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On Inequalities and Partial Orderings for Weighted Reliability Measures

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Inequalities, relations and partial ordering for weighted reliability measures are presented. Inequalities for Lévy distance measure for weighted distributions are obtained in terms of the parent distributions. Reliability inequalities and stability results are established for weighted distributions with monotone hazard and mean residual life functions.

Keywords: Weighted distribution functions; Stochastic Ordering; Hazard function.

Classification: *AMS (MOS) Subject Classification:* 62N05.

1 INTRODUCTION

When observations generated from a stochastic process are recorded with some weight, the resulting distribution is a weighted distribution. Weighted models are widely used in several areas including biometry, ecology, forestry, and reliability (Gupta and Keating [2], Gupta and Kirmani [3], Patil and Rao [5]). In reliability, the so called equilibrium distribution is a weighted distribution. The purpose of this paper is to establish bounds and inequalities for weighted distributions with monotone hazard and mean residual life functions. We give new partial ordering for weighted distributions and obtain inequalities for Lévy distance

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measure between weighted distributions in terms of the parent distributions.

The direct comparisons of random variables or their survival functions is usually more informative than through their moments (Jain *et al.* [4]). It is therefore paramount to investigate relations between weighted random variables via some useful notions of partial ordering that relates to the original random variables or survival functions. In Section 2, some basic definitions and utility notions are presented. Section 3 contains orderings as it relates to the notions of ageing and age smoothness for weighted distribution functions. Results on Lévy distance measure as well as useful inequalities for weighted reliability measures are presented in Section 4. Section 5 contains inequalities for weighted reliability measures including the class of harmonic used better than aged in expectation (*HUBAE*), and harmonic new worse than used (*HNWUE*), increasing hazard rate (*IHR*), decreasing hazard rate (*DHR*), increasing mean residual life (*IMRL*), and decreasing mean residual life (*DMRL*) life distributions. For a review of classes of life distributions and their implications see [1], [6] and [7].

2 UTILITY NOTIONS AND SOME DEFINITIONS

Let X be a nonnegative random variable with absolutely continous distribution function $F(x)$ and probability density function (pdf) $f(x)$. The weighted distribution of X has a pdf given by

$$\bar{F}_W(x) = \bar{F}(x)\{W(x) + M_F(x)\}/E(W(X)), \tag{1}$$

where $M_F(x) = \int_x^\infty \{\bar{F}(t)W'(t)dt\}/\bar{F}(x)$, assuming $W(x)\bar{F}(x) \to 0$ as $x \to \infty$. The corresponding pdf of the weighted random variable X_W is

$$f_W(x) = W(x)f(x)/E(W(X)), \tag{2}$$

$x \geq 0$, where $0 < E(W(X) < \infty$. The mean residual life function (*MRLF*) of X is given by $\delta_F(x) = E(X - x|X > x) = \int_x^\infty \bar{F}(y)dy/\bar{F}(x)$, for $\bar{F}(x) > 0$ and $x \geq 0$. It is well known that the hazard function $\lambda_F(x)$, survival function $\bar{F}(x)$ and MRLF $\delta_F(x)$ are equivalent. We now give some basic and important definitions.

DEFINITION 1 *Let X and Y be two random variables with distribution functions F and G respectively. We say* $F <_{st} G$ *if* $\bar{F}(x) \leq \bar{G}(x)$, *for* $x \geq 0$ *or equivalently, for any increasing function* $\Phi(x)$,

$$E(\Phi(X)) \leq E(\Phi(Y)). \tag{3}$$

DEFINITION 2 *The mean residual life function is decreasing in convex order if*

$$\int_{x+t_1}^{\infty} \bar{F}(y)dy/\bar{F}(t_1) \geq \int_{x+t_2}^{\infty} \bar{F}(y)dy/\bar{F}(t_2), \tag{4}$$

for all $x \geq 0, 0 \leq t_1 \leq t_2$. *This is denoted by* $X_{t_1} \leq_c X_{t_2}$, *for all* $0 \leq t_1 \leq t_2$.

DEFINITION 3 *Let* $M_F(x) = \gamma_F(x)/\bar{F}(x)$, *where*

$$\gamma_F(x) = \int_x^{\infty} \bar{F}(y)W'(y)dy, \; W(x) > 0,$$

and $W'(x) = dW(x)/dx$. *The weighted mean residual life function* $M_F(x) = \gamma_F(x)/\bar{F}(x)$ *is decreasing in convex order if*

$$\int_{x+t_1}^{\infty} \bar{F}(y)W'(y)dy/\bar{F}(t_1) \geq \int_{x+t_2}^{\infty} \bar{F}(y)W'(y)dy/\bar{F}(t_2) \tag{5}$$

for all $x \geq 0, 0 \leq t_1 \leq t_2$, *provided* $W(x)\bar{F}(x) \to 0$ *as* $x \to \infty$.

Clearly if $W'(x) \geq 0$, then $M_F(x) \geq 0$ for all $x \geq 0$ where $M_F(x)$ is given above. If $W(x)$ is increasing, then $\lambda_{F_W}(x) \leq \lambda_F(x)$ for all $x \geq 0$, where

$$\lambda_{F_W}(x) = \{W(x)\lambda_F(x)\}/\{W(x) + M_F(x)\} \tag{6}$$

is the hazard function of the weighted distribution function F_W with survival function given by (1). The mean residual life function of the weighted distribution function F_W is given by

$$\delta_{F_W}(x) = \{\bar{F}(x)\{W(x) + M_F(x)\}\}^{-1} \int_x^{\infty} \bar{F}(y)\{W(y) + M_F(y)\}dy. \tag{7}$$

DEFINITION 4 *Let X be a random variable with distribution functions F in $\mathcal{F}$, where $\mathcal{F}$ is the set of absolutely continous random variables whose distribution function satisfy* $F(0), \lim_{x\to\infty} F(x) = 1$ *and* $Sup\{x : F(x) < 1\} = \infty$. *The distribution function $F(x)$ is said to be finitely and positively smooth if a number $\beta \in (0, \infty)$ exits such that*

$$\lim_{t\to\infty} [\bar{F}(x+t)/\bar{F}(t)] = e^{-\beta x} \tag{8}$$

for all $x \geq 0$, where β is called the asymptotic decay coefficient of X.

If X_t is the life time of a device at age $t \geq 0$, then its survival function is given by

$$\bar{F}_t(x) = P(X > x + t | X > t) = \bar{F}(x+t)/\bar{F}(t), \tag{9}$$

$x \geq 0$.

DEFINITION 5 *Let X be in $\mathcal{F}$ with distribution function $F(x)$ that is finitely and positively smooth with asymptotic decay coefficient β. Then X is said to be*

(i) used better then aged (UBA) if

$$\bar{F}_t(x) \geq e^{-\beta x} \tag{10}$$

for all $t, x \geq 0$;

(ii) used better than aged in expectation (UBAE) if $E(X)$ is positive and

$$E(X_t) \geq \beta^{-1} \tag{11}$$

for all $t \geq 0$;

(iii) harmonic used better than aged in expectation (HUBAE) if

$$\int_x^{\infty} \bar{F}(y)dy \geq \mu e^{-\beta x} \tag{12}$$

for all $x \geq 0$, *where* $\mu = \int_0^{\infty} \bar{F}(y)dy < \infty$;

(iv) harmonic new worst than used in expectation (HNWUE) if

$$\int_0^\infty \bar{F}(y)dy \geq \mu e^{-x/\mu} \tag{13}$$

for all $x \geq 0$. *The inequalities in (i), (ii), (iii), and (iv) are reversed for used worse than aged (UWA), used worst than aged in expectation (UWAE), harmonic used worst than aged in expectation (HUWAE), and harmonic new better than used in expectation (HNBUE) respectively. Note that the inequality in (iii) can be stated as*

$$\left\{x^{-1}\int_0^x \delta_F^{-1}(y)dy\right\}^{-1} \geq \beta^{-1} \tag{14}$$

for all $x > 0$. *That is, the integral harmonic of* $\delta_F(y)$ *over* $(0, x)$ *is greater than or equal to* β^{-1}. *This leads to the next definition.*

DEFINITION 6 *Let F and G be two distribution functions in* $\mathcal{F}$ *that are finitely and positively smooth with asymptotic decay coefficient* β. *We say F is integral harmonic mean (IHM) larger than G if*

$$\left\{x^{-1}\int_0^x \delta_F^{-1}(y)dy\right\}^{-1} \geq \left\{x^{-1}\int_0^x \delta_G^{-1}(y)dy\right\}^{-1} \tag{15}$$

for all $x > 0$.

DEFINITION 7 *A distribution function F is an increasing hazard rate (IHR) distribution if* $\bar{F}(x+t)/\bar{F}(t)$ *is decreasing in* $0 < t < \infty$ *for each* $x \geq 0$. *Similarly, a distribution function F is a decreasing hazard rate (DHR) distribution if* $\bar{F}(x+t)/\bar{F}(t)$ *is increasing in* $0 < t < \infty$ *for each* $x \geq 0$. *It is well known that IHR (DHR) implies DMRL (IMRL).*

PROPOSITION 1 $\bar{F}_W(x) = \bar{F}(x)$ *if and only if* $W(x) + M_F(x) = E(W(X)$, that is $\bar{F}_W(x) = \bar{F}(x)$ *if and only if* $W(x)$ *is a constant.*

3 SOME PARTIAL ORDER FOR RELIABILITY MEASURES

In this section, we obtain present inequalities for the purpose of comparisons of weighted distributions. This is accomplished via the use of some partial order in the moments and reliability functions of the weighted random variables. Let X_W and Y_W be weighted random variables with distribution functions F_W and G_W respectively. The corresponding original distribution are F and G respectively. For size-biased distributions, $W(x) = x$ and the moments of F_W and F are related by

$$E_{F_W}(X^r) = E_F(X^{r+1})/\mu_F \tag{16}$$

$r \geq 1$, where E_{F_W} and F_F denote expectations with respect to F_W and F respectively.

A partial ordering of the weighted random variables X_W and Y_W or their distribution functions F_W and G_W is given by $E(X_W) - E(X) \geq E(Y_W) - E(Y)$ or equivalently $\mathrm{Var}(X)/E(X) \geq \mathrm{Var}(Y)/E(Y)$, where $\mathrm{Var}(X)$ denote the variance of the non-degenerate random variable X. If $E(X) = E(Y)$, the inequalities reduces to $E(X^2) \geq E(Y^2)$. It is well known that for weighted distribution function $F_W, E(X_W) - E(X) \geq (\leq 0)$ if and only if $\mathrm{Cov}(X, X_W) \geq (\leq 0)$. This leads to a partial ordering of the random variables X_W and Y_W.

DEFINITION 8 *Let X_{W_1} and Y_{W_2} be weighted random variables with distribution functions $F_{W_1}(x)$ and $G_{W_2}(x)$ respectively. We say X_{W_1} is larger than Y_{W_2} in weighted order if*

$$\mathrm{Cov}(X, X_{W_1}) \geq \mathrm{Cov}(Y, Y_{W_2}). \tag{17}$$

This is denoted by $X_{W_1} \geq_{lw} Y_{W_2}$. The inequality is reversed for smaller in weighted ordering.

PROPOSITION 2 *For size-biased distribution functions $X_{W_1} \geq_{lw} Y_{W_2}$ if and only if $E(X_W) - E(X) \geq E(Y_W) - E(Y)$.*

Proof This follows from the fact that

$$E(X_{W_1}) = \{\mathrm{Var}(X)/E(X)\} + E(X). \tag{18}$$

■

Note also that if $E(X) = E(Y)$, then $X_{W_1} \geq_{lw} Y_{W_2}$ if and only if

$$E(XW_1(X))/E(YW_2(Y) \geq E(W_1(X))/E(W_2(Y)). \tag{19}$$

THEOREM 1 *Let X_{W_1} and Y_{W_2} be weighted random variables with distribution functions $F_{W_1}(x)$ and $G_{W_2}(x)$ respectively. Suppose $0 < E(W_i(X)) < \infty, i = 1, 2$, and $E(X) = E(Y)$, then $X_{W_1} \geq_{lw} Y_{W_2}$ if and only if $W_1(x)/W_2(x)$ is increasing in x.*

Proof The proof follows from the fact that

$$\begin{aligned}\operatorname{Cov}(Y, W_2(Y)) &= E(YW_2(Y))/E(W_2(Y)) \\ &= E(Yh(Y)W_1(Y))/E(h(Y)W_1(Y)),\end{aligned} \tag{20}$$

where $h(Y) = W_2(Y)/W_1(Y)$.

Consequently, $\operatorname{Cov}(Y, W_2(Y)) \geq \operatorname{Cov}(X, W_1(X))$ if and only if $h(y) = W_2(y)/W_1(y)$ is
increasing in y.

THEOREM 2 *If $\lambda_F(x) \geq \lambda_G(x)$ for all $x \geq 0$, and $W(x)$ is increasing in x, then $\lambda_{F_W}(x) \geq \lambda_{G_W}(x)$ and*

$$\left\{x^{-1} \int_0^x \delta_{F_W}^{-1}(y)dy\right\}^{-1} \leq \left\{x^{-1} \int_0^x \delta_{G_W}^{-1}(y)dy\right\}^{-1} \tag{21}$$

for all $x > 0$. ■

Proof By virtue of the fact that $\lambda_F(x) \geq \lambda_G(x)$ for all $x \geq 0$, we have

$$(\bar{F}(x))^{-1} \int_x^\infty \bar{F}(y)W'(y)dy \leq (\bar{G}(x))^{-1} \int_x^\infty \bar{G}(y)W'(y)dy, \tag{22}$$

for all $x \geq 0$.

Consequently,

$$W(x)\lambda_F(x)/\{W(x) + M_F(x)\} \geq W(x)\lambda_G(x)/\{W(x) + M_G(x)\}, \tag{23}$$

for all $x \geq 0$, where $M_F(x) = \gamma(x)/\bar{F}(x)$ and $\gamma(x) = \int_x^\infty \bar{F}(y)W'(y)dy$. It follows therefore that $\lambda_{F_W}(x) \geq \lambda_{G_W}(x)$ for all $x \geq 0$. This implies $\delta_{F_W}(x) \leq \delta_{G_W}(x)$ for all $x \geq 0$, so that $\delta_{F_W}^{-1}(x) \leq \delta_{G_W}^{-1}(x)$ for all $x \geq 0$.

Consequently,

$$\left\{x^{-1}\int_0^x \delta_{F_W}^{-1}(y)dy\right\}^{-1} \leq \left\{x^{-1}\int_0^x \delta_{G_W}^{-1}(y)dy\right\}^{-1} \tag{24}$$

for all $x > 0$. ■

4 INEQUALITIES FOR LÉVY DISTANCE MEASURE

In this section we obtain useful inequalities and discuss the problem of Lévy distance measure between weighted distribution functions F_W and G_W respectively. Let $H_F^*(x) = \mu_F H_F(x)$, and $H_G^*(x) = \mu_G H_F(x)$, where

$$H_F(x) = \mu_F^{-1}\int_0^x W(t)dF(t)$$

and

$$H_G(x) = \mu_G^{-1}\int_0^x W(t)dG(t)$$

for $x \geq 0$. We assume $H_F^*(0) = H_F^*(0) = 0$. It is clear that $H_F^*(x)$ and $H_G^*(x)$ are bounded nondecreasing functions. The Lévy distance between $H_F^*(x)$ and $H_G^*(x)$ denoted by $L(H_F^*, H_G^*)$ is the infimun of the numbers $c > 0$ satisfying

$$H_F^*(x+c) \geq H_G^*(x) \geq H_F^*(x-c) - c, \tag{25}$$

for all $x \geq 0$, where $W(x)$ is continous, nonnegative, and nondecreasing on $[0, \infty)$. Let the distribution functions F and G of the unweighted random variables X and Y satisfy

$$\int_0^\infty |W(x+c) - W(x)|dF(x) + \int_0^\infty |W(x+c) - W(x)|dG(x) \leq kc^\alpha, \tag{26}$$

for any $0 \leq c \leq a$, and some $k = k(F, G, W)$, $\alpha = \alpha(W) > 0$. Furthermore, we assume that

$$\psi = \int_0^\infty \{(\bar{F}(x))^{1/s} + (\bar{G}(x))^{1/s}\} W'(x)dx < \infty, \tag{27}$$

for some $s > 1$, where $W'(x) = dW(x)/dx$, $W(x) \leq \varepsilon|x|^r \leq \varepsilon a^r$, $a < \infty$ for some $\varepsilon > 0$, $r > 0$.

THEOREM 3 *The Lévy distance* $L(H_F^*, H_G^*)$ *satisfies*

$$L(H_F^*, H_G^*) \leq \tau L^\beta(F, G), \tag{28}$$

where $\beta = min\{r, \alpha, 1 - s^{-1}\}$, $\tau = \{a(\gamma + \psi) + ka^{\alpha-\beta} + \varepsilon a^{r-\beta}\} + 1$, *and*

$$\gamma = Sup_{x\geq 0}\{W(x)[(\bar{F}(x))^{1/s} + (\bar{G}(x))^{1/s}]\}.$$

Proof Let $x \geq 0$, then

$$\begin{aligned}
&H_F^*(x) - H_G^*(x - c)\\
&= \int_0^x W(t)dF(t) - \int_0^{x-c} W(t)dG(t)\\
&= \int_0^x W(t)dF(t) - \int_0^{x-c} [W(t) - W(t+c)]dG(t) - \int_0^x W(t+c)dG(t)\\
&= \int_0^x W(t)dF(t) + \int_0^{x-c} [W(t+c) - W(t)]dG(t) - \int_0^x W(t+c)dG(t)\\
&\leq \int_0^x W(t)dF(t) + \int_0^{\infty} [W(t+c) - W(t)]dG(t) - \int_0^x W(t+c)dG(t)\\
&= \int_0^{\infty} [W(t+c) - W(t)]dG(t) + \int_0^x W(t)dF(t) - \int_0^{x-c} [W(t)]dG(t-c)\\
&\leq kc^\alpha + \int_0^x W(t)dF(t) - \int_0^{x-c} [W(t)]dG(t-c)\\
&= kc^\alpha + W(x)[F(x) - G(x-c)] - \int_0^x [F(t) - G(t-c)]W'(t)dt\\
&\leq kc^\alpha + \mathrm{Sup}_{x\geq 0}\{W(x)[(\bar{F}(x))^{1/s} + (\bar{G}(x))^{1/s}]\}\\
&\quad + \left\{\int_0^{\infty} \{(\bar{F}(t))^{1/s} + (\bar{G}(t))^{1/s}\right\} W'(x)dx\}c^{1-1/s}.
\end{aligned} \tag{29}$$

The first inequality is straightforward. The second inequality follows from equation (26). The line before the last inequality is obtained via integration by parts and the last inequality follows from (27) and the fact that $(|u| + |v|)^b \leq |u|^b + |v|^b$ for $0 \leq b \leq 1$ and for all real numbers u and v.

Noting that $W(x) \leq \varepsilon|x|^r$ for some $r > 0, \varepsilon > 0$, we have for $x \geq 0$, and $|x| \leq a$, $H_F^*(x) - H_G^*(x - c)$

$$\begin{aligned} &\leq kc^\alpha + \mathrm{Sup}_{x\geq 0}\{W(x)[(\bar{F}(x))^{1/s}c^{1-1/s} + (\bar{G}(x))^{1/s}]\} \\ &\quad + \left\{\int_0^\infty \{(\bar{F}(t))^{1/s} + (\bar{G}(t))^{1/s}\right\} W'(x)dx\} c^{1-1/s} \\ &= kc^\alpha + c^{1-1/s}(\gamma + \psi) + a\varepsilon c^r. \end{aligned} \tag{30}$$

Consequently,

$$H_F^*(x) - H_G^*(x - \tau c) \leq \tau c^\beta, \tag{31}$$

and

$$H_G^*(x + \tau c) - H_F^*(x) \leq \tau c^\beta, \tag{32}$$

for all $x \geq 0$, where we let $0 \leq c$ decrease to $L(F, G) \leq 1$ in (31) and (32) with $0 \leq \beta \leq 1$ ■

5 INEQUALITIES FOR RELIABILITY MEASURES

In this section we present reliability inequalities for weighted distributions. Inequalities for the comparisons of notions of ageing including harmonic used better than aged (*HUBAE*) are established. In this regard we consider the class $\mathcal{M}$ of distributions having IHR or DHR and present results on how close the weighted distributions F_W, with monotone weight functions are to the size-biased exponential distribution. We present bounds on the distance between a weighted distribution in the class $\mathcal{M}$ and the size-biased distribution in terms of the moments of F.

The weighted conditional probability of survival is given by

$$F_{W_t}(x) = P(X_W > x + t | X_W > t) = \bar{F}_W(x + t)/\bar{F}_W(t), \tag{33}$$

$x \geq 0$, where $\bar{F}_W(x) = 1 - F_W(x)$ is the reliability function of X_W.

PROPOSITION 3 *Let F and F_W be in $\mathcal{F}$. If $W(x)$ is increasing and F is HNWUE, then* $\int_0^\infty \bar{F}_W(y)dy \geq \mu e^{-x/\mu}$ *for all* $x \geq 0$, *where* $0 < \mu < \infty$.

Proof This result follows from the fact that the random variable X_W and X are stochastically ordered, that is $\bar{F}_W(x) \geq \bar{F}(x)$ for all $x \geq 0$, whenever $W(x)$ is increasing in x.

PROPOSITION 4 *Let F and F_W be distribution functions in $\mathcal{F}$ that are finitely and positively smooth with asymptotic decay coefficient β. If $W(x)$ is increasing in x, then*

$$x^{-1}\int_0^x \delta_{F_W}^{-1}(y)dy \geq x^{-1}\int_0^x \delta_F^{-1}(y)dy \tag{34}$$

for all $x \geq 0$. Furthermore, if F is HUBAE, then $x^{-1}\int_0^x \delta_{F_W}^{-1}(y)dy \geq \beta^{-1}$ *for all* $x \geq 0$.

Proof Note that

$$\delta_F^{-1}(x) = -\left[d\ln\left\{\int_x^\infty \bar{F}(y)dy\right\}/dx\right]^{-1}, \tag{35}$$

so that $\int_0^x \delta_F^{-1}(y)dy = -\ln[\{\int_x^\infty \bar{F}(y)dy\}/\mu]$. Also, $f_W(x)/f(x)$ is increasing in x, so that $\lambda_{F_W}(x) \leq \lambda_F(x)$ for all $x \geq 0$, and $\delta_{F_W}(x) \geq \delta_F(x)$ for all $x \geq 0$. It follows therefore that

$$\int_0^x \delta_{F_W}^{-1}(y)dy \leq \int_0^x \delta_F^{-1}(y)dy \tag{36}$$

and

$$\left\{\int_0^x \delta_{F_W}^{-1}(y)dy\right\}^{-1} \geq \left\{\int_0^x \delta_F^{-1}(y)dy\right\}^{-1} \tag{37}$$

for all $x \geq 0$. The condition that F is *HUBAE* is equivalent to

$$\left\{\int_0^x \delta_F^{-1}(y)dy\right\}^{-1} \geq \beta^{-1} \tag{38}$$

for all $x \geq 0$, by using equation (34).

Consequently,

$$\left\{\int_0^x \delta_{F_W}^{-1}(y)dy\right\}^{-1} \geq \beta^{-1}$$

for all $x \geq 0$. ■

Let the distribution function F possess moments of order J, that is $\mu_j = E(X^j), j = 1, 2, \ldots, J$ and $\{S_j(x)\}, j = 1, 2, \ldots, J$ be a sequence of decreasing functions given by

$$S_j(x) = \begin{cases} \bar{F}(x) & \text{if } j = 0, \\ \int_0^\infty \bar{F}(x+t)t^{j-1}dt/(j-1)!, & \text{if } j = 1, 2, \ldots, J. \end{cases}$$

We let $S_{-1}(x) = f(x)$ be the probability density function of F if it exists. Then $S_j(0) = \mu^j/j!, S_j'(x) = -S_{j-1}(x), j = 1, 2, \ldots, J$. The ratio $S_{j-1}(x)/S_j(x)$ is the hazard function of a distribution function with survival function $S_j(x)/S_j(0)$.

LEMMA 1 *If F has decreasing mean residual life (DMRL), then*

$$S_k(x) \leq S_k(0)e^{-x/\mu}, \tag{39}$$

$k = 1, 2, \ldots, J$. ■

LEMMA 2 *If F has increasing mean residual life (IMRL), and* $\lim_{x\to\infty} S_0(x)/S_1(x) =$
$\alpha > 0$, *then*

$$S_1(x) \geq \mu S_0(x). \tag{40}$$

■

The survival function corresponding to the size-biased residual life distribution function $\bar{F}_{l_t}(x)$ is given by

$$\bar{F}_{l_t}(x) = \bar{F}_l(x+t)/\bar{F}_l(x), \tag{41}$$

where $\bar{F}_l(x) = \bar{F}(x)\{x + \delta_F(x)\}/\mu_F$.

THEOREM 4 *Let $\bar{F}_{W_t}(x)$ be an IHR distribution function with increasing weight function. Then*

$$\int_0^\infty |\bar{F}_{W_t}(x) - \{1 + x/(\mu + t)\}e^{-x/\mu}|dx \le 2\mu|1 + \mu/(\mu + t) - \mu_2/2\mu^2|. \tag{42}$$

Proof Let $A = \{x|\bar{F}_{W_t} \le \{1 + x/(\mu + t)\}e^{-x/\mu}\}$. Then for fixed $t > 0$ and $x \ge 0$, we have for $W(x)$ increasing in x,

$$\begin{aligned}
&\int_0^\infty |\bar{F}_{W_t}(x) - \{1 + x/(\mu + t)\}e^{-x/\mu}|dx \\
&\le 2\int_A (\{1 + x/(\mu + t)\}e^{-x/\mu} - \bar{F}_{W_t}(x))dx \\
&\le 2\int_0^\infty (\{1 + x/(\mu + t)\}e^{-x/\mu} - \bar{F}_{W_t}(x))dx \\
&\le 2\left\{\int_0^\infty (\{1 + x/(\mu + t)\}e^{-x/\mu} - \bar{F}_t(x))\right\}dx \\
&\le 2\left\{\int_0^\infty (\{1 + x/(\mu + t)\}e^{-x/\mu} - \bar{F}(x + t))\right\}dx \\
&\le 2\int_0^\infty (\{1 + x/(\mu + t)\}e^{-x/\mu} - S_1(x + t)/\mu)dx \\
&= 2\mu(1 + \mu/(\mu + t) - \mu_2/2\mu^2).
\end{aligned} \tag{43}$$

The first two inequalities are straightforward, the third inequality follows from the fact that $W(x)$ is increasing, so that $\bar{F}_{W_t}$ and $\bar{F}_t$ are stochastically ordered. The fourth and fifth inequalities follow from Lemma 1.

THEOREM 5 *If $\bar{F}_{W_t}(x)$ is an DHR distribution function in $\mathcal{F}$, then*

$$\int_0^\infty |\bar{F}_{W_t}(x) - \{1 + x/(\mu + t)\}e^{-x/\mu}|dx \ge 2e^{-\varepsilon/\mu}\, max\{0, |\eta + \phi|\}, \tag{44}$$

where $\eta = \mu(e^{-t/\mu} - (\varepsilon/(\mu + t)) - 1)$ *and* $\phi = -(\mu + t)^{-1}$ *provided* $W(x)$ *is an increasing weight function.*

Proof Let $\bar{F}_{W_t}$ be a DHR survival function, then for fixed $t > 0$, there exist $\varepsilon \geq \mu$ such that $\bar{F}_{W_t} \leq \{1 + x/(\mu + t)\}e^{-x/\mu}$ or $\bar{F}_{W_t} \geq \{1 + x/(\mu + t)\}e^{-x/\mu}$ as $x \leq \varepsilon$ or $x \geq \varepsilon$.

Now,

$$\begin{aligned}
&\int_0^\infty |\bar{F}_{W_t}(x) - \{1 + x/(\mu + t)\}e^{-x/\mu}|dx \\
&= 2\int_\varepsilon^\infty (\bar{F}_{W_t}(x) - \{1 + x/(\mu + t)\}e^{-x/\mu})dx \\
&\geq 2\int_\varepsilon^\infty (\bar{F}_t(x) - \{1 + x/(\mu + t)\}e^{-x/\mu})dx \\
&\geq 2\int_\varepsilon^\infty (\bar{F}(x + t) - \{1 + x/(\mu + t)\}e^{-x/\mu})dx \\
&= 2S_1(\varepsilon + t) - 2\mu e^{-\varepsilon/\mu} - \{2/(\mu + t)\}\{\mu\varepsilon e^{-\varepsilon/\mu} + e^{-\varepsilon/\mu}\} \\
&\geq 2\mu e^{-\varepsilon/\mu}\{e^{-t/\mu} - \varepsilon/(\mu + t) - 1\} - 2e^{-\varepsilon/\mu}/(\mu + t) \\
&= 2e^{-\varepsilon/\mu}\{\mu\{e^{-t/\mu} - (\varepsilon/(\mu + t)) - 1\} - 1/(\mu + t)\} \\
&= 2e^{-\varepsilon/\mu}(\eta + \phi). \qquad (45)
\end{aligned}$$

The first inequality follows from the fact that $W(x)$ is increasing, so that $\bar{F}_{W_t}(y) \geq \bar{F}_t(y)$ for all $y \geq 0$. The second inequality is due to the fact that $\bar{F}_t(y) \geq \bar{F}(y + t)$ for all $y \geq 0$, and for $t > 0$. The last inequality follow from Lemma 1.

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