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EQUIVARIANT FORMALITY OF TRANSVERSELY SYMPLECTIC FOLIATIONS AND FROBENIUS MANIFOLDS

YI LIN AND XIANGDONG YANG

ABSTRACT. Consider the Hamiltonian action of a compact connected Lie group on a transversely symplectic foliation whose basic cohomology satisfies the Hard Lefschetz property. We establish an equivariant formality theorem and an equivariant symplectic $d\delta$-lemma in this setting. As an application, we show that there exists a natural Frobenius manifold structure on the equivariant basic cohomology of the given foliation. In particular, this result provides a class of new examples of $dGBV$-algebras whose cohomology carries a Frobenius manifold structure.

CONTENTS

1. Introduction

Reinhart [R59] introduced the basic cohomology of foliations in late 1950's as a cohomology theory for the space of leaves. It has become one of fundamental topological invariants for foliations, especially for Riemannian foliations. An important sub-class of Riemannian foliations are Killing foliations, as any Riemanian foliation on a simply-connected manifold is Killing. According to Molino's structure theory [Mo88], for Killing foliations, the leaf closures are the orbits of leaves under the action of an abelian Lie algebra of transverse Killing fields, called the structure Killing algebra. Goertsches and Töben [GT10] introduced the notion of equivariant basic cohomology, and used it to study the transverse actions of structure Killing algebras on Killing foliations. Among other things, they proved a Borel type localization theorem, and established the equivariant formality in the presence of a basic Morse-Bott function whose critical set is the union of closed leaves. As a result, they are able to compute the basic Betti number in many concrete examples, and relate the basic cohomology to the dynamical aspects of a foliation.

Let (M, η, g) be a compact K-contact manifold with a Reeb vector field ξ, and let T be the closure of ξ in $\mathrm{Isom}(M, g)$. Then T is a compact connected torus; moreover, the characteristic Reeb foliation is Killing, with a structure Killing algebra isomorphic to $\mathrm{Lie}(T)/\mathrm{span}\{\xi\}$. It is well known that in this situation a generic component of the contact moment map $\Phi : M \to \mathfrak{t}^*$ is a Morse-Bott function, whose critical set is the union of closed Reeb orbits. In particular, the results established in [GT10] apply to the transverse actions of structure Killing algebra on K-contact manifolds, c.f. [GNT12], and establish the equivariant formality in this case.

It is noteworthy that the characteristic Reeb foliation of a K-contact manifold (M, η, g) is transversely symplectic; moreover, according to the definition of Hamiltonian actions on transversely symplectic foliations recently introduced by Lin and Sjamaar [LS16], the transverse action of the structure Killing algebra is Hamiltonian with respect to the transverse symplectic form $d\eta$. In view of Goertsches and Töben's equivariant formality result on K-contact manifolds, one naturally wonder if the equivariant formality theorem would continue to hold for a more general class of Hamiltonian actions on transversely symplectic foliations.

On symplectic manifolds, there are two approaches to proving the Kirwan-Ginzburg equivariant formality theorem of the Hamiltonian action of a compact connected Lie

group. The first approach ([Kir84], [Gin87]) is Morse theoretic, which works for arbitrary compact Hamiltonian symplectic manifolds. The second approach [LS04] is symplectic Hodge theoretic, which needs to assume the symplectic manifold to have the Hard Lefschetz property. On the upside, it provides an improved version of the equivariant formality theorem, which asserts that any de Rham cohomology class has a canonical equivariant extension.

In an accompanying paper, the first author extended symplectic Hodge theory to any transversely symplectic foliation with the transverse s-Lefschetz property, and established the symplectic $d\delta$-lemma in this framework. In the present article, for Hamiltonian actions of compact Lie groups on transversely symplectic foliation with the transverse Hard Lefschetz property, we establish an equivariant version of the symplectic $d\delta$-Lemma, and extend the equivariant formality result to this general setup. As explained in [LS16], on transversely symplectic foliations, components of a moment map are in general not Morse-Bott functions, unless the action is clean. A striking feature of our Hodge theoretic approach is that it would continue to work, even when the Morse theoretic method fails, as long as the transverse Hard Lefschetz property is satisfied.

Dubrovin [Du96] introduced the notion of Frobenius manifolds in his study of Topological Field Theory. Barannikov and Kontsevich [BK98] showed that the formal moduli space of solutions to the Maurer-Cartan equation of moduli gauge equivalence, related to a special class of differential Gerstenhaber- Batalin-Vilkoviski (dGBV) algebras, naturally admits a structure of a Frobenius manifold. In addition, they also constructed an important example of such a special dGBV algebra out of the Dolbeault complex of an arbitrary Calabi-Yau manifold.

In [Mer98], the $d\delta$-lemma was used by Merkulov to produce a formal Frobenius manfold structure on the de Rham cohomology of any compact symplectic manifold with the Hard Lefschetz property. Independently, Cao and Zhou proved ([CZ99], [CZ00]) similar results on the de Rham cohomology and equivariant de Rham cohomology of Kähler manifolds. As an initial application of the equivariant $d\delta$-lemma established in the present paper, we prove the following result, which simultaneously generalizes the constructions of Merkulov and of Cao and Zhou.

Theorem 1.1. *Assume that $(M, \mathcal{F}, \omega)$ is a transversely symplectic manifold that satisfies the transverse Hard Lefschetz property, and that a compact connected Lie group G*

acts on M in a Hamiltonian fashion. If $\mathcal{F}$ is also a Riemannian foliation, then there is a canonical formal Frobenius manifold structure on the equivariant basic cohomology $\widetilde{H}^*_{G,B}(M)$ as defined in (4.12).

Transversely symplectic foliations are naturally related to different areas in differential geometry. Reeb characteristic foliations in both contact and co-symplectic geometries are clearly transversely symplectic. Moreover, leaf spaces of transversely symplectic foliations include symplectic orbifolds (in the sense of Satake [Sa57]) and symplectic quasi-folds [Pra01] as special examples. In many known examples, transversely symplectic foliations arise as Kähler foliations that are homologically orientable, which are known to have the transverse hard Lefschetz property (c.f. [Ka90]). The results proved in this paper apply to these situations, and yield a rich class of new examples of $dGBV$-algebra whose cohomology carries the structure of a formal Frobenius manifold.

This paper is organized as follows. In Section 2 we review symplectic Hodge theory on transversely symplectic manifolds. In Section 3, we establish an equivariant formality theorem for the Hamiltonian action of a compact connected Lie group on a transversely symplectic manifold with the transverse Hard Lefschetz property. We also obtain an equivariant version of symplectic $d\delta$-lemma on transversely symplectic manifolds. In Section 4, we show that there exists a formal Frobenius manifold structure on the equivariant basic cohomology of a Hamiltonian transversely symplectic manifold that satisfies the transverse Hard Lefschetz property. In Section 5, we present examples of transversely symplectic foliations, which are also Riemannian, and which satisfy the transverse Hard Lefschetz property.

2. Hodge theory on transversely symplectic foliations

In this section, we review the elements of Hodge theory on transversely symplectic foliations to set up the stage. We refer to [Br88] and [Y96] for general background on

symplectic Hodge theory, and to [L16] for a detailed exposition on symplectic Hodge theory on foliations.

Definition 2.1. ([H70]) Let $\mathcal{F}$ be a foliation on a smooth manifold M, and let P be the integrable subbundle of TM associated to $\mathcal{F}$. $\mathcal{F}$ is said to be a *transversely symplectic foliation*, if there exists a closed 2-form ω, called the *transversely symplectic form* with respect to $\mathcal{F}$, such that $\forall\, x \in M$, the kernel of ω_x coincides with P_x, the fiber of P at x. The triple $(M, \mathcal{F}, \omega)$ is called a *transversely symplectic manifold.*

Let $(M, \mathcal{F}, \omega)$ be a transversely symplectic foliation. The space of *basic forms* is defined as follows

$$\Omega_{\text{bas}}(M) = \{\alpha \in \Omega(M) \mid \iota_X \alpha = 0, \quad \mathcal{L}_X \alpha = 0, \forall \text{ vector field } X \text{ tangent to leaves}\}.$$

Since the exterior differential operator d preserves basic forms, we obtain a subcomplex of the de Rham complex $\{\Omega^*(M), d\}$, called the *basic de Rham complex*

$$\cdots \longrightarrow \Omega_{\text{bas}}^{k-1}(M) \xrightarrow{d} \Omega_{\text{bas}}^{k}(M) \xrightarrow{d} \Omega_{\text{bas}}^{k+1}(M) \xrightarrow{d} \cdots$$

The cohomology of the basic de Rham complex $\{\Omega_{\text{bas}}^*(M), d\}$, denoted by $H_B^*(M)$, is called the *basic cohomology* of M. If M is connected then $H_B^0(M) \cong \mathbb{R}^1$. In general, the group $H_B^k(M)$ may be infinite-dimensional for $k \geq 2$; however, if M is a closed oriented manifold and $\mathcal{F}$ is a Riemannian foliation, then the basic cohomology are finite-dimensional, moreover, $H_B^{2n}(M) = 0$ or $H_B^{2n}(M) = \mathbb{R}$ (cf. [T97, Corollary 7.57]). In particular, when $H_B^{2n}(M) = \mathbb{R}$, we say that $\mathcal{F}$ is *homologically orientable.*

The closed 2-form ω induces a non-degenerate bi-linear paring $B(\cdot, \cdot)$ on $\Omega_{\text{bas}}^p(M)$, which in turn gives rise to the *symplectic Hodge star operator* $\star$ on $\Omega_{\text{bas}}^p(M)$ as follows

$$\beta \wedge \star \alpha = B(\alpha, \beta)\frac{\omega^n}{n!},$$

for any $\alpha, \beta \in \Omega_{\text{bas}}^p(M)$. The bi-linear pairing $B(\cdot, \cdot)$ is symmetric when p is even, and skew-symmetric when p is odd. As an easy consequence, we have that

$$\beta \wedge \star \alpha = \star \beta \wedge \alpha, \quad \star^2 = \text{id}. \tag{2.1}$$

The transpose operator δ of d is defined by

$$\delta : \Omega_{\text{bas}}^p(M) \longrightarrow \Omega_{\text{bas}}^{p-1}(M), \quad \alpha \longmapsto (-1)^p \star d \star \alpha.$$

By definition, it is easy to see that $\delta^2 = 0$ and $d\delta + \delta d = 0$. In this context, a basic form α is called (symplectic) *harmonic* if it satisfies $d\alpha = \delta\alpha = 0$. Set

$$\Omega_{\text{har}}(M) = \{\alpha \in \Omega_{\text{bas}}(M) \mid d\alpha = \delta\alpha = 0\}.$$

Since d anti-commutes with δ, $\{\Omega_{\text{har}}(M), d\}$ is a sub-complex of $\{\Omega_{\text{bas}}(M), d\}$. The cohomology of this sub-complex, denoted by $H^*_{\text{har}}(M)$, is called the *harmonic cohomology* of M.

There are three important operators acting on the space of basic forms:

(1) $L : \Omega^*_{\text{bas}}(M) \longrightarrow \Omega^{*+2}_{\text{bas}}(M), \quad \alpha \longmapsto \alpha \wedge \omega,$

(2) $\Lambda : \Omega^*_{\text{bas}}(M) \longrightarrow \Omega^{*-2}_{\text{bas}}(M), \quad \alpha \longmapsto \star L \star \alpha,$

(3) $H : \Omega^k_{\text{bas}}(M) \longrightarrow \Omega^k_{\text{bas}}(M), \quad \alpha \longmapsto (n - k)\alpha.$

Proposition 2.2. *([L16]) The operators d, δ, L, Λ, and H satisfy the following commutator relations*

$$[L, d] = 0, \quad [d, \Lambda] = \delta, \quad [\Lambda, \delta] = 0, \quad [L, \delta] = -d;$$
$$[L, \Lambda] = H, \quad [H, L] = 2L, \quad [H, \Lambda] = -2\Lambda, \qquad .$$

Lemma 2.3. *Let f be a basic function, and X a foliate vector field [1] such that*

$$\iota_X \omega = df.$$

Then for any basic form α we have that

a) $[\Lambda, \iota_X] = 0.$

b) $\delta(f\alpha) = f\delta\alpha - \iota_X\alpha.$

c) $\delta(df \wedge \alpha) = -df \wedge \delta\alpha + \mathcal{L}_X\alpha.$

Proof. a) is an easy consequence of [L16, Lemma 3.2]. b) can be proved by the same argument as the one used in [LS04, Prop. 2.3]

---

[1] The precise definition of foliate vector fields can be found in the second paragraph of Section 3

Using b) and the identity $d\delta + \delta d = 0$, we have that

$$\delta(df \wedge \alpha) = \delta\left(d(f\alpha) - f d\alpha\right) = -d\delta(f\alpha) - \delta(f d\alpha)$$

$$= -d\left(f\delta\alpha - \iota_X \alpha\right) - f\delta d\alpha + \iota_X d\alpha$$

$$= -d(f\delta\alpha) - f\delta d\alpha + (d\iota_X + \iota_X d)\alpha$$

$$= -df \wedge \delta\alpha - f\left(d\delta + \delta d\right)\alpha + \mathcal{L}_X \alpha$$

$$= -df \wedge \delta\alpha + \mathcal{L}_X \alpha.$$

This proves c).

□

Definition 2.4. Let $(M, \mathcal{F}, \omega)$ be a transversely symplectic foliation of co-dimension $2n$. It is said to satisfy the transverse hard Lefschetz property, if and only if for any $0 \leq k \leq n$, the map

$$L^k : H_B^{n-k}(M) \longrightarrow H_B^{n+k}(M)$$

is an isomorphism.

Brylirski [Br88] conjectured that every de Rham cohomology class of a compact symplectic manifold has a symplectic harmonic representative. However, Mathieu [Ma95] proved that this conjecture is true if and only if the manifold satisfies the hard Lefschetz property. Mathieu's theorem was sharpened by Merkulov [Mer98] and Guillemin [Gui01], who independently established the symplectic $d\delta$-lemma. The symplectic $d\delta$-lemma was first extended to transverse symplectic flows by Zhenqi He [H10], and more recently, by the first author [L16] to arbitrary transversely symplectic foliations. The following results are reformulations of [L16, Theorem 4.1, 4.8].

Theorem 2.5. *([L16, Theorem 4.1]) Let $(\mathcal{F}, \omega)$ be a transversely symplectic foliation on a closed oriented smooth manifold M. Then M satisfies the transverse hard Lefschetz property if and only if every basic cohomology class has a symplectic harmonic representative.*

Theorem 2.6. *([L16, Theorem 4.8]) Assume that $(M, \mathcal{F}, \omega)$ is a transversely symplectic manifold that satisfies the transverse hard Lefschetz property. Then on the space of basic forms we have*

$$im\, d \cap \ker\, \delta = \ker\, d \cap im\, d = im\, d\delta.$$

Let $\Omega_\delta(M) = \ker \delta \cap \Omega_{\mathrm{bas}}(M)$. Since d anti-commutes with δ, $\Omega_\delta(M)$ forms a sub-complex of the basic de Rham complex $\Omega_{\mathrm{bas}}(M)$, the cohomology of which we denote by $H_\delta(M)$. The following result is an easy consequence of Theorem 2.6. Here $H(\Omega_{\mathrm{bas}}(M), \delta)$ denotes the homology of $\Omega_{\mathrm{bas}}(M)$ with respect to δ.

Theorem 2.7. *Assume that $(M, \mathcal{F}, \omega)$ is a transversely symplectic manifold that satisfies the transverse hard Lefschetz property. Then the d-chain maps in the diagram*

$$\Omega_{bas}(M) \longleftarrow \Omega_\delta(M) \longrightarrow H(\Omega_{bas}(M), \delta)$$
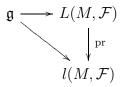
are quasi-isomorphisms that induce isomorphisms in cohomology.

3. EQUIVARIANT FORMALITY AND BASIC $d_G\delta$-LEMMA

In this section we study the equivariant basic cohomology of Hamiltonian actions on transversely symplectic manifolds using the Hodge theoretic approach. We begin by recalling the general notion of transverse actions on a foliated manifold.

Let $\mathcal{F}$ be a foliation on a smooth manifold M, let $\Xi(M)$ be the Lie algebra of smooth vector fields on M, and let $\Xi(\mathcal{F}) \subset \Xi(M)$ be the Lie sub-algebra of vector fields which are tangent to the leaves of $\mathcal{F}$. We say that a field $X \in \Xi(M)$ is *foliate*, if $[X, Y] \in \Xi(\mathcal{F})$ for any $Y \in \Xi(\mathcal{F})$. In particular, the set of foliate fields $L(M, \mathcal{F})$ is a Lie sub-algebra of $\Xi(M)$, since it is the normalizer of $\Xi(\mathcal{F})$ in $\Xi(M)$. A *transverse vector field* is a smooth section of $TM/T\mathcal{F}$ that is induced by a foliate vector field. It is easy to see that the set of transverse fields $l(M, \mathcal{F}) = L(M, \mathcal{F})/\Xi(\mathcal{F})$ also forms a Lie algebra with the induced Lie bracket from $L(M, \mathcal{F})$.

Definition 3.1. (c.f. [GT10]) The infinitesimal action of a finite-dimensional Lie algebra $\mathfrak{g}$ on a foliated manifold $(M, \mathcal{F})$ is said to be transverse, if there is a commutative diagram of Lie algebra homomorphisms

$$
\begin{array}{ccc}
\mathfrak{g} & \longrightarrow & L(M, \mathcal{F}) \\
 & \searrow & \downarrow {\scriptstyle \mathrm{pr}} \\
 & & l(M, \mathcal{F})
\end{array}
$$

Here the vertical map is the natural projection. The action of a compact connected Lie G on a foliated manifold $(M, \mathcal{F})$ is said to be transverse, if the infinitesimal action of its Lie algebra $\mathfrak{g}$ is transverse.

Lemma 3.2. *Consider the transverse action of a compact connected Le group G on a foliated manifold $(M, \mathcal{F})$. If α is a basic form, and if X_M is a fundamental vector field induced by an element $X \in \mathfrak{g}$, then $\iota_{X_M}\alpha$ and $\mathcal{L}_{X_M}\alpha$ are also basic forms.*

Proof. Let $Y \in \Xi(\mathcal{F})$. Since the action of G is transverse, $[Y, X_M] \in \Xi(\mathcal{F})$. It follows that $\iota_Y(\iota_{X_M}\alpha) = -\iota_{X_M}(\iota_Y\alpha) = 0$, and that $\mathcal{L}_Y(\iota_{X_M}\alpha) = \iota_{[Y,X_M]}\alpha + \iota_{X_M}(\mathcal{L}_Y\alpha) = 0$. This proves that $\iota_{X_M}\alpha$ is a basic form. A similar calculation shows that $\mathcal{L}_{X_M}\alpha$ is also basic. $\square$

Suppose that there is a transverse action of a compact connected Lie group G on a foliated manifold $(M, \mathcal{F})$. As an immediate consequence of Lemma 3.2, we see that $\Omega_{\text{bas}}(M)$ is a $G^\star$-module in the sense of [GS99, Definition 2.3.1]. Therefore, it has a well-defined equivariant Cartan model

$$\Omega_{G,\text{bas}}(M) := [S(\mathfrak{g}^*) \otimes \Omega_{\text{bas}}(M)]^G,$$

which we call the equivariant *basic Cartan complex* of the transverse G-manifold M.

To lighten up notations, let us write $\Omega_{\text{bas}} = \Omega_{\text{bas}}(M)$, and $\Omega_{G,\text{bas}} = \Omega_{G,\text{bas}}(M)$. Elements of $\Omega_{G,\text{bas}}$ can be regarded as equivariant polynomial maps from $\mathfrak{g}$ to Ω_{bas}, and are called equivairant basic differential forms on M. The bi-grading on $\Omega_{G,\text{bas}}$ is defined by

$$\Omega_{G,\text{bas}}^{i,j} = [S^i(\mathfrak{g}^*) \otimes \Omega_{\text{bas}}^{j-i}]^G;$$

moreover, it is quipped with the vertical differential $1 \otimes d$, which we abbreviate to d, and the horizontal differential ∂, which is defined by

$$\partial(\alpha(\xi)) = -\iota(\xi)\alpha(\xi), \quad \forall \xi \in \mathfrak{g}.$$

Here $\iota(\xi)$ denotes inner product with the fundamental vector field on M induced by $\xi \in \mathfrak{g}$. As a single complex, $\Omega_{G,\text{bas}}$ has a grading given by $\Omega_{G,\text{bas}}^k = \bigoplus_{i+j=k} \Omega_{G,\text{bas}}^{i,j}$, and a total differential $d_G = d + \partial$, which is called the equivariant exterior differential. We say an equivariant differential form α is equivariantly closed, resp., equivariantly exact, if $d_G\alpha = 0$, resp. $\alpha = d_G\beta$.

Definition 3.3. The total cohomology $\ker d_G/\text{im}\, d_G$ of the equivariant basic Cartan model is defined to be the equivariant basic cohomology $H_{G,B}^\bullet(M)$ of a transverse G-action on $(M, \mathcal{F})$.

We would like to point out that the above definition of equivariant basic cohomology was first introduced by Goertsches and Töben in [GT10] using the language of equivariant cohomology of $\mathfrak{g}^\star$-algebras. Following them, we give the following definition of equivariant formality for transverse G-actions.

Definition 3.4. ([GT10]) A transverse G-action on $(M, \mathcal{F})$ is equivariantly formal if

$$H^\bullet_{G,B}(M) \cong (S\mathfrak{g}^*)^G \otimes H^\bullet_B(M)$$

as a $(S\mathfrak{g}^*)^G$-module.

Lin and Sjamaar [LS16] introduced the following definition of Hamiltonian actions on transversely symplectic manifolds.

Definition 3.5. ([LS16]) Consider the action of a compact connected Lie group G with a Lie algebra $\mathfrak{g}$ on a transversely symplectic manifold $(M, \mathcal{F}, \omega)$. We say that the G-action on M is Hamiltonian, if the G-action preserves the transversely symplectic form ω, and if there exists an equivariant map,

$$\Psi : M \longrightarrow \mathfrak{g}^*,$$

called a moment map, such that $d\langle \Psi, \xi \rangle = \iota(\xi)\omega, \, \forall \xi \in \mathfrak{g}$.

From now on, we assume that $(M, \mathcal{F}, \omega)$ is a transversely symplectic manifold that satisfies the transverse hard Lefschetz property, and that there is a compact connected Lie group G acting on M in a Hamiltonian fashion with a moment map $\Phi : M \to \mathfrak{g}^*$, where $\mathfrak{g} = \mathrm{Lie}(G)$. The symplectic Hodge theory gives rise to a fourth differential $1 \otimes \delta$ on $\Omega_{G,\mathrm{bas}}$, which we will abbreviate to δ.

Lemma 3.6. *On the space of equivariant basic differential forms $\Omega_{G,bas}$, the following identities hold.*

$$\partial\delta = -\delta\partial, \; d_G\delta = -\delta d_G.$$

Proof. It was shown in [LS04, Lemma 3.1] that $\partial\delta = -\delta\partial$ and $d_G\delta = -\delta d_G$ hold on the space of equivariant differential forms. Since d_G, δ and ∂ map basic forms to basic forms, these two identities also hold on the space of equivariant basic differential forms. $\square$

This implies that $\Omega_{G,\delta} = \ker \cap \Omega_{G,\mathrm{bas}}$ is a double subcomplex of $\Omega_{G,\mathrm{bas}}$, and that the homology $H(\Omega_{G,\mathrm{bas}}, \delta)$ with respect to δ is a double complex with differentials induced

by d and δ. Thus we have a diagram of morphisms of double complexes.

$$\Omega_{G,\mathrm{bas}} \longleftarrow \Omega_{G,\delta} \longrightarrow H(\Omega_{G,\mathrm{bas}}, \delta). \tag{3.1}$$

Since δ acts trivially on the polynomial part, these morphisms are actually morphisms of $(S\mathfrak{g}^*)^G$-modules.

We first establish a preliminary result about the action of $\iota(\xi)$ on invariant basic forms. Let Ω^G_{bas} be the space of G-invariant basic forms on M. The identity $\mathcal{L}_\xi = \iota(\xi)d + d\iota(\xi)$ implies that $\iota(\xi) : \Omega^G_{\mathrm{bas}} \to \Omega^G_{\mathrm{bas}}$ is a chain map with respect to d. Here $\mathcal{L}_\xi$ denotes the Lie derivative of the vector field on M induced by $\xi \in \mathfrak{g}$. Similarly, an application of the identity $\delta\partial + \partial\delta = 0$ to the zeroth column of $\Omega_{G,\mathrm{bas}}$ implies that $\iota(\xi)$ is a chain map with respect to δ.

Lemma 3.7. *Let $\xi \in \mathfrak{g}$ and $\alpha \in \Omega^G_{bas}$. If α is closed, then $\iota(\xi)\alpha$ is d-exact. If α is δ-closed, then $\iota(\xi)\alpha$ is δ-exact.*

Proof. Since the action of G is Hamiltonian, it follows from [LS04, Prop. 2.5] that

$$\iota(\xi)\alpha = \Phi^\xi(\delta\alpha) - \delta(\Phi^\xi\alpha). \tag{3.2}$$

where Φ^ξ is the ξ-component of the moment map $\Phi : M \to \mathfrak{g}^*$. If α is δ-closed, then we have that $\iota(\xi)\alpha = -\delta(\Phi^\xi\alpha)$. Since Φ^ξ is a basic function, $\iota(\xi)\alpha$ is δ-exact in Ω^G_{bas}.

It remains to show that if $\alpha \in \Omega^G_{\mathrm{bas}}$ is a closed basic k-form, then $\iota(\xi)\alpha$ is d-exact. Since M satisfies the transverse Hard Lefschetz property, by [L16, Theorem 4.3], $[\alpha]_B \in H^k_B(M)$ has a unique primitive decomposition

$$[\alpha]_B = \sum_r L^r[\alpha_r]_B.$$

Here $[\alpha_r]_B \in H^{k-2r}_B(M)$ is a primitive basic cohomomology class, i.e., $L^{n-k+2r+1}[\alpha]_B = 0$. However, since the action is Hamiltonian, we have that $\iota(\xi)(\omega \wedge \alpha) = d\Phi^\xi \wedge \alpha + \omega \wedge \iota(\xi)\alpha$. Thus to finish the proof, it suffices to show that $\iota(\xi)\alpha$ is exact when $[\alpha]_B$ is a primitive basic chomology class. We note that he argument given in [LS04, Lemma 3.2] continues to hold in the present situation to show the exactness of $\iota(\xi)\alpha$. $\qquad\square$

Note that the symplectic $d\delta$-lemma, Theorem 2.6, holds for equivariant basic differential forms as well as for ordinary basic differential forms, and that the inclusion $\Omega^G_{bas} \hookrightarrow \Omega_{bas}$ is a deformation retraction for δ as well as for d. The same argument as given in [LS04, Lemma 3.3.] leads to the following result.

Lemma 3.8. *The double complex $H_\delta^\bullet(\Omega_{G,bas})$ is trivial. Moreover,*

$$H(\Omega_{G,bas}, \delta) \cong (S\mathfrak{g}^*)^G \otimes H_B(M). \tag{3.3}$$

Theorem 3.9. *Let $(M, \mathcal{F}, \omega)$ be a transversely symplectic manifold that satisfies the transverse hard Lefschetz property, and let a compact connected Lie group G act on M in a Hamiltonian fashion. Then the morphisms (3.1) induces isomorphisms of $(S\mathfrak{g}^*)^G$-modules*

$$H_{G,B}(M) \overset{\cong}{\longleftarrow} H_{G,\delta}(M) \overset{\cong}{\longrightarrow} H(\Omega_G, \delta).$$

Proof. We first note that since G is connected, the identity $\mathcal{L}_\xi = d\iota(\xi) + \iota(\xi)d$ together with the identity (3.2) implies that G acts trivially on both $H_B(M)$ and $H_\delta(M)$. Let E be the spectral sequence of $\Omega_{G,\text{bas}}$ relative to the filtration associated to the horizontal grading and E_δ that of $\Omega_{G,\delta}$. The first terms are

$$E_1 = \ker d/\operatorname{im} d = (S\mathfrak{g}^* \otimes H_B(M))^G = (S\mathfrak{g}^*)^G \otimes H_B(M)$$

$$(E_\delta)_1 = (\ker d \cap \ker \delta)/(\operatorname{im} d \cap \ker \delta) = (S\mathfrak{g}^* \otimes H_\delta(M))^G = (S\mathfrak{g}^*)^G \otimes H_B(M). \tag{3.4}$$

Here we used the observation we made in the paragraph right before Lemma 3.8, as well as the isomorphism $H_\delta(M) \cong H_B(M)$ of Theorem 2.7. By Lemma 3.8, $H(\Omega_{\text{bas}}, \delta)$ is a trivial double complex, its spectral sequence is therefore constant with trivial differentials at each stage. The two morphisms (3.1) induce morphisms of spectral sequences

$$E \longleftarrow E_\delta \longrightarrow H(\Omega_{bas}, \delta).$$

It follows from (3.3) and (3.4) that these morphisms induce isomorphisms at the first stage. Thus they must induce isomorphisms at every stage. In particular, these three spectral sequences converge to the same limit, and so the morphisms (3.1) induce isomorphisms on total cohomology. This completes the proof of Theorem 3.9. □

The exactly same argument as used in [LS04, Theorem 3.9] gives us the following equivariant version of the symplectic $d\delta$-lemma on transversely symplectic manifolds.

Theorem 3.10. *Let $\alpha \in \Omega_{G,bas}$ be an equivariant basic form satisfying $d_G\alpha = 0$ and $\delta\alpha = 0$. If α is either d_G-exact or δ-exact, then there exists $\beta \in \Omega_{G,bas}$ such that $\alpha = d_G\delta\beta$.*

We now discuss the implications of Theorem 3.9. Observe that $\Omega^{0,k}_{G,\mathrm{bas}} = (\Omega^k_{\mathrm{bas}})^G$, the space of invariant basic k-forms on M. Thus the zeroth column of the equivairant basic Cartan model is the invariant basic de Rham complex Ω^G_{bas}, which is a deformation retract of the basic de Rham complex because G is connected. Therefore $H(\Omega^G_{\mathrm{bas}}) = H_B(M)$. The natural projection map $\overline{p} : \Omega_{G,\mathrm{bas}} \to \Omega^G_{\mathrm{bas}}$, defined by $\overline{p}(\alpha) = \alpha(0)$, is a chain map with respect to the equivariant exterior derivative d_G on $\Omega_{G,\mathrm{bas}}$ and the ordinary exterior derivative d on Ω_{bas}. It therefore induces a morphism of cohomology groups $p : H_{G,B}(M) \to H_B(M)$. Theorem 3.9 implies that the spectral sequence E degenerates at the first stage, and that the map p is surjective. In other words, every basic cohomology class can be extended to an equivariant basic cohomology class. However, Theorem 3.9 would also imply that there is a canonical choice of such an extension. Let

$$s : H_B(M) \to H_{G,B}(M) \tag{3.5}$$

be the composition of the map $H_B(M) \to (S\mathfrak{g}^*)^G \otimes H_B(M)$ which sends a cohomology class a to $1 \otimes a$, and the isomorphism $(S\mathfrak{g}^*)^G \otimes H_B(M) \to H_{G,B}(M)$ given by Theorem 3.9. The following result is an easy consequence of Theorem 2.7 and Theorem 3.9. We refer to [LS04, Corollary 3.5] for more details.

Corollary 3.11. *s is a section of p. Thus every basic cohomology class can be extended to a equivariant basic cohomology class in a canonical way.*

4. Formal Frobenius manifold structure on equivariant basic cohomology

Consider the Hamiltonian action of a compact connected Lie group on a transversely symplectic foliation. In this section, following the approach initiated by Barannikov and Kontsevich [BK98], we show that if the foliation satisfies the transverse Hard Lefschetz property, and if it is also a Riemannian foliation, then there exists a formal Frobenius manifold structure on its equivariant basic cohomology. We first give a quick review of differential *Gerstenhaber-Batalin-Vilkovisky (GBV) algebra.*

Suppose that $(\mathscr{A}, \wedge)$ is a supercommutative graded algebra with identity over a field k, and that there is a k-linear operator $\delta : \mathscr{A}^* \longrightarrow \mathscr{A}^{*-1}$. Define the bracket $[\bullet]$ by setting

$$[a \bullet b] = (-1)^{|a|}\left(\delta(a \wedge b) - (\delta a) \wedge b - (-1)^{|a|} a \wedge (\delta b) \right),$$

where a and b are homogeneous elements and $|a|$ is the degree of $a \in \mathscr{A}$. We say that $(\mathscr{A}, \wedge, \delta)$ forms a *Gerstenhaber-Batalin-Vilkovisky (GBV) algebra* with odd bracket $[\bullet]$, if it satisfies:

(1) δ is a differential, i.e. $\delta^2 = 0$;

(2) for any homogeneous elements a, b and c we have

$$[a \bullet (b \wedge c)] = [a \bullet b] \wedge c + (-1)^{(|a|+1)|b|} b \wedge [a \bullet c]. \tag{4.1}$$

Definition 4.1 (dGBV-algebra). A GBV-algebra $(\mathscr{A}, \wedge, \delta)$ is called a differentiable Gerstenhaber-Batalin-Vilkovisky (dGBV) algebra, if there exists a differential operator $d : \mathscr{A}^* \longrightarrow \mathscr{A}^{*+1}$ such that

(1) d is a derivation with respect to the product $\wedge$, i.e. $d(a \wedge b) = da \wedge b + (-1)^{|a|} a \wedge db$ for any homogeneous elements a and b;

(2) $d\delta + \delta d = 0$.

An *integral* on a dGBV algebra $\mathscr{A}$ is a k-linear functional

$$\int : \mathscr{A} \longrightarrow k \tag{4.2}$$

such that for all $a, b \in \mathscr{A}$, the following equations hold

$$\int (da) \wedge b = (-1)^{|a|+1} \int a \wedge db,$$

$$\int (\delta a) \wedge b = (-1)^{|a|+1} \int a \wedge \delta b.$$

Clearly, an integral $\int$ induces a bi-linear pairing on $H_d^*(\mathscr{A})$ as follows.

$$(\cdot, \cdot) : H_d^*(\mathscr{A}) \times H^*(\mathscr{A}) \to k, \quad ([a], [b]) = \int a \wedge b.$$

In particular, if the above bi-linear pairing is non-degenerate, then we say that the integral is *nice*.

The following theorem enables us to use a dGBV algebra as an input to produce a formal Frobenius manifold(cf. [BK98],[Man00]).

Theorem 4.2. *Let $(\mathscr{A}, \wedge, \delta, d, [\bullet])$ be a dGBV algebra satisfying the following conditions*

(1) *the dimension of $H_d^*(\mathscr{A})$ is finite;*

(2) *there exists a nice integral on $\mathscr{A}$;*

(3) *the inclusions $(\ker \delta, d) \hookrightarrow (\mathscr{A}, d)$ and $(\ker d, \delta) \hookrightarrow (\mathscr{A}, \delta)$ are quasi-isomorphisms.*

Then there is a canonical construction of a formal Frobenius manifold structure on
$H_d^*(\mathscr{A})$

As an initial step, we first prove that the equivariant basic Cartan complex of a transversely symplectic foliation carries the structure of a dGBV algebra.

Proposition 4.3. *Suppose that there is a transverse action of a compact connected Lie group G on a transversely symplectic manifold M. Let δ be the differential on equivariant basic differential forms as introduced in Section 3, and let $\wedge$ denote the wedge product. Then the quadruple $(\Omega_{G,bas}, \wedge, \delta, d_G)$ is a dGBV algebra.*

Proof. The only thing that requires a proof is that (4.1) holds on equivariant basic differential forms. To this end, it suffices to show that (4.1) holds for ordinary basic differential forms a, b, c on a foliated coordinate neighborhood. So without loss of generality, we may assume that $b = f_0 df_1 \wedge \cdots \wedge df_k$, and that $\forall 0 \leq i \leq k$, f_i is a basic functions such that $df_i = \iota_{X_i}\omega$ for some foliated vector field X_i. However, it is easy to see that if $b_1, \cdots, b_s$ are basic forms such that $\forall 1 \leq i \leq s$, (4.1) holds for $b = b_i$ and arbitrarily given basic forms a and c, then (4.1) holds for $b = b_1 \wedge \cdots \wedge b_s$ and arbitrarily given basic forms a and c. Therefore it is enough to show that (4.1) is true in the following two cases.

Case 1) $b = f$ is a basic function such that $df = \iota_X\omega$ for some foliate vector X. Applying ii) in Lemma 2.3, we have that

$$
\begin{aligned}
[a \bullet fc] &= (-1)^{|a|} \left(\delta(a \wedge fc) - \delta(a) \wedge fc - (-1)^{|a|} a \wedge \delta(fc) \right) \\
&= (-1)^{|a|} \left(f\delta(a \wedge c) - (\iota_X a) \wedge c - \delta(a) \wedge fc - (-1)^{|a|} a \wedge f\delta c \right) \\
&= f[a \bullet c] - (-1)^{|a|} \iota_X a \wedge c \\
&= f[a \bullet c] + (-1)^{|a|} (\delta(fa) - f\delta a) \wedge c \\
&= f[a \bullet c] + [a \bullet f] \wedge c
\end{aligned}
$$

Case 2) $b = df$ for a basic function f such that $df = \iota_X \omega$ for some foliate vector X.

Applying iii) in Lemma 2.3, we have that

$$
\begin{aligned}
[a \bullet (df \wedge c)] &= (-1)^{|a|} \left(\delta(a \wedge df \wedge c) - \delta a \wedge df \wedge c - (-1)^{|a|} a \wedge \delta(df \wedge c) \right) \\
&= \mathcal{L}_X(a \wedge c) - df \wedge \delta(a \wedge c) - (-1)^{|a|} \delta a \wedge df \wedge c + a \wedge df \wedge \delta c - a \wedge \mathcal{L}_X c \\
&= \mathcal{L}_X a \wedge c - df \wedge \delta(a \wedge c) + df \wedge \delta a \wedge c + a \wedge df \wedge \delta c \\
&= \mathcal{L}_X a \wedge c - df \wedge \left(\delta(a \wedge c) - \delta a \wedge c - (-1)^{|a|} a \wedge \delta c \right) \\
&= \mathcal{L}_X a \wedge c + (-1)^{|a|+1} df \wedge [a \bullet c].
\end{aligned}
$$
(4.3)

On the other hand, applying iii) in Lemma 2.3 again, we have that

$$
\begin{aligned}
[a \bullet df] &= (-1)^{|a|} \left(\delta(a \wedge df) - \delta a \wedge df - (-1)^{|a|} a \wedge \delta df \right) \\
&= \delta(df \wedge a) - (-1)^{|a|} \delta a \wedge df + a \wedge d\delta f \\
&= -df \wedge \delta a + \mathcal{L}_X a + df \wedge \delta a \\
&= \mathcal{L}_X a
\end{aligned}
$$
(4.4)

It follows immediately from (4.3) and (4.4) that (4.1) holds in this case.

$\square$

Recall that a foliation $\mathcal{F}$ on a smooth manifold M is said to be *Riemannian*, if there exists a $(2,0)$ tensor g on M, called a *transverse Riemannian metric*, such that: 1) $g(X,X) \geq 0$, $\forall X \in C^\infty(TM)$; 2) $g(X,\xi) = 0$, $\forall X \in C^\infty(TM)$, $\forall \xi \in C^\infty(P)$; 3) $\mathcal{L}_\xi g = 0$, $\forall \xi \in C^\infty(P)$. From now on, we assume that $(\mathcal{F}, \omega)$ is a transversely symplectic foliation on a closed connected oriented Riemannian manifold M which satisfies the transverse Hard Lefschetz property, that $\mathcal{F}$ is also a Riemannian foliation, and that there is a compact connected Lie group acting on M in a Hamiltonian fashion.

Let P be the integrable subbundle of TM that is associated to the foliation $\mathcal{F}$. By our assumption, both the tangent bundle TM and the normal bundle $Q = TM/P$ are oriented. So P is also oriented with the induced orientation. Suppose that $\dim M = 2n + l$ and $\dim \mathcal{F} = l$. The transverse Hard Lefschetz yields that $H_B^{2n}(M) \cong H_B^0(M) \cong \mathbb{R}$. Thus the Riemannian foliation $\mathcal{F}$ is taut. As a result, there exists a G-invariant bundle-like metric g on M such that the mean curvature 1-form $\kappa = 0$ (cf. [PAW09, Theorem 1.4.6.]). Henceforth, we will assume that M is equipped with a G-invariant bundle-like metric g such that the mean curvature one form $\kappa = 0$.

Let $\chi_{\mathcal{F}} \in \Omega^l(M)$ be the characteristic form of $\mathcal{F}$ with respect to the metric g. Since g is G-invariant, $\chi_{\mathcal{F}}$ is also G-invariant, and therefore can be regarded as an equivariant basic differential form. Using the usual equivariant integral ([GS99]), we define a $S(\mathfrak{g}^*)^G$-linear operator as follows.

$$\int : \Omega_{G,\text{bas}} \longrightarrow S(\mathfrak{g}^*)^G, \quad \alpha \longmapsto \int_M \alpha \wedge \chi_{\mathcal{F}}. \tag{4.5}$$

Lemma 4.4. *For any $\alpha \in \Omega_{G,bas}^s$ and $\beta \in \Omega_{G,bas}^t$ we have*

a)

$$\int (d_G \alpha) \wedge \beta = (-1)^{s+1} \int \alpha \wedge d_G \beta, \tag{4.6}$$

b)

$$\int (\delta \alpha) \wedge \beta = (-1)^{s+1} \int \alpha \wedge \delta \beta. \tag{4.7}$$

Proof. a) We first prove a preliminary result that for any two ordinary basic differential forms $\alpha \in \Omega_{\text{bas}}^s(M)$ and $\beta \in \Omega_{\text{bas}}^t(M)$,

$$\int_M (d\alpha) \wedge \beta \wedge \chi_{\mathcal{F}} = (-1)^{s+1} \int_M \alpha \wedge d\beta \wedge \chi_{\mathcal{F}}, \tag{4.8}$$

By Leibniz rule,

$$d(\alpha \wedge \beta \wedge \chi_{\mathcal{F}}) = d\alpha \wedge \beta \wedge \chi_{\mathcal{F}} + (-1)^s \alpha \wedge (d\beta) \wedge \chi_{\mathcal{F}} + (-1)^{s+t} \alpha \wedge \beta \wedge d\chi_{\mathcal{F}}.$$

Since

$$\int_M d(\alpha \wedge \beta \wedge \chi_{\mathcal{F}}) = 0, \tag{4.9}$$

to prove (4.8) it suffices to show that

$$\int_M \alpha \wedge \beta \wedge d\chi_{\mathcal{F}} = 0.$$

First we observe that since $\chi_{\mathcal{F}}$ is of degree l, we may assume that $s+t = 2n-1$, for otherwise (4.9) holds for degree reasons. Next recall that by our choice of the bundle-like metric, the mean curvature form $\kappa = 0$. So applying [T97, Formula 4.26], we have that $d\chi_{\mathcal{F}} = \varphi^0$, where φ^0 is a $(l+1)$-form such that for any sections $X_1, \cdots, X_l$ of P, the 1-form $\iota_{X_1} \cdots \iota_{X_l} \varphi^0$ vanishes. Since α and β are basic, this implies that $\alpha \wedge \beta \wedge \varphi^0 = 0$, from which (4.8) follows immediately.

Since d does not act on the polynomial part of an equivariant basic form, (4.8) also holds for equivariant basic forms. On the other hand, $\forall \alpha \in \Omega_{G,\text{bas}}^s(M)$ and

$\beta \in \Omega_{G,\mathrm{bas}}^t(M)$, a simple degree counting shows that

$$\int_M \partial\alpha \wedge \beta \wedge d\chi_{\mathcal{F}} = \int_M \alpha \wedge \partial\beta \wedge d\chi_{\mathcal{F}} = 0. \tag{4.10}$$

(4.6) follows immediately from (4.8) and (4.10).

b) It suffices to show that for any ordinary basic forms $\alpha \in \Omega_{\mathrm{bas}}^s(M)$ and $\beta \in \Omega_{\mathrm{bas}}^t(M)$,

$$\int_M (\delta\alpha) \wedge \beta \wedge \chi_{\mathcal{F}} = (-1)^{s+1} \int \alpha \wedge (\delta\beta) \wedge \chi_{\mathcal{F}}.$$

Without loss of generality, we may assume that $s + t = 2n + 1$. Using (2.1) and (4.8), we have that

$$\begin{aligned}
\int_M (\delta\alpha) \wedge \beta \wedge \chi_{\mathcal{F}} &= (-1)^{s+1} \int_M (\star d \star \alpha) \wedge \beta \wedge \chi_{\mathcal{F}} \\
&= (-1)^{s+1} \int_M (d \star \alpha) \wedge \star\beta \wedge \chi_{\mathcal{F}} \\
&= -\int_M (\star\alpha) \wedge d \star \beta \wedge \chi_{\mathcal{F}} \\
&= (-1)^{s+1} \int_M \alpha \wedge \delta\beta \wedge \chi_{\mathcal{F}}.
\end{aligned} \tag{4.11}$$

$\square$

It is clear that $(S\mathfrak{g}^*)^G$ is an integral domain. Let $\mathbb{F} = \{\frac{f}{g} \mid f, g \in (S\mathfrak{g}^*)^G\}$ be the fractional field of $(S\mathfrak{g}^*)^G$. Define

$$\widetilde{\Omega}_{G,\mathrm{bas}} = \Omega_{G,\mathrm{bas}} \otimes_{(S\mathfrak{g}^*)^G} \mathbb{F}.$$

Extend $d_G, \wedge$ and δ to $\widetilde{\Omega}_{G,\mathrm{bas}}$, and define

$$\widetilde{H}_{G,B}^*(M) = H^*(\widetilde{\Omega}_{G,\mathrm{bas}}, d_G). \tag{4.12}$$

As an easy consequence of Theorem 3.9, we have

$$\widetilde{H}_{G,B}^*(M) = H_{G,B}^*(M) \otimes_{(S\mathfrak{g}^*)^G} \mathbb{F}.$$

Applying Proposition 4.3, we see that $(\widetilde{\Omega}_{G,\mathrm{bas}}, \delta, \wedge, d_G)$ is a $dGBV$-algebra. Moreover, the operator defined in (4.5) naturally extends to a $\mathbb{F}$-linear operator

$$\int : \widetilde{\Omega}_{G,\mathrm{bas}} \to \mathbb{F}. \tag{4.13}$$

It follows easily from Lemma 4.4 that (4.13) defines an integral on the dGBV algebra $(\widetilde{\Omega}_{G,bas}, \wedge, \delta, d_G)$. The following result on the Poincaré duality of a Riemannian foliation would enable us to show that this integral is nice .

Theorem 4.5. *([T97, Corollary 7.58]) Let $\mathcal{F}$ be a taut and transversally oriented Riemannian foliation on a closed oriented manifold M. The the pairing*

$$\alpha \otimes \beta \longmapsto \int_M \alpha \wedge \beta \wedge \chi_{\mathcal{F}}$$

induces a non-degenerate pairing

$$H_B^r(M) \times H_B^{q-r}(M) \longrightarrow \mathbb{R}$$

on finite-dimensional vector spaces, where $q = codim\,\mathcal{F}$.

Lemma 4.6. *The integral operator defined in (4.13) is nice, i.e., it induces a $\mathbb{F}$-bi-linear non-degenerate pairing*

$$\widetilde{H}_{G,B}^*(M) \times \widetilde{H}_{G,B}^*(M) \to \mathbb{F}$$

Proof. Let $[\alpha]$ be an arbitrary class in $H_{G,B}^*(M)$ such that

$$\int_M \alpha \wedge \beta \wedge \chi_{\mathcal{F}} = 0, \quad \forall\, [\beta] \in H_{G,B}^*(M).$$

To prove Lemma 4.6, it suffices to show $[\alpha]$ has to vanish.

Let $\{f_1, \cdots, f_k, \cdots\}$ be a basis of the real vector space $(S\mathfrak{g}^*)^G$. Then by Theorem 3.9, there exist finitely many unique non-zero elements $[\gamma_i]$'s in $H_B^*(M)$ such that

$$[\alpha] = \sum_i f_i \otimes s([\gamma_i]).$$

Here $s : H_B(M) \to H_{G,B}(M)$ is the canonical section introduced in (3.5). Let k_i be the degree of the basic form γ_i. After a reshuffling of the index, we may assume that $k_1 \geq k_2 \geq \cdots$. Then for any $[\zeta] \in H_B^{2n-k_1}(M)$,

$$\sum_i f_i \otimes \left(\int_M s([\gamma_i]) \wedge s([\zeta]) \wedge \chi_{\mathcal{F}} \right) = 0.$$

This implies in particular that $\int_M \gamma_1 \wedge \zeta \wedge \chi_{\mathcal{F}} = 0$. Since $[\zeta] \in H_B^{2n-k_1}(M)$ is arbitrarily chosen, by Theorem 4.5 we have that $[\gamma_1] = 0$. Thus $s([\gamma_1]) = 0$. Repeating this argument, we see that $[\gamma_i] = 0$ for all i. Therefore $[\alpha]$ must be zero as well.

$\square$

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1. It remains to show that the following maps induced by the inclusions

$$\rho : H(\ker \delta, d_G) \longrightarrow H(\Omega_{G,\text{bas}}, d_G) \tag{4.14}$$

$$\mu : H(\ker d_G, \delta) \longrightarrow H(\Omega_{G,\text{bas}}, \delta) \qquad (4.15)$$

are isomorphisms. The fact that the map (4.14) is an isomorphism is a direct consequence of Theorem 3.9. Let $\alpha \in \ker d_G$ be a δ-closed form which represents a class $[\alpha]$ in $H(\ker d_G, \delta)$. Suppose that $[\alpha]$ is trivial in $H(\Omega_{G,\text{bas}}, \delta)$, then there exists a $\beta \in \Omega_{G,\text{bas}}$ such that $\alpha = \delta\beta$. By Theorem 3.10, we have that $\alpha = d_G\delta\gamma$ for some $\gamma \in \Omega_{G,\text{bas}}$. This implies that α represents a trivial class in $H(\ker d_G, \delta)$, and that the map (4.15) is injective.

To see that (4.15) is surjective, suppose that $\alpha \in \Omega_{G,\text{bas}}$ such that $\delta\alpha = 0$, i.e., $[\alpha]$ is a class in $H(\Omega_{G,\text{bas}}, \delta)$. Let $\gamma = d_G\alpha$. Then γ is both d_G-exact and δ-closed. By Theorem 3.10, there exists a $\beta \in \Omega_{G,\text{bas}}$ such that $\gamma = d_G\delta\beta$. Set $\tilde{\alpha} = \alpha - \delta\beta$. Then we have that $\tilde{\alpha} \in \ker d_G$, and that $[\tilde{\alpha}] = [\alpha]$ in $H(\Omega_{G,\text{bas}}, \delta)$. This proves that (4.15) is surjective. By Theorem 4.2, there is a formal Frobenius manifold structure on $H^*_{G,B}(M)$ over $S(\mathfrak{g}^*)^G$. $\qquad\qquad\square$

When G is a trivial group consisting of one single element, Theorem 1.1 has the following important consequence.

Corollary 4.7. *Assume that $(M, \mathcal{F}, \omega)$ is a transversely symplectic manifold that satisfies the transverse Hard Lefschetz property. If $\mathcal{F}$ is also a Riemannian foliation, then there is a canonical formal Frobenius manifold structure on the basic cohomology $H^*_B(M)$.*

Remark 4.8. When the foliation $\mathcal{F}$ is zero dimensional, we recover from Corollary 4.7 Merkulov's construction [Mer98] of a Frobenius manifold structure on the de Rham cohomology of a symplectic manifold with the Hard Lefschetz property. When the foliation $\mathcal{F}$ is zero dimensional, and when M is a closed Kähler manifold, we recover from Theorem 1.1 the construction by Cao and Zhou [CZ99], which produces a Frobenius manifold structure on the equivariant cohomology of a Hamiltonian action of a compact connected Lie group on a Kähler manifold. Moreover, we are able to remove the assumption in [CZ99] that the action is holomorphic.

5. EXAMPLES OF FROBENIUS MANIFOLD STRUCTURES FROM TRANSVERSELY SYMPLECTIC FOLIATIONS

In this section we present examples of transversely symplectic foliations which are Riemannian, and which satisfy the transverse Hard Lefschetz property. Theorem 1.1

and Corollary 4.7 apply to these cases, and provide us new examples of $dGBV$-algebras in various geometries, whose cohomology admits a formal Frobenius manifold structure. We begin this section by making the following simple observation, which will be used to give examples of transversely symplectic foliations that are Riemannian.

Lemma 5.1. *Consider the locally free action of a compact connected Lie group G on a manifold M. Let $\mathfrak{g} = Lie(G)$, let $\mathfrak{h}$ be a sub-Lie-algebra of $\mathfrak{g}$, and let $\mathcal{F}$ be the foliation generated by the infinitesimal action of $\mathfrak{h}$ on M. Then $\mathcal{F}$ must be a Riemannian foliation.*

Proof. Let P be the integrable sub-bundle of TM associated to the foliation $\mathcal{F}$, and let Q be the orthogonal complement of P in TM with respect to a given G-invariant metric on M. Then it is easy to see that both P and Q are G-bundles. By averaging over G, we may assume that there exists a G-invariant Riemannian metric g_Q over the vector bundle Q. Now define a $(2,0)$ tensor on M as follows.

$$g(X + Y, X' + Y') = g_Q(X, X'), \forall\, X, X' \in C^\infty(Q), \, \forall\, Y, Y' \in C^\infty(P).$$

Then g is a G-invariant $(2,0)$-tensor over M. By definition, it is straightforward to check that g is a transverse Riemannian metric. $\square$

Now we discuss examples of transversely symplectic foliations to which Theorem 1.1 and Corollary 4.7 apply.

Example 5.2 (Co-oriented contact manifolds). Let M be a $2n+1$ dimensional co-oriented contact manifold with a contact one form η and a Reeb vector ξ. Then the Reeb characteristic foliation $\mathcal{F}_\xi$ induced by ξ is transversely symplectic, with $d\eta$ being the transversely symplectic form. If there exists a contact metric g such that ξ is a Killing vector field, then (M, η, g) is called a K-contact manifold. It is well known that the Reeb characteristic foliation of a K-contact manifold (M, η, g) is Riemannian. By Corollary 4.7, when M satisfies the transverse Hard Lefschetz property, its basic cohomology will carry the structure of a formal Frobenius manifold. In particular, this is the case when (M, η, g) is a Sasakian manifold (c.f. [BG08]). However, it is also noteworthy that it was recently discovered in [CNMY15] that there exist examples of compact K-contact manifolds which do not admit any Sasakian structures, and which satisfy the Hard Lefschetz property as introduced in [CNY13]. By [L13, Theorem 4.4], these non-Sasakian K-contact manifolds also satisfy the transverse Hard Lefschetz property.

Example 5.3 (Hamiltonian actions on contact manifolds). Let M be a $2n+1$ dimensional compact contact manifold M with a contact one form η and a Reeb vecto ξ, and G a compact connected Lie group with Lie algebra $\mathfrak{g}$. Suppose that G acts on M preserving the contact one form η. Then the η-contact moment map $\Phi : M \to \mathfrak{g}^*$, given by

$$< \Phi, X >= \eta(X_M), \ \ \forall X \in \mathfrak{t} = \mathrm{Lie}(T),$$

is also a moment map for the G-action on the transversely symplectic manifold $(M, d\eta, \mathcal{F}_\xi)$. Here $< \cdot, \cdot >$ is the dual pairing between $\mathfrak{t}$ and $\mathfrak{t}^*$, and X_M is the fundamental vector field generated by X.

According to [BG08, Definition 8.4.28], if the Reeb vector ξ is generated by the infinitesimal action of an element in $\mathfrak{g}$, then the action of G is said to be of *Reeb type*. It is clear from Lemma 5.1 that in this case $\mathcal{F}_\xi$ is also a Riemannian foliation. If in addition, there is a Sasakian metric g such that (M, η, g) is a Sasakian manifold, then $\mathcal{F}_\xi$ satisfies the transverse Hard Lefschetz property. In particular, these observations apply to the case when G is an $n + 1$ dimensional torus, and when the action of G is of Reeb type, i.e., when M is a compact toric contact manifold of Reeb type (c.f. [BG08]). Therefore by Theorem 1.1, there is a formal Frobenius manifold structure on the equivariant basic cohomology of toric contact manifolds of Reeb type.

Example 5.4 (Co-symplectic manifolds, c.f. [Li08]). Let (M, η, ω) be a $2n + 1$ dimensional compact co-symplectic manifold. By definition, η is a closed one form, and ω a closed two form ω, such that $\eta \wedge \omega^n$ is a volume form. Then the Reeb characteristic foliation $\mathcal{F}_\xi$ induced by the Reeb vector field ξ (defined by the equations $\iota_\xi \eta = 1$ and $\iota_\xi \omega = 0$) is transversely symplectic with ω being the transversely symplectic form.

We claim that for any $1 \leq k \leq n$, ω^k represents a non-trivial basic cohomology class in $H_B^{2k}(M)$. Assume to the contrary that $[\omega^k] = 0 \in H_B^{2k}(M)$ for some $1 \leq k \leq n$. Then there exists a basic $(2n - 1)$-form β such that $\omega^n = d\beta$. Since $d\eta = 0$, we have that

$$\int_M \eta \wedge \omega^n = \int_M \eta \wedge d\beta = \int_M -d(\eta \wedge \beta) = 0,$$

which contradicts the fact that $\eta \wedge \omega^n$ is a volume form. This proves our claim.

The co-symplectic manifold M is called a co-Kähler manifold, if one can associate to (M, η, ω) an almost contact structure (ϕ, ξ, η, g), where ϕ is an $(1, 1)$-tensor, and g a Riemannian metric, such that ϕ is parallel with respect to the Levi-Civita connection of g. It is straightforward to check that if M is co-Kähler, then the Reeb characteristic

foliation $\mathcal{F}_\xi$ is Kähler. Due to the claim established in the previous paragraph, it is indeed a homologically oriented Kähler foliation, and therefore satisfies the transverse Hard Lefschetz property. By Corollary 4.7, the basic cohomology of M has a structure of a formal Frobenius manifold.

Example 5.5 (Symplectic orbifolds). Let (X, σ) be an effective symplectic orbifold (cf. [Sa57]) of dimension $2n$. Then the total space of the orthogonal frame orbi-bundle $\pi : P \longrightarrow X$ is a smooth manifold on which the structure group $O(2n)$ acts locally free. The form $\omega := \pi^*\sigma$ is a closed 2-form on P whose kernel gives rise to a transversely symplectic foliation $\mathcal{F}$. If g is a Riemannian metric on X, then π^*g is a transverse Riemannian metric on P. Thus $\mathcal{F}$ is also Riemannian. When ω is a Kähler two form, it was shown in [WZ09] that P satisfies the transverse Hard Lefschetz property. Since in this case, the basic differential complex of $\mathcal{F}$ is isomorphic to the de Rham differential complex on X, Corollary 4.7 implies that there is a formal Frobenius manifold structure on the de Rham cohomology of X.

Now suppose that a compact connected Lie group G acts on (X, σ) in a Hamiltonian fashion with a moment map $\Phi : X \to \mathfrak{t}^*$, where $\mathfrak{t} = \mathrm{Lie}(G)$. By averaging, we may assume that there is a G-invariant Riemannian metric g that is compatible with σ. Then the action of G maps an orthogonal frame to another orthogonal frame, and therefore, lifts to an action on the orthogonal frame bundle P. It is easy to see that the lifted G-action on $(P, \pi^*\sigma)$ is Hamiltonian with a moment map $\Psi = \pi^*\Phi$. Analogous to the discussion in the previous paragraph, when X is Kähler orbifold, Theorem 1.1 implies that there is a formal Frobenius manifold structure on the equivariant de Rham cohomology of X.

Example 5.6 (Symplectic quasi-folds [Pra01]). Assume that (X, σ) is a symplectic manifold on which the torus T acts in a Hamiltonian fashion. We denote the moment map by $\phi : X \longrightarrow \mathfrak{t}^*$. Let $N \subset T$ be a non-closed subgroup with Lie algebra $\mathfrak{n}$ and let a be regular value of the corresponding moment map $\varphi : X \longrightarrow \mathfrak{n}^*$. Consider the submanifold $M = \varphi^{-1}(a) \subset X$. The N-action on M yields a transversely symplectic foliation $\mathcal{F}$ with $\omega := i^*\sigma$ being the transversely symplectic form, where i is the inclusion map of M in X. In this case, the leaf space of $\mathcal{F}$ is the symplectic quasi-fold as introduced by Prato [Pra01], at least when N is a connected subgroup of T. It is straightforward to check that the induced T-action on $(M, \mathcal{F}, \omega)$ is Hamiltonian.

It is clear from Lemma 5.1 that $\mathcal{F}$ is also a Riemannian foliation. Moreover, using an argument similar to the one given in Example 5.4, it can be shown that $\mathcal{F}$ is homologically oriented. The leaf space of $\mathcal{F}$ is called a toric quasi-fold when $\dim T$ is half of the dimension of the leaf space. It can be shown that in this case $\mathcal{F}$ is a Kähler foliation[2]. Thus there exist formal Frobenius manifold structures on the basic cohomology and equivariant basic cohomology of toric quasi-folds.

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[2] We will study the equivariant basic cohomology theory of toric quasi-folds more carefully in a forthcoming paper. In particular, we will provide details on the two claims we made here.

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