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A ZARISKI-LOCAL NOTION OF F-TOTAL ACYCLICITY FOR COMPLEXES OF SHEAVES

LARS WINTHER CHRISTENSEN, SERGIO ESTRADA, AND ALINA IACOB

ABSTRACT. We study a notion of total acyclicity for complexes of flat sheaves over a scheme. It is Zariski-local—i.e. it can be verified on any open affine covering of the scheme—and for sheaves over a quasi-compact semi-separated scheme it agrees with the categorical notion. In particular, it agrees, in their setting, with the notion studied by Murfet and Salarian for sheaves over a noetherian semi-separated scheme. As part of the study we recover, and in several cases extend the validity of, recent results on existence of covers and precovers in categories of sheaves. One consequence is the existence of an adjoint to the inclusion of these totally acyclic complexes into the homotopy category of complexes of flat sheaves.

INTRODUCTION

This paper is part of a thrust to extend Gorenstein homological algebra to schemes. The first major advance was made by Murfet and Salarian [21], who introduced an operational notion of total acyclicity over noetherian semi-separated schemes. Total acyclicity has its origin in Tate cohomology of finite group representations, which is computed via, what we now call, totally acyclic complexes of projectives. The contemporary terminology was introduced in works of Avramov and Martsinkovsky [3] and Veliche [27]: Given a commutative ring R, a chain complex $P_\bullet$ of projective R-modules is called *totally acyclic* if it is acyclic and $\mathrm{Hom}_R(P_\bullet, Q)$ is acyclic for every projective R-module Q.

Categories of sheaves do not, in general, have enough projectives, so it is not obvious how to define an interesting notion of total acyclicity in this setting. Murfet and Salarian's approach was to focus on flat sheaves: They say that a complex $\mathscr{F}_\bullet$ of flat quasi-coherent sheaves over a noetherian semi-separated scheme X is **F***-totally acyclic* if it is acyclic and $\mathscr{I} \otimes \mathscr{F}_\bullet$ is acyclic for every injective quasi-coherent sheaf $\mathscr{I}$ on X. This notion has its origin in the work of Enochs, Jenda, and Torrecillas, who in [11] introduced it for complexes of modules. The assumptions on X ensure that a quasi-coherent sheaf on X is (categorically) injective if and only if every section of the sheaf is an injective module. In fact, **F**-total acyclicity as defined in [21] is a *Zariski-local* property. That is, a complex $\mathscr{F}_\bullet$ of flat quasi-coherent sheaves on X is **F**-totally acyclic if there is an open affine covering $\mathcal{U}$ of X such that $\mathscr{F}_\bullet(U)$ is an **F**-totally acyclic complex of flat modules for every $U \in \mathcal{U}$.

In this paper we give a definition of **F**-total acyclicity for complexes of flat quasi-coherent sheaves without placing any assumptions on the underlying scheme. Our

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definition is Zariski-local, and it agrees with the one from [21] when the latter applies. In fact we prove more, namely (Proposition 2.10) that over any quasi-compact semi-separated scheme X, an acyclic complex $\mathscr{F}_\bullet$ of flat quasi-coherent sheaves is $\mathbf{F}$-totally acyclic per our definition if and only if $\mathscr{I} \otimes \mathscr{F}_\bullet$ is acyclic for every injective quasi-coherent sheaf $\mathscr{I}$ on X.

The key to the proof of Zariski-localness (Corollary 2.8) is the next result (Proposition 2.7), which says that $\mathbf{F}$-total acyclicity for complexes of modules is a so-called *ascent–descent* property. By a standard argument (Lemma 2.4), this implies that the corresponding property of complexes of quasi-coherent sheaves is Zariski-local.

Proposition. *Let* $\varphi\colon R \to S$ *be a flat homomorphism of commutative rings.*
   (1) *If* $F_\bullet$ *is an* $\mathbf{F}$*-totally acyclic complex of flat R-modules, then* $S \otimes_R F_\bullet$ *is an* $\mathbf{F}$*-totally acyclic complex of flat S-modules.*
   (2) *If* φ *is faithfully flat and* $F_\bullet$ *is a complex of R-modules such that* $S \otimes_R F_\bullet$ *is an* $\mathbf{F}$*-totally acyclic complex of flat S-modules, then* $F_\bullet$ *is an* $\mathbf{F}$*-totally acyclic complex of flat R-modules.*

In lieu of projective sheaves one can focus on vector bundles—not necessarily finite dimensional. In Section 3 we touch on a notion of total acyclicity for complexes of vector bundles. By comparing it to $\mathbf{F}$-total acyclicity we prove that it is Zariski-local for locally coherent locally d-perfect schemes (Theorems 3.8–3.9).

The keystone result of Section 4 is Theorem 4.2, which says that for any scheme X, the class of $\mathbf{F}$-totally acyclic complexes is covering in the category of chain complexes of quasi-coherent sheaves on X. It has several interesting consequences. The homotopy category of chain complexes of flat quasi-coherent sheaves over X is denoted $\mathbf{K}(\mathrm{Flat}\,X)$. It is a triangulated category, and the full subcategory $\mathbf{K}_{\mathrm{tac}}(\mathrm{Flat}\,X)$ of $\mathbf{F}$-totally acyclic complexes in $\mathbf{K}(\mathrm{Flat}\,X)$ is a triangulated subcategory. Theorem 4.2 allows us to remove assumptions on the scheme in [21, Corollary 4.26] and obtain (Corollary 4.6):

Corollary. *For any scheme X, the inclusion* $\mathbf{K}_{\mathrm{tac}}(\mathrm{Flat}\,X) \to \mathbf{K}(\mathrm{Flat}\,X)$ *has a right adjoint.*

A *Gorenstein flat* quasi-coherent sheaf is defined as a cycle sheaf in an $\mathbf{F}$-totally acyclic complex of flat quasi-coherent sheaves. Enochs and Estrada [9] prove that every quasi-coherent sheaf over any scheme has a flat precover. As another consequence (Corollary 4.8) of Theorem 4.2 we obtain a Gorenstein version of this result; the affine case was already proved by Yang and Liang [29].

Corollary. *Let X be a scheme. Every quasi-coherent sheaf on X has a Gorenstein flat precover. If X is quasi-compact and semi-separated, then the Gorenstein flat precover is an epimorphism.*

Finally, Theorem 4.2 combines with Theorem 3.8 to yield (Corollary 4.9):

Corollary. *Let R be a commutative coherent d-perfect ring. Every R-module has a Gorenstein projective precover.*

This partly recovers results of Estrada, Iacob, and Odabaşı [14, Corollary 2] and of Bravo, Gillespie, and Hovey [6, Proposition 8.10].

1. Preliminaries

Let κ be a cardinal, by which we shall always mean a regular cardinal. An object A in a category $\mathcal{C}$ is called κ-*presentable* if the functor $\mathrm{Hom}_\mathcal{C}(A, -)$ preserves κ-directed colimits. A category $\mathcal{C}$ is called *locally κ-presentable* if it is cocomplete and there is a set $\mathcal{S}$ of κ-presentable objects in $\mathcal{C}$ such that every object in $\mathcal{C}$ is a κ-directed colimit of objects in $\mathcal{S}$.

An $\aleph_0$-directed colimit is for short called a *direct limit.*

1.1. κ-**pure morphisms.** Let κ be a cardinal and $\mathcal{C}$ be a category. We recall from the book of Adámek and J. Rosický [1, Definition 2.27] that a morphism $\varphi\colon A \to B$ in $\mathcal{C}$ is called κ-*pure* if for every commutative square

$$
\begin{array}{ccc}
A' & \xrightarrow{\ \varphi'\ } & B' \\
{\scriptstyle\alpha}\big\downarrow & & \big\downarrow \\
A & \xrightarrow{\ \varphi\ } & B
\end{array}
$$

where the objects A' and B' are κ-presentable, there exits a morphism $\gamma\colon B' \to A$ with $\gamma \circ \varphi' = \alpha$. A subobject $A \subseteq B$ is called κ-*pure* if the monomorphism $A \to B$ is κ-pure.

1.2. **Complexes.** In the balance of this section, $\mathcal{G}$ denotes a Grothendieck category, and $\mathrm{Ch}(\mathcal{G})$ denotes the category of chain complexes over $\mathcal{G}$. It is elementary to verify that $\mathrm{Ch}(\mathcal{G})$ is also a Grothendieck category. We use homological notation, so a complex $M_\bullet$ in $\mathrm{Ch}(\mathcal{G})$ looks like this

$$
M_\bullet \ = \ \cdots \longrightarrow M_{i+1} \xrightarrow{\ \partial_{i+1}\ } M_i \xrightarrow{\ \partial_i\ } M_{i-1} \longrightarrow \cdots
$$

We denote by $\mathrm{Z}_n(M_\bullet)$ and $\mathrm{B}_n(M_\bullet)$ the nth cycle and nth boundary object of $M_\bullet$.

Let M be an object in $\mathcal{G}$. We denote by $\mathrm{S}_n(M)$ the complex with M in degree n and 0 elsewhere. By $\mathrm{D}_n(M)$ we denote the complex with M in degrees n and $n-1$, differential $\partial_n = \mathrm{Id}^M$, the identity map, and 0 elsewhere.

1.3. **(Pre)covers.** Let $\mathcal{F}$ be a class of objects in $\mathcal{G}$. A morphism $\phi\colon F \to M$ in $\mathcal{G}$ is called an $\mathcal{F}$-*precover* if F is in $\mathcal{F}$ and

$$
\mathrm{Hom}_\mathcal{G}(F', F) \longrightarrow \mathrm{Hom}_\mathcal{G}(F', M) \longrightarrow 0
$$

is exact for every $F' \in \mathcal{F}$. Further, if $\phi\colon F \to M$ is a precover and every morphism $\sigma\colon F \to F$ with $\phi\sigma = \phi$ is an automorphism, then ϕ is called an $\mathcal{F}$-*cover.* If every object in $\mathcal{G}$ has an $\mathcal{F}$-(pre)cover, then the class $\mathcal{F}$ is called *(pre)covering.*

The dual notions are *(pre)envelope* and *(pre)enveloping.*

1.4. **Orthogonal classes and cotorsion pairs.** Let $\mathcal{F}$ be a class of objects in $\mathcal{G}$ and consider the orthogonal classes

$$
\mathcal{F}^\perp \ = \ \{G \in \mathcal{G} \mid \mathrm{Ext}^1_\mathcal{G}(F, G) = 0 \text{ for all } F \in \mathcal{F}\} \ \text{ and}
$$

$$
{}^\perp\mathcal{F} \ = \ \{G \in \mathcal{G} \mid \mathrm{Ext}^1_\mathcal{G}(G, F) = 0 \text{ for all } F \in \mathcal{F}\}\,.
$$

Let $\mathcal{S} \subseteq \mathcal{F}$ be a set. The pair $(\mathcal{F}, \mathcal{F}^\perp)$ is said to be *cogenerated by the set* $\mathcal{S}$ if an object G belongs to $\mathcal{F}^\perp$ if and only if $\mathrm{Ext}^1_\mathcal{G}(F, G) = 0$ holds for all $F \in \mathcal{S}$.
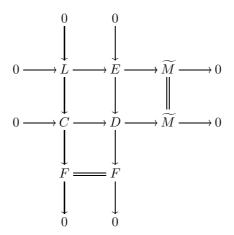
A pair $(\mathcal{F}, \mathcal{C})$ of classes in $\mathcal{G}$ with $\mathcal{F}^\perp = \mathcal{C}$ and ${}^\perp\mathcal{C} = \mathcal{F}$ is called a *cotorsion pair.* A cotorsion pair $(\mathcal{F}, \mathcal{C})$ in $\mathcal{G}$ is called *complete* provided that for every $M \in \mathcal{G}$ there are short exact sequences $0 \to C \to F \to M \to 0$ and $0 \to M \to C' \to F' \to 0$ with $F, F' \in \mathcal{F}$ and $C, C' \in \mathcal{C}$. Notice that for every complete cotorsion pair $(\mathcal{F}, \mathcal{C})$, the class $\mathcal{F}$ is precovering and the class $\mathcal{C}$ is preenveloping.

1.5. **Kaplansky classes.** Let $\mathcal{F}$ be a class of objects in $\mathcal{G}$ and κ be a cardinal. One says that $\mathcal{F}$ is a κ-*Kaplansky class* if for every inclusion $Z \subseteq F$ of objects in $\mathcal{G}$ such that F is in $\mathcal{F}$ and Z is κ-presentable, there exists a κ-presentable object W in $\mathcal{F}$ with $Z \subseteq W \subseteq F$ and such that F/W belongs to $\mathcal{F}$. We say that $\mathcal{F}$ is a *Kaplansky class* if it is a κ-Kaplansky for some cardinal κ.

Proposition 1.6. *Every Kaplansky class in $\mathcal{G}$ that is closed under extensions and direct limits is covering.*

Proof. Let κ be a cardinal; let $\mathcal{F}$ be a κ-Kaplansky class in $\mathcal{G}$ and assume that it is closed under extensions and direct limits. It now follows from Eklof's lemma [8, Lemma 1] that the pair $(\mathcal{F}, \mathcal{F}^\perp)$ is cogenerated by a set.

Let M be an object in $\mathcal{G}$. Denote by $\widetilde{M}$ the sum of all images in M of morphisms with domain in $\mathcal{F}$. That is, $\widetilde{M} = \sum_{\varphi \in \mathrm{Hom}(F,M),\, F \in \mathcal{F}} \mathrm{Im}(\varphi)$; as $\mathcal{G}$ is well-powered the sum is well-defined. Since $\mathcal{F}$ is closed under coproducts, there exists a short exact sequence $0 \to L \to E \to \widetilde{M} \to 0$, with $E \in \mathcal{F}$. By a result of Enochs, Estrada, García Rozas, and Oyonarte [10, Theorem 2.5] there is a short exact sequence $0 \to L \to C \to F \to 0$ with $C \in \mathcal{F}^\perp$ and $F \in \mathcal{F}$. Consider the push-out diagram,

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & & & \\
& & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & L & \longrightarrow & E & \longrightarrow & \widetilde{M} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & C & \longrightarrow & D & \longrightarrow & \widetilde{M} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & & & \\
& & F & =\!\!=\!\!= & F & & & & \\
& & \downarrow & & \downarrow & & & & \\
& & 0 & & 0 & & & &
\end{array}
$$

Since $\mathcal{F}$ is closed under extensions, one has $D \in \mathcal{F}$, and since $C \in \mathcal{F}^\perp$ it follows that $D \to \widetilde{M}$ is an $\mathcal{F}$-precover. By the definition of $\widetilde{M}$, it immediately follows that M has an $\mathcal{F}$-precover, so the class $\mathcal{F}$ is precovering. Finally, any precovering class that is closed under direct limits is covering; see Xu [28, Theorem 2.2.12] for an argument in a module category that carries over to Grothendieck categories. $\square$

1.7. **Kaplansky classes and filtrations.** Recall that a well ordered direct system, $\{M_\alpha \mid \alpha \leq \lambda\}$, of objects in $\mathcal{G}$ is called *continuous* if one has $M_0 = 0$ and, for each limit ordinal $\beta \leq \lambda$, one has $M_\beta = \varinjlim_{\alpha < \beta} M_\alpha$. If all morphisms in the system are monomorphisms, then the system is called a *continuous directed union.*

Let $\mathcal{S}$ be a class of objects in $\mathcal{G}$. An object M in $\mathcal{G}$ is called $\mathcal{S}$-*filtered* if there is a continuous directed union $\{M_\alpha \mid \alpha \leq \lambda\}$ of subobjects of M such that $M = M_\lambda$ and for every $\alpha < \lambda$ the quotient $M_{\alpha+1}/M_\alpha$ is isomorphic to an object in $\mathcal{S}$. We denote by $\mathrm{Filt}(\mathcal{S})$ the class of all $\mathcal{S}$-filtered objects in $\mathcal{G}$.

Let κ be a cardinal and $\mathcal{F}$ be a κ-Kaplansky class in $\mathcal{G}$ that is closed under direct limits. It is standard to verify that there exists a set $\mathcal{S}$ of κ-presentable objects in $\mathcal{F}$ with $\mathcal{F} \subseteq \mathrm{Filt}(\mathcal{S})$; see for example the proof of [15, Lemma 4.3]. In general the classes $\mathcal{F}$ and $\mathrm{Filt}(\mathcal{S})$ need not be equal, but if $\mathcal{F}$ is closed under extensions, then equality holds. An explicit example of strict containment is provided by the (Kaplansky) class of phantom morphisms in the (Grothendieck) category of representations of the $\mathbb{A}_2$ quiver; see Estrada, Guil Asensio, and Ozbek [12].

Šťovíček proves in [26, Corollary 2.7(2)] that every Kaplansky class $\mathcal{F}$ that is closed under direct limits (and extensions) is *deconstructible,* which per [26, Definition 1.4] means precisely that there exists a set $\mathcal{S}$ with $\mathcal{F} = \mathrm{Filt}(\mathcal{S})$. However, the assumption about closedness under extensions is not stated explicitly.

2. Faithfully flat descent for **F**-total acyclicity

Definition 2.1. Let X be a scheme with structure sheaf $\mathcal{O}_X$, and let $\mathfrak{P}$ be a property of modules over commutative rings.

(1) A quasi-coherent sheaf $\mathscr{M}$ on X is said to *locally* have property $\mathfrak{P}$ if for every open affine subset $U \subseteq X$, the $\mathcal{O}_X(U)$-module $\mathscr{M}(U)$ has property $\mathfrak{P}$.

(2) As a (local) property of quasi-coherent sheaves on X, the property $\mathfrak{P}$ is called *Zariski-local* if the following conditions are equivalent for every quasi-coherent sheaf $\mathscr{M}$ on X.

- The sheaf $\mathscr{M}$ locally has property $\mathfrak{P}$.
- There exists an open affine covering $\mathcal{U}$ of X such that for every $U \in \mathcal{U}$ the $\mathcal{O}_X(U)$-module $\mathscr{M}(U)$ has property $\mathfrak{P}$.

That is, Zariski-localness of a property of sheaves means that it can be verified on any open affine covering. A useful classic tool for verifying Zariski-localness is based on flat ascent and descent of the underlying module property. We make it explicit in Lemma 2.4; see also [20, §34.11].

Definition 2.2. Let $\mathfrak{P}$ be a property of modules over commutative rings and let $\mathcal{R}$ be a class of commutative rings.

(1) $\mathfrak{P}$ is said to *ascend* in $\mathcal{R}$, if for every flat epimorphism $R \to S$ of rings in $\mathcal{R}$ and for every R-module M with property $\mathfrak{P}$, the S-module $S \otimes_R M$ has property $\mathfrak{P}$.

(2) $\mathfrak{P}$ is said to *descend* in $\mathcal{R}$ if an R-module M has property $\mathfrak{P}$ whenever there exists a faithfully flat homomorphism $R \to S$ of rings in $\mathcal{R}$ such that the S-module $S \otimes_R M$ has property $\mathfrak{P}$.

If $\mathfrak{P}$ ascends and descends in $\mathcal{R}$, then it is called an *ascent–descent property*, for short an *AD-property*, in $\mathcal{R}$.

Definition 2.3. A property $\mathfrak{P}$ of modules over commutative rings is said to be *compatible with finite products* if the following conditions are equivalent for all commutative rings R_1 and R_2, all R_1-modules M_1, and all R_2-modules M_2.

- M_1 and M_2 have property $\mathfrak{P}$.
- The $R_1 \times R_2$-module $M_1 \times M_2$ has property $\mathfrak{P}$.

Lemma 2.4. *Let X be a scheme with structure sheaf $\mathcal{O}_X$ and let $\mathfrak{P}$ be a property of modules over commutative rings. If $\mathfrak{P}$ is compatible with finite products and an AD–property in the class $\mathcal{R} = \{\mathcal{O}_X(U) \mid U \subseteq X \text{ is an open affine subset}\}$ of commutative rings, then $\mathfrak{P}$ as a property of quasi-coherent sheaves on X is Zariski-local.*

Proof. Let $\mathcal{U} = \{U_i \mid i \in I\}$ be an open affine covering of X such that for every $i \in I$ the $\mathcal{O}_X(U_i)$-module $\mathscr{M}(U_i)$ has property $\mathfrak{P}$. Let U be an arbitrary open affine subset of X. There exists a standard open covering $U = \bigcup_{j=1}^n D(f_j)$ such that for every j there is a $U_j \in \mathcal{U}$ with the property that $D(f_j)$ is a standard open subset of U_j; that is, $f_j \in \mathcal{O}_X(U_j)$ and $D(f_j) = \mathrm{Spec}(\mathcal{O}_X(U_j)_{f_j})$; see [20, Lemma 25.11.5]. In particular, one has $\mathscr{M}(D(f_j)) \cong \mathscr{M}(U_j) \otimes_{\mathcal{O}_X(U_j)} \mathcal{O}_X(U_j)_{f_j}$, and it follows that $\mathscr{M}(D(f_j))$ has property $\mathfrak{P}$ as it ascends in $\mathcal{R}$. The compatibility of $\mathfrak{P}$ with direct products now ensures that the module $\prod_{j=1}^n \mathscr{M}(D(f_j))$ over $\prod_{j=1}^n \mathcal{O}_X(D(f_j))$ has property $\mathfrak{P}$. As the canonical morphism $\mathcal{O}_X(U) \to \prod_{j=1}^n \mathcal{O}_X(D(f_j))$ is faithfully flat, it follows that the $\mathcal{O}_X(U)$-module $\mathscr{M}(U)$ has property $\mathfrak{P}$. $\square$

While ascent of a module property is usually easy to prove, it can be more involved to prove descent. For instance, it is easy to see that flatness is an AD-property. Also the flat Mittag-Leffler property is known to be an AD-property:

Ascent is easy to prove, while descent follows from Raynaud and Gruson [25, II.5.2]; see also Perry [24, §9] for correction of an error in [25]. The AD-property is also satisfied by the κ-restricted flat Mittag-Leffler modules, where κ is an infinite cardinal (see Estrada, Guil Asensio, and Trlifaj [13]).

We are also concerned with properties of complexes of sheaves and modules; it is straightforward to extend Definitions 2.1–2.3 and Lemma 2.4 to the case where $\mathfrak{P}$ is a property of complexes.

Next we introduce the property for which we will study Zariski-localness.

Definition 2.5. A complex of flat R-modules $F_\bullet = \cdots \to F_{i+1} \to F_i \to F_{i-1} \to \cdots$ is called **F-totally acyclic** if it is acyclic and $I \otimes_R F_\bullet$ is acyclic for every injective R-module I.

Definition 2.6. Let X be a scheme with structure sheaf $\mathcal{O}_X$. A complex $\mathscr{F}_\bullet = \cdots \to \mathscr{F}_{i+1} \to \mathscr{F}_i \to \mathscr{F}_{i-1} \to \cdots$ of flat quasi-coherent sheaves on X is called **F-totally acyclic** if for every open affine subset $U \subseteq X$ the complex $\mathscr{F}_\bullet(U)$ of flat $\mathcal{O}_X(U)$-modules is **F**-totally acyclic.

The next lemma shows, in particular, that **F**-total acyclicity is an AD-property.

Proposition 2.7. *Let $\varphi\colon R \to S$ be a flat homomorphism of commutative rings.*
  (1) *If $F_\bullet$ is an **F**-totally acyclic complex of flat R-modules, then $S \otimes_R F_\bullet$ is an **F**-totally acyclic complex of flat S-modules.*
  (2) *If φ is faithfully flat and $F_\bullet$ is a complex of R-modules such that $S \otimes_R F_\bullet$ is an **F**-totally acyclic complex of flat S-modules, then $F_\bullet$ is an **F**-totally acyclic complex of flat R-modules.*

Proof. (1) Since φ is flat and $F_\bullet$ is an acyclic complex of flat R-modules, it follows that $S \otimes_R F_\bullet$ is an acyclic complex of flat S-modules. Now, let E be an injective S-module, by flatness of φ it is also injective as an R-module. Indeed, there are isomorphisms $\mathrm{Hom}_R(-, E) \cong \mathrm{Hom}_R(-, \mathrm{Hom}_S(S, E)) \cong \mathrm{Hom}_S(S \otimes_R -, E)$. It follows that $E \otimes_S (S \otimes_R F_\bullet) \cong E \otimes_R F_\bullet$ is acyclic.

(2) Since φ is faithfully flat and $S \otimes_R F_\bullet$ is an **F**-totally acyclic complex of flat S-modules, it follows that $F_\bullet$ is an acyclic complex of flat R-modules. Let I be an injective R-module; it must be shown that $I \otimes_R F_\bullet$ is acyclic. The S-module $\mathrm{Hom}_R(S, I)$ is injective, so it follows from the next chain of isomorphisms that $\mathrm{Hom}_R(S, I) \otimes_R F_\bullet$ is acyclic:

$$\mathrm{Hom}_R(S, I) \otimes_R F_\bullet \cong (\mathrm{Hom}_R(S, I) \otimes_S S) \otimes_R F_\bullet \cong \mathrm{Hom}_R(S, I) \otimes_S (S \otimes_R F_\bullet)\,;$$

here the last complex is acyclic by the assumption that $S \otimes_R F_\bullet$ is **F**-totally acyclic.

As φ is faithfully flat, the exact sequence of R-modules $0 \to R \to S \to S/R \to 0$ is pure. Hence the induced sequence

$$0 \longrightarrow \mathrm{Hom}_R(S/R, I) \longrightarrow \mathrm{Hom}_R(S, I) \longrightarrow \mathrm{Hom}_R(R, I) \longrightarrow 0$$

is split exact. It follows that $\mathrm{Hom}_R(R, I) \cong I$ is a direct summand of $\mathrm{Hom}_R(S, I)$ as an R-module. Hence the complex $I \otimes_R F_\bullet$ is a direct summand of the acyclic complex $\mathrm{Hom}_R(S, I) \otimes_R F_\bullet$ and, therefore, acyclic. $\square$

The proposition above together with (the complex version of) Lemma 2.4 ensure that the property of being **F**-totally acyclic is Zariski-local as a property of complexes of quasi-coherent sheaves.

Corollary 2.8. *Let X be a scheme with structure sheaf $\mathcal{O}_X$. A complex $\mathscr{F}_\bullet$ of flat quasi-coherent sheaves on X is **F**-totally acyclic if there exists an affine open covering $\mathcal{U}$ of X such that the complex $\mathscr{F}_\bullet(U)$ of flat $\mathcal{O}_X(U)$-modules is **F**-totally acyclic for every $U \in \mathcal{U}$.* $\square$

Remark 2.9. Our definition 2.6 is different from [21, Definition 4.1], but as shown in [21, Lemma 4.5] the two are equivalent if X is Noetherian and semi-separated, which is the blanket assumption in [21]. In the next proposition we substantially relax the hypothesis on X and show that our definition coincides with the one from [21] if X is quasi-compact and semi-separated.

Proposition 2.10. *Let X be a quasi-compact and semi-separated scheme and let $\mathscr{F}_\bullet$ be an acyclic complex of flat quasi-coherent sheaves on X. Conditions (i) and (ii) below are equivalent and imply (iii).*

(i) The complex $\mathscr{J} \otimes_{\mathcal{O}_X} \mathscr{F}_\bullet$ is acyclic for every injective object $\mathscr{J}$ in $\mathfrak{Qcoh}(X)$.

*(ii) There exists a semi-separating open affine covering $\mathcal{U}$ of X such that for every $U \in \mathcal{U}$, the complex $\mathscr{F}_\bullet(U)$ of flat $\mathcal{O}_X(U)$-modules is **F**-totally acyclic.*

*(iii) For every $x \in X$ the complex $(\mathscr{F}_\bullet)_x$ of flat $\mathcal{O}_{X,x}$-modules is **F**-totally acyclic.*

Proof. Let $\mathcal{U}$ be a semi-separating open affine covering of X. For every $U \in \mathcal{U}$, the inclusion $U \to X$ gives an adjoint pair (j_U^*, j_*^U), where $j_U^* \colon \mathfrak{Qcoh}(X) \to \mathfrak{Qcoh}(U)$ and $j_*^U \colon \mathfrak{Qcoh}(U) \to \mathfrak{Qcoh}(X)$ are the inverse and direct image functor respectively. Since j_U^* is an exact functor and j_*^U is a right adjoint of j_U^*, it follows that j_*^U preserves injective objects. Now the implication $(i) \Rightarrow (ii)$ follows.

$(ii) \Rightarrow (i)$: Let $\mathcal{U}$ be a semi-separating affine covering of X, such that $\mathscr{F}_\bullet(U)$ is **F**-totally acyclic for every $U \in \mathcal{U}$. Without loss of generality, assume that $\mathcal{U}$ is finite. Given a quasi-coherent sheaf $\mathscr{J}$ there exists, since the scheme is semi-separated, a monomorphism

$$0 \longrightarrow \mathscr{J} \longrightarrow \prod_{U \in \mathcal{U}} j_*^U(\widetilde{E_U}) \,,$$

where E_U denotes the injective hull of $\mathscr{J}(U)$ in the category of $\mathcal{O}_X(U)$-modules, and $\widetilde{E_U} \in \mathfrak{Qcoh}(U)$ is the corresponding sheaf. Recall that each quasi-coherent sheaf $j_*^U(\widetilde{E_U})$ is injective per the argument above. We assume that $\mathscr{J}$ is injective in $\mathfrak{Qcoh}(X)$, so it is a direct summand of the finite product $\prod_{\mathcal{U}} j_*^U(\widetilde{E_U})$. It is thus sufficient to prove that $j_*^U(\widetilde{E_U}) \otimes_{\mathcal{O}_X} \mathscr{F}_\bullet$ is acyclic for every $U \in \mathcal{U}$, as that will imply that $\mathscr{J} \otimes_{\mathcal{O}_X} \mathscr{F}_\bullet$ is a direct summand of an acyclic complex and hence acyclic. Fix $U \in \mathcal{U}$; for every $W \in \mathcal{U}$ there are isomorphisms

$$
\begin{aligned}
(j_*^U(\widetilde{E_U}) \otimes_{\mathcal{O}_X} \mathscr{F}_\bullet)(W) &\cong j_*^U(\widetilde{E_U})(W) \otimes_{\mathcal{O}_X(W)} \mathscr{F}_\bullet(W) \\
&= (E_U \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(U \cap W)) \otimes_{\mathcal{O}_X(W)} \mathscr{F}_\bullet(W) \\
&\cong E_U \otimes_{\mathcal{O}_X(U)} (\mathcal{O}_X(U \cap W) \otimes_{\mathcal{O}_X(W)} \mathscr{F}_\bullet(W)) \\
&\cong E_U \otimes_{\mathcal{O}_X(U)} \mathscr{F}_\bullet(U \cap W) \\
&\cong E_U \otimes_{\mathcal{O}_X(U)} (\mathcal{O}_X(W \cap U) \otimes_{\mathcal{O}_X(U)} \mathscr{F}_\bullet(U)) \\
&\cong (E_U \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(W \cap U)) \otimes_{\mathcal{O}_X(U)} \mathscr{F}_\bullet(U) \\
&\cong (\mathcal{O}_X(W \cap U) \otimes_{\mathcal{O}_X(U)} E_U) \otimes_{\mathcal{O}_X(U)} \mathscr{F}_\bullet(U) \\
&\cong \mathcal{O}_X(W \cap U) \otimes_{\mathcal{O}_X(U)} (E_U \otimes_{\mathcal{O}_X(U)} \mathscr{F}_\bullet(U)) \,.
\end{aligned}
$$

The last complex is acyclic as $\mathcal{O}_X(W \cap U)$ is a flat $\mathcal{O}_X(U)$-module and the complex $E_U \otimes_{\mathcal{O}_X(U)} \mathscr{F}_\bullet(U)$ is acyclic by the assumption that $\mathscr{F}_\bullet(U)$ is **F**-totally acyclic.

$(i) \Rightarrow (iii)$: Given an $x \in X$ consider a subset $U \in \mathcal{U}$ with $x \in U$. Let E be an injective $\mathcal{O}_{X,x}$-module; it is injective over $\mathcal{O}_X(U)$ as well as one has $\mathcal{O}_{X,x} \cong (\mathcal{O}_X(U))_x$. By (i) the complex $j_*^U(\widetilde{E}) \otimes_{\mathcal{O}_X} \mathscr{F}_\bullet$ is acyclic, and hence so is

$$(j_*^U(\widetilde{E}) \otimes_{\mathcal{O}_X} \mathscr{F}_\bullet)_x \cong (E \otimes_{\mathcal{O}_{X,x}} \mathscr{F}_\bullet)_x \,.$$

Thus, the $\mathcal{O}_{X,x}$-complex $(\mathscr{F}_\bullet)_x$ is **F**-totally acyclic. $\qquad\square$

If the scheme is noetherian and semi-separated, then all three conditions in Proposition 2.10 are equivalent; see [21, Lemma 4.4] for the remaining implication.

3. Total acyclicity vs. **F**-total acyclicity

Let X be a scheme with structure sheaf $\mathcal{O}_X$. Recall from Drinfeld [7, Section 2] that a not necessarily finite dimensional vector bundle on X is a quasi-coherent sheaf $\mathscr{P}$ such that the $\mathcal{O}_X(U)$-module $\mathscr{P}(U)$ is projective for every open affine subset $U \subseteq X$; i.e. it is locally projective per Definition 2.1. This is a Zariski-local notion because projectivity of modules is an AD-property and compatible with finite products; see [24]. We take a special interest in **F**-totally acyclic complexes of infinite dimensional vector bundles; to this end we recall:

Definition 3.1. A complex $P_\bullet = \cdots \to P_{i+1} \to P_i \to P_{i-1} \to \cdots$ of projective R-modules is called *totally acyclic* if it is acyclic, and $\mathrm{Hom}(P_\bullet, Q)$ is acyclic for every projective R-module Q.

As opposed to **F**-total acyclicity, it is not clear to us that total acyclicity leads to a Zariski-local property of complexes of quasi-coherent sheaves on an arbitrary scheme. The purpose of this section is to identify conditions on schemes that ensure that total acyclicity coincides with **F**-total acyclicity for complexes of vector bundles.

We need a few definitions parallel to those in Section 2.

Definition 3.2. Let $\mathfrak{P}$ be a property of commutative rings.
   (1) A scheme X with structure sheaf $\mathcal{O}_X$ is said to *locally* have property $\mathfrak{P}$ if for every open affine subset $U \subseteq X$ the ring $\mathcal{O}_X(U)$ has property $\mathfrak{P}$.
   (2) As a (local) property of schemes, the property $\mathfrak{P}$ is called *Zariski-local* if the following conditions are equivalent for every scheme X.
   - X locally has property $\mathfrak{P}$.
   - There exists an open affine covering $\mathcal{U}$ of X such that for every $U \in \mathcal{U}$ the ring $\mathcal{O}_X(U)$ has property $\mathfrak{P}$.

Recall that a commutative ring R is called *d-perfect* if every flat R-module has projective dimension at most d. Bass [4, Theorem P] described the 0-perfect rings.

Example 3.3. Every locally Noetherian scheme of Krull dimension d is locally coherent and locally d-perfect.

If R is a commutative coherent ring of global dimension 2, then the polynomial ring in n variables over R is coherent of global dimension $n + 2$; see Glaz [17, Theorem 7.3.14]. Thus, the scheme $\mathbb{P}_R^n$ is locally coherent and locally $(n + 2)$-perfect.

Definition 3.4. Let $\mathfrak{P}$ a property of commutative rings and let $\mathcal{R}$ be a class of commutative rings.
   (1) $\mathfrak{P}$ is said to *ascend* in $\mathcal{R}$, if for every flat epimorphism $R \to S$ of rings in $\mathcal{R}$, the ring S has property $\mathfrak{P}$ if R has property $\mathfrak{P}$.
   (2) $\mathfrak{P}$ is said to *descend* in $\mathcal{R}$ if for every faithfully flat homomorphism $R \to S$ of rings in $\mathcal{R}$, the ring S has property $\mathfrak{P}$ only if R has property $\mathfrak{P}$.

If $\mathfrak{P}$ ascends and descends in $\mathcal{R}$, then it is called an *ascent–descent property*, for short an *AD-property*, in $\mathcal{R}$.

Definition 3.5. A property $\mathfrak{P}$ of commutative rings is said to be *compatible with finite products* if for all commutative rings R_1 and R_2, the product ring $R_1 \times R_2$ has property $\mathfrak{P}$ if and only if R_1 and R_2 have property $\mathfrak{P}$.

The proof of the next lemma is parallel to that of Lemma 2.4.

Lemma 3.6. *Let X be a scheme with structure sheaf $\mathcal{O}_X$ and let $\mathfrak{P}$ be a property of commutative rings. If $\mathfrak{P}$ is compatible with finite products and an AD-property in the class of commutative rings $\{\mathcal{O}_X(U) \mid U \subseteq X \text{ is an open affine subset}\}$, then $\mathfrak{P}$ as a property of schemes is Zariski-local.* $\qquad\square$

Proposition 3.7. *The properties local coherence and local d-perfectness of schemes are Zariski-local.*

Proof. Coherence and d-perfectness are properties of rings that are compatible with finite products, so by Lemma 3.6 it is enough to prove that they are AD-properties.

Harris [18, Corollary 2.1] proves that coherence descends along faithfully flat homomorphisms of rings. To prove ascent, let $R \to S$ be a flat epimorphism and assume that R is coherent. Let $\{F_i \mid i \in I\}$ be a family of flat S-modules. As S is flat over R, every flat S-module is a flat R-module. Since R is coherent the R-module $\prod_{i \in I} F_i$ is flat, and as flatness ascends so is the S-module $S \otimes_R \prod_{i \in I} F_i$. There are isomorphisms of S-modules

$$S \otimes_R \prod_{i \in I} F_i \;\cong\; S \otimes_R \prod_{i \in I}(S \otimes_S F_i) \;\cong\; (S \otimes_R S) \otimes_S \prod_{i \in I} F_i \;\cong\; \prod_{i \in I} F_i \,,$$

where the last isomorphism holds as $R \to S$ is an epimorphism of rings. Thus, $\prod_{i \in I} F_i$, is a flat S-module, and it follows that S is coherent; see [17, Theorem 2.3.2].

To see that d-perfectness ascends, let $R \to S$ be a flat epimorphism and assume that R is d-perfect. Let F be a flat S-module; it is also flat over R, so there is an exact sequence of R-modules $0 \to P_d \to \cdots \to P_1 \to P_0 \to F \to 0$ with each P_i projective. As S is flat over R, it induces an exact sequence of S-modules

$$0 \longrightarrow S \otimes_R P_d \longrightarrow \cdots \longrightarrow S \otimes_R P_1 \longrightarrow S \otimes_R P_0 \longrightarrow S \otimes_R F \longrightarrow 0 \,.$$

Each S-module $S \otimes_R P_i$ is projective, so one has $\mathrm{pd}_S(S \otimes_R F) \le d$. Finally, as $R \to S$ is an epimorphism one has $S \otimes_R F \cong F$, so d-perfectness ascends. To prove descent, let $R \to S$ be a faithfully flat ring homomorphism where S is d-perfect. Let F be a flat R-module, and consider a projective resolution $\cdots \to P_1 \to P_0 \to F \to 0$ over R. As above it yields a projective resolution over S,

$$\cdots \longrightarrow S \otimes_R P_d \longrightarrow \cdots \longrightarrow S \otimes_R P_1 \longrightarrow S \otimes_R P_0 \longrightarrow S \otimes_R F \longrightarrow 0 \,.$$

The inequality $\mathrm{pd}_S(S \otimes_R F) \le d$ implies that the S-module

$$\mathrm{Coker}(S \otimes_R P_{d+1} \longrightarrow S \otimes_R P_d) \;\cong\; S \otimes_R \mathrm{Coker}(P_{d+1} \to P_d)$$

is projective, and it follows from [24] that the R-module $\mathrm{Coker}(P_{d+1} \to P_d)$ is projective, whence one has $\mathrm{pd}_R F \le d$. $\qquad\square$

Theorem 3.8. *Let X be locally coherent and locally d-perfect scheme, and let $\mathscr{P}_\bullet$ be a complex of vector bundles on X. If there exists an open affine covering $\mathcal{U}$ of X such that $\mathscr{P}_\bullet(U)$ is totally acyclic in $\mathrm{Ch}(\mathscr{R}(U))$ for every $U \in \mathcal{U}$, then $\mathscr{P}_\bullet$ is **F**-totally acyclic.*

Proof. Over a coherent d-perfect ring, every totally acyclic complex is **F**-totally acyclic; see Holm [19, Proposition 3.4]. Thus, every complex $\mathscr{P}_\bullet(U)$ is **F**-totally acyclic and the result follows as that is a Zariski-local property by Corollary 2.8. $\quad\square$

Theorem 3.9. *Let X be locally coherent and $\mathscr{P}_\bullet$ be a complex of possibly infinite dimensional vector bundles. If $\mathscr{P}_\bullet$ is **F**-totally acyclic then $\mathscr{P}_\bullet(U)$ is totally acyclic for every open affine subset $U \subseteq X$.*

Proof. Let $U \subseteq X$ be an open affine subset. The complex $\mathscr{P}_\bullet(U)$ is an **F**-totally acyclic complex of projective $\mathcal{O}_X(U)$-modules. Since $\mathcal{O}_X(U)$ is a coherent ring, $\mathscr{P}_\bullet(U)$ is totally acyclic by a result of Bravo, Gillespie, and Hovey [6, Theorem 6.7]. $\square$

For a different proof of the theorem, one could verify that the proof of [21, Lemma 4.20(ii)] extends to coherent rings.

Remark 3.10. Let X be a scheme with structure sheaf $\mathcal{O}_X$. A complex $\mathscr{P}_\bullet$ of vector bundles on X would be called totally acyclic if for every open affine subset $U \subseteq X$ the $\mathcal{O}_X(U)$-complex $\mathscr{P}_\bullet(U)$ is totally acyclic as defined in 3.1. We have not explicitly addressed that property, because we do not know if it is Zariski-local. However, if X is locally coherent and locally d-perfect, then the property is Zariski-local. Indeed, assume that there exists an open affine covering $\mathcal{U}$ of X such that $\mathscr{P}_\bullet(U)$ is a totally acyclic complex of projective $\mathcal{O}_X(U)$-modules for every $U \in \mathcal{U}$. It follows from Theorem 3.8 that $\mathscr{P}_\bullet$ is **F**-totally acyclic, and then for every open affine subset $U \subseteq X$ the complex $\mathscr{P}_\bullet(U)$ is totally acyclic by Theorem 3.9.

4. EXISTENCE OF ADJOINTS

Definition 4.1. Let X be a scheme; by $\mathbf{F}_{\mathrm{tac}}(\mathrm{Flat}\, X)$ we denote the class of **F**-totally acyclic complexes of quasi-coherent sheaves on X.

Theorem 4.2. *Let X be a scheme. The class $\mathbf{F}_{\mathrm{tac}}(\mathrm{Flat}\, X)$ is covering in the category $\mathrm{Ch}(\mathfrak{Qcoh}(X))$, and if $\mathfrak{Qcoh}(X)$ has a flat generator, then every such cover is an epimorphism.*

The assumption about existence of a flat generator for $\mathfrak{Qcoh}(X)$ is satisfied if the scheme X is quasi-compact and semi-separated; see Alonso Tarrío, Jeremías López, and Lipman [2, (1.2)]. We prepare for the proof with a couple of lemmas.

Lemma 4.3. *Let X be a scheme, $\mathscr{M}_\bullet$ be a complex in $\mathbf{F}_{\mathrm{tac}}(\mathrm{Flat}\, X)$, and $\mathscr{M}'_\bullet$ be a subcomplex of $\mathscr{M}_\bullet$. If conditions (1) and (2) below are satisfied, then the complexes $\mathscr{M}'_\bullet$ and $\mathscr{M}_\bullet/\mathscr{M}'_\bullet$ belong to $\mathbf{F}_{\mathrm{tac}}(\mathrm{Flat}\, X)$.*

(1) *$\mathscr{M}'_\bullet$ is acyclic.*
(2) *There exists an open covering $\mathcal{U}$ of X such that the $\mathcal{O}_X(U)$-submodules $\mathscr{M}'_n(U) \subseteq \mathscr{M}_n(U)$ and $\mathrm{Z}_n(\mathscr{M}'_\bullet(U)) \subseteq \mathrm{Z}_n(\mathscr{M}_\bullet(U))$ are pure for every $U \in \mathcal{U}$ and all $n \in \mathbb{Z}$.*

Proof. Fix $U \in \mathcal{U}$. For every $n \in \mathbb{Z}$ the submodule $\mathscr{M}'_n(U) \subseteq \mathscr{M}_n(U)$ is pure and $\mathscr{M}_n(U)$ is flat, so the modules $\mathscr{M}'_n(U)$ and $(\mathscr{M}_n/\mathscr{M}'_n)(U)$ are flat. By assumption, there is for every $n \in \mathbb{Z}$ a commutative diagram with exact rows

$$(*) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Z}_n(\mathscr{M}'_\bullet(U)) & \longrightarrow & \mathscr{M}'_n(U) & \longrightarrow & \mathrm{Z}_{n-1}(\mathscr{M}'_\bullet(U)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Z}_n(\mathscr{M}_\bullet(U)) & \longrightarrow & \mathscr{M}_n(U) & \longrightarrow & \mathrm{Z}_{n-1}(\mathscr{M}_\bullet(U)) & \longrightarrow & 0 \,, \end{array}$$

where the vertical homomorphisms are pure embeddings. Let I be an injective $\mathcal{O}_X(U)$-module. In the commutative diagram obtained by applying $I \otimes_{\mathcal{O}_X(U)} -$ to $(*)$, the vertical homomorphisms are pure embeddings, and the bottom row is exact as the complex $\mathscr{M}_\bullet(U)$ is **F**-totally acyclic. By commutativity it follows that the homomorphism $I \otimes_R \mathrm{Z}_n(\mathscr{M}'_\bullet(U)) \to I \otimes_R \mathscr{M}'_\bullet(U)$ is injective, whence $I \otimes_{\mathcal{O}_X(U)} \mathscr{M}'_\bullet(U)$ is acyclic; i.e. $\mathscr{M}'_\bullet(U)$ is **F**-totally acyclic. Now, as $(\mathscr{M}_\bullet/\mathscr{M}'_\bullet)(U)$ is a complex of flat $\mathcal{O}_X(U)$-modules, the sequence

$$0 \longrightarrow I \otimes_{\mathcal{O}_X(U)} \mathscr{M}'_\bullet(U) \longrightarrow I \otimes_{\mathcal{O}_X(U)} \mathscr{M}_\bullet(U) \longrightarrow I \otimes_{\mathcal{O}_X(U)} (\mathscr{M}_\bullet/\mathscr{M}'_\bullet)(U) \longrightarrow 0$$

is exact for every injective $\mathcal{O}_X(U)$-module I; since the left-hand and middle complexes are acyclic, so is the right-hand complex, that is, $(\mathscr{M}_\bullet/\mathscr{M}'_\bullet)(U)$ is **F**-totally acyclic. Finally, since $U \in \mathcal{U}$ is arbitrary, it follows that $\mathscr{M}'_\bullet$ and $\mathscr{M}_\bullet/\mathscr{M}'_\bullet$ are complexes of flat quasi-coherent sheaves and **F**-totally acyclic per Corollary 2.8. $\qquad\square$

For the next lemma, recall from 1.1 the notion of a κ-pure morphism.

Lemma 4.4. *Let X be a scheme and κ be a cardinal. If $\tau\colon \mathscr{F} \to \mathscr{G}$ is a κ-pure morphism in $\mathfrak{Qcoh}(X)$, then $\tau(U)\colon \mathscr{F}(U) \to \mathscr{G}(U)$ is pure monomorphism of $\mathcal{O}_X(U)$-modules for every open affine subset $U \subseteq X$.*

Proof. The category $\mathfrak{Qcoh}(X)$ is a Grothendieck category; see [9, Corollary 3.5]. By a result of Beke [5, Proposition 3] there exists an infinite cardinal κ, such that $\mathfrak{Qcoh}(X)$ is locally κ-presentable. The pure morphism τ is by [1, Proposition 2.29] a monomorphism, so it yields a κ-pure exact sequence

$$\mathbb{E} \;=\; 0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0\,.$$

By [1, Proposition 2.30] it is the colimit of a κ-directed system $(\mathbb{E}_\alpha)$ of splitting short exact sequences in $\mathfrak{Qcoh}(X)$. For every open affine subset $U \subseteq X$, one gets a κ-directed system $(\mathbb{E}_\alpha(U))$ of split exact sequences of $\mathcal{O}_X(U)$-modules. Since κ is infinite, every κ-directed system is an $\aleph_0$-directed system; see for example [15, Fact A.2]. Therefore, $\mathbb{E}(U)$ is a direct limit of splitting short exact sequences of $\mathcal{O}_X(U)$-modules. Hence, the sequence $\mathbb{E}(U)$ of $\mathcal{O}_X(U)$-modules is pure. $\qquad\square$

Lemma 4.5. *For every scheme X, the class $\mathbf{F}_{\mathrm{tac}}(\mathrm{Flat}\,X)$ is a Kaplansky class.*

Proof. Let κ be a cardinal such that the categories $\mathfrak{Qcoh}(X)$ and $\mathrm{Ch}(\mathfrak{Qcoh}(X))$ are locally κ-presentable; see [9, Corollary 3.5], [5, Proposition 3], and 1.2. Let $\mathscr{M}_\bullet \neq 0$ be a complex in $\mathrm{Ch}(\mathfrak{Qcoh}(X))$; by [1, Theorem 2.33] there is a cardinal γ such that every γ-presentable subcomplex $\mathscr{M}''_\bullet \subseteq \mathscr{M}_\bullet$ can be embedded in a γ-presentable κ-pure subcomplex $\mathscr{M}'_\bullet \subseteq \mathscr{M}_\bullet$. To prove that $\mathbf{F}_{\mathrm{tac}}(\mathrm{Flat}\,X)$ is a Kaplansky class, it is enough to verify that $\mathscr{M}'_\bullet$ satisfies conditions (1) and (2) in Lemma 4.3. Now, for any κ-presentable sheaf $\mathscr{L} \in \mathfrak{Qcoh}(X)$, the complexes $\mathrm{S}_n(\mathscr{L})$ and $\mathrm{D}_n(\mathscr{L})$ are κ-presentable. It follows that the κ-pure exact sequence $0 \to \mathscr{M}'_\bullet \to \mathscr{M}_\bullet \to \mathscr{M}_\bullet/\mathscr{M}'_\bullet \to 0$ induces exact sequences

$$0 \to \mathrm{Mor}(\mathrm{S}_n(\mathscr{L}), \mathscr{M}'_\bullet) \to \mathrm{Mor}(\mathrm{S}_n(\mathscr{L}), \mathscr{M}_\bullet) \to \mathrm{Mor}(\mathrm{S}_n(\mathscr{L}), \mathscr{M}_\bullet/\mathscr{M}'_\bullet) \to 0$$

and

$$0 \to \mathrm{Mor}(\mathrm{D}_n(\mathscr{L}), \mathscr{M}'_\bullet) \to \mathrm{Mor}(\mathrm{D}_n(\mathscr{L}), \mathscr{M}_\bullet) \to \mathrm{Mor}(\mathrm{D}_n(\mathscr{L}), \mathscr{M}_\bullet/\mathscr{M}'_\bullet) \to 0\,,$$

where Mor is short for $\mathrm{Hom}_{\mathrm{Ch}(\mathfrak{Qcoh}(X))}$. For every $\mathscr{N}_\bullet \in \mathrm{Ch}(\mathfrak{Qcoh}(X))$ and $n \in \mathbb{Z}$ there are standard isomorphisms

$$\mathrm{Mor}(\mathscr{L}, \mathrm{Z}_n(\mathscr{N}_\bullet)) \;\cong\; \mathrm{Mor}(\mathrm{S}_n(\mathscr{L}), \mathscr{N}_\bullet) \quad \text{and} \quad \mathrm{Mor}(\mathscr{L}, \mathscr{N}_n) \;\cong\; \mathrm{Mor}(\mathrm{D}_n(\mathscr{L}), \mathscr{N}_\bullet)\,,$$

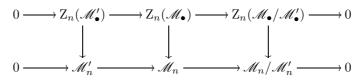which allow us to rewrite the exact sequences above as

$$0 \to \mathrm{Mor}(\mathscr{L}, \mathrm{Z}_n(\mathscr{M}'_\bullet)) \to \mathrm{Mor}(\mathscr{L}, \mathrm{Z}_n(\mathscr{M}_\bullet)) \to \mathrm{Mor}(\mathscr{L}, \mathrm{Z}_n(\mathscr{M}_\bullet/\mathscr{M}'_\bullet)) \to 0$$

and

$$0 \to \mathrm{Mor}(\mathscr{L}, \mathscr{M}'_n) \to \mathrm{Mor}(\mathscr{L}, \mathscr{M}_n) \to \mathrm{Mor}(\mathscr{L}, \mathscr{M}_n/\mathscr{M}'_n) \to 0\,.$$

As the category $\mathfrak{Qcoh}(X)$ is locally κ-presentable, it has a generating set of κ-presentable objects; it follows that $0 \to \mathrm{Z}_n(\mathscr{M}'_\bullet) \to \mathrm{Z}_n(\mathscr{M}_\bullet) \to \mathrm{Z}_n(\mathscr{M}_\bullet/\mathscr{M}'_\bullet) \to 0$ and $0 \to \mathscr{M}'_n \to \mathscr{M}_n \to \mathscr{M}_n/\mathscr{M}'_n \to 0$ are exact, even κ-pure exact, sequences in $\mathfrak{Qcoh}(X)$. Let $U \subseteq X$ be an open affine subset; by Lemma 4.4, the sequences of $\mathcal{O}_X(U)$-modules $0 \to \mathrm{Z}_n(\mathscr{M}'_\bullet(U)) \to \mathrm{Z}_n(\mathscr{M}_\bullet(U)) \to \mathrm{Z}_n((\mathscr{M}_\bullet/\mathscr{M}'_\bullet)(U)) \to 0$ and $0 \to \mathscr{M}'_n(U) \to \mathscr{M}_n(U) \to (\mathscr{M}_n/\mathscr{M}'_n)(U) \to 0$ are pure exact. Thus, condition (2)

in Lemma 4.3 is satisfied. It remains to show that the subcomplex $\mathscr{M}'_\bullet$ is acyclic. To this end, apply the snake lemma to the canonical diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Z}_n(\mathscr{M}'_\bullet) & \longrightarrow & \mathrm{Z}_n(\mathscr{M}_\bullet) & \longrightarrow & \mathrm{Z}_n(\mathscr{M}_\bullet/\mathscr{M}'_\bullet) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{M}'_n & \longrightarrow & \mathscr{M}_n & \longrightarrow & \mathscr{M}_n/\mathscr{M}'_n & \longrightarrow & 0
\end{array}
$$

to get an exact sequence $0 \to \mathrm{B}_{n-1}(\mathscr{M}'_\bullet) \to \mathrm{B}_{n-1}(\mathscr{M}_\bullet) \to \mathrm{B}_{n-1}(\mathscr{M}_\bullet/\mathscr{M}'_\bullet) \to 0$ for every $n \in \mathbb{Z}$. Now apply the snake lemma to

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{B}_n(\mathscr{M}'_\bullet) & \longrightarrow & \mathrm{B}_n(\mathscr{M}_\bullet) & \longrightarrow & \mathrm{B}_n(\mathscr{M}_\bullet/\mathscr{M}'_\bullet) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Z}_n(\mathscr{M}'_\bullet) & \longrightarrow & \mathrm{Z}_n(\mathscr{M}_\bullet) & \longrightarrow & \mathrm{Z}_n(\mathscr{M}_\bullet/\mathscr{M}'_\bullet) & \longrightarrow & 0
\end{array}
$$

One has $\mathrm{B}_n(\mathscr{M}_\bullet) = \mathrm{Z}_n(\mathscr{M}_\bullet)$ for every $n \in \mathbb{Z}$ as $\mathscr{M}_\bullet$ is acyclic. It follows that $\mathrm{B}_n(\mathscr{M}'_\bullet) = \mathrm{Z}_n(\mathscr{M}'_\bullet)$ holds for all $n \in \mathbb{Z}$, so $\mathscr{M}'_\bullet$ is acyclic. $\qquad\square$

We can prove now Theorem 4.2.

Proof of Theorem 4.2. The goal is to apply Proposition 1.6. The class $\mathbf{F}_{\mathrm{tac}}(\mathrm{Flat}\, X)$ is Kaplansky by Lemma 4.5, and it remains to prove that it is closed under extensions and direct limits. Let $0 \to \mathscr{M}'_\bullet \to \mathscr{M}_\bullet \to \mathscr{M}''_\bullet \to 0$ be an exact sequence in $\mathrm{Ch}(\mathfrak{Qcoh}(X))$ with $\mathscr{M}'_\bullet$ and $\mathscr{M}''_\bullet$ in $\mathbf{F}_{\mathrm{tac}}(\mathrm{Flat}\, X)$. For every open affine subset $U \subseteq X$, there is an exact sequence $0 \to \mathscr{M}'_\bullet(U) \to \mathscr{M}_\bullet(U) \to \mathscr{M}''_\bullet(U) \to 0$ of complexes of $\mathcal{O}_X(U)$-modules. As $\mathscr{M}'_\bullet$ and $\mathscr{M}''_\bullet$ are complexes of flat modules, so is $\mathscr{M}_\bullet$. The sequence remains exact when tensored by an injective $\mathcal{O}_X(U)$-module, and since $\mathscr{M}'_\bullet(U)$ and $\mathscr{M}''_\bullet(U)$ are $\mathbf{F}$-totally acyclic, so is $\mathscr{M}_\bullet(U)$. It follows that $\mathscr{M}_\bullet$ belongs to $\mathbf{F}_{\mathrm{tac}}(\mathrm{Flat}\, X)$; that is, $\mathbf{F}_{\mathrm{tac}}(\mathrm{Flat}\, X)$ is closed under extensions.

Let $\{\mathscr{F}^i_\bullet \mid i \in I\}$ be a direct system of complexes in $\mathbf{F}_{\mathrm{tac}}(\mathrm{Flat}\, X)$. For every open affine subset $U \subseteq X$ one then has a direct system $\{\mathscr{F}^i_\bullet(U) \mid i \in I\}$ of $\mathbf{F}$-totally acyclic complexes of $\mathcal{O}_X(U)$-modules. Now, $\varinjlim_{i\in I} \mathscr{F}^i_\bullet(U)$ is an acyclic complex of flat $\mathcal{O}_X(U)$-modules, and since direct limits commute with tensor products and homology, it is $\mathbf{F}$-totally acyclic. The quasi-coherent sheaf $\mathscr{F}_\bullet = \varinjlim_{i\in I} \mathscr{F}^i_\bullet$ satisfies $\mathscr{F}_\bullet(U) = \varinjlim_{i\in I} \mathscr{F}^i_\bullet(U)$, so we conclude that $\mathscr{F}_\bullet$ belongs to $\mathbf{F}_{\mathrm{tac}}(\mathrm{Flat}\, X)$. Now it follows from Proposition 1.6 that $\mathbf{F}_{\mathrm{tac}}(\mathrm{Flat}\, X)$ is covering.

Now assume that the category $\mathfrak{Qcoh}(X)$ has a flat generator $\mathscr{F}$. It follows that the complexes of flat sheaves $\mathrm{D}_n(\mathscr{F})$, $n \in \mathbb{Z}$, generate the category $\mathrm{Ch}(\mathfrak{Qcoh}(X))$. Evidently, each complex $\mathrm{D}_n(\mathscr{F})$ is $\mathbf{F}$-totally acyclic, so every $\mathbf{F}_{\mathrm{tac}}(\mathrm{Flat}\, X)$-cover is an epimorphism. $\qquad\square$

For any scheme X, one can consider the homotopy category of flat quasi-coherent sheaves, which we denote $\mathbf{K}(\mathrm{Flat}\, X)$. It is a triangulated category, and the full subcategory $\mathbf{K}_{\mathrm{tac}}(\mathrm{Flat}\, X)$ of $\mathbf{F}$-totally acyclic complexes in $\mathbf{K}(\mathrm{Flat}\, X)$ is a triangulated subcategory. The next result generalizes [21, Corollary 4.26] to arbitrary schemes.

Corollary 4.6. *For any scheme X, the inclusion $\mathbf{K}_{\mathrm{tac}}(\mathrm{Flat}\, X) \to \mathbf{K}(\mathrm{Flat}\, X)$ has a right adjoint.*

Proof. The full subcategory $\mathbf{K}_{\mathrm{tac}}(\mathrm{Flat}\, X)$ is closed under retracts, so per Neeman [22, Definition 1.1] it is a thick subcategory of $\mathbf{K}(\mathrm{Flat}\, X)$. By Theorem 4.2 every complex in $\mathbf{K}(\mathrm{Flat}\, X)$ has a $\mathbf{K}_{\mathrm{tac}}(\mathrm{Flat}\, X)$-precover; now [22, Proposition 1.4] yields the existence of a right adjoint to the inclusion $\mathbf{K}_{\mathrm{tac}}(\mathrm{Flat}\, X) \to \mathbf{K}(\mathrm{Flat}\, X)$. $\qquad\square$

Definition 4.7. Let X be a scheme. A quasi-coherent sheaf $\mathscr{G}$ on X is called *Gorenstein flat* if there exists an **F**-totally acyclic complex $\mathscr{F}_{\bullet}$ of flat quasi-coherent sheaves on X with $\mathscr{G} \cong \mathrm{Z}_n(\mathscr{F}_{\bullet})$ for some $n \in \mathbb{Z}$.

The affine case of the next result was proved by Yang and Liang [29, Theorem A]. Recall from [2, (1.2)] that $\mathfrak{Qcoh}(X)$ has a flat generator—in particular, a Gorenstein flat generator—if the scheme X is quasi-compact and semi-separated.

Corollary 4.8. *Let X be a scheme. Every quasi-coherent sheaf on X has a Gorenstein flat precover. Moreover, if the category $\mathfrak{Qcoh}(X)$ has a Gorenstein flat generator, then the Gorenstein flat precover is an epimorphism.*

Proof. Let $\mathscr{M} \in \mathfrak{Qcoh}(X)$; by Theorem 4.2 there exists an **F**-totally acyclic cover $\varphi \colon \mathscr{F}_{\bullet} \to \mathrm{S}_1(\mathscr{M})$ in $\mathrm{Ch}(\mathfrak{Qcoh}(X))$. For every acyclic complex $\mathscr{A}_{\bullet}$ in $\mathrm{Ch}(\mathfrak{Qcoh}(X))$ there is an isomorphism

$$(*) \qquad \mathrm{Hom}_{\mathrm{Ch}(\mathfrak{Qcoh}(X))}(\mathscr{A}_{\bullet}, \mathrm{S}_1(\mathscr{M})) \cong \mathrm{Hom}_{\mathfrak{Qcoh}(X)}(\mathrm{Z}_0(\mathscr{A}_{\bullet}), \mathscr{M}) .$$

Thus, the cover induces a morphism $\phi \colon \mathrm{Z}_0(\mathscr{F}_{\bullet}) \to \mathscr{M}$, where $\mathrm{Z}_0(\mathscr{F}_{\bullet})$ is Gorenstein flat. We now argue that ϕ is a Gorenstein flat precover. Consider a morphism $\theta \colon \mathscr{G} \to \mathscr{M}$ in $\mathfrak{Qcoh}(X)$ with $\mathscr{G}$ Gorenstein flat. There exists an **F**-totally acyclic complex $\mathscr{F}'_{\bullet}$ of flat quasi-coherent sheaves with $\mathrm{Z}_0(\mathscr{F}'_{\bullet}) \cong \mathscr{G}$. The morphism θ corresponds by $(*)$ to a morphism $\vartheta \colon \mathscr{F}'_{\bullet} \to \mathrm{S}_1(\mathscr{M})$ in $\mathrm{Ch}(\mathfrak{Qcoh}(X))$. Since φ is an **F**-totally acyclic cover, there exists a morphism $\varkappa \colon \mathscr{F}'_{\bullet} \to \mathscr{F}_{\bullet}$ such that $\varphi \circ \varkappa = \vartheta$ holds. It is now straightforward to verify that the induced morphism $\kappa \colon \mathrm{Z}_0(\mathscr{F}'_{\bullet}) \to \mathrm{Z}_0(\mathscr{F}_{\bullet})$ satisfies $\phi \circ \kappa = \theta$. Thus ϕ is a Gorenstein flat precover of $\mathscr{M}$. Finally, a flat generator of $\mathfrak{Qcoh}(X)$ is trivially Gorenstein flat, so if such a generator exists, then every Gorenstein flat precover is surjective. $\square$

The next corollary partly recovers [6, Proposition 8.10] and [14, Corollary 2]; the argument is quite different from the one given in [6] and [14].

Corollary 4.9. *Let R be a coherent and d-perfect ring. Every R-module has a Gorenstein projective precover.*

Proof. Let $\mathrm{Proj}\, R$ denote the class of projective R-modules. In the category of complexes of R-modules, $(\mathrm{Ch}(\mathrm{Proj}\, R), \mathrm{Ch}(\mathrm{Proj}\, R)^{\perp})$ is a complete cotorsion pair; see Gillespie [16, Section 5.2]. Let F be a complex in $\mathrm{Ch}(\mathrm{Flat}\, R)$; there is an exact sequence

$$(*) \qquad\qquad 0 \longrightarrow E \longrightarrow P \longrightarrow F \longrightarrow 0$$

with P is in $\mathrm{Ch}(\mathrm{Proj}\, R)$ and E in $\mathrm{Ch}(\mathrm{Proj}\, R)^{\perp}$. As the class of flat R-modules is resolving, E belongs to the intersection $\mathrm{Ch}(\mathrm{Proj}\, R)^{\perp} \cap \mathrm{Ch}(\mathrm{Flat}\, R)$, which by Neeman [23, 2.14] is the class of pure acyclic complexes of flat R-modules. Notice that P is **F**-totally acyclic if and only if F is **F**-totally acyclic. It thus follows from Theorem 3.9 that every **F**-totally acyclic complex has a totally acyclic precover.

Let M be any complex of R-modules. By Theorem 4.2 it has an **F**-totally acyclic cover $\phi \colon F \to M$, and as noted above F has a totally acyclic precover $\psi \colon P \to F$. We argue that the composite $\phi \circ \psi$ is a totally acyclic precover of M. To this end, let $\delta \colon P' \to M$ be a morphism, where P' is a totally acyclic complex; by Theorem 3.8 it is **F**-totally acyclic. Since ϕ is an **F**-totally acyclic cover of M, there exists a morphism $\theta \colon P' \to F$ with $\phi \circ \theta = \delta$. Now, since ψ is a totally acyclic precover of F, there exists $\gamma \colon P' \to P$ such that $\psi \circ \gamma = \theta$. Finally, one has $(\phi \circ \psi) \circ \gamma = \phi \circ \theta = \delta$. So $\phi \circ \psi$ is a totally acyclic precover of M. Finally, arguing as in the proof of Corollary 4.8, we infer that every module has a Gorenstein projective precover. $\square$

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