initial condition are $x_i(0)=x_{i0}$, $v_i(0)=v_{i0}$. Let $x_i = [x_{i0} \quad x_{20} \cdots x_{n0}]^T$, $v_i = [v_{i0} \quad v_{20} \cdots v_{n0}]^T$.
Let $p_i=h_{2x_i}$, $q_i=h_{2y_i}$ $[x_{m+1}, x_{m+2}, \dots, x_n].$
 *, v*₀= $[y_{10}, y_{20}, \dots, y_{n0}]$ ^{*r*}.

Recous multi-agent system (1) with protocol (2) \mathbf{x}_{s} $[x_{1}, x_{2}, \dots, x_{m}], \mathbf{v}_{s}$ $[v_{1}, v_{2}, \dots, v_{m}], \mathbf{x}_{f}$ $[x_{m+1}, x_{m+2}, \dots, x_{n}]$.

The initial condition are $x(0)=x_{00}, v_{i}(0)=v_{00}$. Let $x_{0} = [x_{i0}, x_{20}, \dots, x_{n0}]^{T}$, $v_{0} = [v_{i0}, v_{20}, \dots, v_{n0}]^{T}$.

Let $p_{i}=h_{i}x_{i}, q_{$ $\mathbf{x}_s \quad [x_1, x_2, \dots, x_m], \mathbf{v}_s \quad [v_1,$

The initial condition are $x_i(0)=x_{i0}, v_i(0)=v_{i0}$. Let $x_0 = [x]$

Let $p_i=h_2x_i, q_i=h_2v_i, i \in I_m$, and $p_i=h_1x_i, i \in I_w/I_M$. Thu

can be rewritten as follows,
 $\begin{cases} p_i(t) = q_i(t), i \in I_m, \end{cases}$

$$
x_x = [x_1, x_2, ..., x_m], v_y = [v_1, v_2, ..., v_m], x_y = [x_{m+1}, x_{m+2}, ..., x_n].
$$
\nThe initial condition are *x_i*(0)=*x_0*, *v_i*(0)=*v_0*. Let *x_0* = [*x_0*, *x_0* ···, *x_0*]^{-*t*}, *v_0* = [*x_0*, *x_0* ···, *x_0*]^{-*t*}, *v_0* = [*x_0*, *x_0* ···, *x_0*]^{-*t*}],
\nLet *p* = *hxx*, *q* = *hxy*, *i* ∈ I_m, and *p* = *hxx*, *i* ∈ I_n*y* |*x*. Thus, the heterogeneous multi-agent system (1) with protocol (2)
\nbe rewritten as follows,
\n
$$
\begin{cases}\np_i(t) = q_i(t), i \in I_m, \\
q_i(t) = h_i \sum_{j=1}^{k} a_{ij}(p_j - p_i), i \in I_n / I_m.\n\end{cases}
$$
\nTable (1-3)
\nObviously, system (2-1) with protocol (2-2) can solve the group consensus if and only if the system (2-3) can achieve
\ngroup consensus.
\nFor the undirected graph, under the symmetry condition of the weight matrices *A* and *B*, that is, *A* = *A*^T and *B* = *B*^T, one
\nobtain the following result.
\n**Theorem 1** For the connected undirected G with *A* = *A*^T and *B* = *B*^T, system (1) succeed by (2) can solve the group
\nsensus problem.
\nProof. Take the Lyapunov function,
\n
$$
V(t) = \sum_{i=1}^{k} \sum_{j=1}^{k} a_{ij} \frac{(p_j(t) - p_i(t))^2}{2} + \sum_{i=1}^{k} \frac{(q_i(t))^2}{h_i},
$$
\nwhich is positive definite with respect to

Table $(1-3)$

 $P_i(t) = q_i(t)$
 $q_i(t) = h_2 \sum_{j=1}^{n}$
 $p_i(t) = k_2 h_1$

Obviously, system (2-1) with protocol (2-2) c

the group consensus.

For the undirected graph, under the symmetry

can obtain the following result. hieve
, one
group Between the space of the connected undirected G with $A=A^T$ and consensus problem.
Theorem 1 For the connected undirected G with $A=A^T$ and consensus problem. **Theorem 1** For the connected undirected G with $A=A^T$ and $B=B^T$, system (1) see the connected graph, under the symmetry condition of the weight matrices *A* and *A* botain the following result.
Theorem 1 For the conne maximum is if and only if the system (2-3) can achieve
t matrices A and B, that is, $A = A^T$ and $B = B^T$, one
i, system (1) steered by (2) can solve the group Obviously, system (2-1) with protocol (2-2) ca
the group consensus.
For the undirected graph, under the symmetry
can obtain the following result.
Theorem 1 For the connected undirected C
consensus problem.
Proof. Take th Doviously, system (2-1) with protocol (2-2) can solve the group consensus if and only if the system (2-3) group consensus.

For the undirected graph, under the symmetry condition of the weight matrices A and B, that is, can solve the group consensus if and only if the system (2-3) can
 ny condition of the weight matrices *A* and *B*, that is, $A=A^T$ and $B=$
 G with $A=A^T$ and $B=B^T$, system (1) steered by (2) can solve th
 $\sum_{j=1}^{$ the group consensus if and only if the system (2-3)
 i of the weight matrices *A* and *B*, that is, $A = A^T$ are
 $= A^T$ and $B = B^T$, system (1) steered by (2) can sol
 $\frac{f(t) - p_i(t))^2}{2} + \sum_{i=1}^m \frac{(q_i(t))^2}{h_2}$,
 $\forall i \neq j$ can solve the group consensus if and only if the system (2-3) can achieve
 hy condition of the weight matrices *A* and *B*, that is, $A=A^T$ and $B=B^T$, one
 G with $A=A^T$ and $B=B^T$, system (1) steered by (2) can solv

$$
V(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \frac{(p_j(t) - p_i(t))^2}{2} + \sum_{i=1}^{m} \frac{(q_i(t))^2}{h_2}
$$

$$
\oint_{P_i}(t) = q_i(t), i \in I_{\infty},
$$
\n
$$
q_i(t) = h_i \sum_{j=1}^{n} a_{ij}(p_j - p_j) + k_i h_2 \sum_{j=1}^{n} b_{ij}(q_j - q_j), i \in I_{\infty}.
$$
\n
$$
p_i(t) = k_i \sum_{j=1}^{n} a_{ij}(p_j - p_j), i \in I_{\infty} \cap I_{\infty}.
$$
\nTable (1-3)

\nObviously, system (2-1) with protocol (2-2) can solve the group consensus if and only if the system (2-3) can achieve the group consensus.

\nFor the undirected graph, under the symmetry condition of the weight matrices *A* and *B*, that is, $A = A^T$ and $B = B^T$, one can obtain the following result.

\nTheorem 1 For the connected undirected G with $A = A^T$ and $B = B^T$, system (1) steered by (2) can solve the group consensus problem.

\nProof. Take the Lyapunov function,

\n
$$
V(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \frac{(p_j(t) - p_i(t))^2}{2} + \sum_{i=1}^{n} \frac{(q_i(t))^2}{h_i},
$$
\nwhich is positive definite with respect to

\n
$$
p_j(t) - p_j(t)(\forall i \neq j, i, j \in I_{\infty})
$$
\nDifferentiating

\n
$$
V(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(p_j - p_i)(p_j - p_j) + \sum_{i=1}^{n} \frac{2q_i(t)}{h_i},
$$
\n
$$
= \sum_{i=1}^{n} 2q_i \left(\sum_{j=1}^{n} a_{ij}(p_j - p_j) + k_i \sum_{j=1}^{n} b_{ij}(q_j - q_j) \right)
$$
\n
$$
+ \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(p_j - p_j)(q_j - q_i) + \sum_{i=n+1}^{n} \sum_{j=n+1}^{n} a_{ij}(p_j - p_j)(p_j - p_i).
$$
\nMethod of Section 7. Show that, $A = \sum_{i=1}^{n} a_{ij}(p_i - p_i)$, where $A = \sum_{i=1}^{n} a_{ij}(p$

Since $A = A^T$ and $B = B^T$, then

Since
$$
A = A^T
$$
 and $B = B^T$, then
\n
$$
\sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} (p_j - p_i)(q_j - q_i) = -2 \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} (p_j - p_i) q_i
$$
\n
$$
\sum_{i=m+1}^{n} \sum_{j=1}^{m} a_{ij} (p_j - p_i)(q_j - p_i) = \sum_{i=1}^{m} \sum_{j=m+1}^{n} a_{ij} (p_j - p_i)(p_j - q_i)
$$
\n
$$
\sum_{i=m+1}^{n} \sum_{j=m+1}^{n} a_{ij} (p_j - p_i)(p_j - p_i) = 2 \sum_{i=m+1}^{n} \sum_{j=m+1}^{n} a_{ij} (p_j - p_i) p_j
$$
\n
$$
\sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} (q_j - q_i)^2 = -\sum_{i=1}^{m} \sum_{j=1}^{m} b_{ij} (q_j - q_i) q_i
$$
\nHence,
\n
$$
V(t) = -2k_1 \sum_{i=1}^{m} b_{ij} (q_j - q_i)^2 + 2 \sum_{i=1}^{m} \sum_{j=m+1}^{n} a_{ij} (p_j - p_i) p_j + 2 \sum_{i=m+1}^{n} \sum_{j=m+1}^{n} a_{ij} (p_j - p_i) p_j
$$
\n
$$
= -2k_1 \sum_{i=1}^{m} b_{ij} (q_j - q_i)^2 + 2 \sum_{i=1}^{n} \sum_{j=m+1}^{n} a_{ij} (p_j - p_i) p_j
$$
\n
$$
= -2k_1 \sum_{i=1}^{m} b_{ij} (q_j - q_i)^2 - 2 \sum_{i=m+1}^{n} \sum_{j=1}^{n} a_{ij} (p_j - p_i)
$$

Since
$$
A=A^T
$$
 and $B=B^T$, then
\n
$$
\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (p_j - p_i) (q_j - q_i) = -2 \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (p_j - p_i) q_j
$$
\n
$$
\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (p_j - p_i) (q_j - p_i) = \sum_{j=1}^{n} \sum_{j=1}^{n} a_{ij} (p_j - p_i) (p_j - q_i)
$$
\n
$$
\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (p_j - p_i) (p_j - p_i) = 2 \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (p_j - p_i) p_j
$$
\n
$$
\sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} (q_j - q_i)^2 = -\sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} (q_j - q_i) q_i.
$$
\nHence,
\n
$$
V(t) = -2k \sum_{i=1}^{n} b_{ij} (q_i - q_i)^2 + 2 \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (p_j - p_i) p_j + 2 \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (p_j - p_i) p_j
$$
\n
$$
= -2k \sum_{i=1}^{n} b_{ij} (q_i - q_i)^2 + 2 \sum_{i=1}^{n} \sum_{j=n+1}^{n} a_{ij} (p_j - p_i) p_j
$$
\n
$$
= -2k \sum_{i=1}^{n} b_{ij} (q_i - q_i)^2 - 2 \sum_{i=1}^{n} \sum_{j=n+1}^{n} a_{ij} (p_j - p_i)
$$
\n
$$
= -2k \sum_{i=1}^{n} b_{ij} (q_i - q_i)^2 - \sum_{i=1}^{n} \sum_{j=n+1}^{n} a_{ij} (p_j - p_i)
$$
\n
$$
= -2k \sum_{i=1}^{n} b_{ij} (q_i - q_i)^2 - \sum_{i=1}^{n} \sum_{j=n+1}^{n} a_{ij} (p_j - p_i)
$$
\n
$$
= -2k \sum_{i=1}^{
$$

$$
\begin{cases}\n\lim_{t \to \infty} \| p_i(t) - p_j(t) \| = 0, i, j \in I_n, \\
\lim_{t \to \infty} \| q_i(t) - q_j(t) \| = 0, i, j \in I_m.\n\end{cases}
$$

1.
$$
\lim_{t \to 1} \left| \int_{x}^{t} f(t) dt \right|^{2} = \frac{1}{2} \left| \int_{t}^{t} f(t) dt \right|^{2}
$$
\n1.
$$
\lim_{t \to \infty} \left\| \int_{t}^{t} \left| \int_{t}^{t} f(t) dt \right|^{2} = 0, i, j \in I_{n},
$$
\n
$$
\lim_{t \to \infty} \left\| q_{i}(t) - q_{j}(t) \right\| = 0, i, j \in I_{m}.
$$
\n
$$
h_{l}x_{i}, i \in I_{n}/I_{m}, \text{ we get}
$$
\n
$$
\lim_{t \to \infty} h_{2} \left\| x_{i}(t) - x_{j}(t) \right\| = 0, i, j \in I_{m}
$$
\n
$$
\lim_{t \to \infty} h_{1} \left\| x_{i}(t) - x_{j}(t) \right\| = 0, i, j \in I_{n}/I_{m},
$$
\n
$$
\lim_{t \to \infty} h_{2} \left\| v_{i}(t) - v_{j}(t) \right\| = 0, i \in I_{m}.
$$
\n
$$
\lim_{t \to \infty} h_{2} \left\| v_{i}(t) - v_{j}(t) \right\| = 0, i \in I_{m}.
$$
\n
$$
\lim_{t \to \infty} \left| \int_{t}^{24} |f(x_{i})|^{2} = 0, i \in I_{m}.
$$

Due to $p_i = h_2x_i$, $q_i = h_2v_i$, $i \in I_m$, and $p_i = h_1x_i$, $i \in I_m / I_m$, we get
 $\begin{cases}\n\lim_{t \to \infty} h_2 ||x_i(t) - x_j(t)|| = 0, i, j \in I_m \\
\lim_{t \to \infty} h_1 ||x_i(t) - x_j(t)|| = 0, i, j \in I_n / I_m,\n\end{cases}$
 $\begin{cases}\n\lim_{t \to \infty} h_2 ||x_i(t) - x_j(t)|| = 0, i, j \in I_n / I_m, \\
\lim_{t \to \infty}$

under directed graph. A necessary lemma $1^{[24]}$ is needed.
 Lemma 1^[24] Suppose that $x = [x_1, x_2, ..., x_n]^T \in \mathbb{R}^n$ and $L = [l_{ij}] \in \mathbb{R}^{n \times n}$ satisfy that $l_{ij} \le 0, i \ne j$ and $\sum_{j=1}^N l_{ij} = 0, i = 1, 2, ..., N$. Then, the Due to $p_i = h_2x_i$, $q_i = h_2v_i$, $i \in I_m$, and $p_i = h_1x_i$, $i \in I_n/I_m$, we get
 $\begin{cases}\n\lim_{t \to \infty} h_2 ||x_i(t) - x_j(t)|| = 0, i, j \in I_m \\
\lim_{t \to \infty} h_1 ||x_i(t) - x_j(t)|| = 0, i, j \in I_n / I_m, \\
\lim_{t \to \infty} h_2 ||v_i(t) - v_j(t)|| = 0, i \in I_m.\n\end{cases}$

This completes the proo **Let**
 $L = -2k_1 \sum_{n=1}^{\infty} b_n (q_1 - q_1)^2 - \frac{2}{k_2 h_1} \sum_{n=1}^{\infty} p_1^2 \le 0,$

if $k_i > 0, k_i > 0, h_i > 0$. It follows from Lasalle's invariance principle ^[33] that
 $\begin{cases} \lim_{n \to \infty} |p_i(t) - p_j(t)| = 0, i, j \in \mathbb{I}_n, \\ \lim_{n \to \infty} |q_i(t) -2k_1 \sum_{i=1}^{n} b_i (a_i - a_j)^2 - 2 \sum_{i=1}^{n} p_i \sum_{j=1}^{n} a_j (p_j - p_i)$
 $= -2k_1 \sum_{i=1}^{n} b_i (a_i - a_j)^2 - \frac{2}{k_2 h_i} \sum_{i=1}^{n} p_i^2 \le 0,$
 *if k*_i>≈0, *k*_i>≈0, *h*, >⇒0. It follows from Lasable's invariance principle ^[23] that
 $\sum_{j=1}^{i} i_{ij} = 0, i = 1, 2,$, , $\mathbb{E} = -\mathbb{Z}\kappa_i \sum_{i=1}^n b_i (q_j - q_i) f^{-1} \frac{1}{k_2 h_i} \sum_{i=1}^n p_i^{n_i - 2\alpha_i}$
 l $\mathbf{f} k_i > 0, k_i > 0, h_i > 0$. It follows from Lasalle's invariance principle ^[23] that
 $\begin{cases} \lim_{t \to \infty} ||p_i(t) - p_j(t)|| = 0, i, j \in \mathbb{I}_n, \\ \lim_{t \to \infty} ||$ $= -2k_i \sum_{i=1}^{\infty} b_i (q_i - q_i)^2 - \frac{2}{k_i h_i} \sum_{i=1}^{\infty} p_i^2 \le 0,$

if $k_i > 0, k_i > 0, h_i > 0$. It follows from Lasalle's invariance principle ^[23] that
 $\lim_{t \to \infty} ||p_i(t) - p_j(t)|| = 0, i, j \in I_*,$

Due to $p_i - h_i x_i, q_i = h_i y_i, i \in I_m$, and p

(1) Consensus is reached asymptotically for the system $x = -Lx$;

(2) The directed graph of L has a directed spanning tree, e^{Lt} is a row stochastic matrix with positive diagonal entries
 $\forall t \ge 0$, there exists a nonneg (1) Consensus is reached asymptotically for the system $x = -Lx$;

(2) The directed graph of *L* has a directed spanning tree, e^L is a row stochastic matrix with positive diagonal entries
 $\forall t \ge 0$, there exists a nonne (1) Consensus is reached asymptotically for the system $x = -Lx$;

(2) The directed graph of *L* has a directed spanning tree, e^{Lt} is a row stochastic matrix with positive diagonal entries

for ∀*t* ≥ 0, there exists a n stic matrix with positive diagonal entries
 $e^{-Lt} \rightarrow 1$ as $t \rightarrow \infty$ where $c^T L = 0^T$ and

al parts; 1; where the contract of the c c^T **1** = 1; (1) Consensus is reached asymptotically for the syst

(2) The directed graph of *L* has a directed spanning
 $\forall t \ge 0$, there exists a nonnegative column vector
 1 = 1;

(3) The rank of *L* is *n-1*;

(4) *L* has a simp (1) Consensus is reached asymptotically for the system $x = -Lx$;

(2) The directed graph of *L* has a directed spanning tree, e^{kt} is a row stochastic matrix with positive diagonal entries $\forall t \ge 0$, there exists a nonne (1) Consensus is reached asymptotically for the system $x = -Lx$;

(2) The directed graph of *L* has a directed spanning tree, e^{Lt} is a row stochastic matrix with positive $\forall t \ge 0$, there exists a nonnegative column vec

-
-
-

 $\forall t \ge 0$, there exists a nonnegative column vector **c** ∈ Rⁿ, such that $e^{-Lt} \rightarrow 1$ as $t \rightarrow \infty$ where $c^TL = 0^T$ and
 Theorem 2 3) The rank of L is n-1;

(3) The rank of L is n-1;

(4) L has a simple zero eigenvalue c'1=1;

(3) The rank of L is n-1;

(4) L has a simple zero eigenvalue and all other eigenvalues have positive real parts;

(5) Lx=0 implies that $x_j = x_2 = ... = x_n$.

From Lemma 1, we have the following result for the directed $\frac{1}{h_2}$) \bar{d}_s , $k_2 > 0$, where $\bar{d}_s = \max_{i=1, ..., m} \sum_{j=1}^{n} a_{ij}$ is w stochastic matrix with positive diagonal entries

that $e^{-Lt} \rightarrow 1$ as $t \rightarrow \infty$ where $c^T L = 0^T$ and

sitive real parts;

sensus asymptotically if and only if the graph G
 $k_1 > (1 + \frac{1}{h_2})\overline{d}_s$, $k_2 > 0$, where $\overline{$ stochastic matrix with positive diagonal entries

hat $e^{-Lt} \rightarrow 1$ as $t \rightarrow \infty$ where $c^TL = 0^T$ and

tive real parts;

nsus asymptotically if and only if the graph G
 $> (1 + \frac{1}{h_2})\overline{d}_s$, $k_2 > 0$, where $\overline{d}_s = \max_{i=1,$ the graph G
 $max_{a_1, ..., m} \sum_{j=1}^{n} a_{ij}$ is *n s* if the graph G
 $\overline{S}_s = \max_{i=1, \dots, m} \sum_{j=1}^n a_{ij}$ is *d* is diagonal entries
 $c^T L = 0^T$ and
 $\overline{d}_s = \max_{i=1, \dots, m} \sum_{j=1}^n a_{ij}$ is
 $d = 0$ and $\overline{d}_s = \max_{i=1, \dots, m} \sum_{j=1}^n a_{ij}$ is $= \max_{i=1, ..., m} \sum_{j=1}^{n} a_{ij}$ is (3) The rank of *L* is *n*-*I*;

(4) *L* has a simple zero eigenvalue and all other eigenvalues have positive real parts;

(5) *Lx*=0 implies that $x_i = x_2 = ... = x_n$.

From Lemma 1, we have the following result for the directe (4) *L* has a simple zero eigenvalue and all other eigenvalues have positive real parts;

(5) *L*x=0 implies that $x_I = x_2 = ... = x_n$.

From Lemma 1, we have the following result for the directed graph.
 Theorem 2 System (1) (1) Consensus is reached asymptotically for the system $x = -Lx$;

(2) The directed graph of *L* has a directed spanning tree, e^2 is a row stochastic matrix with positive diagonal entries
 $\forall t \ge 0$, there exists a nonne consensus is reached asymptotically for the system $x = -Ix$;

the directed graph of *L* has a directed spanning tree, e¹ is a row stochastic matrix with positive diagonal entries
 >0 , there exists a nonnegative column From Lemma 1, we have the following result for the directed graph.
 Theorem 2 System (1) with protocol (2) can achieve group consensus asymptotically if and only

contains a directed spanning tree, if the feedbacks gain feedbacks gains satisfy $k_1 > (1 + \frac{1}{h_2})\overline{d}_s$, $k_2 > 0$, where \overline{d}_s
corresponding Laplician matrix \overline{L}_s .
 $q_m \overline{d}_s$, \mathbf{p}_f $[p_{m+1}, p_{m+2}, p_n]$, \overline{p}_s $[\mathbf{p}_s, \mathbf{q}_s, \mathbf{p}_f]$, then system
 $Q = -I_m$ Q alue and all other eigenvalues have positive real parts;
 πx_n .

following result for the directed graph.

h protocol (2) can achieve group consensus asymptotically if and only if the graph G

e. if the feedbacks gains tumn vector $\mathbf{c} \in \mathbb{R}^n$, such that $e^{-2t} \to 1$ as $t \to \infty$ where $c^T L = \mathbf{0}^T$ and

other eigenvalues have positive real parts;

Sult for the directed graph.

(2) can achieve group consensus asymptotically if and and all other eigenvalues have positive real parts;
 $-\overline{x}_n$.

Blowing result for the directed graph.
 t protocol (2) can achieve group consensus asymptotically if and only if the graph G
 t e., if the feedbacks gain **Example 11** other eigenvalues have positive real parts;
 k result for the directed graph.
 A (2) can achieve group consensus asymptotically if and only if the graph G
 E feedbacks gains satisfy $k_i > (1 + \frac{1}{h_2})\vec{a$ megative column vector $e \in R^{\alpha}$, such that $e^{-L} \rightarrow 1$ as $t \rightarrow \infty$ where $c^{T}L = 0^{T}$ and

value and all other eigenvalues have positive real parts;

...= x_n.

following result for the directed graph.

iith protocol (nive column vector $e \in \mathbb{R}^{n}$, such that $e^{-kt} \rightarrow 1$ as $t \rightarrow \infty$ where $e^{rt}L = 0^{T}$ and
 e and all other eigenvalues have positive real parts;
 \therefore
 \therefore

wing result for the directed graph.

rotocol (2) can achi *fs f O I O* **1.** We have the following result for the directed graph.
 1, we have the following result for the directed graph.
 System (1) with protocol (2) can achieve group consensus asymptotically if and only if

ted spanning 1, we have the following result for the directed graph.

System (1) with protocol (2) can achieve group consensus asymptotically if and only if

ted spanning tree, if the feedbacks gains satisfy $k_i > (1 + \frac{1}{h_2})\overline{d}_i$, satisfy $k_1 > (1 + \frac{1}{h_2})\overline{d}_x$, $k_2 > 0$, where $\overline{d}_x = \max_{i=1, \dots, m} \sum_{j=1}^{\infty} a_{ij}$ is

uplician matrix \overline{L}_x .
 p_{m+2}, \dots, p_n]⁷, ξ [**p**₁, **q**₁, **p**₁], then system (3) can be
 $\frac{O}{h_1 L_y}$
 $\left(\frac{h_2 A$

 q_2 , q_m]^T, \mathbf{p}_f [p_{m+1} , p_{m+2} , p_n]^T, ξ [\mathbf{p}_s , \mathbf{q}_s , \mathbf{p}_f], then system (3) can b
 $\begin{vmatrix}\nO & -I_m & O \\
h_2\overline{L}_s & k_1h_2\hat{L} & -h_2A_{sf} \\
-k_2h_1A_{fs} & O & k_2h_1\overline{L}_f\n\end{vmatrix} \xi(t) - \Phi \xi(t)$

Table $D_{n+1}, D_{n+2}, \quad D_n \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [p_1, \mathbf{q}_2, p_2]$, then system (3) can be
 $\left(\frac{\partial}{\partial t} - h_2 A_{\mathbf{y}}\right) \xi(t)$ $-\Phi \xi(t)$
 $\left(\frac{\partial}{\partial t} h_1 L_f\right)$
 $\left(\frac{\partial}{\partial t} h_2 L_f\right)$
 $\left(\frac{\partial}{\partial t} h_1 L_f\right)$
 $\left(\frac{\partial}{\partial t} h_2 L_f\right)$
 ponding Laplician matrix \overline{L}_s .
 \mathbf{p}_f $[p_{m+1}, p_{m+2}, p_{n}]^T$, ξ $[\mathbf{p}_s, \mathbf{q}_s, \mathbf{p}_f]$, then system (3) can be
 $-\overline{L}_w$ Q
 $k_2h_1\overline{L}_d$ $-h_2A_{N_f}$ $\xi(t)$ $-\Phi \xi(t)$
 \overline{C} \overline{C} \overline{C} \overline{C}

$$
p_{2}, \quad , p_{m} \right]^{T}, \mathbf{q}_{s} \quad [q_{1}, q_{2}, \quad , q_{m} \right]^{T}, \mathbf{p}_{f} \quad [p_{m+1}, p_{m+2}, \quad , p_{n} \right]^{T}, \xi \quad [p_{s}, \mathbf{q}_{s}, \mathbf{p}_{f}] \quad \text{then \ system}
$$
\n
$$
\xi(t) = -\begin{pmatrix} 0 & -I_{m} & 0 \\ h_{2} \overline{L}_{s} & k_{1} h_{2} \hat{L} & -h_{2} A_{s} \\ -k_{2} h_{1} A_{s} & 0 & k_{2} h_{1} \overline{L}_{f} \end{pmatrix} \xi(t) \quad -\Phi \xi(t)
$$
\n
$$
\text{Table}(1-4)
$$
\n
$$
0 \quad -I_{m} \quad 0 \quad \text{Right} \quad \text{and} \quad \Theta \text{ is the zero matrix.}
$$
\n
$$
h_{2} \overline{L}_{s} \quad k_{1} h_{2} \hat{L} \quad -h_{2} A_{s} \quad 0 \quad k_{2} h_{1} \overline{L}_{f} \text{,}
$$
\n
$$
h_{2} \overline{L}_{s} \quad k_{1} h_{2} \hat{L} \quad -h_{2} A_{s} \quad 0 \quad k_{2} h_{1} \overline{L}_{f} \text{,}
$$
\n
$$
\text{Fix } \Phi \text{ using the following nonsingular matrix}
$$
\n
$$
(I \quad 0 \quad 0 \quad 0)
$$

Table $(1-4)$

where $\begin{array}{ccc} & \circ & & \stackrel{-I_m}{\circ} \\ \circ & & \end{array}$ $\binom{n_1}{2}$ $\binom{n_2}{2}$ $\binom{n_3}{3}$ $\begin{pmatrix} 0 & -I_m & 0 \end{pmatrix}$ I is an identity matrix and O is the p Φ $\left| h_i \overline{L}_s - k_l h_j \hat{L} - h_i A_s \right|$, I is an identity matrix and O is the zero map $\left(-k_2 h_1 A_{f_2} \quad O \quad k_2 h_1 \overline{L}_f\right)$ $\zeta(t) = -\begin{pmatrix} O & -I_m & O \\ h_2 \overline{L}_s & k_1 h_2 \hat{L} & -h_2 A_{sf} \\ -k_2 h_1 A_{fs} & O & k_2 h_1 \overline{L}_f \end{pmatrix} \zeta(t) \quad -\Phi \zeta(t)$

Table(1-4)

where Φ $\begin{pmatrix} O & -I_m & O \\ h_2 \overline{L}_s & k_1 h_2 \hat{L} & -h_2 A_{sf} \\ -k_2 h_1 A_{fs} & O & k_2 h_1 \overline{L}_f \end{pmatrix}$, *I* is an $\left[L -h_2 A_{sf} \atop k_2 h_1 L_f \right] \xi(t)$ $-\Phi \xi(t)$

able(1-4)

matrix and *O* is the zero matrix.

r matrix
 $\left[\begin{array}{ccc} m & 0 & 0 \\ m & I_m & 0 \\ 0 & 0 & I_{n-m} \end{array} \right],$ *n m* where Φ $\begin{pmatrix} O & -I_m & O \\ h_2 \overline{L}_s & k_1 h_2 \hat{L} & -h_2 A_g \\ -k_2 h_1 A_s & O & k_2 h_1 \overline{L}_f \end{pmatrix}$, *I* is an identity matrix and *O* is the zero m

Transform matrix Φ using the following nonsingular matrix
 $Q = \begin{pmatrix} I_m & 0 & 0 \\ I_m$

$$
Q = \begin{pmatrix} I_m & 0 & 0 \\ I_m & I_m & 0 \\ 0 & 0 & I_{n-m} \end{pmatrix},
$$

$$
\mathbf{L} = \mathbf{P}_s \quad (P_1, P_2, \dots, P_m) \cdot \mathbf{q}_m \cdot \mathbf{q}_m \cdot \mathbf{q}_m \cdot \mathbf{p}_f \quad (P_{m+1}, P_{m+2}, \dots, P_n) \cdot \mathbf{s} \quad (P_s, \mathbf{q}_s, P_f) \quad \text{when } s \text{, so} \quad (z) \text{ can be}
$$
\n
$$
\xi(t) = -\begin{pmatrix} 0 & -I_m & 0 \\ h_2 \overline{L}_s & h_1 h_2 \hat{L}_s & -h_2 A_g \\ -h_2 h_1 A_s & 0 & k_2 h_1 \hat{L}_s \end{pmatrix} \xi(t) & -\Phi \xi(t)
$$
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$$
\text{Table}(1-4)
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$$
\text{where } \Phi \begin{pmatrix} 0 & -I_m & 0 \\ h_2 \overline{L}_s & k_1 h_2 \hat{L}_s + h_2 A_g \\ -k_2 h_1 A_s & 0 & k_2 h_1 \hat{L}_s \end{pmatrix}, I \text{ is an identity matrix and } O \text{ is the zero matrix.}
$$
\n
$$
Q = \begin{pmatrix} I_m & 0 & 0 \\ I_m & I_m & 0 \\ 0 & 0 & I_{n-m} \end{pmatrix}.
$$
\n
$$
\text{Then we can get}
$$
\n
$$
\Phi^* = Q \Phi Q^{-1} = \begin{pmatrix} I_m & -I_m & 0 \\ h_2 \overline{L}_s - k_1 h_2 \hat{L}_m + I_m & k_1 h_2 \hat{L}_m - I_m & -h_2 A_g \\ -k_2 h_1 I_{m-m} k A_s & 0 & k_2 h_1 I_{m-m} \hat{L}_f \end{pmatrix}.
$$
\n
$$
\text{Table (1-5)}
$$

Table $(1-5)$

$$
\eta(t)=-\Phi^*\eta(t).
$$

Let $\eta(t) = Q\xi(t)$, then we can rewrite system (6) as follows:
 $\eta(t) = -\Phi^* \eta(t)$.

Obviously, Φ^* and Φ have the same eigenvalues. Since $k_1 > (1 + \frac{1}{h_2})\overline{d}_s$, $k_2 > 0$, it follows that all dia

matrix Φ^* are $\frac{1}{h_2}$) \overline{d}_s , $k_2 > 0$, it follows that all diagonal elemer ories:
 $k_1 > (1 + \frac{1}{h_2})\overline{d_s}, k_2 > 0$, it follows that all diagonal elements

are nonpositive. Hence, the matrix Φ^* can be seen as a $> (1 + \frac{1}{h_2})\overline{d}_s$, $k_2 > 0$, it follows that all diagonal elements
are nonpositive. Hence, the matrix Φ^* can be seen as a Let $\eta(t) = Q\xi(t)$, then we can rewrite system (6) as follows:
 $\eta(t) = -\Phi^*\eta(t)$.

Obviously, Φ^* and Φ have the same eigenvalues. Since $k_1 > (1 + \frac{1}{h_2})\overline{d}_s$, $k_2 > 0$, it follows that all diagonal elements

of ma Let $\eta(t) = Q\xi(t)$, then we can rewrite system (6) as follows:
 $\eta(t) = -\Phi^t \eta(t)$.

Obviously, Φ^t and Φ have the same eigenvalues. Since $k_1 > (1 + \frac{1}{h_2})\overline{d}_s$, $k_2 > 0$, it follows that all diagonal ele

of matrix *I* of the matrix of can be seen as a
I_m O
I_m O
 O *L O*, it follows that all diagonal elements

Hence, the matrix ϕ^* can be seen as a
 $\begin{pmatrix} I_m & O \\ O & L \end{pmatrix}$

$$
Φ \quad \text{have the same eigenvalues. Since } k_1 > (1 + \frac{1}{h_2})\overline{d}_s, k_2 > 0 \text{, it follows that all diagonal elements}
$$
\n
$$
n\n\text{mnegative and all nondiagonal elements are nonpositive. Hence, the matrix $Φ^*$ can be seen as a of graph $θ^*$.\n
$$
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$$
a_n = \int_0^{\pi} e^{i\theta} \cdot e^{i\theta} \, d\theta
$$
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a_n = \int_0^{\pi} e^{i\theta} \cdot e^{i\theta} \, d\theta
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a_n = \int_0^{\pi} e^{i\theta} \, d\theta
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a_n = \int_0^{\pi} e^{i\theta} \, d\theta
$$
\n
$$
a_n = \int_0^{\
$$

Table $(1-6)$

Therefore,

Let $\eta(t) = Q\xi(t)$, then we can rewrite system (6) as follows:
 $\eta(t) = -\Phi^t \eta(t)$.

Obvoously, Φ^t and Φ have the same eigenvalues. Since $k_i > (1 + \frac{1}{h_2})\vec{a}_i$, $k_i > 0$, it follows that all diagonal elements

matrix Let $\eta(t) = Q\xi(t)$, then we can rewrite system (6) as follows:
 $\eta(t) = -\Phi^*\eta(t)$.

Obviously, Φ^* and Φ^- have the same eigenvalues. Since $k_i > (1 + \frac{1}{\hbar_i})\overline{d}_{\rho_i}, k_i > 0$, it follows that all diagonal elements

volid $\Phi^* \to \begin{bmatrix} 0 & h_2\overline{L}_s & -h_2A_{sf} \\ 0 & -k_2h_1A_{\beta} & k_2h_1\overline{L}_f \end{bmatrix} \to \begin{bmatrix} 0 & \overline{L}_s & -A_{sf} \\ 0 & -A_{\beta} & \overline{L}_f \end{bmatrix} = \begin{bmatrix} I_m & O \\ O & L \end{bmatrix}$

Table (1-6)

Therefore,
 $rank(\Phi^*) = m + rank(L)$.

From Lemma 1, we have know that $rank(L$ consensus. Therefore,
 $rank(\Phi^*) = m + rank(L)$.

From Lemma 1, we have know that $rank(L) = n - 1$. Since G contains a directed spanning tree, and from (2-6), we

nothain that $rank(\Phi^*) = m + n - 1$. Hence, graph G^{*} also has a directed spanning tree. Ther Table $(1-6)$
 $rank(\Phi^*) = m + rank(L)$.

From Lemma 1, we have know that $rank(L) = n-1$. Since G contains a directed spanning tree, and from (2-6), we

can obtain that $rank(\Phi^*) = m + n - 1$. Hence, graph G° also has a directed spanning tree. T Therefore,
 rank(Φ^*) = $m + rank(L)$.

From Lemma 1, we have know that $rank(L) = n - 1$. Since G contains a direct

can obtain that $rank(\Phi^*) = m + n - 1$. Hence, graph G^{*} also has a directed s

reaches consensus asymptotically, and *rank*(Φ) = $m + rank(L)$.

From Lemma 1, we have know that $rank(L) = n - 1$. Since G contains a directed spanning tree, and from (2-6), we

a obtain that $rank(\Phi^*) = m + n - 1$. Hence, graph G' also has a directed spanning tree. Therefor From Lemma 1, we have know that $rank(L) = n-1$. Since G contains a directed spanning tree, and from (2-6), we
can obtain that $rank(\Phi^*) = m + n-1$. Hence, graph G^{*} also has a directed spanning tree. Therefore, system (2-5)
reaches **Theorem 3** If system (1) with protocol (2) can achieve group consensus asymptotically, and then the heterogeneous multi-agent system (2-3) can achieve the group sensus.
Necessity: By contradiction, assuming that G has no

consensus.

Necessity: By contradiction, assuming that G has no a directed spanning tree, then G also has no a directed tree. From

Lemma 1, we can easily know system (5) can not reaches consensus asymptotically, which co

Therefore,
\n
$$
rank(\Phi^*) = m + rank(L)
$$
.
\nFrom Lemma 1, we have know that $rank(L) = n - 1$. Since G contains a directed spanning tree, and from (2-6), we
\ncan obtain that $rank(\Phi^*) = m + n - 1$. Hence, graph G⁺ also has a directed spanning tree. Therefore, system (2-5)
\nreaches consensus asymptotically, and then the heterogeneous multi-agent system (2-3) can achieve the group
\nconsensus.
\nNecessity: By contradiction, assuming that G has no a directed spanning tree, then G also has no a directed tree. From
\nLemma 1, we can easily know system (5) can not reaches consensus asymptotically, which contradicts to the condition
\nof Theorem 2. This completes the proof of necessity.
\nIn general, most of the results of existing literatures only show that the convergence of the system (1), but don't pay
\nmuch attention to the specific convergence points. In the following, we will consider this problem.
\n**Theorem 3** If system (1) with protocol (2) can achieve group consensus asymptotically, then the conver-gence points
\nare
\n
$$
\lim_{t \to \infty} x_s = \frac{1}{h_2} (k_1 k_2 \ln_n h_1 \ln_n t_1^T x_s(0) + k_2 \ln_n h_1 \ln_n t_1^T x_s(0) + 1_n k_2 \ln_n t_2^T x_s(0)) + 1
$$
\n
$$
\lim_{t \to \infty} x_s = \frac{1}{h_1} (k_1 k_2 \ln_n h_1 \ln_n t_1^T x_s(0) + k_2 \ln_n h_1 \ln_n t_1^T x_s(0) + 1_{n = h_1} (k_2^T x_s(0)) + 1
$$
\nProof. First of all, based on the meaning of h_1 , h_2 of the group consensus in this paper, we have that system (1) with
\nprotocol (2) can achieve group consensus, as $t \to \infty$ if
\n
$$
\sum_{t \to \infty} \sum_{s=1}^{\infty} (k_1 k_2 \ln_n h_1 \ln_n t_1^T x_s(0) + k_2 \ln_n h_1 \ln_n t_1^T x_s(0) + 1_{n = h_1} (k_2^T x_s(0)) + 1
$$
\nProof. First of all, based on the meaning of h_1 , h_2 of the group consensus in this paper, we have that system (1) with
\nprotocol (2) can achieve group consensus, as $t \to \infty$

20| Zhang YX. 20| Zhang YX. 20| Zhang YX. **20|** $\sum_{n=1}^{n} M_1 D_2 L_1 \Delta x_n(0) + \Delta_{2,n=m} M_1 P_3 (0) + \Delta_{2,n=m} M_1 P_4 (1 - \Delta_{3,n})$
20| Zhang YX. 20| Zhang YX. **20|** Zhang YX. **20|** Zhang YX. **20|** Zhang YX. **20|** Zhang YX. **20**

$$
x_i - h_i \to c, l \in \{1, 2\}, i \in I_n
$$
\n
$$
v_i \to 0, i \in I_m
$$
\nTable (1-7)

\nodes in the two groups can reach a consensus state asymptotically while

Table $(1-7)$

 $, l \in \{1, 2\}, i \in I_n$,
 I_m

able (1-7)

two groups can reach a consensus state asymptotically while $x_i - h_i \rightarrow c, l \in \{1, 2\}, i \in I_n$
 $v_i \rightarrow 0, i \in I_m$

Table (1-7)
 i and the two groups can reach a consensus state asymptotically while $x_i - h_i \rightarrow c, l \in \{1, 2\}, i \in I_n$
 $v_i \rightarrow 0, i \in I_m$

Table (1-7)
 des in the two groups can reach a consensus state asymptotically while $-h_i \rightarrow c, l \in \{1, 2\}, i \in I_n$
 $\rightarrow 0, i \in I_m$

Table (1-7)

as in the two groups can reach a consensus state asymptotically while $x_i - h_i \rightarrow c, l \in \{1, 2\}, i \in I_n$
 $v_i \rightarrow 0, i \in I_m$

Table (1-7)

where *c* is a constant. It means that the nodes in the two groups can reach a consensus state asymptotically while

there is no consensus among different groups.

 $x_i - h_i \rightarrow c, l \in \{1, 2\}, i \in I_n$,
 $v_i \rightarrow 0, i \in I_m$

Table (1-7)

where *C* is a constant. It means that the nodes in the two groups can re

there is no consensus among different groups.

According to Lemma 1, we can know that m $x_i - h_i \rightarrow c, l \in \{1, 2\}, i \in I_n$

Table (1-7)

Table 0, and the real part of non-ze $x_i - h_i \rightarrow c, l \in \{1, 2\}, i \in I_n$

Table (1-7)

Where *C* is a constant. It means that the nodes in the two groups can reach a consensus state asymptotically while

there is no consensus among different groups.

According to Le reach a consensus state asymptotically while

e eigenvalue 0, and the real part of non-zero

cal form of matrix $-\Phi$, then we have
 $\int_{1\times(n+m-1)}^{1} \left(\begin{array}{c} \gamma_1^T \\ \gamma_1^T \\ \gamma_2^T \end{array} \right)$. each a consensus state asymptotically while

eigenvalue 0, and the real part of non-zero

ul form of matrix $-\Phi$, then we have
 $\begin{pmatrix} \gamma_1^r \\ \gamma_2^r \\ \gamma_{n+m}^r \end{pmatrix}$. -7)

oups can reach a consensus state asymptotically v

only one eigenvalue 0, and the real part of non-

in canonical form of matrix $-\Phi$, then we have
 $\left.\begin{array}{cc} 0 & 0_{1\times(n+m-1)} \\ 0 & J' \end{array}\right| \begin{pmatrix} \gamma_1^T \\ \gamma_{n+m}^T \end{pmatrix}$.
 reach a consensus state asymptotical

: eigenvalue 0, and the real part of a

cal form of matrix $-\Phi$, then we have
 $\begin{pmatrix} x_{(n+m-1)} \\ y' \\ y' \\ y_{n+m} \end{pmatrix}$. [1,2}, $i \in I_n$,

groups can reach a consensus state asymptotically while

has only one eigenvalue 0, and the real part of non-zero

ordan canonical form of matrix $-\Phi$, then we have
 $\begin{pmatrix} 0 & 0_{1\times(n+m-1)} \\ 0_{(n+m-1)} & J' \end{pm$

$$
x_i - h_i \rightarrow c, l \in \{1, 2\}, i \in I_n
$$

\n
$$
v_i \rightarrow 0, i \in I_m
$$

\nTable (1-7)
\nThis that the nodes in the two groups can reach a consensus state asymptotically while
\nferent groups.
\ncan know that matrix Φ has only one eigenvalue 0, and the real part of non-zero
\nrequal to 0. Let J is the Jordan canonical form of matrix $-\Phi$, then we have
\n
$$
-\Phi = PJP^{-1} = (\omega_1, \quad, \omega_{n+m}) \begin{pmatrix} 0 & 0_{1 \times (n+m-1)} \\ 0_{(n+m-1)} & J' \end{pmatrix} \begin{pmatrix} \gamma_1^T \\ \gamma_1^T \\ \gamma_{n+m}^T \end{pmatrix}.
$$

\ne choose $\omega_1 = \begin{bmatrix} 1_n^T, 0_n^T, 1_{n-m}^T \end{bmatrix}^T$, and easily know ω_1 is a right eigenvector of eigenvalue

 $j-h_i \rightarrow c, l \in \{1,2\}, i \in I_n$

Table (1-7)

Es in the two groups can reach a consensus state asymptotically while

matrix Φ has only one eigenvalue 0, and the real part of non-zero
 J is the Jordan canonical form of matri $i \in I_n$,

(i) $i \in I_n$,

(i) only one eigenvalue 0, and the real part of non-zero

(i) only one eigenvalue 0, and the real part of non-zero

(i) canonical form of matrix $-\Phi$, then we have

(i) $\begin{pmatrix} \gamma_1^T \\ \gamma_{n+m-1}^T \end{pm$ $h_i \rightarrow c, l \in \{1, 2\}, i \in I_n$

Table (1-7)

in the two groups can reach a consensus state asymptotically while

trix Φ has only one eigenvalue 0, and the real part of non-zero
 I is the Jordan canonical form of matrix $-\Phi$ each a consensus state asymptotically while

eigenvalue 0, and the real part of non-zero

1 form of matrix $-\Phi$, then we have
 $\begin{pmatrix} r_1^T \\ r_2^T \\ r_{n+m}^T \end{pmatrix}$
 J'
 $\begin{pmatrix} r_1^T \\ r_{n+m}^T \end{pmatrix}$.
 $\begin{pmatrix} r_1^T \\ r_{n+m}^T \end{$ $-h_i \rightarrow c, l \in \{1, 2\}, i \in I_n$

Table (1-7)

So in the two groups can reach a consensus state asymptotically while

matrix Φ has only one eigenvalue 0, and the real part of non-zero
 J is the Jordan canonical form of matri $x_i - b_i \rightarrow c_i, l \in \{1, 2\}, i \in I_n$
 $v_i \rightarrow 0, i \in I_n$

Table (1-7)

ere c is a constant. It means that the nodes in the two groups can reach a consensus state asymptotically while

re is no consensus among different groups.

Accor Without loss of generality, we choose $\omega_1 = \left[\mathbf{1}_m^T, \mathbf{0}_m^T, \mathbf{1}_{n-m}^T\right]^T$, and easily know ω_1 is a right eigenvector of eigenvalue *T*_{*x*} $-h_i \rightarrow c, l \in \{1, 2\}, i \in I_n$
 *T*able (1-7)
 Table (1-7)
 Table 1-7)
 Table 1-7)
 Table 1-8
 Table 1-8 is a right eigenvector of eigenvalue

and the real part of non-zero
 γ_1^T
 γ_1^T

is a right eigenvector of eigenvalue

as one 0 eigenvalue. And we know According to Lemma 1, we can know that matrix Φ has only one eigenvalue 0, and the real part of non-zero
eigenvalues are all greater than or equal to 0. Let J is the Jordan canonical form of matrix $-\Phi$, then we have $x_i - h_i \rightarrow c, l \in \{1, 2\}, t \in \mathbb{T}_n$
 $y_i \rightarrow 0, l \in \mathbb{T}_n$

The $(1-7)$

where c is a constant. It means that the nodes in the two groups can reach a consensus state asymptotically while

there is no consensus among different gr is the Jordan canonical form of matrix $-\Phi$, then we have
 $\left.\rho_{n+m}\right)\left(\begin{array}{cc} 0 & 0_{1\times(n+m-1)} \\ 0 & J' \end{array}\right)\left(\begin{array}{c} \gamma_1^T \\ \gamma_{n+m}^T \end{array}\right)$.
 $\left.\rho_{n+m}^T\right]$, and easily know ω_1 is a right eigenvector of eigenvalus

span $\left\{\n\begin{aligned}\n &\text{(0)} \quad 0_{1 \times (n+m-1)} \\
 &\text{(0)} \quad \text{(0)} \quad \text{($ $\left(\begin{array}{cc} 0 & 0_{1\times(n+m-1)} \\ 0 & J' \end{array}\right) \begin{pmatrix} \gamma_1' \\ \gamma_1 \\ \gamma_{n+m} \end{pmatrix}.$
 $\left(\begin{array}{cc} 1_{n+m}^T \\ 1_{n+m}^T \end{array}\right)^T$, and easily know ω_1 is a right eigenvector

value ω_1 is a right eigenvector
 ω_2 is a right eigenvector
 Table (1-7)

the two groups can reach a consensus state asymptotically while
 $\mathbf{x} \Phi$ has only one eigenvalue 0, and the real part of non-zero

is the Jordan canonical form of matrix $-\Phi$, then we have
 $\mathbf{a}_{n+m} \begin{pmatrix}$ Table (1-7)

ne two groups can reach a consensus state asymptotically while
 Φ has only one eigenvalue 0, and the real part of non-zero

the Jordan canonical form of matrix $-\Phi$, then we have
 ω_{n+m} $\begin{pmatrix} 0 & 0_{1\times(n$ $T_i \rightarrow c, t \in \{1, 2\}, t \in I_n$,
 $0, i \in I_m$

Table (1-7)

in the two groups can reach a consensus state asymptotically v

trix Φ has only one eigenvalue 0, and the real part of non
 J is the Jordan canonical form of matrix Table (1-7)

the two groups can reach a consensus state asymptotically wh

rix Φ has only one eigenvalue 0, and the real part of non-z

is the Jordan canonical form of matrix $-\Phi$, then we have
 $\binom{0}{a_{n+m}}\begin{pmatrix}0&0_{1$ $n_i \rightarrow c, i \in \{1, 2\}, i \in I_n$,

Table (1-7)

in the two groups can reach a consensus state asymptotically while

natrix Φ has only one eigenvalue 0, and the real part of non-zero
 J is the Jordan canonical form of matrix **Fig. 1.** Table (1-7)

in the two groups can reach a consensus state asymptotically while

hatrix Φ has only one eigenvalue 0, and the real part of non-zero
 J is the Jordan canonical form of matrix $-\Phi$, then we hav ere C is a constant. It means that the nodes in the two groups can reach a consensus state asymptotically w

re is no consensus among different groups.

According to 1 cmma 1, we can know that matrix Φ has only one eig s.

that matrix Φ has only one eigenvalue 0, and the real part of non-zero
 Let J is the Jordan canonical form of matrix $-\Phi$, then we have
 ${}^{i} = (\omega_1, \dots, \omega_{\text{even}}) \begin{pmatrix} 0 & 0_{\nu(\text{even}-1)} & \gamma^r \\ 0_{\text{even}-1} & J & J \end{pmatrix}$.
 Table (1-7)

tim. It means that the nodes in the two groups can reach a consensus state asymptotically while

among different groups.

the real part of non-zero

reader than or equal to 0. Let J' is the Jordan cunonical Without loss of generality, we choose $\omega_1 = [\Gamma_n^T, 0_n^T, \Gamma_{n,m}^T]^T$, and easily know ω_1 is a right eigenvector of eigenvalue

0 of matrix Φ . The directed topology G contains a spanning tree, so matrix L only has $\left[I_n^T, 0_n^T, I_{n=m}^T\right]^T$, and easily know ω_i is a right eigenvector of eigenvalue

ains a spanning tree, so matrix *L* only has one 0 eigenvalue. And we know
 $\mu_i^T J \in \mathbb{R}^n$ with $\mu_i^T L = 0$, then
 $\left[\mu_i^T \overline{L}_$ $\left(\begin{array}{cc} 0_{(q+m-1)} & J & J_{(q+m)} \end{array} \right)$
 $\left(\begin{array}{c} 0_{m+1}^T \end{array} \right)$, $\left(\begin{array}{c} 0_{m+1}^T \end{array} \right)$, $\left(\begin{array}{c} 0_{m+1}^T \end{array} \right)$, $\left(\begin{array}{c} 0_{m+1}^T \end{array} \right)$, and easily know ω_1 is a right eigenvector of eigenval

$$
\begin{cases} \mu_1^T \overline{L}_s - \mu_2^T A_{fs} = 0, \\ -\mu_1^T A_{sf} + \mu_2^T \overline{L}_f = 0. \end{cases}
$$

$$
\gamma_1^T = \left[k_1 k_2 h_1 h_2 \mu_1^T \hat{L}, k_2 h_1 \mu_1^T, h_2 \mu_2^T \right],
$$

there exists an non-negative vector $\mu' = [\mu'_1, \mu'_2] \in \mathbb{R}^n$ with $\mu' L = 0$, then
 $\begin{cases} \mu''_1 L^2 - \mu''_1 A_{ij} + \mu''_2 L^2 = 0. \end{cases}$

According to $\gamma_1 \Phi = \gamma_1 \lambda$, we can get
 $\gamma_1^T = [\lambda_1 k_2 h_1 h_2 \mu'_1 \hat{L} , k_2 h_1 \mu'_1 \hat{h}_2 \mu$ as a column vector of *P*, and then γ_1^T is
ne real part $Re(\lambda_i) < 0 (i = 1, 2, ..., n)$ of $\left\{\mu_1^T \overline{L}_s - \mu_2^T A_{fs} = 0, \ -\mu_1^T A_{sf} + \mu_2^T \overline{L}_f = 0.\right\}$
According to $\gamma_1 \Phi = \gamma_1 \lambda$, we can get
 $\gamma_1^T = \left[k_1 k_2 h_1 h_2 \mu_1^T \hat{L}_s k_2 h_1 \mu_1^T h_2 \mu_2^T\right]$
it is a left eigenvector corresponding to eigenvalue a row vector of P^{-1} . In addition, matrix $-\Phi$ has only one eigenvalue 0 and the real part $Re(\lambda) < 0$ ($i = 1, 2, \dots, n$) of $\gamma_1^{\text{T}} = \left[k_1 k_2 h_1 h_2 \mu_1^{\text{T}} h_2 \mu_2^{\text{T}} \right],$
it is a left eigenvector corresponding to eigenvalue 0 of matrix Φ . Choose ω_1 as a column vector of *P*, and then γ_1^{T} is
a row vector of *P*⁻¹. In addit $A_{jk} = 0$,
 $A_{jk} = 0$,
 $A_{k}^{T} \overline{L}_{j} = 0$.
 $\hat{L}_{k} k_{j} h_{k} \mu_{i}^{T} h_{j} \mu_{j}^{T}$,
 \overline{L}_{k}^{T} is eigenvalue 0 and the real part $Re(\lambda_{i}) < 0$ (*i* = 1, 2, , *n*) of
 $P\begin{pmatrix} 1 & 0 \\ 0 & e^{j\pi} \end{pmatrix} P^{-1}$

to non-zero ei $A_{\beta} = 0$,
 \hat{L} , $k_2 h_1 \mu_1^T$, $h_2 \mu_2^T$, $\left[\frac{1}{2}$, $\frac{1}{2}$, \frac Final matrix **Φ**. Choose $ω_1$ as a column vector
 J one eigenvalue 0 and the real part $Re(λ_i)$
 $P^{-1} = P\begin{pmatrix} 1 & 0 \\ 0 & e^{Jt} \end{pmatrix} P^{-1}$

ading to non-zero eigenvalue. From ^[27], we k
 $J' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
 D Ing tree, so matrix L only has one 0 eigenvalue. And we know

th $\mu^T L = 0$, then
 $\mu_2^T A_{\beta} = 0$,
 $\mu_1^T \hat{L} \hat{k}_2 h_i \mu_1^T h_2 \mu_2^T$,
 $\mu_1^T \hat{L} \hat{k}_2 h_i \mu_1^T h_2 \mu_2^T$,

atrix Φ . Choose ω_1 as a column vecto $\int_{\mu_2}^{T} A_{\beta} = 0,$
 $\mu_2^{T} \overline{L}_{f} = 0.$
 $\int_{\mu_2}^{T} \hat{L}_{\beta} h_{\mu_1} h_{\beta} \mu_2^{T} \Big]$,
 $\text{trix } \Phi. \text{ Choose } \omega_1 \text{ as a column vector of } P, \text{ and then } \gamma_1^{T} \text{ is}$
 $\text{eigenvalue 0 and the real part } Re(\lambda_i) < 0 \text{ (}i = 1, 2, \dots, n \text{) of}$
 $\int_{0}^{1} \begin{pmatrix} 1 & 0 \\ 0 & e^{j\eta$ $\left[\mu_1^T L_i - \mu_1^T A_k - 0,$

According to $\gamma_1 \Phi = \gamma_1 \lambda$, we can get
 $\gamma_1^T = \left[k_1 k_1 h_1 \mu_1^T L_k h_1 \mu_2^T h_2 \mu_1 \mu_2^T\right],$

as a left eigenvector corresponding to eigenvalue 0 of matrix Φ . Choose ω_k as a column vect

$$
e^{-\Phi t} = P e^{Jt} P^{-1} = P \begin{pmatrix} 1 & 0 \\ 0 & e^{Jt} \end{pmatrix} P^{-1}
$$

$$
\lim_{t\to\infty}e^{J't}=\begin{pmatrix}1&&0\\&&\\&&0\\0&&&0\end{pmatrix}
$$

It is easy to know the solution of system (2-4) is $\xi(t) = e^{-4t/\xi}(0)$, then
 Electronic Science Technology and Application
 Volume 4 | 2017 | 21

$$
\lim_{t \to \infty} e^{-\omega_t} = [\omega^T, y^T]^T = \begin{pmatrix} k_1 k_2 \mathbf{1}_m h_1 h_2 \mu_1^T \hat{L} & k_2 \mathbf{1}_m h_1 \mu_1^T & \mathbf{1}_{n=m} h_2 \mu_2^T \\ \mathbf{0}_{m \times m} & \mathbf{0}_{m \times (n-m)} & \mathbf{0}_{m \times (n-m)} \end{pmatrix},
$$

and then

$$
\lim_{t \to \infty} \xi(t) = \begin{pmatrix} k_1 k_2 \mathbf{1}_m h_1 h_2 \mu_1^T \hat{L} & k_2 \mathbf{1}_m h_1 \mu_1^T & \mathbf{1}_{n=m} h_2 \mu_2^T \\ \mathbf{0}_{m \times m} & \mathbf{0}_{m \times m} & \mathbf{0}_{m \times (n-m)} \\ \mathbf{0}_{m \times m} & \mathbf{0}_{m \times (n-m)} & \mathbf{0}_{m \times (n-m)} \\ k_1 k_2 \mathbf{1}_m h_1 h_2 \mu_1^T \hat{L} & k_2 \mathbf{1}_m h_1 \mu_1^T & \mathbf{1}_{n=m} h_2 \mu_2^T \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix}
$$
that is,
that is,

$$
\lim_{t \to \infty} \begin{pmatrix} \rho_s \\ \rho_s \\ \rho_f \end{pmatrix} = \begin{pmatrix} k_1 k_2 \mathbf{1}_m h_1 h_2 \mu_1^T \hat{L} & k_2 \mathbf{1}_m h_1 \mu_1^T & \mathbf{1}_{n=m} h_2 \mu_2^T \\ \mathbf{0}_{m \times m} & \mathbf{0}_{m \times (n-m)} \\ \mathbf{0}_{m \times m} & \mathbf{0}_{m \times m} & \mathbf{0}_{m \times (n-m)} \\ \mathbf{0}_{m \times m} & \mathbf{0}_{m \times m} \\ \mathbf{0}_{m \times m} & \mathbf{0}_{m \times m} \\ \mathbf{0}_{m \times m} & \mathbf{0}_{m \times m} \end{pmatrix} \begin{pmatrix} x_1(0) \\ y_2(0) \\ y_3(0) \\ x_4(0) \end{pmatrix}
$$
that is,
Based on $p_i = h_$

$$
\lim_{t \to \infty} \xi(t) = \begin{pmatrix} k_1 k_2 \mathbf{1}_m h_1 h_2 \mu_1^T \hat{L} & k_2 \mathbf{1}_m h_1 \mu_1^T & \mathbf{1}_{n-m} h_2 \mu_2^T \\ \mathbf{0}_{m \times m} & \mathbf{0}_{m \times m} & \mathbf{0}_{m \times (n-m)} \\ k_1 k_2 \mathbf{1}_m h_1 h_2 \mu_1^T \hat{L} & k_2 \mathbf{1}_m h_1 \mu_1^T & \mathbf{1}_{n-m} h_2 \mu_2^T \end{pmatrix} \begin{pmatrix} x_s(0) \\ y_s(0) \\ x_f(0) \end{pmatrix}
$$

$$
\lim_{t \to \infty} e^{-\omega_t} = [\omega^T, y^T]^T = \begin{pmatrix} k_1 k_2 \mathbf{1}_m h_1 h_2 \mu_1^T \hat{L} & k_2 \mathbf{1}_m h_1 \mu_1^T & \mathbf{1}_{n=m} h_2 \mu_2^T \\ \mathbf{0}_{m \times m} & \mathbf{0}_{m \times (n-m)} & \mathbf{0}_{m \times (n-m)} \end{pmatrix},
$$

and then

$$
\lim_{t \to \infty} \xi(t) = \begin{pmatrix} k_1 k_2 \mathbf{1}_m h_1 h_2 \mu_1^T \hat{L} & k_2 \mathbf{1}_m h_1 \mu_1^T & \mathbf{1}_{n=m} h_2 \mu_2^T \\ \mathbf{0}_{m \times m} & \mathbf{0}_{m \times m} & \mathbf{0}_{m \times (n-m)} \\ \mathbf{0}_{m \times m} & \mathbf{0}_{m \times (n-m)} & \mathbf{0}_{m \times (n-m)} \\ k_1 k_2 \mathbf{1}_m h_1 h_2 \mu_1^T \hat{L} & k_2 \mathbf{1}_m h_1 \mu_1^T & \mathbf{1}_{n=m} h_2 \mu_2^T \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix}
$$
that is,
that is,

$$
\lim_{t \to \infty} \begin{pmatrix} \rho_s \\ \rho_s \\ \rho_f \end{pmatrix} = \begin{pmatrix} k_1 k_2 \mathbf{1}_m h_1 h_2 \mu_1^T \hat{L} & k_2 \mathbf{1}_m h_1 \mu_1^T & \mathbf{1}_{n=m} h_2 \mu_2^T \\ \mathbf{0}_{m \times m} & \mathbf{0}_{m \times (n-m)} \\ \mathbf{0}_{m \times m} & \mathbf{0}_{m \times m} & \mathbf{0}_{m \times (n-m)} \\ \mathbf{0}_{m \times m} & \mathbf{0}_{m \times m} \\ \mathbf{0}_{m \times m} & \mathbf{0}_{m \times m} \\ \mathbf{0}_{m \times m} & \mathbf{0}_{m \times m} \end{pmatrix} \begin{pmatrix} x_1(0) \\ y_2(0) \\ y_3(0) \\ x_4(0) \end{pmatrix}
$$
that is,
Based on $p_i = h_$

 $\lim_{t \to \infty} \xi(t) = \begin{pmatrix} k_1 k_2 \mathbf{1}_m h_1 h_2 \mu_1^T \\ \mathbf{0}_{m \times m} \\ k_1 k_2 \mathbf{1}_m h_1 h_2 \mu_1^T \end{pmatrix}$
t is,
 $\lim_{t \to \infty} \begin{pmatrix} p_s \\ p_s \\ p_f \end{pmatrix} = \begin{pmatrix} k_1 k_2 \mathbf{1}_m h_1 h_2 \mu_1^T \hat{L} \\ \mathbf{0}_{m \times m} \\ k_1 k_2 \mathbf{1}_m h_1 h_2 \mu_1^T \hat{L} \end{pmatrix}$
Bas $, q_i = h_i v_i, i \in I_m$, and $p_i = h_i x_i, i \in I_n / I_m$, the converg

$$
\lim_{t\to\infty}e^{-\delta x}=[\omega^T,y^T]^T=\begin{pmatrix}\hat{k}_1k,1_n,\hat{h}_1k,\hat{p}_1l,\hat{L} & \hat{k}_11_n,\hat{h}_2l,\hat{L} & 1_{n,n}\hat{h}_2l,\hat{L} \\
0_{n,m}& 0_{n,m}& 0_{n,m+1},\\ \hat{k}_1k,1_n,\hat{h}_1l,\hat{L} & \hat{k}_11_n,\hat{h}_2l,\hat{L} & \hat{k}_11_n,\hat{h}_2l,\hat{L} & \hat{k}_1\hat{h}_2l,\hat{L} \\
0_{n,m}& 0_{n,m+1},\\ \lim_{t\to\infty}\hat{\xi}(t)=\begin{pmatrix}\hat{k}_1k,1_n,\hat{h}_1k,\hat{L} & \hat{k}_11,\hat{h}_1k,\hat{L} & 1_{n,m}\hat{h}_2l,\hat{L} \\
0_{n,m}& 0_{n,m+1},\\ \hat{k}_1k,1_n,\hat{h}_2k,\hat{L} & \hat{k}_21,\hat{h}_1k,\hat{L} & 1_{n,m}\hat{h}_2l,\hat{L} \\
0_{n,m}& 0_{n,m+1},\\ \hat{L} & 0_{n,m}\end{pmatrix}=\begin{pmatrix}\hat{x}_1k,1_n,\hat{h}_1l,\hat{L} & \hat{k}_11,\hat{h}_2l,\hat{L} & 1_{n,m}\hat{h}_2l,\hat{L} \\
0_{n,m}& 0_{n,m+1},\\ \hat{L} & 0_{n,m}\end{pmatrix}=\begin{pmatrix}\hat{x}_1(0)\\ \hat{x}_1(0)\\ \hat{x}_2(0)\\ \hat{x
$$

 $\frac{1}{h_2}$) \overline{d}_s , $k_2 > 0$, then system (1) with protocol (2)

$$
\lim_{t \to \infty} x = k_1 k_2 h_1 h_2 \mu_1^T \hat{L} x_s(0) + k_2 h_1 \mu_1^T v_s(0) + h_2 \mu_2^T x_f(0) ,
$$

$$
\lim_{t \to \infty} v_s = \mathbf{1}_m 0.
$$

where $\overline{d}_s = \max_{i=1, \dots, m} \sum_{j=1}^{n} a_{ij}$ is maximum value of the diagonal element of corresponding Laplician matrix \overline{L}_s .
 Remark 3 Notice that $h_i = h_2$ means that there is only one group in the heterogeneous mul sus if and only if G contains a directed spann

re
 $\lim_{t \to \infty} x = k_1 k_2 h_1 h_2 \mu_1^T \hat{L} x_s(0)$
 $\lim_{t \to \infty} v_s = \mathbf{1}_m 0$.
 $\lim_{t \to \infty} x_s = \mathbf{1}_m 0$.

Max $\sum_{j=1}^n a_{ij}$ is maximum value of the diagonal expans that there *n* **i**) is if and only if G contains a directed spanni

(2) are
 $\lim_{t \to \infty} x = k_1 k_2 h_1 h_2 \mu_1^T \hat{L} x_s(0)$
 $\lim_{t \to \infty} v_s = \mathbf{1}_m 0$.
 $\int_{s = \lim_{i=1, \dots, m} \sum_{j=1}^{n} a_{ij}$ is maximum value of the diagonal extrappose that $h_1 =$ $=1, m \leftrightarrow w$ is maximum value of the diagonal element of corresponding Laplician matrix \overline{L}_s .
 $\lim_{t \to \infty} x_s = 1_n 0$.
 $\lim_{t \to \infty} v_s = 1$ is maximum value of the $\lim_{t \to \infty} x = k_1 k_2 h_1 h_2 \mu_1^T \hat{L}x_s(0) + k_2 h_1 \mu_1^T v_s(0) + h_2 \mu_2^T x_s(0)$.
 $\lim_{t \to \infty} y_s = 1_m 0$.

ere $\overline{d}_s = \max_{i=1, \dots, m} \sum_{j=1}^n a_{ij}$ is maximum value of the diagonal element of corresponding Laplician matrix $\overline{$

3. Simulation
In this section, some simulation examples are presented
continuous-time heterogeneous multi-agent systems. **3. Simulation**
In this section, some simulation examples are presented to demonstrate the effectiveness of the theoretic results for
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Example 1 Consider a ten-agent undi

3. Simulation
In this section, some simulation examples are presented to demonstrate the effecontinuous-time heterogeneous multi-agent systems.
Example 1 Consider a ten-agent undirected network with four first-order in **Example 1** Consider a ten-agent undirected network with four first-order integrator agents (agents 7-10) and six **Example 1** Consider a ten-agent undirected network with four first-order integrator agents (agents 7-10) an **3. Simulation**
In this section, some simulation examples are presented to demonstrate the effectiveness of the theoretic results for
continuous-time heterogeneous multi-agent systems.
Example 1 Consider a ten-agent und as

 $A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0$ when $h_1 = h_2 > 0$.

For the continuous-time heterogeneous multi-agent system, we choose $h_i=1$, $h_2=6$, $k_i=3$, $k_2=1$, $x_0=[2, 5, 3, -1, 1, 5, -3, 10, -2, 8, 2, 4]$, $v_{\overline{e}}[2, 5, 8, 3, -9, -4, 1]$. By calculating, the conditions of Theorem For the continuous-time heterogeneous multi-agent system, we choose $h_1=1$, $h_2=6$, $k_1=3$, $k_2=1$, $x_0=[2,5,3,-1,1.5,-3,10,-2.8,2,4]$, $v_{\overline{\theta}}=[2,5,8,3,-9,-4,1]$. By calculating, the conditions of Theorem 3 are satisfied For the continuous-time heterogeneous multi-agent system, we choose $h_i=1$, $h_2=6$, $k_i=3$, $k_2=1$, $x_i=$ $[2, 5, 3, -1, 1.5, -3, 10, -2.8, 2, 4]$, $v_i=$ $[2, 5, 3, 3, -9, -4, 1]$. By calculating, the conditions of Theorem 3 For the continuous-time heterogeneous multi-
[2, 5, 3, -1, 1. 5, -3, 10, -2. 8, 2, 4], $v_{\overline{C}}$ [2. 5, 8, satisfied, then such system can achieve consensus frequencies in the velocity trajectories, respectively. Special

Conclusion

Conclusion
In this paper, we have investigated the group consensus of heterogeneous multi-agent systems under undirected and
directed fixed topology in continuous-time. we get the heterogeneous multi-agent systems can re **Conclusion**
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