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Supersymmetric Origins of the Properties of sech-Pulses and sine-Gordon Solitons

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A Thesis Presented

by

ANDREW KOLLER

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ABSTRACT

SUPERSYMMETRIC ORIGINS OF THE PROPERTIES OF SECH-PULSES AND SINE-GORDON SOLITONS

June 2011

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In this thesis, we show that the members of a class of reflectionless Hamiltonians, namely, Akulin's Hamiltonians [1], are connected via a supersymmetric (SUSY) chain. While the reflectionless property in question (vanishing reflection coefficients at *all* values of the spectral parameter, e.g. energy) has been mentioned in the literature for over two decades [1, 2], the enabling algebraic mechanism was previously unknown. We show that the supersymmetric connection of the Akulin's Hamiltonians to a potential-free Hamiltonian is the origin of this property. As the first application for our findings, we show that the SUSY decomposition of Akulin's Hamiltonians explains a well-known effect in laser physics: when a two-level atom, initially in the ground state, is subjected to a laser pulse of the form $V(t) = (n\hbar/\tau)/\cosh(t/\tau)$, with n an integer and τ the pulse duration, it remains in the ground state after the pulse has been applied, for *any* choice of the laser detuning. The second application concerns the sine-Gordon equation: we demonstrate that the first member of the Akulin's chain is related to the L-operator of the Lax pair for the one-soliton solution of the sine-Gordon equation: its reflectionless nature is now explained by supersymmetry.

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CHAPTER 1

INTRODUCTION

In general, waves reflect when they encounter a change in their medium. Two well-known examples are electromagnetic waves reflecting at the interface of two dielectrics, and quantum-mechanical plane waves reflecting off of a potential. In both cases, destructive interference can lead to the absence of reflected waves, i.e., perfect transmission. A Fabry-Perot interferometer with thickness a, for example, will not reflect waves of wavelengths $\lambda_n = 4a/n$, with n an integer, and can thus be used as an anti-reflection coating [3]. The exact same phenomenon occurs when scattering plane waves off of a finite square well potential in 1D quantum mechanics [4]. In these cases, perfect transmission only occurs for specific wavelengths.

It is natural to ask whether it is possible to engineer a system that perfectly transmits waves of *all* wavelengths. In other words, does *reflectionless scattering* exist? Is it possible to construct a piece of glass whose index of refraction varies continuously throughout its thickness, or a quantum-mechanical potential whose depth changes smoothly in space, in such a way that every reflected wave is destroyed by interference, regardless of its wavelength?

Kay and Moses addressed this problem in their classic paper [5] and found that the answer was *yes*, reflectionless scattering is possible, and that the problem is equivalent for classical electromagnetic waves and for the scattering of quantum-mechanical plane waves. They found that the simplest reflectionless potentials in quantum mechanics are

$$V_N(x) = -\frac{N(N+1)}{\cosh^2(x)}, \tag{1.1}$$

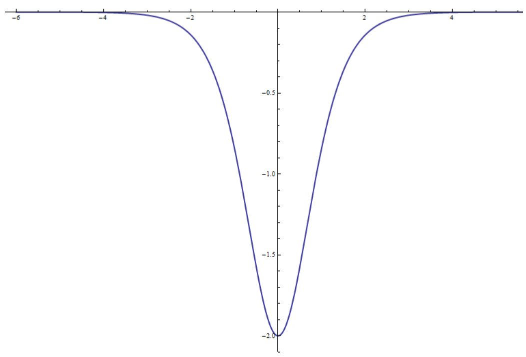

Figure 1.1: Reflectionless Potential $V(x) = -2\text{sech}^2(x)$: Waves of all energies are perfectly transmitted without reflection.

with N an integer, and $\hbar = 2m = 1$, the first of which is shown in Figure 1.1. In fact, they discovered a more general family of reflectionless potentials, all related to the basic shape of (1.1). These generalized reflectionless potentials, which we refer to as “Schrödinger Camels" due to their shape, are discussed in detail in Section 2.2.2.

The next natural question is if reflectionless potentials are simply an interesting coincidence, or whether something more fundamental is responsible for their unique properties. The answer is that they have a deep algebraic connection to “free space" via what is called quantum-mechanical supersymmetry (QMSUSY or just SUSY) [6]. Free space is inherently reflectionless, and the supersymmetric connection between free space and these special potentials guarantees that they will exhibit reflectionless scattering.

It was later discovered that reflectionless potentials of the form (1.1) are intimately connected to the inverse scattering method (ISM) used to solve the initial value problem for the Korteweg-de Vries (KdV) equation [7]. The KdV equation is a nonlinear partial differential equation (NPDE) describing the time evolution of a field $U(x,t)$. Interestingly, an initial value of $U(x,0) = -V_N(x)$, where $V_N(x)$ is one of the reflectionless potentials given in (1.1), leads to an N-soliton solution of the KdV equation, where at $t = 0$ all of the solitons are located at the origin. Solitons are self-reinforcing nonlinear waves which maintain their shape and scatter elastically off one another as they propagate through space, and are a central feature of integrable nonlinear PDE’s. In general, when the inverse scattering method is used to solve the initial value problem for a NPDE, a reflectionless *direct* scattering problem leads to soliton solutions for the nonlinear PDE in question [7]. This connection will be explored in detail in Section 2.3.

The case described above can be summarized as follows. A supersymmetric connection to free space is responsible for potentials exhibiting reflectionless scattering in 1D quantum mechanics. The same phenomenon of reflectionless scattering leads to multi-soliton solutions of the KdV equation, and is thus deeply connected to KdV's integrability:

Known Case:

SUSY $\rightarrow$ Reflectionless Scattering for Potentials $V_N(x)$ $\rightarrow$ Soliton Solutions of KdV

The central objective of this thesis is to present a second, previously unknown case that parallels this connection between supersymmetry, reflectionless scattering, and integrable nonlinear PDE's. We consider a family of Hamiltonians H_n, which we refer to as Akulin's Hamiltonians. Akulin's Hamiltonians are a family of 2×2 matrix differential operators given by

$$H_n = \sigma_z \partial_x - \sigma_x n / \cosh(x) \tag{1.2}$$
$$n = \ldots, -3, -2, -1, 0, +1, +2, +3, \ldots$$

When a scattering problem is defined for these Hamiltonians, as described in Section 2.2.1, it is found that their reflection coefficients vanish for every eigenvalue, thus the H_n's are reflectionless. Like the example of reflectionless potentials in quantum mechanics (1.1), we show that this property of the Hamiltonians H_n can be explained by a supersymmetric connection to free space (Section 3.2). It is important to note that Akulin's Hamiltonians represent a new case of reflectionless scattering, and cannot be mapped to the known reflectionless potentials in 1D quantum mechanics.

To complete the analogy with the previously-known case, Akulin's Hamiltonians H_n generate reflectionless direct scattering initial conditions for the inverse scattering method applied to the sine-Gordon equation. In particular, the Hamiltonians $H_{\mp 1}$ lead to the single kink and anti-kink soliton solutions of sine-Gordon. We were unable to verify if each H_n leads to a multi-soliton solution at the time of the writing of this thesis, but such a connection is highly suspected:

New Case:

$$\boxed{\text{SUSY} \rightarrow \text{Reflectionless Scattering for } H_n \rightarrow \text{Soliton Solution(s) of sine-Gordon}}$$

This thesis begins by developing the relevant background material related to supersymmetry in Section 2.1, from its field theory origin to its application to quantum mechanics. The details of the phenomenon of reflectionless scattering are examined in Section 2.2, including a description of the cases for which SUSY is the known origin of the reflectionless scattering (Section 2.2.2), and those cases for which the cause of the reflectionless scattering is unknown (Section 2.2.3). The implicit question is whether SUSY will eventually be understood as the generator of *all* cases of reflectionless scattering. The inverse scattering method is then examined in detail in Section 2.3, with emphasis on the cases of KdV and sine-Gordon.

We then examine the scattering problem for Akulin's Hamiltonians in Section 3.1, showing explicitly that they are reflectionless. Their connection to free space via a (non-unique) supersymmetric chain is given in Section 3.2. An application to laser physics is examined in Section 4.1, and the connection to sine-Gordon solitons is explained in Section 4.2. We conclude with a (long) list of unanswered questions and directions for future work in Chapter 5. It should be clear by the end of this thesis that there is a deep connection between supersymmetry, reflectionless scattering, and integrability that is far from completely-understood.

CHAPTER 2

BACKGROUND

2.1 Supersymmetry

Supersymmetry (SUSY) is a symmetry between bosonic and fermionic degrees of freedom that first arose in field theory. The fundamental properties of bosons and fermions are determined by their creation and annihilation operators, which commute for bosonic degrees of freedom and anti-commute for fermionic degrees of freedom. Supersymmetry unites bosons and fermions via a Lie superalgebra, which contains both commutation and anti-commutation relations [6, 8, 9]. In the past 25 years there has been extensive research applying supersymmetry to problems in quantum mechanics, where it is referred to as quantum mechanical supersymmetry (QMSUSY). In quantum mechanical supersymmetry, two Hamiltonians, $\hat{H}_0$ and $\hat{H}_1$ are unified in the same way that bosons and fermions are unified in the original field theory SUSY. The two Hamiltonians, known as supersymmetric partners, share spectra (except for a single bound state energy), and are transformed into each other via linear differential operators [6]. These powerful relationships between $\hat{H}_0$ and $\hat{H}_1$ are what have generated so much interest in QMSUSY. Outside of the initial description of supersymmetry in field theory, we use the terms “supersymmetry" and “quantum mechanical supersymmetry" interchangeably.

2.1.1 SUSY Formalism

Creation and annihilation operators for bosons and fermions are at the foundation of the algebraic structure of supersymmetry. Consider such operators b^+, b^-, f^+, and f^-, which change the occupation numbers of a bosonic or fermionic state by ± 1. They obey [8]:

$$[b^-, b^+] = 1; \qquad \{f^-, f^+\} \ = \ 1; \qquad [b, f] = 0 \tag{2.1}$$

$$\left(f^+\right)^2 = \left(f^-\right)^2 \ = \ 0. \tag{2.2}$$

Relation (2.2) is known as nilpotency, and is the property of fermionic creation and annihilation operators that allows for construction of a supersymmetric algebra.

Imagine we have operators Q^+ and Q^-, known as supercharges, that transform fermionic states to bosonic states and vice versa:

$$Q^+ : f \Rightarrow b, \sim b^+ f^-; \qquad Q^- : b \Rightarrow f, \sim b^- f^+. \tag{2.3}$$

Since the supercharges contain fermionic creation or annihilation operators, they are also nilpotent:

$$\left(Q^+\right)^2 = \left(Q^-\right)^2 \ = \ 0 \tag{2.4}$$

Now define new charges Q_1, Q_2, and a Hamiltonian $\mathcal{H}$ as :

$$Q_1 \ \equiv \ Q^+ + Q^-; \quad Q_2 \equiv -i\left(Q^+ - Q^-\right) \tag{2.5}$$

$$\mathcal{H} \ \equiv \ \left(Q_1\right)^2 = \left(Q_2\right)^2 = \{Q^+, Q^-\}. \tag{2.6}$$

It is easy to verify that this Hamiltonian commutes with *all* the charge operators $Q^{\pm}_{1,2}$,

$$[Q, \mathcal{H}] = 0, \tag{2.7}$$

and is therefore invariant under transformations between fermions and bosons, as a consequence of (2.6) and the nilpotency of the supercharges (2.4). We now have a closed algebra defined by

$$\{Q_i, Q_j\} = 2\delta_{ij}\mathcal{H}; \qquad [Q_i, \mathcal{H}] = 0. \tag{2.8}$$

The algebra defined in (2.8) is known as a *Lie superalgebra*, and contains both commutator relations describing continuous symmetries and anti-commutator relations describing discrete symmetries. In contrast, a Lie algebra contains only commutator relations describing continuous symmetries [8]. The unification of discrete and continuous symmetries into a single superalgebra is how supersymmetry unifies bosonic and fermionic degrees of freedom.

We now consider a simple realization of supersymmetry:

$$\begin{aligned}
Q^+ &= \begin{pmatrix} 0 & \hat{B} \\ 0 & 0 \end{pmatrix}; \qquad Q^- = \begin{pmatrix} 0 & 0 \\ \hat{A} & 0 \end{pmatrix}; \\
Q_1 &= \begin{pmatrix} 0 & \hat{B} \\ \hat{A} & 0 \end{pmatrix}; \qquad Q_2 = -i \begin{pmatrix} 0 & \hat{B} \\ -\hat{A} & 0 \end{pmatrix}; \\
\mathcal{H} &= (Q_1)^2 = (Q_2)^2 = \{Q^+, Q^-\} \\
&= \begin{pmatrix} \hat{B}\hat{A} & 0 \\ 0 & \hat{A}\hat{B} \end{pmatrix} = \begin{pmatrix} \hat{H}_0 & 0 \\ 0 & \hat{H}_1 \end{pmatrix}.
\end{aligned} \tag{2.9}$$

Hamiltonian $\mathcal{H}$ contains Hamiltonians $\hat{H}_0$ and $\hat{H}_1$, known as supersymmetric partners. $\mathcal{H}$ acts on two components objects, the first component representing a bosonic degree of freedom and the second component representing a fermionic degree of freedom. $\hat{H}_0$ is the Hamiltonian for the bosonic degree of freedom and $\hat{H}_1$ is the Hamiltonian for the fermionic degree of freedom. Note that the action of the supercharges on a state behaves in the way expected by the original definition of the supercharges (2.3):

$$Q^+ \begin{bmatrix} 0 \\ \alpha \end{bmatrix} = \begin{bmatrix} \hat{B}\alpha \\ 0 \end{bmatrix} : f \Rightarrow b \tag{2.10}$$
$$Q^- \begin{bmatrix} \beta \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \hat{A}\beta \end{bmatrix} : b \Rightarrow f,$$

that is, Q^+ transforms a fermionic degree of freedom into a bosonic degree of freedom, and Q^- transforms a bosonic degree of freedom into a fermionic degree of freedom [6, 8, 9].

It is straightforward to show that the levels of the supersymmetric Hamiltonian $\mathcal{H}$ are doubly degenerate as long as the eigenvalue $\lambda \neq 0$. This degeneracy will play a fundamental role in supersymmetric quantum mechanics. Since Q_1 commutes with $\mathcal{H}$ they can be simultaneously diagonalized:

$$Q_1\psi_1 = \lambda\psi_1; \qquad \mathcal{H}\psi_1 = Q_1^2\psi_1 = \lambda^2\psi_1. \tag{2.11}$$

Define $\psi_2 \equiv Q_2\psi_1$. We use $\{Q_1, Q_2\} = 0$, i.e., $Q_1Q_2 = -Q_2Q_1$ to diagonalize ψ_2:

$$\begin{aligned} Q_1\psi_2 &= Q_1Q_2\psi_1 = -Q_2Q_1\psi_1 \\ &= -\lambda Q_2\psi_1 = -\lambda\psi_2. \end{aligned} \tag{2.12}$$

Thus $\psi_1 \neq \psi_2$ as long as $\lambda > 0$. But since $[\mathcal{H}, Q_2] = 0$,

$$\mathcal{H}\psi_2 = \mathcal{H}Q_2\psi_1 = Q_2\mathcal{H}\psi_1 = \lambda^2 Q_2\psi_1 = \lambda^2\psi_2, \tag{2.13}$$

meaning ψ_1 and ψ_2 are both eigenvectors of $\mathcal{H}$ with the same eigenvalue λ^2, i.e, $\mathcal{H}$ is doubly degenerate for eigenvalues greater than zero [6, 8, 9].

2.1.2 Supersymmetric Quantum Mechanics

We now apply the results of Section 2.1.1 to non-relativistic quantum mechanics, following the conventions of Sukumar [6]. Two deviations will be made from the formalism developed in Section 2.1.1. First, the Hamiltonians $\hat{H}_0$ and $\hat{H}_1$ are necessarily Hermitian, so we send $\hat{B} \to \hat{A}^\dagger$. It is important to note, however, that the underlying algebra of SUSY does not require $\hat{H}_0$ and $\hat{H}_1$ to be Hermitian. Later in this thesis we will relax the Hermiticity of $\hat{H}_0$ and $\hat{H}_1$. Additionally, (2.11) and (2.13) imply that the energy levels of $\mathcal{H}$ are greater than or equal to zero. This is true in field theory, but not in non-relativistic quantum mechanics where the absolute energy scale is arbitrary. We will thus add a constant, called the factorization energy, to the factorizations of $\hat{H}_0$ and $\hat{H}_1$.

In one dimensional stationary quantum mechanics, the Hamiltonian and corresponding energy eigenvalue equation ares given by

$$\hat{H} = -\frac{1}{2}\partial_x^2 + V(x); \qquad \hat{H}\psi \;=\; E\psi \tag{2.14}$$

with $\hbar = m = 1$. The Hamiltonian can always be factored[1] into

$$\hat{H} = \hat{A}^\dagger \hat{A} + \epsilon, \tag{2.15}$$

where ϵ is the factorization energy and

$$\hat{A}^\dagger = \frac{1}{\sqrt{2}}\left(+\partial_x + W(x)\right); \qquad \hat{A} = \frac{1}{\sqrt{2}}\left(-\partial_x + W(x)\right). \tag{2.16}$$

The function $W(x)$ is known as the superpotential, and combining (2.14), (2.15), and (2.16) we see that it satisfies the nonlinear differential equation

$$W'(x) + W^2(x) = 2\left(V(x) - \epsilon\right). \tag{2.17}$$

[1]This result can be generalized to higher dimensions. See [6].

It can be shown that (2.17) has a family of solutions $W(x, \epsilon, \lambda)$ given by

$$W(x, \epsilon, \lambda) = \frac{d}{dx} \ln \psi(x, \epsilon) + \frac{1/\psi^2(x, \epsilon)}{\lambda + \int_{-\infty}^{x} dz/\psi^2(z, \epsilon)} \tag{2.18}$$

where λ is an arbitrary parameter, and $\psi(x, \epsilon)$ satisfies the Schrödinger equation with $E = \epsilon$ [6]. The factorization energy must be less than or equal to every energy eigenvalue of $\hat{H}$, since

$$\begin{aligned} E &= \langle \psi | \hat{H} | \psi \rangle = \langle \psi | \hat{A}^\dagger \hat{A} + \epsilon | \psi \rangle = \langle \psi | \hat{A}^\dagger \hat{A} | \psi \rangle + \epsilon \\ &= \langle \hat{A} \psi | \hat{A} \psi \rangle + \epsilon, \end{aligned} \tag{2.19}$$

and the quantity $\langle \hat{A}\psi | \hat{A}\psi \rangle \geq 0$. Thus $\epsilon \leq E_g$, where E_g is the ground state energy of $\hat{H}$. This result is distinct from supersymmetric field theory, where the Hamiltonian is required to be positive semi-definite, leading to a zero ground state energy. In quantum mechanics, however, operators of the form $\hat{A}^\dagger \hat{A}$ are positive semi-definite, but the Hamiltonian is not, because the energy scale can be shifted arbitrarily.

Often the factorization energy is chosen so that $\epsilon = E_g$. In this case the operator $\hat{A}$ annihilates the ground state of $\hat{H}$:

$$\hat{A} | \psi^g \rangle = 0, \tag{2.20}$$

which follows immediately from (2.19). We will use the convention that $\epsilon = E_g$ for the remainder of the discussion of supersymmetry in one-dimensional quantum mechanics.

The most important results of supersymmetry are the relationships between the factorized Hamiltonian $\hat{H}_0$ and its supersymmetric partner $\hat{H}_1$. If $\hat{H}_0$ is factorized in the way described above, then its superpartner is found by combining $\hat{A}$ and $\hat{A}^\dagger$ in the opposite order:

$$\hat{H}_0 = \hat{A}^\dagger\hat{A} + \epsilon = -\frac{1}{2}\partial_x^2 + V_0(x); \qquad \hat{H}_1 = \hat{A}\hat{A}^\dagger + \epsilon = -\frac{1}{2}\partial_x^2 + V_1(x). \tag{2.21}$$

If $\hat{A}$ and $\hat{A}^\dagger$ do not commute, then $\hat{H}_0 \neq \hat{H}_1$, i.e., $V_0(x) \neq V_1(x)$. It is straightforward to show that if $|\psi_0^\alpha\rangle$ is an eigenstate of $\hat{H}_0$ with energy α, $\hat{H}_0|\psi_0^\alpha\rangle = \alpha|\psi_0^\alpha\rangle$, then $\hat{A}|\psi_0^\alpha\rangle$ is an eigenstate of $\hat{H}_1$ with the same energy:

$$\begin{aligned}
\hat{H}_1\left(\hat{A}|\psi_0^\alpha\rangle\right) &= \left(\hat{A}\hat{A}^\dagger + \epsilon\right)\left(\hat{A}|\psi_0^\alpha\rangle\right) && (2.22)\\
&= \hat{A}\left(\hat{A}^\dagger\hat{A}\right)|\psi_0^\alpha\rangle + \epsilon\hat{A}|\psi_0^\alpha\rangle && (2.23)\\
&= \hat{A}\left(\hat{H}_0 - \epsilon\right)|\psi_0^\alpha\rangle + \epsilon\hat{A}|\psi_0^\alpha\rangle\\
&= \hat{A}\left(\hat{H}_0|\psi_0^\alpha\rangle\right) = \alpha\left(\hat{A}|\psi_0^\alpha\rangle\right)\\
\Rightarrow |\psi_1^\alpha\rangle \propto \hat{A}|\psi_0^\alpha\rangle. & && (2.24)
\end{aligned}$$

In (2.24), equality is withheld since $\hat{A}|\psi_0^\alpha\rangle$ is not guaranteed to be normalized. Similar arguments show that if $|\psi_1^\beta\rangle$ is an eigenstate of $\hat{H}_1$ with energy β, $\hat{H}_1|\psi_1^\beta\rangle = \beta|\psi_1^\beta\rangle$, then $\hat{A}^\dagger|\psi_1^\beta\rangle$ is an eigenstate of $\hat{H}_0$ with the same energy:

$$|\psi_0^\beta\rangle \propto \hat{A}^\dagger|\psi_1^\beta\rangle. \tag{2.25}$$

From (2.24) and (2.25) we see that $\hat{H}_0$ and $\hat{H}_1$ have the same spectrum of energy eigenvalues, and the operators $\hat{A}$ and $\hat{A}^\dagger$ transform energy eigenstates of $\hat{H}_0$ to $\hat{H}_1$ and vice versa. The exception is the ground state of $\hat{H}_0$ since (2.20) implies that $\hat{A}|\psi_0^g\rangle = 0$ and thus $\hat{H}_1$ does not share this eigenvalue with $\hat{H}_0$. The picture that emerges from the analysis is that $\hat{H}_0$ and $\hat{H}_1$ have identical spectra except that $\hat{H}_0$ has an extra bound state, as illustrated in Figure 2.1. This is exactly the two-fold degeneracy of $\mathcal{H}$ found in section Section 2.1.1 [6, 8].

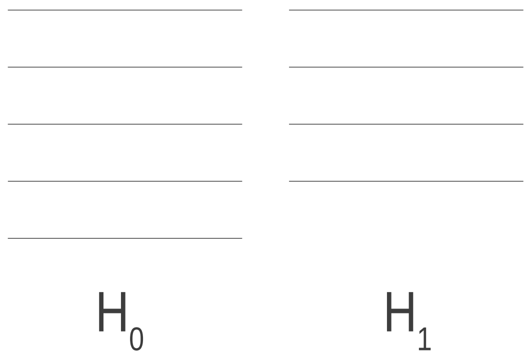

Figure 2.1: Equivalent Spectra of H_0 and H_1: A possible alignment of the eigenvalues of supersymmetric partners H_0 and H_1: their spectra are equivalent except for the ground state of H_0.

From the factorizations (2.21) it follows that $\hat{H}_0$, $\hat{H}_1$ are related by

$$\hat{H}_1\hat{A} = \hat{A}\hat{H}_0 \tag{2.26}$$

$$\hat{A}^\dagger\hat{H}_1 = \hat{H}_0\hat{A}^\dagger. \tag{2.27}$$

When two Hamiltonians are related by (2.26) and (2.27), we refer to $\hat{A}$ and $\hat{A}^\dagger$ as the *intertwiners* of $\hat{H}_0$ and $\hat{H}_1$. More specifically, $\hat{A}$ in (2.26) is the right intertwiner and $\hat{A}^\dagger$ in (2.27) is the left intertwiner[2]. Interestingly, equation (2.26) alone implies that $\hat{A}$ transforms eigenstates of $\hat{H}_0$ to eigenstates of $\hat{H}_1$:

$$\hat{H}_1\left(\hat{A}|\psi_0^\alpha\rangle\right) = \left(\hat{H}_1\hat{A}\right)|\psi_0^\alpha\rangle = \left(\hat{A}\hat{H}_0\right)|\psi_0^\alpha\rangle = \alpha\left(\hat{A}|\psi_0^\alpha\rangle\right), \tag{2.28}$$

and similarly, equation (2.27) alone implies that $\hat{A}^\dagger$ transforms eigenstates of $\hat{H}_1$ to eigenstates of $\hat{H}_0$:

$$\hat{H}_0\left(\hat{A}^\dagger|\psi_1^\beta\rangle\right) = \left(\hat{H}_0\hat{A}^\dagger\right)|\psi_1^\beta\rangle = \left(\hat{A}^\dagger\hat{H}_1\right)|\psi_1^\beta\rangle = \beta\left(\hat{A}^\dagger|\psi_1^\beta\rangle\right), \tag{2.29}$$

without any reference to the underlying supersymmetric structure of $\hat{H}_0$ and $\hat{H}_1$. Super-

[2]Unless stated otherwise, in this thesis the term “intertwiner" refers to the right intertwiner, which is essential to understanding the origin of scattering without reflection.

symmetry implies the existence of intertwiners, but it is unclear whether intertwiners imply the existence of supersymmetry. This question will be addressed in Section 2.1.7. The existence of a right intertwiner, (2.26), is essential for understanding why SUSY partners of potential-free Hamiltonians (i.e. free space) show reflectionless scattering at all energies.

Quantum mechanical supersymmetry can be thought of as a generalization of the ladder operator method for solving the quantum harmonic oscillator. The harmonic oscillator is the simplest nontrivial case of quantum mechanical supersymmetry: if $\hat{H}_0$ is the Hamiltonian for a harmonic oscillator, then its supersymmetric partner $\hat{H}_1$ is another harmonic oscillator with its energy scale shifted upwards by $\hbar\omega$, i.e. $V_1(x) = V_0(x) + \hbar\omega$ [9].

The harmonic oscillator is the only nontrivial case where the potentials differ by only a constant: typically the functional forms of $V_0(x)$ and $V_1(x)$ are quite different. The power of QMSUSY is apparent given the small number of exactly solvable quantum mechanical systems. For every case where an exact solution is known, supersymmetric factorization yields another exactly solvable Hamiltonian. In short, SUSY doubles the number of quantum mechanical systems that can be solved exactly.

2.1.3 QMSUSY Factorization of Non-Hermitian Operators

Outside of the usual quantum mechanics, the Hamiltonians $\hat{H}_0$ and $\hat{H}_1$ need not be Hermitian. For these situations we can have $\hat{B} \neq \hat{A}^\dagger$, but most of the relationships developed in 2.1.2 will remain. In particular, if $\hat{H}_0$ and $\hat{H}_1$ are given by

$$\hat{H}_0 = \hat{B}\hat{A} + \epsilon; \qquad \hat{H}_1 = \hat{A}\hat{B} + \epsilon, \tag{2.30}$$

then the intertwining and eigenstate relationships of $\hat{H}_0$ and $\hat{H}_1$ will be the same as in Section 2.1.2:

$$\hat{H}_1 \hat{A} = \hat{A}\hat{H}_0; \qquad \hat{H}_0 \hat{B} = \hat{B}\hat{H}_1 \tag{2.31}$$
$$\hat{A} \mid \psi_0\rangle \propto \mid \psi_1\rangle; \qquad \hat{B} \mid \psi_1\rangle \propto \mid \psi_0\rangle$$

where $\mid \psi_0\rangle$ is an eigenstate of $\hat{H}_0$ and $\mid \psi_1\rangle$ is an eigenstate of $\hat{H}_1$ with the same eigenvalue. Relaxing the Hermiticity of the factorized operators opens up a larger class of problems to which QMSUSY can be applied. In particular, in this thesis we study the SUSY factorization of a family of non-Hermitian operators given by

$$H_n = \sigma_z \partial_x - \sigma_x n / \cosh(x) \tag{2.32}$$
$$n = \ldots, -3, -2, -1, 0, +1, +2, +3, \ldots$$

We refer to these operators as the Akulin Hamiltonians, after a set of exactly-solvable two-level time-dependent quantum-mechanical systems described by Akulin [1]. In this case the label of "Hamiltonian" does not imply Hermiticity[3].

[3]Additionally, we often refer to factorized operators as "Hamiltonians" $\hat{H}$ to distinguish them from their SUSY factors $\hat{A}$ and $\hat{B}$.

2.1.4 SUSY Chains

Consider a sequence of supersymmetric relationships:

$$\begin{aligned}
\hat{H}_0 &= \hat{B}_0\hat{A}_0 + \epsilon_0 = \hat{A}_1\hat{B}_1 + \epsilon_1 \\
\hat{H}_1 &= \hat{B}_1\hat{A}_1 + \epsilon_1 = \hat{A}_2\hat{B}_2 + \epsilon_2 \\
&\vdots \\
\hat{H}_m &= \hat{B}_m\hat{A}_m + \epsilon_m = \hat{A}_{m+1}\hat{B}_{m+1} + \epsilon_{m+1} \\
&\vdots \\
\hat{H}_n &= \hat{B}_n\hat{A}_n + \epsilon_n = \dots .
\end{aligned} \tag{2.33}$$

Eigenstates of $\hat{H}_n$ will be linked to eigenstates of $\hat{H}_m$ via the intertwiner $\hat{\Upsilon}_{n\leftarrow m}$:

$$\begin{aligned}
\mid \psi_n\rangle &\propto \hat{B}_n\hat{B}_{n-1}\cdots\hat{B}_{m+1} \mid \psi_m\rangle \\
\hat{\Upsilon}_{n\leftarrow m} &= \hat{B}_n\hat{B}_{n-1}\cdots\hat{B}_{m+1} \quad .
\end{aligned} \tag{2.34}$$

2.1.5 An Example: Free Space Hierarchy

We now illustrate an important example, which is the hierarchy obtained by starting with $\hat{H}_0 = -\frac{d^2}{dx^2}$, representing a potential-free Hamiltonian, i.e., free space (with $\hbar = 2m = 1$). Letting $\epsilon_n = -n^2$, $\hat{A}_n = \partial_x + n\tanh(x)$, $\hat{B}_n = \hat{A}_n^\dagger$, and setting an arbitrary position shift equal to zero at each stage generates the chain [6]

$$\hat{H}_n = -\frac{d^2}{dx^2} - \frac{n\,(n+1)}{\cosh^2(x)} \quad ,$$

where, for any positive integer n, the potentials

$$V_n(x) = -\frac{n\,(n+1)}{\cosh^2(x)} \tag{2.35}$$

are reflectionless at all energies [10]. As described earlier, and explained in detail in Section 2.2.1, a reflectionless potential is one for which the reflection coefficient, defined in terms of asymptotic solutions of the eigenvalue problem, is identically zero for any eigenvalue (energy). The case $n = 1$ for the potentials described above (2.35) is the famous Pöschl-Teller potential. As discussed in Chapter 1, potentials of the form (2.35) were also discovered by Kay and Moses [5] by considering the possibility of perfect transmission of electromagnetic waves through a dielectric at all wavelengths. The first three members of this hierarchy are shown in Figure 2.2. Interestingly, the potential $V_n(x)$ admits n bound-states, which are also plotted in Figure 2.2.

The eigenstates of each potential (2.35) are linked to eigenstates of $\hat{H}_0$ (which are simply plane-waves) via the intertwiner

$$| \psi_n\rangle \propto \hat{B}_n\hat{B}_{n-1}\cdots\hat{B}_1|\psi_0\rangle = \hat{\Upsilon}_{n\leftarrow 0}|\psi_0\rangle. \tag{2.36}$$

Each of the operators $\hat{B}_m = -\frac{d}{dx} + m\tanh(x)$ asymptotically becomes a differential operator with constant coefficients at $x \to \pm\infty$, as does the product $\hat{\Upsilon}_{n\leftarrow 0}$. Therefore, the map between eigenstates locally converts plane waves to plane waves conserving the direction of momentum. If one of the members of the supersymmetric chain exhibits a lack of reflection at all energies, as does $\hat{H}_0$ in this case, then every member of the chain will likewise be reflectionless. In general, if a Hamiltonian is linked to a potential-free Hamiltonian via a supersymmetric chain, it is reflectionless because of the intertwiner (2.36).

2.1.6 Cases Solved via QMSUSY

Supersymmetry has been used to solve the 1D and 3D quantum harmonic oscillator (as discussed in Section 2.1.2), the Coulomb, Morse, Scarf II (hyperbolic), Rosen-Morse II (hyperbolic), Eckart, Scarf I (trigonometric), Pöschl-Teller (discussed in Section 2.1.5), and Rosen-Morse I (trigonometric) potentials [9], as well as additional reflectionless potentials

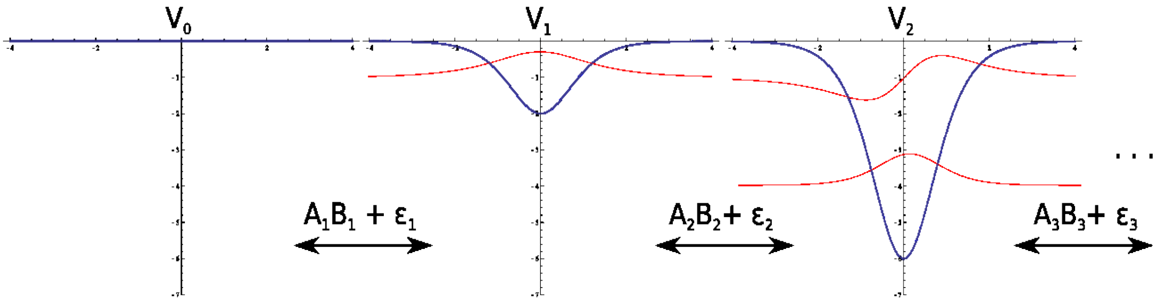

Figure 2.2: First Three Members of Free Space Hierarchy: Read multiplication order "away" from potential in question. (a) Free space Hamiltonian $H_0 = -\partial_x^2$, (b) $H_1 = -\partial_x^2 - 2\text{sech}^2(x)$, (c) $H_2 = -\partial_x^2 - 6\text{sech}^2(x)$. The potential $V_n(x)$ admits n bound-state solutions, plotted in red.

discussed in section 2.2. SUSY has been used to solve the problem of an electron in a magnetic field [8], as well as a spin-$1/2$ electron bound to a magnetic wire [11]. In the latter case, SUSY is extended to multi-component wavefunctions and the associated operators are 2×2 matrices. This is similar to the SUSY factorization of the Akulin Hamiltonians described in Section 3.1, where the Hamiltonian and SUSY factors are 2×2 matrices of differential operators.

2.1.7 Do SUSY-Free Intertwiners Exist?

It is clear that if two Hamiltonians are part of the same SUSY chain, then they are intertwined via (2.35). It is unknown, however, if SUSY-free intertwiners exist, i.e., if two Hamiltonians can be intertwined without being linked via a supersymmetric chain. The question of whether SUSY-free intertwiners exist is important for several reasons. First, it is often easier to find intertwiners than it is to SUSY-factorize Hamiltonians. In our investigation of the Akulin Hamiltonians, for example, we found the intertwiner between $\hat{H}_1$ and $\hat{H}_0$ before finding their SUSY connection. If SUSY-free intertwiners do not exist, then finding an intertwiner immediately implies the existence of supersymmetry. Furthermore, a proof that SUSY-free intertwiners do not exist would represent a major step towards understanding why supersymmetry is so intimately related to reflectionlessness and integrability.

2.2 Reflectionless Scattering

2.2.1 Definition of the Scattering Problem

The direct scattering problem is one of cataloging solutions of the eigenvalue problem for a linear differential operator H:

$$H\psi = \lambda\psi. \tag{2.37}$$

The operator H contains one or more potentials $u(x)$, any order of derivatives, and any number of components. For example, if H is of the form of the Schrödinger operator

$$H = -\partial_x^2 + u(x), \tag{2.38}$$

then our eigenvalue equation (2.37) is the Sturm-Liouville problem for $u(x)$ [7]:

$$\psi_{xx} + (\lambda - u)\psi = 0. \tag{2.39}$$

This is the situation analyzed in quantum mechanics. Asymptotic solutions to (2.39) will take the form of linear combinations of $e^{\pm ikx}$ with $\lambda = k^2$. The case when $\lambda > 0$ is known as the continuous spectrum. We are free to define the scattering problem as follows:

$$\psi \sim \begin{cases} e^{ikx} + R(k)e^{-ikx} & x \to -\infty \\ T(k)e^{ikx} & x \to +\infty, \end{cases} \tag{2.40}$$

that is, a wave a modulus one is incident from $x = -\infty$ on the potential, and $T(k)$ and $R(k)$ give the reflected and transmitted amplitudes, respectively. Physically, from conservation of norm we have

$$|T(k)|^2 + |R(k)|^2 = 1. \tag{2.41}$$

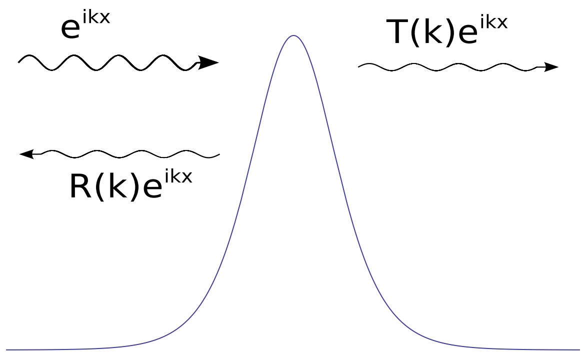

Figure 2.3: Definition of the Scattering Problem: e^{ikx} is incident on a potential from $x = -\infty$; $R(k)$ and $T(k)$ give the amplitudes of the reflected and transmitted waves, respectively.

In the mathematics literature the continuous scattering problem is often defined as a plane wave of modulus one, moving to the left, incident from $x = +\infty$. We choose the above convention because it is more in line with the physics literature.

Similarly, eigenvalues $\lambda < 0$ permit a set decaying solutions as $x \to \pm\infty$, known as the discrete spectrum, cataloged by their eigenvalues and asymptotic amplitudes:

$$\psi_n \sim \begin{cases} \tilde{c}_n e^{\kappa_n x} & x \to -\infty \\ c_n e^{-\kappa_n x} & x \to \infty. \end{cases} \tag{2.42}$$

Physically, these are the tails of the bound-state solutions [7]. The bound-state spectrum is discrete because the boundary conditions place heavy restrictions on the form of the eigenfunctions with $\lambda < 0$, that is, these solutions must decay exponentially instead of growing exponentially.

The second type of scattering problem we are interested in is when the differential operator H is a 2×2 matrix and the eigenfunctions ψ have two components. In particular, the Akulin Hamiltonians have the form $H_n = \sigma_z \partial_x - \sigma_x q(x)$:

$$H = \begin{pmatrix} \partial_x & -q(x) \\ -q(x) & -\partial_x \end{pmatrix}, \qquad \psi = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}. \tag{2.43}$$

We wish to solve the eigenvalue problem

$$H\psi = i\lambda\psi, \tag{2.44}$$

where the eigenvalue is defined as $i\lambda$ because of a transformation of the original statement of the eigenvalue problem (see Section 4.1). We take $k = \lambda$ and write four asymptotic solutions of (2.44) [7]:

$$\begin{aligned}
\psi_+(x) &\sim e^{ikx}\begin{pmatrix} 1 \\ 0 \end{pmatrix}; \qquad \tilde{\psi}_+(x) \sim e^{-ikx}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad x \to +\infty \\
\psi_-(x) &\sim e^{ikx}\begin{pmatrix} 1 \\ 0 \end{pmatrix}; \qquad \tilde{\psi}_-(x) \sim e^{-ikx}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad x \to -\infty.
\end{aligned} \tag{2.45}$$

These solutions are linearly independent, so we can write the asymptotic solutions at $+\infty$ as linear combinations of the asymptotic solutions at $-\infty$:

$$\psi_+ = T(k)\psi_- + R(k)\tilde{\psi}_-; \qquad \tilde{\psi}_+ = \tilde{R}(k)\psi_- + \tilde{T}(k)\tilde{\psi}_-, \tag{2.46}$$

which we can write in terms of a scattering matrix S:

$$\Psi_+ = \Psi_- S; \qquad S = \begin{pmatrix} T & \tilde{R} \\ R & \tilde{T} \end{pmatrix}, \tag{2.47}$$

and

$$\Psi_+ = \begin{pmatrix} \psi_{+1} & \tilde{\psi}_{+1} \\ \psi_{+2} & \tilde{\psi}_{+2} \end{pmatrix}; \qquad \Psi_- = \begin{pmatrix} \psi_{-1} & \tilde{\psi}_{-1} \\ \psi_{-2} & \tilde{\psi}_{-2} \end{pmatrix}. \tag{2.48}$$

Here the subscripts 1 and 2 on $\psi_\pm$ refer to the first and second components. Notice that the H in (2.43) contains two *first-order* derivatives, differing in sign, acting on different components of ψ. This implies that the sign of k (the direction of momentum), is uniquely tied to a component of ψ (its internal state). Indeed, from (2.45) wee see that right-moving waves e^{ikx} are tied to the internal state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and left-moving waves e^{-ikx} are tied to the internal state $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Reflectionless scattering is a unique phenomenon where reflection coefficients vanish for all values of k, that is, $R(k) = 0$ and $\tilde{R}(k) = 0$. The classical analogy is perfect transmission of electromagnetic waves through a dielectric medium at every wavelength. As discussed in Chapter 1, this was the problem investigated by Kay and Moses [5], who discovered potentials of the form (2.35).

2.2.2 SUSY as the Mechanism for Reflectionless Scattering: Known Cases

Supersymmetry is responsible for reflectionless scattering in most of the known cases [6, 12, 13, 14, 15]. As described in Section 2.1.5, a Hamiltonian will be reflectionless if it is linked to a potential-free Hamiltonian via a supersymmetric chain. We now discuss the known cases where SUSY is the mechanism for reflectionless scattering.

We have already seen that in stationary quantum mechanics, potentials of the form (2.35) are reflectionless due to their SUSY connection to free-space. This problem is worth examining in more detail. We can start with a potential for the Schrödinger equation

$$u(x) = -U_0 \text{sech}^2(x), \tag{2.49}$$

where U_0 is a constant. The Sturm-Liouville equation

$$\psi'' + (\lambda + U_0 \text{sech}^2(x))\psi = 0 \tag{2.50}$$

becomes the associated Legendre equation

$$\frac{d}{dT}\left\{(1-T^2)\frac{d\psi}{dT}\right\} + \left(U_0 + \frac{\lambda}{(1-T^2)}\right)\psi = 0, \tag{2.51}$$

under the substitution $T = \tanh(x)$. The normalizable solutions of the associated Legendre equation are the associated Legendre functions, $P_N^n(T)$, with the condition that $U_0 = N(N+1)$, where N is a positive integer, $\lambda = -n^2$, and $n = 1, 2, \ldots, N$.

For the continuous spectrum, $\lambda = k^2$, and the solutions to (2.51) which behave like $\psi \sim e^{ikx}$ as $x \to \infty$ are

$$\psi(x) = T(k)2^{ik}(\text{sech}(x))^{-ik}F(\tilde{a}, \tilde{b}; \tilde{c}; z), \tag{2.52}$$

where $F(\tilde{a}, \tilde{b}; \tilde{c}; z)$ is the hypergeometric function, and

$$\begin{aligned} \tilde{a} &= \tfrac{1}{2} - ik + (U_o + \tfrac{1}{4})^{1/2}; \qquad \tilde{b} = \tfrac{1}{2} - ik - (U_0 + \tfrac{1}{4})^{1/2} \\ \tilde{c} &= 1 - ik; \qquad z = \tfrac{1}{2}(1 - T). \end{aligned} \tag{2.53}$$

Here $T(k)$ is the transmission coefficient (*not* $T \equiv \tanh(x)$). The asymptotic solutions as $x \to -\infty$ are

$$\psi(x) \sim \frac{\Gamma(\tilde{c})\Gamma(\tilde{a} + \tilde{b} - \tilde{c})}{\Gamma(\tilde{a})\Gamma(\tilde{b})}e^{ikx} + \frac{\Gamma(\tilde{c})\Gamma(\tilde{c} - \tilde{a} - \tilde{b})}{\Gamma(\tilde{c} - \tilde{a})\Gamma(\tilde{c} - \tilde{b})}e^{-ikx}, \tag{2.54}$$

and thus from the definition of the reflection and transmission coefficients (2.40)

$$T(k) = \frac{\Gamma(\tilde{a})\Gamma(\tilde{b})}{\Gamma(\tilde{c})\Gamma(\tilde{a} + \tilde{b} - \tilde{c})}; \qquad R(k) = \frac{T(k)\Gamma(\tilde{c})\Gamma(\tilde{c} - \tilde{a} - \tilde{b})}{\Gamma(\tilde{c} - \tilde{a})\Gamma(\tilde{c} - \tilde{b})}. \tag{2.55}$$

Using the identity that $\Gamma(\frac{1}{2} - z)\Gamma(\frac{1}{2} + z) = \pi / \cos(\pi z)$ it follows that the reflection coefficient $R(k) = 0$ for all k if $U_0 = N(N+1)$, where N is a positive integer [7]. Thus we see that the class of potentials $u(x) = -U_0\text{sech}^2(x)$ has scattering states that can be solved exactly in terms of hypergeometric functions, and that, for a discrete set when $U_0 = N(N+1)$ the reflection coefficient vanishes for all values of k. $U_0 = N(N+1)$ are also the special values when the bound states of $u(x)$ are expressible in terms of associated Legendre functions $P_N^n(x)$ and the bound state energies are given by $\lambda = -n^2$. Mathematically, the existence of this set of integrable potentials appears as a curious coincidence. Beneath the mess of special functions, however, is a supersymmetric mechanism supplying the integrability. A similar story will play out in the discussion of the Akulin Hamiltonians.

There is another class of reflectionless potentials for the 1-D Schrödinger equation, which we refer to as the "Schrödinger camels" after their shape (see Fig. 2.4). The Schrödinger camels are obtained by allowing a spatial shift at every step of the supersymmetric chain originating from free-space (see Section 2.1.5). They are also intimately connected to the Korteweg-de Vries (KdV) equation [16, 17]. For the KdV equation, it is known that an initial condition of

$$u(x, 0) = -N(N+1)\text{sech}^2(x), \tag{2.56}$$

with N a positive integer, results in an N-soliton solution[4], where each of the N solitons moves at a different speed. The Schrödinger camels are simply $u(x, t)$, where those N solitons are allowed to evolve for time t and separate spatially, with a general asymptotic form [7]

$$u(x, t) \sim -2\sum_{n=1}^{N} n^2\text{sech}^2\{n(x - 4n^2 t) \mp x_n\}, \tag{2.57}$$

and asymptotic phases

$$x_n = \frac{1}{2}\ln\left\{\prod_{\substack{m=1\\m\neq n}}^{N} |\frac{n-m}{n+m}|^{\text{sgn}(n-m)}\right\}, \tag{2.58}$$

where $\mp x_n$ is the phase as $t \to \pm\infty$. Properties of the inverse scattering transform guarantee that the N-soliton solution $u(x, t)$ of the KdV equation will *remain* a reflectionless

[4]Normally for KdV, the field $u(x, t)$ is the *negative* of the potential for the Schrödinger equation. To avoid confusion, in this discussion we use the alternate form of KdV where $u \to -u$ so that $u(x, t)$ is equal to the potential for the Schrödinger equation, with no minus sign.

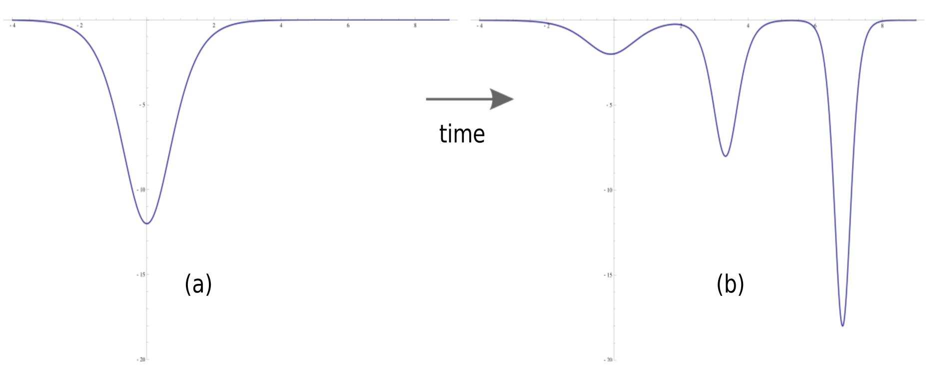

Figure 2.4: Reflectionless "Schrödinger Camel" Potential and 3-Soliton Solution of KdV: The reflectionless potential $u(x, 0) = -12\text{sech}^2(x)$, shown in (a), evolves in time via KdV to a 3-soliton solution $u(x, t)$, shown in (b), an example of a reflectionless "Schrödinger camel" potential. Notice the nonlinear effect: the amplitude of (a) is much less than the combined amplitudes in (b).

potential for the Schrödinger equation for all times t [16, 17, 18], as will be discussed in more detail in Section 2.3.2.

The relativistic Dirac equation in scalar and pseudoscalar external potentials is also shown to exhibit cases of reflectionless scattering [15, 19, 20]. These reflectionless systems have a SUSY mechanism equivalent to SUSY in one-dimensional quantum mechanics.

Lastly, the reflectionless family of the Akulin Hamiltonians, analyzed in this thesis, are another example of Hamiltonians whose reflectionless nature is explained by a SUSY factorization.

2.2.3 Cases of Reflectionless Scattering Without Known SUSY Mechanism

The first case of reflectionless scattering without a known SUSY interpretation was studied by Ablowitz. Reflectionless time-dependent perturbations to the time-dependent Schrödinger equation were used to generate multi-soliton solutions of the Kadomtsev-Petviashvili-I equation [21]. Note that in this case, unlike in all other known cases, the reflectionless problem is set in two spatial dimensions.

There is also a reflectionless system describing excitations on top of the ground state of a 1D attractive Bose condensate, known as the bosonic Bogoliubov-de Gennes system

[22, 23]. The dynamics of the gas is described by the Non-Linear Schrödinger Equation (NLSE):

$$i\hbar\frac{\partial}{\partial t}\psi(z,\,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + gN|\psi(z,\,t)|^2\right]\psi(z,\,t) \tag{2.59}$$
$$\int_{-\infty}^{+\infty} dz\,\psi(z,\,t) = 1; \qquad g < 0,$$

where N is the number of bosons in the condensate. Bosons interact via $V(z_1,\,z_2) = g\delta(z_1 - z_2)$. Here we use $\hbar = m = 1$. If we define $\tilde{G} \equiv |g|N$, we get

$$i\frac{\partial}{\partial t}\psi(z,\,t) = \left[-\frac{1}{2}\frac{\partial^2}{\partial z^2} - \tilde{G}|\psi(z,\,t)|^2\right]\psi(z,\,t) \tag{2.60}$$
$$\int_{-\infty}^{+\infty} dz\,\psi(z,\,t) = 1; \qquad \tilde{G} > 0.$$

The NLSE has a solitonic steady state:

$$\phi(z) = \frac{1}{\sqrt{2l}}\frac{1}{\cosh(z/l)}, \tag{2.61}$$

where $l = \frac{2}{\tilde{G}}$, $\mu = -\frac{1}{8}\tilde{G}^2$, and $\phi(z)$ obeys the stationary Nonlinear Schrödinger Equation,

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial z^2} - \tilde{G}|\phi(z)|^2\right]\phi(z) = \mu\phi(z). \tag{2.62}$$

The steady state generates the following solution of the time-dependent NLSE (2.60):

$$\psi(z,\,t) = \phi(z)e^{-i\mu t}. \tag{2.63}$$

Around the stationary solution (2.61), solutions of the time-dependent NLSE can be decomposed onto a sum of the stationary solution and a small correction,

$$\psi(z,\,t) = [\phi(z) + \delta\psi(z,\,t)]e^{-i\mu t}, \tag{2.64}$$

where the correction obeys

$$i\frac{\partial}{\partial t}\begin{pmatrix} \delta\psi(z,\, t) \\ \delta\psi^*(z,\, t) \end{pmatrix} \approx \hat{\mathcal{L}} \begin{pmatrix} \delta\psi(z,\, t) \\ \delta\psi^*(z,\, t) \end{pmatrix}, \tag{2.65}$$

with

$$\hat{\mathcal{L}} = \begin{pmatrix} \hat{L} & \hat{M} \\ -\hat{M}^\dagger & -\hat{L} \end{pmatrix} \tag{2.66}$$

called the Bogoliubov-de Gennes Liouvillian (BdG), and

$$\begin{aligned} \hat{L} &= -\frac{1}{2}\frac{\partial^2}{\partial z^2} - 2\tilde{G}|\phi(z)|^2 = -\frac{1}{2}\frac{\partial^2}{\partial z^2} - \frac{2}{l^2\cosh(z/l)} \\ \hat{M} &= -\tilde{G}|\phi(z)|^2 = -\frac{1}{l^2\cosh(z/l)}. \end{aligned} \tag{2.67}$$

The eigenstate-eigenvalue problem for the BdG system is:

$$\hat{\mathcal{L}}|w_{\aleph}\succ = \epsilon_{\aleph}|w_{\aleph}\succ\ . \tag{2.68}$$

Its solutions are positive "energy" eigenstates:

$$|w_{k+}\succ = \frac{1}{\sqrt{2\pi}}\frac{1}{(kl)^2+1}\begin{pmatrix} (kl + i\tanh[z/l])^2 \\ 1/\cosh^2[z/l] \end{pmatrix} \times\ \exp[+ikz], \tag{2.69}$$

with $\epsilon_{k+} = \frac{k^2}{2} - \mu$, and negative "energy" eigenstates:

$$|w_{k-}\succ = \frac{1}{\sqrt{2\pi}}\frac{1}{(kl)^2+1}\begin{pmatrix} 1/\cosh^2[z/l] \\ (kl - i\tanh[z/l])^2 \end{pmatrix} \times\ \exp[-ikz], \tag{2.70}$$

with $\epsilon_{k-} = -\frac{k^2}{2} + \mu = -\epsilon_{k+}$. These solutions demonstrate that $\hat{\mathcal{L}}$ is reflectionless, since they do not “mix" left and right-moving waves. Physically this reflectionless property means that small excitations pass through the ground state of the gas unperturbed.

A supersymmetric decomposition is of the Bogoliubov-de Gennes Liouvillian is currently unknown, but suspected. We have found an intertwiner connecting $\hat{\mathcal{L}}$ to a potential-free Liouvillian (discussed further in Section 5.3), which is strong evidence of a SUSY connection.

The BdG system is interesting because it represents the third stage of simplification at which a system of attractive bosons exhibits scattering without reflection. In the full, many-body interaction the integrability is due to the Bethe ansatz. In the mean-field approximation, the system obeys the Non-Linear Schrödinger Equation, a known integrable PDE whose soliton solutions provide the lack of reflection. And finally, the linearized mean-field approximation, the BdG system, still has the reflectionless property. Whether SUSY is responsible for the reflectionless nature of the system at the level of linearized mean-field is not yet known.

2.3 Integrable Partial Differential Equations, Solitons, and the Inverse Scattering Method

2.3.1 The Inverse Scattering Method

The inverse scattering method (ISM) is a method developed for solving the initial value problem of integrable nonlinear partial differential equations (NPDE's). Given the initial profile $u(x, 0)$ one can use the ISM to find the time-evolved profile $u(x, t)$.

Associated with each integrable NPDE are two linear differential operators, $\hat{L}$ and $\hat{M}$. The solution $u(x, t)$ of the NPDE appears as a parameterizing field in $\hat{L}$ and $\hat{M}$ [18, 7]:

$$\hat{L} = \hat{L}(u(x,\, t)); \qquad \hat{M} = \hat{M}(u(x,\, t)). \tag{2.71}$$

Associated with $\hat{L}$ is a Hamiltonian $\hat{H}$ that defines a spectral problem

$$\hat{H}\psi = \lambda\psi; \qquad \frac{\partial}{\partial t}\lambda = 0, \tag{2.72}$$

i.e., $\hat{H}$ is *time-independent* in the Schrödinger representation. In the case of KdV, $\hat{L} = \hat{H}$. For sine-Gordon, $\hat{L} = \hat{\sigma}_z(\hat{H} - \lambda)$ [24] . The fact that the eigenvalues of ψ do not change in time, even as the potential $u(x,t)$ in $\hat{L}$ evolves in time, was one of the key discoveries leading to the ISM [18]. The eigenvector ψ evolves in time through $\hat{M}$:

$$\psi_t = \hat{M}\psi, \tag{2.73}$$

i.e., $\hat{M}$ is *time-dependent* in the Schrödinger representation. The Lax equation, or Lax representation of the integrable NPDE, is the Heisenberg equation of motion for $\hat{L}$, generated by $\hat{M}$:

$$\frac{\partial}{\partial t}\hat{L} = [\hat{M},\, \hat{L}] \quad \Longrightarrow \quad \text{PDE-in-question}[u(x,\, t)]. \tag{2.74}$$

These relationships suggest a procedure for finding $u(x,t)$ given $u(x,0)$. First, find the scattering data at $t = 0$, $S(0)$, for $\hat{H}(u(x,0))$, consisting of the set $\{R(k;0), \kappa_n, c_n(0)\}$, where $R(k;0)$ are the reflection coefficients at time $t = 0$, $c_n(0)$ are the tails of the bound states at $t = 0$, and κ_n are the discrete eigenvalues (independent of time), as described in Section 2.2.1:

$$S(0) = \{R(k;0), \kappa_n, c_n(0)\}. \tag{2.75}$$

Next, use the time evolution of ψ, $\psi_t = \hat{M}\psi$, to find the scattering data at time t, $S(t)$:

$$S(t) = \{R(k;t), \kappa_n, c_n(t)\}. \tag{2.76}$$

Lastly, invert the scattering data at time t to find $u(x,t)$. This last step is the true “inverse scattering" problem, that is, determining the potential from the scattering data. As Drazin and Johnson explain, “in physical terms the problem is essentially finding the shape (or perhaps mass distribution) of an object which is mechanically vibrated, from a knowledge of all the sounds that it makes, i.e., from the energy or amplitude at each frequency [7]." The inversion formulae were discovered by Gel’fand and Levitan in 1955 [18], and recast in the form presented here by Marchenko [7]. Inversion of the scattering data involves solving a nontrivial integral equation, usually referred to as the Marchenko equation, whose kernel is constructed from the scattering data [7]:

$$K(x,z;t) + F(x+z;t) + \int_x^\infty K(x,y;t)F(y+z;t)dy = 0. \tag{2.77}$$

The function F is defined as

$$F(X;t) = \sum_{n=1}^{N} c_n(t)^2 e^{-\kappa_n X} + \frac{1}{2\pi}\int_{-\infty}^{\infty} R(k;t)e^{ikX}dk, \tag{2.78}$$

and the potential at time t is:

$$u(x,t) = -2\frac{d}{dx}K(x,x;t). \tag{2.79}$$

The above analysis is correct for the KdV equation which has one component (first-order in time). For two-component (second-order in time) nonlinear PDE’s like the sine-Gordon equation, the Marchenko integral equation becomes a vector equation with two-components [7].

The inverse scattering method is very similar to the use of Fourier transforms to solve linear partial differential equations [7]. To solve a linear PDE, the initial profile $u(x,0)$ is projected onto a Fourier basis $A(k)$ using a Fourier Transform (FT). Each Fourier component evolves in time as $A(k)e^{-i\omega(k)t}$, where $\omega(k)$ is given by the dispersion relation for the

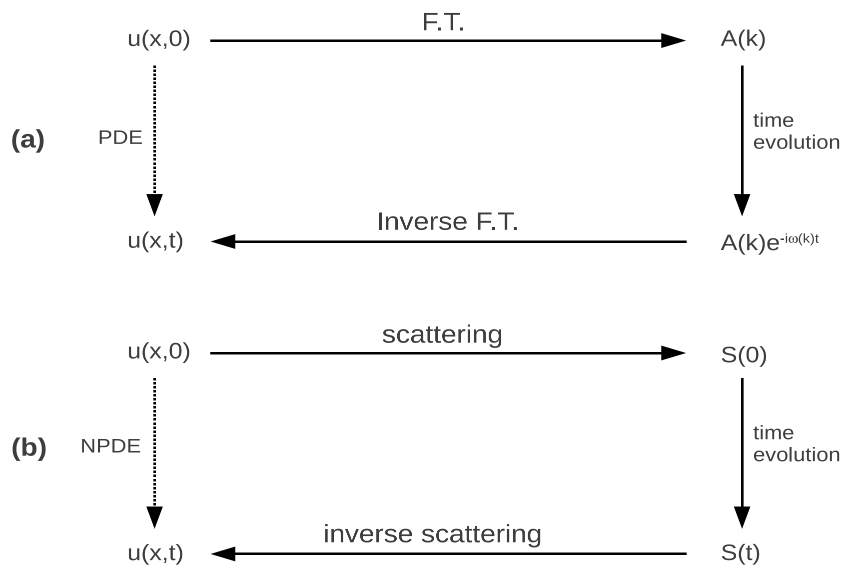

Figure 2.5: Comparison of Fourier Transform and Inverse Scattering Method: (a) Fourier transform method for solving linear PDE's and (b) inverse scattering method for solving nonlinear PDE's.

PDE in question. The inverse Fourier transform (IFT) can then be used to find $u(x,t)$. A comparison of the two methods is illustrated Figure 2.5

2.3.2 Korteweg-de Vries equation

The KdV equation is given by:

$$\frac{\partial}{\partial t}U + 6U\frac{\partial}{\partial x}U + \frac{\partial^3}{\partial x^3}U = 0; \qquad U = U(x,\, t), \tag{2.80}$$

where we have sent $u \to U$. The $\hat{H}$ operator is the Hamiltonian of the time-independent Schrödinger equation, and $\hat{H} = \hat{L}$:

$$\hat{L} = \hat{H} = -(d^2/dx^2) + V(x\,|t); \qquad V(x\,|t) = -U(x,\, t). \tag{2.81}$$

The notation $V(x\,|t)$ and $\psi(x|t)$ signifies that t is simply a *parameter* of the potential $V(x|t)$ and eigenvector $\psi(x|t)$, unlike the field $U(x,t)$ which is truly evolving in time under the KdV equation. Furthermore, "time" here is not the "time" that appears in the time-dependent Schrödinger equation.

The direct scattering problem for $\hat{H}$ is equivalent to the energy-eigenvalue problem for 1D quantum mechanics:

$$\hat{H}\psi = \lambda\psi; \qquad \psi = \psi(x\,|t). \tag{2.82}$$

The $\hat{M}$-operator for the KdV equation is given by:

$$\hat{M} = -4\frac{\partial^3}{\partial x^3} - 3\left\{U\frac{\partial}{\partial x} + \frac{\partial}{\partial x}(U\cdot)\right\}, \tag{2.83}$$

which time-evolves the scattering states as

$$\frac{\partial}{\partial t}\psi(x,\,t) = \hat{M}\psi(x,\,t); \qquad \psi(x,\,t) \equiv \psi(x\,|t). \tag{2.84}$$

The Lax representation of the KdV equation is [18, 7, 24]:

$$\frac{\partial}{\partial t}\hat{L} = [\hat{M},\,\hat{L}] \quad \Longrightarrow \quad \frac{\partial}{\partial t}U + 6U\frac{\partial}{\partial x}U + \frac{\partial^3}{\partial x^3}U = 0. \tag{2.85}$$

The inverse scattering problem for the KdV equation is particularly confusing for those familiar with quantum mechanics. In quantum mechanics, the Hamiltonian determines the energy eigenvalues *and* generates the time evolution of eigenfunctions. In the Lax formulation for KdV, $\hat{H}$ for the time-independent Schrödinger equation is the $\hat{H}$ operator which defines the scattering problem, but the scattering states are evolved in "time" by another operator $\hat{M}$ which has nothing to do with quantum mechanics. Again, "time" does not even mean the same thing here as it does in non-stationary quantum mechanics.

2.3.3 sine-Gordon equation

The sine-Gordon equation is given by:

$$\frac{\partial^2}{\partial\zeta\partial\eta}\Phi = \sin(\Phi), \tag{2.86}$$

where we have sent $u \to \Phi$; $x \to \zeta$; $t \to \eta$. The covariant form of sine-Gordon is:

$$\frac{\partial^2}{\partial\tilde{x}^2}U - \frac{\partial^2}{\partial\tilde{t}^2}U = \sin(U), \tag{2.87}$$

with the connection between the two versions being:

$$U(\tilde{x},\, \tilde{t}) = \Phi(\zeta = \frac{1}{2}(\tilde{x}+\tilde{t}),\, \eta = \frac{1}{2}(\tilde{x}-\tilde{t})). \tag{2.88}$$

The $\hat{H}$ operator for sine-Gordon is

$$\hat{H} = (d/d\zeta)\,\hat{\sigma}_z - v(\zeta\,|\eta)\,\hat{\sigma}_x; \qquad v(\zeta\,|\eta) = \frac{1}{2}\frac{\partial}{\partial\eta}\Phi(\zeta,\, \eta), \tag{2.89}$$

defining the scattering problem:

$$\hat{H}\psi = \lambda\psi; \qquad \psi = \begin{pmatrix} \psi_1(\zeta\,|\eta) \\ \psi_2(\zeta\,|\eta) \end{pmatrix}. \tag{2.90}$$

For sine-Gordon there is a family of L-operators:

$$\hat{L} = \hat{L}(\lambda) = \hat{\sigma}_z(\hat{H} - \lambda), \tag{2.91}$$

and a family of M-operators:

$$\hat{M} = \hat{M}(\lambda) = \frac{1}{4\lambda}\left\{\cos[\Phi]\hat{\sigma}_z - \sin[\Phi]\hat{\sigma}_x\right\}, \tag{2.92}$$

since $\hat{L}$ and $\hat{M}$ depend on λ. The Lax equation is:

$$\frac{\partial}{\partial\eta}\hat{L}(\lambda) = [\hat{M}(\lambda),\, \hat{L}(\lambda)] \quad \overset{\forall\lambda}{\Longrightarrow} \quad \frac{\partial^2}{\partial\zeta\partial\eta}\Phi = \sin(\Phi), \tag{2.93}$$

and the scattering states evolve in time as:

$$\frac{\partial}{\partial\eta}\psi(\zeta,\,\eta) = \hat{M}\psi(\zeta,\,\eta); \qquad \psi(\zeta,\,\eta) \equiv \psi(\zeta\,|\eta), \tag{2.94}$$

where $\hat{M}$ and ψ are taken at the same λ [24].

2.3.4 Reflectionless $\hat{H}$ Operators, Solitons, and Supersymmetry

If the $\hat{H}$ operator for a NPDE is reflectionless, then $R(k) = 0$ for all time in the direct scattering problem. In this case, the Marchenko integral equation (2.77) greatly simplifies [7, 18]. The function $F(X,t)$ (2.78) contains only a sum over the discrete eigenvalues of ψ, and the Marchenko equation reduces to a system of *algebraic* equations.

Furthermore, the solutions to the nonlinear PDE in question are *solitons*, which are particle-like solitary waves which propagate without dispersion and scatter elastically off one another. For the KdV equation, for example, an initial profile $u(x,0) = N(N+1)\text{sech}^2(x)$, generating a reflectionless $\hat{H}$ operator (the Schrödinger operator with $V(x) = -u(x,t)$), leads to an N-soliton solution. Supersymmetry can thus be used to generate these multi-soliton solutions of the KdV equation, since potentials of the form $V(x) = -N(N+1)\text{sech}^2(x)$ are linked by a supersymmetric chain [6, 12, 13, 14].

It is unknown if supersymmetry can be used to generate N-soliton solutions of other integrable NPDE's. Interestingly, the $n = 1$ member of the Akulin Hamiltonians is a version of the $\hat{H}$ operator for the one-soliton solution of the sine-Gordon equation. We suspect that the nth Akulin Hamiltonian will lead to an n-soliton solution of the sine-Gordon equation, but we were unable to confirm this hypothesis by the time this thesis was written.

CHAPTER 3

AKULIN'S HAMILTONIANS

3.1 Scattering Properties of Akulin's Hamiltonians

We now analyze the family of Hamiltonians

$$H_n = \sigma_z \partial_x - \sigma_x n / \cosh(x) \quad . \tag{3.1}$$
$$n = \ldots, -3, -2, -1, 0, +1, +2, +3, \ldots,$$

which we refer to as Akulin's Hamiltonians. We will show that the family is linked by a supersymmetric chain and since H_0 is reflectionless, every other member of the chain is also reflectionless.

We first consider the scattering problem for H_n when n is *not necessarily* an integer. This is similar to how Akulin first analyzed the problem[1] [1]. We will find the scattering states in terms of hypergeometric functions and demonstrate that when n is an integer, H_n is reflectionless. The purpose is to illustrate the analogy between the Hamiltonians H_n and potentials of the form $V(x) = -U_0 \text{sech}^2(x)$ in normal quantum mechanics, discussed in Section 2.1.5. In the latter case, we also found the scattering states in terms of hypergeometric functions which became reflectionless when $U_0 \to N(N+1)$, where N is a positive integer.

We wish to solve the scattering problem

$$H_n \psi = i\lambda\psi. \tag{3.2}$$

[1]Akulin analyzed a time-dependent two-level system, not a spatial scattering problem, but the two problems are formally analogous.

We let $\lambda \to k$, $q(x) = n/\cosh(x)$, $\psi = \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)$ and rearrange the problem to yield

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} ik & q \\ -q & -ik \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}; \qquad \Rightarrow \qquad \psi' = \mathcal{L}\psi. \tag{3.3}$$

Here we briefly summarize the strategy for finding solutions to (3.3), omitting the details. It is actually much easier to solve the problem by transforming $\mathcal{L} \to \mathcal{L} - ikI$. The solutions will transform $\psi \to e^{-ikx}\psi$. In the new system, we combine the two first-order equations into a single second order equation, and change variables to $z = \frac{1}{1+e^{2x}}$. The resulting differential equation has solutions in terms of hypergeometric functions. Transforming back, we find that the four asymptotic solutions of (3.2) are:

$$\begin{aligned}
\psi_+ &= \begin{pmatrix} u_+ \\ v_+ \end{pmatrix} = \begin{pmatrix} F\left(n, -n; \frac{1}{2} - ik; \frac{1}{1+e^{2x}}\right) e^{ikx} \\ (u_+' - iku_+)/q \end{pmatrix} \\
\psi_- &= \begin{pmatrix} u_- \\ v_- \end{pmatrix} = \begin{pmatrix} F\left(n, -n; \frac{1}{2} + ik; \frac{1}{1+e^{-2x}}\right) e^{ikx} \\ (u_-' - iku_-)/q \end{pmatrix} \\
\tilde{\psi}_+ &= \begin{pmatrix} \tilde{u}_+ \\ \tilde{v}_+ \end{pmatrix} = \begin{pmatrix} -(\tilde{v}_+' + ik\tilde{v}_+)/q \\ F\left(n, -n; \frac{1}{2} + ik; \frac{1}{1+e^{2x}}\right) e^{-ikx} \end{pmatrix} \\
\tilde{\psi}_- &= \begin{pmatrix} \tilde{u}_- \\ \tilde{v}_- \end{pmatrix} = \begin{pmatrix} -(\tilde{v}_-' + ik\tilde{v}_-)/q \\ F\left(n, -n; \frac{1}{2} - ik; \frac{1}{1+e^{-2x}}\right) e^{-ikx} \end{pmatrix},
\end{aligned} \tag{3.4}$$

where F is the hypergeometric function. For each of these solutions, it is cleaner to give the second component in terms of the first component, rather than write the second component out explicitly. The solutions (3.4) are given in the form described in (2.45) so that we can analyze the scattering properties of H_n. We will analyze one of the two sets of reflection and transmission coefficients. Recall that

$$\psi_+ = T(k)\psi_- + R(k)\tilde{\psi}_-; \qquad \Rightarrow \qquad u_+ = T(k)u_- + R(k)\tilde{u}_-. \tag{3.5}$$

From the definition of the asymptotic solutions, we know that as $x \to \infty$, $u_+ \to 1$, and $\tilde{u}_- \to 0$. This implies that

$$T(k) = \frac{1}{\lim\limits_{x\to\infty} u_-} = \frac{1}{F(n,-n;\frac{1}{2}+ik,1)} = \frac{\Gamma(\frac{1}{2}+ik-n)\Gamma(\frac{1}{2}+ik+n)}{\Gamma(\frac{1}{2}+ik)^2}. \tag{3.6}$$

After some trigonometric manipulations [1] we find:

$$\begin{aligned} |T(k)|^2 &= \frac{2\cosh^2(\pi k)}{\cosh(2\pi k)+\cos(2\pi n)} \\ |R(k)|^2 &= 1-|T(k)|^2 = \frac{\sin^2(\pi n)}{\cosh^2(\pi k)}, \end{aligned} \tag{3.7}$$

and we see that if n is an integer, $|T(k)| \to 1$ and $|R(k)| \to 0$, i.e., H_n is reflectionless for all k. Just like the case of the $-N(N+1)\text{sech}^2(x)$ potentials, the "coincidence" of reflectionlessness is a manifestation of an underlying SUSY mechanism.

3.2 Akulin's Chain and its Intertwiners

We found a (non-unique) supersymmetric chain connecting all the Hamiltonians of the form (3.1). These Hamiltonians are not directly linked, however; there is an intermediate Hamiltonian, which we refer to as $H_{n+1/2}$, between H_n and H_{n+1}. In other words, H_n is the SUSY partner of $H_{n+1/2}$, and $H_{n+1/2}$ is the SUSY partner of H_{n+1}:

$$\ldots H_n \Longleftrightarrow H_{n+1/2} \Longleftrightarrow H_{n+1} \ldots \tag{3.8}$$

We found many ambiguities in generating such a chain, especially in the form of the intermediate Hamiltonians $H_{n+1/2}$. We used several symmetries of the Hamiltonians H_n to remove many of these ambiguities. Consider the following four transformations:

$$\mathcal{T}_I \equiv \bigcirc \tag{3.9}$$
$$\mathcal{T}_x \equiv \hat{R}\sigma_x \cdot \bigcirc \cdot \sigma_x^{-1}\hat{R}^{-1}$$
$$\mathcal{T}_y \equiv (i\sigma_x) \cdot \bigcirc \cdot (i\sigma_y)^{-1}$$
$$\mathcal{T}_z \equiv \hat{R}\sigma_z \cdot \bigcirc \cdot \sigma_z^{-1}\hat{R}^{-1}.$$

$\hat{R}$ refers to the operation $x \to -x$. Under these actions, the Hamiltonians transform as

$$H_n \overset{\mathcal{T}_x}{\leftrightarrow} +H_n \tag{3.10}$$
$$H_n \overset{\mathcal{T}_y}{\leftrightarrow} -H_n$$
$$H_n \overset{\mathcal{T}_z}{\leftrightarrow} -H_n \quad .$$

Forming the group $Z_2 \times Z_2 = \mathrm{Dih}_2$, consisting of $180°$ rotations about coordinate axes. If we define

$$s_{\mathcal{T}} = \frac{\mathcal{T}[H]}{H} \quad , \tag{3.11}$$

then we can write the following set of four chains for the Akulin system:

$$\tag{3.12}$$

$$\begin{array}{lllllll} H_n & = \ldots = & s_{\mathcal{T}}\left(\mathcal{T}[B_n^{(+)}]\mathcal{T}[A_n^{(+)}] + \epsilon_n^{(+)}\right) & & & & \\ H_{n+1/2} & = & s_{\mathcal{T}}\left(\mathcal{T}[A_n^{(+)}]\mathcal{T}[B_n^{(+)}] + \epsilon_n^{(+)}\right) & = & s_{\mathcal{T}}\left(\mathcal{T}[A_{n+1}^{(-)}]\mathcal{T}[B_{n+1}^{(-)}] + \epsilon_{n+1}^{(-)}\right) & & \\ H_{n+1} & & & = & s_{\mathcal{T}}\left(\mathcal{T}[B_{n+1}^{(-)}]\mathcal{T}[A_{n+1}^{(-)}] + \epsilon_{n+1}^{(-)}\right) & = & \ldots \end{array}$$

where $\mathcal{T}$ can be chosen from $\{\mathcal{T}_I,\ \mathcal{T}_x,\ \mathcal{T}_y,\ \mathcal{T}_z\}$. The untransformed SUSY factors are

$$\begin{aligned}
B_n^{(+)} &= \begin{pmatrix} 1 & -\frac{1}{2}n/\cosh(x) \\ (-1)^n\cosh(x)-\sinh(x) & -\partial_x - \frac{1}{2}\left((-1)^n(n+1)+\tanh(x)\right) \end{pmatrix} \qquad (3.13)\\
A_n^{(+)} &= \begin{pmatrix} +\partial_x - \frac{1}{2}\left((-1)^n(n+1)+\tanh(x)\right) & -\frac{1}{2}n/\cosh(x) \\ (-1)^n\cosh(x)-\sinh(x) & 1 \end{pmatrix}\\
A_n^{(-)} &= (-1)\times\begin{pmatrix} -\partial_x + \frac{1}{2}\left((-1)^n(n-1)-\tanh(x)\right) & +\frac{1}{2}n/\cosh(x) \\ (-1)^n\cosh(x)+\sinh(x) & 1 \end{pmatrix}\\
B_n^{(-)} &= \begin{pmatrix} 1 & +\frac{1}{2}n/\cosh(x) \\ (-1)^n\cosh(x)+\sinh(x) & \partial_x + \frac{1}{2}\left((-1)^n(n-1)-\tanh(x)\right) \end{pmatrix} \quad ,
\end{aligned}$$

with factorization constants

$$\begin{aligned}
\epsilon_n^{(+)} &= (-1)^n(n+\frac{1}{2}) \\
\epsilon_n^{(-)} &= (-1)^n(n-\frac{1}{2}) \quad .
\end{aligned} \qquad (3.14)$$

We also used the following transformation property of the Hamiltonians to help fix the form of the SUSY factors (3.13):

$$H_n \overset{\mathcal{T}_{\text{inv.}}}{\Leftrightarrow} -H_{-n}; \qquad \mathcal{T}_{\text{inv.}} \equiv \hat{R} \bigcirc \hat{R}^{-1}. \qquad (3.15)$$

Requiring that the A's and B's transform as

$$\begin{aligned}
B_{+n}^{(+)} &\overset{\mathcal{T}_{\text{inv.}}}{\Leftrightarrow} +B_{-n}^{(-)} \\
A_{+n}^{(+)} &\overset{\mathcal{T}_{\text{inv.}}}{\Leftrightarrow} -A_{-n}^{(-)} \quad ,
\end{aligned} \qquad (3.16)$$

makes the symmetry in (3.15) manifest. Including inversion with the transformations (3.9) forms the group $Z_2 \times Z_2 \times Z_2$, whose actions are combinations of inversion and the $180°$ rotations about the coordinate axes.

The intertwiners linking eigenstates of H_n to eigenstates of H_{n+1} $(\Upsilon_{n+1\leftarrow n})$, and the intertwiners linking eigenstates of H_n to eigenstates of H_{n-1} $(\Upsilon_{n-1\leftarrow n})$ have the following defining relationships:

$$H_{n+1}\Upsilon_{n+1\leftarrow n} = \Upsilon_{n+1\leftarrow n}H_n \tag{3.17}$$
$$H_{n-1}\Upsilon_{n-1\leftarrow n} = \Upsilon_{n-1\leftarrow n}H_n \quad .$$

From the chain (3.12) we see that the intertwiners are given by

$$\Upsilon_{n+1\leftarrow n} = v_{\mathcal{T}}\left(\mathcal{T}[B^{(-)}_{n+1}]\mathcal{T}[A^{(+)}_n]\right); \qquad \Upsilon_{n-1\leftarrow n} = v_{\mathcal{T}}\left(\mathcal{T}[B^{(+)}_{n-1}]\mathcal{T}[A^{(-)}_n]\right), \tag{3.18}$$

where

$$v_{\mathcal{T}} = \frac{\mathcal{T}[\Upsilon]}{\Upsilon} \quad . \tag{3.19}$$

The intertwiners have the form:

$$\Upsilon_{n+1\leftarrow n} = \partial_x - (n+\frac{1}{2})\tanh(x) + \frac{1}{2}(i\sigma_y)/\cosh(x)$$
$$\Upsilon_{n-1\leftarrow n} = \partial_x + (n-\frac{1}{2})\tanh(x) - \frac{1}{2}(i\sigma_y)/\cosh(x) \quad , \tag{3.20}$$

and transform under the $\mathcal{T}$ operations as

$$\Upsilon_{n\pm1\leftarrow n} \overset{\mathcal{T}_x}{\leftrightarrow} -\Upsilon_{n\pm1\leftarrow n} \tag{3.21}$$

$$\Upsilon_{n\pm1\leftarrow n} \overset{\mathcal{T}_y}{\leftrightarrow} +\Upsilon_{n\pm1\leftarrow n}$$

$$\Upsilon_{n\pm1\leftarrow n} \overset{\mathcal{T}_z}{\leftrightarrow} -\Upsilon_{n\pm1\leftarrow n}$$

$$\Upsilon_{n\pm1\leftarrow n} \overset{\mathcal{T}_{\text{inv.}}}{\leftrightarrow} -\Upsilon_{-n\mp1\leftarrow -n} \quad . \tag{3.22}$$

Fixing the inversion properties of the SUSY factors (3.16) is also responsible for the inversion symmetry of the intertwiners, (3.22).

The picture that emerges from the preceding discussion is the following. Using the symmetries of our Hamiltonians, we are able to link them all through four different supersymmetric chains, with each individual chain resulting from one of the symmetry operations under which they are invariant (up to a constant ±1). Each of the four chains links H_n to H_{n+1} via a different intermediate Hamiltonian $H_{n+1/2}$. The results are summarized in Fig. 3.1.

It is a mystery why the intertwiners (3.20) have such a simple form, and why there is only *one* intertwiner connecting H_n to H_{n+1}, given that there are *four* supersymmetric chains connecting H_n to H_{n+1}. Since the eigenstates of H_n are doubly-degenerate, a one-to-one map between eigenstates of H_n and H_{n+1} is not necessary. It is possible that the simplicity of the intertwiners is the result of more hidden symmetries or relationships between the SUSY factors.

Discovery of any such relationships between the A's and B's is of the utmost importance for future SUSY work, since it might allow for the reduction of the factorization problem into a smaller number of steps. For instance, factorizing a Hamiltonian in 1D quantum mechanics is greatly simplified by writing the Hamiltonian in the form $H = A^\dagger A + \epsilon$. The problem of factorization is then reduced to finding solutions to a single differential equation for the superpotential $W(x)$ (see Section 2.1.2). We were unable to find any similar

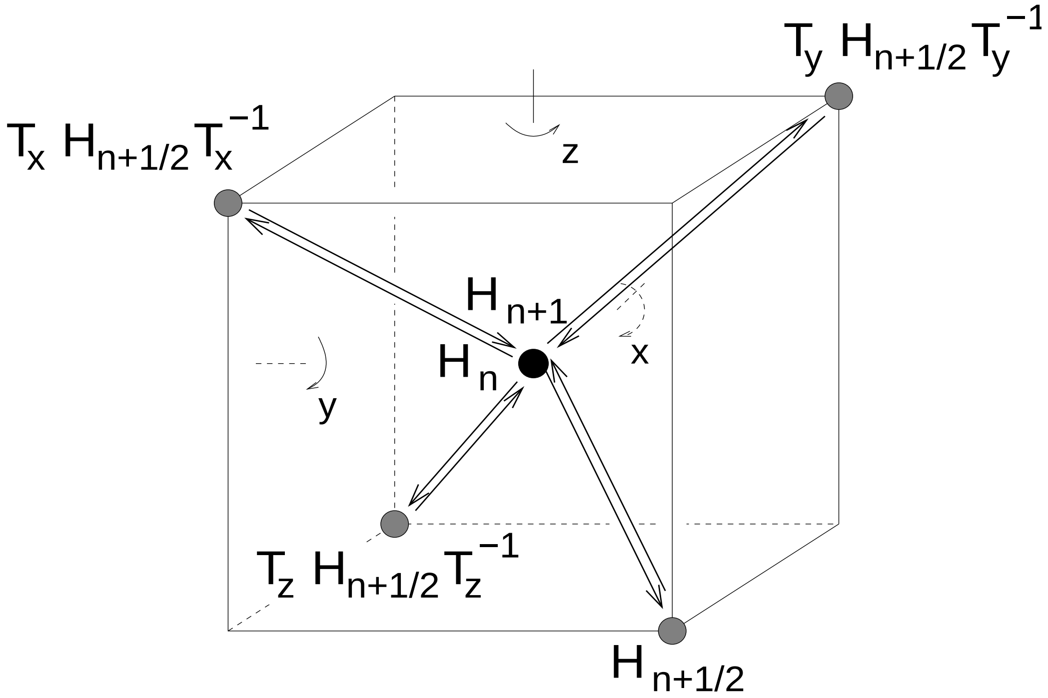

Figure 3.1: Four SUSY Chains for the Akulin Hamiltonians: Thick arrows correspond to the QMSUSY connections. The 180° rotations about the coordinate axes OX, OY, and OZ correspond to the transformations $\mathcal{T}_x \equiv \hat{R}\sigma_x \cdot \bigcirc \cdot \sigma_x^{-1}\hat{R}^{-1}$, $\mathcal{T}_y \equiv (i\sigma_x) \cdot \bigcirc \cdot (i\sigma_y)^{-1}$, and $\mathcal{T}_z \equiv \hat{R}\sigma_z \cdot \bigcirc \cdot \sigma_z^{-1}\hat{R}^{-1}$ respectively.

relationships between the SUSY factors (3.13) for Akulin's SUSY chain, but they may still exist. Certainly the mystery of the intertwiners described above, as well as some ambiguities discussed in Section 5.1 suggest that there is more to Akulin's SUSY chain than we currently understand.

CHAPTER 4

APPLICATIONS

4.1 Application 1: Inversionless Laser Pulse for a Two Level Atom

The first system to which we can apply our SUSY decomposition of the Akulin Hamiltonians comes from atomic physics. This system is the one analyzed by Akulin. Consider a two-level atom subjected to a time-dependent pulse of the form $V_{eg}(t) = V/\cosh(t/\tau)$ and detuning Δ. Here V is the amplitude of the pulse, τ is its duration, and $|e\rangle$ and $|g\rangle$ are the excited and ground states, respectively. The time-dependence of this system can be solved exactly in terms of hypergeometric functions (the analysis is identical to that in Section 3.1), and it is known that for specific values of the pulse amplitude, the transition probability is zero regardless of the detuning choice Δ [1]. These amplitudes are given by $V = n\hbar/\tau$, where n is an integer. If we represent the probability amplitudes of the ground and excited states by ψ_g and ψ_e, respectively, the dynamics of the system will obey

$$\begin{aligned} i\frac{d}{dt}\psi_g &= +\frac{\Delta}{2}\psi_g + \frac{n}{\tau\cosh(t/\tau)}\psi_e \\ i\frac{d}{dt}\psi_e &= +\frac{n}{\tau\cosh(t/\tau)}\psi_g - \frac{\Delta}{2}\psi_e \quad . \end{aligned} \tag{4.1}$$

The remarkable property of this pulse is that if the population is prepared entirely in the ground state ψ_g at $t \to -\infty$, then the whole population will return to the ground state for $t \to +\infty$, for *any* non-zero value of Δ. Now, we can regard the excited state population after the pulse is applied (generally present, but absent in our case) as a reflected wave in a scattering problem. Similarly, the ground state populations before and after the pulse can be regarded as the transmitted and incident waves, respectively (note the order). To formalize the analogy, we make the substitution $x = -t/\tau$, $u = \psi_g$, $v = -\psi_e$, $\lambda = \Delta\tau/2$. (Note that x is a *dimensionless* coordinate.) Now the dynamics of the system can be rewritten as a two-component spatial eigenvalue problem involving a 2×2 Hamiltonian, H_{lp}:

$$\begin{aligned} \begin{pmatrix} -i\frac{d}{dx} & \frac{n}{\cosh(x)} \\ -\frac{n}{\cosh(x)} & i\frac{d}{dx} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} &= \lambda \begin{pmatrix} u \\ v \end{pmatrix} \\ \Rightarrow \qquad H_{lp}\psi_{lp} &= \lambda\psi_{lp}. \end{aligned} \tag{4.2}$$

The subscript lp is meant to indicate the meaningful objects in the laser-pulse problem.

Let us look at the case when $n = 1$. We will classify the eigenstates by their wavevector k and by the eigenvalues of $\hat{H}$. For each k we have two eigenvalues $\lambda = \pm k$:

$$\begin{aligned} |\psi_k\rangle^{(\lambda=+k)} &\propto \begin{pmatrix} ik - \frac{\tanh(x)}{2} \\ \frac{-i}{2\cosh(x)} \end{pmatrix} e^{ikx} \\ |\psi_k\rangle^{(\lambda=-k)} &\propto \begin{pmatrix} \frac{-i}{2\cosh(x)} \\ ik - \frac{\tanh(x)}{2} \end{pmatrix} e^{ikx} \quad . \end{aligned} \tag{4.3}$$

We can see from the eigenstates that $\hat{H}$ is reflectionless. If one replaced the off-diagonal perturbation $\frac{1}{\cosh(x)}$ in (4.2) by a perturbation of a general position, the scattering state $|\psi_k\rangle^{(\lambda=+k)}$ (whose incident internal state is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$) would show a reflected wave, $\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-ikx}$, corresponding to the internal state $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The peculiar property of the $\frac{1}{\cosh(x)}$ perturbation is exactly the absence of the reflected wave. The second scattering state, $|\psi_k\rangle^{(\lambda=-k)}$, shows the same phenomenon, with the internal states reversed.

We can apply a simple transformation to map the laser-pulse eigenvalue problem to the eigenvalue problem associated with the Akulin Hamiltonians, H_n discussed at length in

Section 3.1:

$$H_n = iUH_{lp}U^{-1}; \qquad \psi_n = U\psi_{lp}; \qquad U = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}, \tag{4.4}$$

and $U^{-1} = U^{\dagger}$. Under these transformations the two eigenvalue problems are equivalent:

$$H_{lp}\psi_{lp} = \lambda\psi_{lp} \qquad \Longleftrightarrow \qquad H_n\psi_n = i\lambda\psi_n. \tag{4.5}$$

The supersymmetric chains (3.12) linking each member H_n to the inherently reflectionless Hamiltonian H_0 completely explains the inversionless property of the two-level atomic system discussed above.

4.2 Application 2: One-soliton Solution of Sine-Gordon Equation

As described in Section 2.3.3, the $\hat{H}$ operator for sine-Gordon is

$$\hat{H} = (d/d\zeta)\,\hat{\sigma}_z - v(\zeta\,|\eta)\,\hat{\sigma}_x; \qquad v(\zeta\,|\eta) = \frac{1}{2}\frac{\partial}{\partial\eta}\Phi(\zeta,\,\eta), \tag{4.6}$$

defining the scattering problem:

$$\hat{H}\psi = \lambda\psi; \qquad \psi = \begin{pmatrix} \psi_1(\zeta\,|\eta) \\ \psi_2(\zeta\,|\eta) \end{pmatrix}. \tag{4.7}$$

It is well-known that the sine-Gordon equation is integrable, and it supports many-soliton solutions; each such solution, if substituted to the scattering problem (4.7), generates a family of reflectionless problems, parametrized by η [25]. The simplest example is a single-soliton (antikink) solution:

$$\Phi(\zeta,\,\eta) = 4\arctan(\exp(\alpha\zeta + \eta/\alpha)) \quad .$$

Interestingly, after a trivial substitution $\zeta = x/\alpha$, $\eta = 0$, the single-soliton Hamiltonian for sine-Gordon (4.6), becomes exactly the Akulin Hamiltonian with $n = 1$:

$$\hat{H}_{\text{one-soliton(SG)}} \quad \xrightarrow{\zeta=x/\alpha,\eta=0} \quad \hat{H}_{1(\text{Akulin})}. \tag{4.8}$$

Similarly, the case when $n = -1$ results in the single kink-soliton solution. Recall, from Section 2.3.4 that it is the reflectionless property of particular $\hat{H}$ operators that lead to soliton solutions of NPDE's; (4.8) shows that supersymmetry (3.12) is responsible for the reflectionless nature of the one-soliton $\hat{H}$ operators for sine-Gordon.

CHAPTER 5

AREAS OF FUTURE RESEARCH

5.1 Ambiguities in Factors of SUSY Chain for Akulin's Hamiltonians

We were unable to remove all of the ambiguities in defining the supersymmetric chain (3.12), even after fixing the property (3.16). The following replacements would not affect H_n, the intertwiners, nor the property (3.16): For each group of the for SUSY operators, $B_n^{(+\text{sign}(n))}$, $A_n^{(+\text{sign}(n))}$, $B_{n+\text{sign}(n)}^{(-\text{sign}(n))}$, $A_{n+\text{sign}(n)}^{(-\text{sign}(n))}$,

- $\mathcal{T} \rightarrow \mathcal{T}_{|n+\text{sign}(n)/2|}$;
- $$\begin{aligned} B_n^{(+\text{sign}(n))} &\rightarrow B_n^{(+\text{sign}(n))} P \\ B_{n+\text{sign}(n)}^{(-\text{sign}(n))} &\rightarrow B_{n+\text{sign}(n)}^{(-\text{sign}(n))} P \\ A_n^{(+\text{sign}(n))} &\rightarrow P^{-1} A_n^{(+\text{sign}(n))} \\ A_{n+\text{sign}(n)}^{(-\text{sign}(n))} &\rightarrow P^{-1} A_{n+\text{sign}(n)}^{(-\text{sign}(n))} \end{aligned} ,$$
  where P is any 2×2 constant matrix.

It is also not clear how to make the "rotation" properties of the Hamiltonians and intertwiners, (3.11) and (3.22), manifest; either using the remaining ambiguities, or by a clever analysis of the relationships between the suggested A's and B's (3.13).

5.2 n-Soliton Solutions of Sine-Gordon Equation

As discussed in Section 2.3.4, SUSY chains for reflectionless $\hat{H}$ operators of the KdV equation generate multi-soliton solutions, where at $t = 0$ all of the solitons are located at the

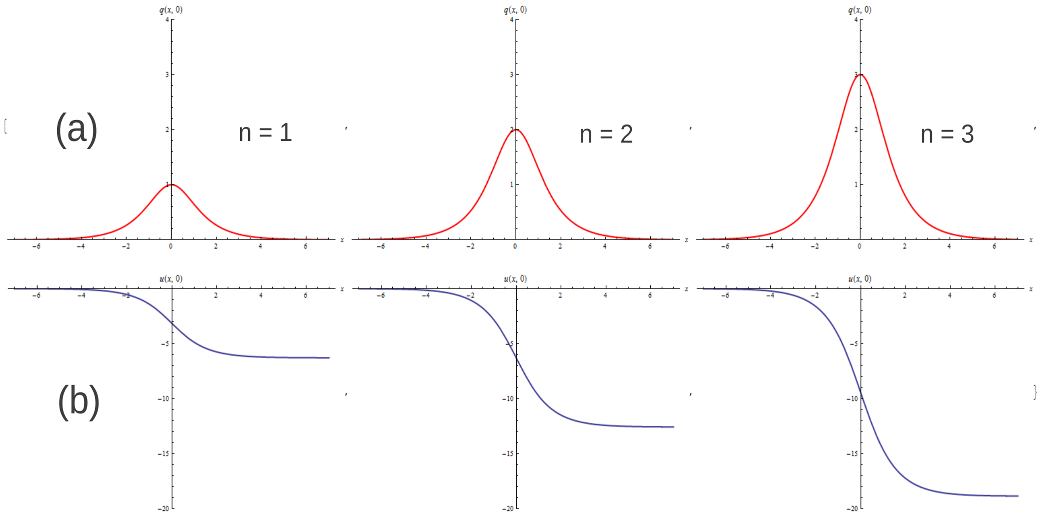

Figure 5.1: Reflectionless initial conditions for sine-Gordon, corresponding to the $n = 1, 2, 3$ Akulin Hamiltonians. (a) Plots the “potentials" $v(\zeta\ 0) = n/\cosh(\zeta)$ in red that appear in the direct-scattering matrix H. (b) Plots the initial values of the field $\Phi(\zeta, 0) = -4n\tan^{-1}(e^{\zeta})$ in blue which will evolve under sine-Gordon. The $n = 1$ case is the well-known anti-kink soliton; it is unknown if higher values of n lead to multi-soliton solutions.

origin. We demonstrate in Section 4.2 that the $n = 1$ member of the Akulin Hamiltonians is the $\hat{H}$ operator for the sine-Gordon equation associated with the one-soliton solution when $\eta = 0$, and the soliton is located at the origin. Using KdV as a guide [6, 12, 13, 14], we suspect that each member of the Akulin chain $\hat{H}_n$ is the $\hat{H}$ operator for n-soliton solutions of sine-Gordon. The first three corresponding “potentials" and initial conditions are shown in Figure 5.1. We were unable to find the appropriate n-soliton solutions (many are known) to match the forms of H_n. Furthermore, we have not yet been able to propagate the initial state contained in H_n via the inverse scattering method to see if an n-soliton solution results. We also suspect that it might be possible to use SUSY at the level of the Lax formulation to generate soliton solutions of any integrable PDE.

5.3 Bogoliubov-de Gennes System for 1D attractive Bose Condensate

As described in Section 2.2.3 the BdG system for a solitonic Bose Condensate is reflectionless [22, 23]. While it is unknown if a SUSY decomposition exists, we have found an intertwiner between the Liouvillian $\hat{\mathcal{L}}$ and a potential-free Liouvillian $\hat{\mathcal{L}}_0$. The Liouvillian is defined in (2.66), and for simplicity we set $l = 1$. We get

$$\hat{\mathcal{L}} = \begin{pmatrix} -\frac{1}{2}\partial_x^2 - 2\mathrm{sech}^2(x) + \frac{1}{2} & -\mathrm{sech}^2(x) \\ \mathrm{sech}^2(x) & \frac{1}{2}\partial_x^2 + 2\mathrm{sech}^2(x) - \frac{1}{2} \end{pmatrix}. \tag{5.1}$$

$\hat{\mathcal{L}}$ is intertwined with a potential-free Liouvillian $\hat{\mathcal{L}}_0$, via

$$\hat{\mathcal{L}}\Upsilon = \Upsilon\hat{\mathcal{L}}_0, \tag{5.2}$$

where

$$\hat{\mathcal{L}}_0 = \begin{pmatrix} -\frac{1}{2}\partial_x^2 + \frac{1}{2} & 0 \\ 0 & \frac{1}{2}\partial_x^2 - \frac{1}{2} \end{pmatrix}, \qquad \Upsilon = \begin{pmatrix} \hat{f} & \hat{g} \\ \hat{g} & \hat{f} \end{pmatrix}, \tag{5.3}$$

and

$$\begin{aligned} \hat{f} &= \partial_x^4 + (1 - 2\tanh(x))\partial_x^3 + (\tanh(x) - 1)^2\partial_x^2 + (1 - \mathrm{sech}^2(x) - 2\tanh(x))\partial_x + \tanh^2(x) \\ \hat{g} &= -\mathrm{sech}^2(x)(\partial_x^2 + \partial_x + 1). \end{aligned} \tag{5.4}$$

5.4 Akulin's Hamiltonians as Linearizations

Just as the bosonic Bogoliubov-de Gennes Liouvillian represents the linearization of the nonlinear Schrödinger equation, it is possible that one or more of the Akulin Hamiltonians are a linearization of an integrable NPDE. It is even possible, of course, that this integrable PDE has not yet been discovered.

5.5 Do all Cases of Scattering Without Reflection Have a SUSY Mechanism?

The ultimate question is, of course, whether SUSY is responsible for all cases of scattering without reflection. As discussed in Section 2.1.7, a major step towards answering this question would be a proof that there are no SUSY-free intertwiners. If supersymmetry is responsible for all cases of scattering without reflection, it might reflect a deeper connection between SUSY and integrability.

CHAPTER 6

CONCLUSION

We have presented a new case demonstrating the connection between supersymmetry, reflectionless scattering, and soliton solutions of an integrable nonlinear partial differential equation. In the previously-known case, a supersymmetric connection to free space explains the reflectionless nature of quantum-mechanical potentials of the form $V_N(x) = -N(N+1)\text{sech}^2(x)$. These potentials lead to N-soliton solutions of the KdV equation when they are used as its initial values $U(x, 0)$. For the case of Akulin's Hamiltonians H_n presented in this thesis, a supersymmetric connection to free space is also the mechanism responsible for their reflectionless scattering. Furthermore, each H_n represents a reflectionless direct-scattering problem for the sine-Gordon equation, and at least in the case of $n = \pm 1$, leads to soliton solutions. We suspect that every other H_n leads to multi-soliton solutions of sine-Gordon. Additionally, the SUSY connection of the Hamiltonians H_n to free space explains why laser pulses of the form $V(t) = (n\hbar/\tau)/\cosh(t/\tau)$ result in no population inversion for a two-level atom, for any value of the laser detuning.

What we have accomplished, however, is only a further *demonstration* of the connection between SUSY, reflectionless scattering, and integrable NPDE's. We still do not know if SUSY is responsible for all cases of reflectionless scattering, or why the cause is supersymmetry and not simply a SUSY-free intertwiner. We understand that SUSY explains reflectionless scattering, and that reflectionless scattering leads to solitons of NPDE's, but we do know if the connection originates at a deeper level. It does appear that supersymmetric chains are related to multi-soliton solutions for integrable nonlinear PDE's; perhaps this mechanism can be understood at the level of the Lax formulation of the inverse scattering method. There are likely more reflectionless Hamiltonians to be discovered, and definitely more integrable nonlinear PDE's to be examined for a SUSY-soliton connection. Certainly, the subject is ripe for future research.

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