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Hypoelliptic multiscale Langevin diffusions: large deviations, invariant measures and small mass asymptotics

Wenqing Hu* Konstantinos Spiliopoulos†

Abstract

We consider a general class of hypoelliptic Langevin diffusions and study two related questions. The first question is large deviations for hypoelliptic multiscale diffusions as the noise and the scale separation parameter go to zero. The second question is small mass asymptotics of (a) the invariant measure corresponding to the hypoelliptic Langevin operator and of (b) related hypoelliptic Poisson equations. The invariant measure corresponding to the hypoelliptic problem and appropriate hypoelliptic Poisson equations enter the large deviations rate function due to the multiscale effects. Based on the small mass asymptotics we derive that the large deviations behavior of the multiscale hypoelliptic diffusion is consistent with the large deviations behavior of its overdamped counterpart. Additionally, we rigorously obtain an asymptotic expansion of the solution to relevant hypoelliptic Poisson equations with respect to the mass parameter, characterizing the order of convergence as the mass parameter goes to zero. The proof of convergence of invariant measures is of independent interest, as it involves an improvement of the hypocoercivity result for the kinetic Fokker-Planck equation. We do not restrict attention to gradient drifts and our proof provides explicit information on the dependence of the bounds of interest in terms of the mass parameter.

Keywords: large deviations; hypoelliptic multiscale diffusions; homogenization; hypocoercivity; non-gradient systems.

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The second order Langevin equation

$$\mu \ddot{q}_t^\tau = f(q_t^\tau) - \lambda \dot{q}_t^\tau + \sigma(q_t^\tau) \dot{W}_t, \quad q_0^\tau = q \in \mathbb{R}^n, \quad \dot{q}_0^\tau = p \in \mathbb{R}^n,$$

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is one of the most classical equations in probability theory as well as in mathematical physics ([18, 11, 26]). It describes, under Newton's law, the motion of a particle of mass τ in a force field $f(q)$, $q \in \mathbb{R}^n$, subject to random fluctuations and to a friction proportional to the velocity. Here W_t is the standard Wiener process (Brownian motion) in \mathbb{R}^n , $\lambda > 0$ is the friction coefficient.

In this paper we are interested in the case where the force field $f(q)$ has multiscale structure and the magnitude of the random fluctuations are small. In particular, our starting object of interest is the second order hypoelliptic multiscale Langevin equation,

$$\tau \dot{q}_t^\varepsilon = \left[\frac{\varepsilon}{\delta} b \left(q_t^\varepsilon, \frac{q_t^\varepsilon}{\delta} \right) + c \left(q_t^\varepsilon, \frac{q_t^\varepsilon}{\delta} \right) - \lambda(q_t^\varepsilon) \dot{q}_t^\varepsilon \right] dt + \sqrt{\varepsilon} \sigma \left(q_t^\varepsilon, \frac{q_t^\varepsilon}{\delta} \right) \dot{W}_t, \quad (0.1)$$

where $\varepsilon, \delta \ll 1$ and $\delta = \delta(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$. Here, $\lambda(q) > 0$ is an inhomogeneous friction coefficient. Moreover, ε represents the strength of the noise, whereas δ is the parameter that separates the scales. We study the homogenization regime where $\frac{\varepsilon}{\delta} \rightarrow \infty$ as $\varepsilon, \delta \downarrow 0$.

It is well known that when $\tau \downarrow 0$, the solution to (0.1) approximates that of a first order equation. In particular, if λ is a constant, then in the overdamped case, i.e. when τ is small, the motion can be approximated by the first order Langevin equation (see for example [12])

$$\dot{\tilde{q}}_t^\varepsilon = \frac{1}{\lambda} \left[\frac{\varepsilon}{\delta} b \left(\tilde{q}_t^\varepsilon, \frac{\tilde{q}_t^\varepsilon}{\delta} \right) + c \left(\tilde{q}_t^\varepsilon, \frac{\tilde{q}_t^\varepsilon}{\delta} \right) \right] + \sqrt{\varepsilon} \frac{\sigma \left(\tilde{q}_t^\varepsilon, \frac{\tilde{q}_t^\varepsilon}{\delta} \right)}{\lambda} \dot{W}_t. \quad (0.2)$$

The situation is much more complex in the case that the friction coefficient depends on the position too, see [15, 13]. In particular, in the setting of (0.1), the motion of q^ε as $\tau \downarrow 0$ is approximated by

$$\dot{\tilde{q}}_t^\varepsilon = \left[\frac{\varepsilon}{\delta} \frac{b \left(\tilde{q}_t^\varepsilon, \frac{\tilde{q}_t^\varepsilon}{\delta} \right)}{\lambda(\tilde{q}_t^\varepsilon)} + \frac{c \left(\tilde{q}_t^\varepsilon, \frac{\tilde{q}_t^\varepsilon}{\delta} \right)}{\lambda(\tilde{q}_t^\varepsilon)} - \varepsilon \frac{\nabla \lambda(\tilde{q}_t^\varepsilon)}{2\lambda^3(\tilde{q}_t^\varepsilon)} \alpha \left(\tilde{q}_t^\varepsilon, \frac{\tilde{q}_t^\varepsilon}{\delta} \right) \right] + \sqrt{\varepsilon} \frac{\sigma \left(\tilde{q}_t^\varepsilon, \frac{\tilde{q}_t^\varepsilon}{\delta} \right)}{\lambda(\tilde{q}_t^\varepsilon)} \dot{W}_t, \quad (0.3)$$

where $\alpha(q, r) = \sigma(q, r)\sigma^T(q, r)$. Clearly, when $\lambda(q) = \lambda = \text{constant}$, (0.3) reduces to (0.2).

The first goal of this paper is to consider the large deviations behavior of the solution to (0.1) q^ε in such a way that, when the mass is small, it is consistent with the large deviations behavior of the solution to the overdamped counterpart (0.3), or equivalently (0.2). In particular we want to investigate the conditions under which the tail behavior of (0.1) and of (0.3) agree, at least in a limiting sense.

It turns out that we get interesting non-trivial behavior when the mass τ relates to ε, δ in a specific way that will be explained in the sequel. For this reason we shall write τ^ε in place of τ when we want to emphasize this dependence. We prove that if the mass of the particle τ scales appropriately with the order of the fluctuations and in particular if it is of order δ^2/ε , i.e., if $\tau = m \frac{\delta^2}{\varepsilon}$ with m small but positive, then the large deviation behaviors of the overdamped and underdamped systems agree. The large deviations result for (0.1) is given in Theorem 1.4 and the agreement in terms of the large deviations behavior of (0.1) and (0.3) is given in Theorem 1.8.

In order to derive the large deviations principle we follow the weak convergence approach, see [6]. This framework transforms the large deviations problem to convergence of a hypoelliptic stochastic control problem. Due to the hypoellipticity one needs certain a-priori bounds that establish compactness, see [14]. We obtain an explicit form of the control (equivalently change of measure) that leads to the proof of large deviations upper bound in the multiscale hypoelliptic case. Even though we do not address this issue in the current paper, we mention that the explicit information on the optimal control can

be used for the construction of provably-efficient Monte Carlo schemes in the spirit of the constructions done in [8, 29] for the corresponding elliptic case.

Under the parameterizations $\tau = m \frac{\delta^2}{\varepsilon}$ and when $\delta \ll \varepsilon$ we derive the large deviations principle for $\{q^\varepsilon, \varepsilon > 0\}$, where q^ε solves (0.1), see Theorem 1.4. The large deviations rate function is derived in closed form and it depends on m . The next natural question is to derive that as $m \downarrow 0$ the large deviations rate function converges to that of the large deviations principle for the overdamped case, i.e., for the solution to (0.3). However, to our surprise, we find that even in the case of constant diffusion the rigorous proof of such a convergence is highly involved. We prove such a convergence in the special case of diffusion coefficient $\sigma(q, r) = \sqrt{2D\lambda(q)}I$, $D > 0$ (which is the parametrization of the fluctuation-dissipation theorem) and we include a discussion for the general variable diffusion coefficient case in Remark 1.9. This result supports the claim that the large deviations behavior of the multiscale second order Langevin diffusion and for its first order counterpart agree, see Theorem 1.8.

The second and related goal of this paper is to rigorously develop small mass asymptotics for the invariant measure, see Theorem 1.6 and for certain Poisson equations, see Theorem 1.7, that appear in the rate function of the large deviation principle (see Theorem 1.4) due to the homogenization effects. Our proof of the convergence as $m \downarrow 0$ of the large deviation rate function requires a thorough analysis of the small mass asymptotic for the invariant measure of the fast motion corresponding to (0.1). In particular, since we will allow the drift term $b(q, r)$ to be a general vector field rather than a gradient field, our proof of the convergence involves a non-trivial improvement of the hypocoercivity result for linear Fokker-Planck equation ([30, Section 1.7], see also [5]). If $b(q, r)$ is not a gradient field, then certain operators that appear in the analysis are not anti-symmetric. This implies that extra terms appear that need to be appropriately handled. Then making use and extending the hypocoercivity results of [30], we prove that the invariant measures corresponding to the $m > 0$ case, converges in L^2 to the invariant measure corresponding to the $m = 0$ problem. Here we make use of the $((\cdot, \cdot))$ inner product introduced in [30] and we combine the different terms in such a way that the desired bounds follow. To accomplish this goal in the general non-gradient case, we use the structure of the hypoelliptic operator in an effective way.

Using the convergence of the invariant measure as the mass parameter goes to zero and Poincaré inequality, we also prove that the solution to the related hypoelliptic Poisson equation converges to the appropriate elliptic Poisson equation (the so-called “cell problems”) in the appropriate L^2 sense as the mass parameter goes to zero. These Poisson equations appear due to the homogenization effects of the drift $b(q, r)$. In addition to that, the proof provides a rigorous justification of the corresponding multiscale expansion of the solutions of the corresponding equations in powers of \sqrt{m} as $m \downarrow 0$. Related heuristic, i.e., without proof, asymptotic expansions can be also found in [25]. We would like to emphasize that our method of proof allows to obtain upper bounds for the norms of interest with detailed dependence on the parameters of interest, such as the mass of the particle.

Partial motivation for our work comes from chemical physics and biology, and in particular from the dynamical behavior of proteins such as their folding and binding kinetics. As it has been suggested long time ago (e.g., [19, 32]) the potential surface of a protein might have a hierarchical structure with potential minima within potential minima. As a consequence, the roughness of the energy landscapes that describe proteins has numerous effects on their kinetic properties as well as on their behavior at equilibrium.

One of the first papers that used a simple model with two separated time scales to model diffusion in rough potentials is [32]. The situation usually investigated [19, 32, 9]

is based on the first order equation (0.2) even though the physical model and what is many times used in molecular simulations is the more complex second order Langevin equation that involves both position and velocity, see for example [20] and would also usually include more than two separated time scales. The usual choice of coefficients is $\lambda(q) = \text{constant}$, $b(q, q/\delta) = -\frac{2D}{k_\beta T} \nabla Q(q/\delta)$, $c(q, q/\delta) = -\frac{2D}{k_\beta T} \nabla V(q)$ and $\sigma(q, q/\delta) = \sqrt{2D\lambda I}$, where k_β is the Boltzmann constant and T is the temperature, in such a way that the fluctuation-dissipation theorem holds. We remark here that our formulation is general and includes the parametrization suggested by the fluctuation-dissipation theorem as a special case. Notice that the choice of the separable drift

$$b(q, q/\delta) = -\nabla Q(q/\delta), \quad c(q, q/\delta) = -\nabla V(q)$$

represents the motion of a massless particle in a rough potential $\varepsilon Q(q/\delta) + V(q)$. In particular, the model of interest in this case becomes

$$\dot{\tilde{q}}_t^\varepsilon = -\frac{\varepsilon}{\delta} \frac{2D}{k_\beta T} \nabla Q\left(\frac{\tilde{q}_t^\varepsilon}{\delta}\right) - \frac{2D}{k_\beta T} \nabla V(\tilde{q}_t^\varepsilon) + \sqrt{\varepsilon} \sqrt{2D} \dot{W}_t. \quad (0.4)$$

The questions of interest in [32, 9] are related to the effect of taking $\delta \downarrow 0$ with ε small but fixed. This is almost the same to requiring that δ goes to 0 much faster than ε does, which is the regime that we study in this paper.

The related mathematical literature is quite rich. For the related hypocoercivity theory the reader is referred to [30]. For the case $\delta = 1$, the large deviations principle of the solutions to (0.1) and (0.2) as $\varepsilon \downarrow 0$ is being compared in [4]. For the case $\varepsilon = 1$, periodic homogenization for a special case of (0.1) (in particular when $c(q, r) = 0$ and $b(q, r) = b(r)$) has been addressed in [14]. Also, when $\varepsilon = 1$ random homogenization for (0.1) when $c(q, r) = 0$ and the special case of gradient drift $b(q, r) = -\nabla Q(r)$ has been addressed in [2, 24]. More is known about the overdamped case (0.2), see [7, 17, 21, 28] where homogenization and large deviation results for the solution to equations of the form (0.2) are obtained under different relations between ε and δ , in both periodic and random environments.

The rest of the paper is structured as follows. In Section 1 we formulate the problem, our assumptions and the main results of this paper in detail. In Sections 2-3 we prove the large deviations principle for the hypoelliptic problem. In Sections 4-6 and in the Appendix we exploit the small mass asymptotics.

In particular, using the weak convergence approach we turn the large deviations principle into a law of large numbers for a stochastic control problem. Section 2 proves the convergence of the controlled stochastic equation and Section 3 proves the convergence of the cost functional, which is the Laplace principle. In Section 4 we prove the small mass limit of the rate function in the diffusion $\sigma(q, r) = \sqrt{2D\lambda(q)}I$ case, using the convergence of the invariant measures as $m \rightarrow 0$ (Section 5) and of the related ‘‘cell problems’’ that are auxiliary Poisson equations that appear in the rate functions due to homogenization effects (Section 6). We emphasize that Section 5 is of independent interest as it is an extension of the hypo-coercivity result for the linear kinetic Fokker-Planck equation [30, Section 1.7], since we do not restrict our attention to drifts that are of gradient form. The method of proofs also yields explicit decay rates of the norms of interest with regards to parameters of interest such as the mass of the particle. Most of the proofs to technical lemmas are deferred to the Appendix.

1 Problem formulation, assumptions and main results

In this section, we formulate more precisely the problem that we are studying in this paper, we state our main assumptions and our main results. In preparation for stating the main results, we recall the concept of a Laplace principle.

Definition 1.1. Let $\{q^\varepsilon, \varepsilon > 0\}$ be a family of random variables taking values on a Polish space \mathcal{S} and let I be a rate function on \mathcal{S} . We say that $\{q^\varepsilon, \varepsilon > 0\}$ satisfies the Laplace principle with rate function I if for every bounded and continuous function $h : \mathcal{S} \rightarrow \mathbb{R}$

$$\lim_{\varepsilon \downarrow 0} -\varepsilon \ln \mathbb{E} \left[\exp \left\{ -\frac{h(q^\varepsilon)}{\varepsilon} \right\} \right] = \inf_{x \in \mathcal{S}} [I(x) + h(x)].$$

If the rate function has compact level sets, then the Laplace principle is equivalent to the corresponding large deviations principle with the same rate function (see Theorems 2.2.1 and 2.2.3 in [6]). Hence, instead of proving a large deviations principle for $\{q^\varepsilon\}$ we prove a Laplace principle for $\{q^\varepsilon\}$.

Our main regularity assumption in regards to the coefficients of (0.1) is given by Condition 1.2.

Condition 1.2. The functions $b(q, r), c(q, r), \sigma(q, r)$ are

1. periodic with period 1 in the second variable in each direction, and
2. $C^1(\mathbb{R}^d)$ in r and $C^2(\mathbb{R}^d)$ in q with all partial derivatives continuous and globally bounded in q and r .

The diffusion matrix $\alpha(q, r) = \sigma(q, r)\sigma^T(q, r)$ is uniformly non-degenerate. There exist constants $0 < \underline{\lambda} < \bar{\lambda}$ such that for every $q \in \mathbb{R}^d$, $\underline{\lambda} < \lambda(q) < \bar{\lambda}$. Moreover, the function $\lambda(q)$ is in $C^1(\mathbb{R}^d)$ with bounded partial derivatives.

Using the parametrization $\tau = m \frac{\delta^2}{\varepsilon}$, the system being considered is

$$m \frac{\delta^2}{\varepsilon} \ddot{q}_t^\varepsilon = \left[\frac{\varepsilon}{\delta} b \left(q_t^\varepsilon, \frac{q_t^\varepsilon}{\delta} \right) + c \left(q_t^\varepsilon, \frac{q_t^\varepsilon}{\delta} \right) - \lambda(q_t^\varepsilon) \dot{q}_t^\varepsilon \right] + \sqrt{\varepsilon} \sigma \left(q_t^\varepsilon, \frac{q_t^\varepsilon}{\delta} \right) \dot{W}_t. \tag{1.1}$$

Setting $p_t^\varepsilon = \sqrt{m \frac{\delta}{\varepsilon}} \dot{q}_t^\varepsilon$ we obtain the following system of equations which we also supplement with initial conditions

$$\dot{q}_t^\varepsilon = \frac{1}{\sqrt{m}} \frac{\varepsilon}{\delta} p_t^\varepsilon, \tag{1.2}$$

$$\begin{aligned} \dot{p}_t^\varepsilon &= \frac{1}{\sqrt{m}} \frac{1}{\delta} \left[\frac{\varepsilon}{\delta} b \left(q_t^\varepsilon, \frac{q_t^\varepsilon}{\delta} \right) + c \left(q_t^\varepsilon, \frac{q_t^\varepsilon}{\delta} \right) \right] - \frac{\lambda(q_t^\varepsilon)}{m} \frac{\varepsilon}{\delta^2} p_t^\varepsilon + \frac{\sqrt{\varepsilon}}{\delta} \frac{\sigma \left(q_t^\varepsilon, \frac{q_t^\varepsilon}{\delta} \right)}{\sqrt{m}} \dot{W}_t, \\ q_0^\varepsilon &= q_o \in \mathbb{R}^d, \quad p_0 = p_o \in \mathbb{R}^d. \end{aligned}$$

Condition 1.2, guarantees that (1.1) and (1.2) too, have a unique strong solution; this is a classical result, see for example [12] or Theorem 5.2.1 of [23]. The infinitesimal generator for the (q, p) process satisfying (1.2) is given by

$$\begin{aligned} \mathcal{L} &= \frac{1}{\sqrt{m}} \left[\frac{\varepsilon}{\delta} p \cdot \nabla_q + \frac{\varepsilon}{\delta^2} b(q, q/\delta) \cdot \nabla_p + \frac{1}{\delta} c(q, q/\delta) \cdot \nabla_p \right] \\ &\quad + \frac{1}{m} \frac{\varepsilon}{\delta^2} \left[-\lambda(q) p \cdot \nabla_p + \frac{1}{2} \alpha(q, q/\delta) : \nabla_p^2 \right], \end{aligned}$$

where we recall that $\alpha(q, r) = \sigma(q, r)\sigma^T(q, r)$.

We can assume that p_o is a random variable, as long as it is independent of the driving Wiener process W_t and as long as $\mathbb{E} \left[e^{\frac{1}{2} |\sigma_{\max}^{-1} p_o|^2} \right] < \infty$ (see Appendix A), where we have defined $\sigma_{\max} = \max_{i,j=1,\dots,d} \sup_{(q,r)} |\sigma_{i,j}(q, r)|$.

Sometimes, we may write $X_t^\varepsilon = (q_t^\varepsilon, p_t^\varepsilon)$. Let $|\bullet|$ be the Euclidean norm in \mathbb{R}^d and introduce the control set

$$\mathcal{A} = \left\{ u = \{u_s \in \mathbb{R}^d : 0 \leq s \leq T\} \text{ progressively } \mathcal{F}_s\text{-measurable and } \mathbb{E} \int_0^T |u_s|^2 ds < \infty \right\}.$$

The result in [3] gives the following representation

$$-\varepsilon \ln \mathbb{E}_{q_0} \left[\exp \left(-\frac{h(q_\bullet^\varepsilon)}{\varepsilon} \right) \right] = \inf_{u \in \mathcal{A}} \mathbb{E}_{q_0} \left[\frac{1}{2} \int_0^T |u_s|^2 ds + h(\bar{q}_\bullet^\varepsilon) \right].$$

Here the process \bar{q}_t^ε is the q -component of the hypoelliptic controlled diffusion process $\bar{X}_t^\varepsilon = (\bar{q}_t^\varepsilon, \bar{p}_t^\varepsilon)$:

$$\begin{aligned} \dot{\bar{q}}_t^\varepsilon &= \frac{1}{\sqrt{m}} \frac{\varepsilon}{\delta} \bar{p}_t^\varepsilon, \\ \dot{\bar{p}}_t^\varepsilon &= \frac{1}{\sqrt{m}} \frac{1}{\delta} \left[\frac{\varepsilon}{\delta} b \left(\bar{q}_t^\varepsilon, \frac{\bar{q}_t^\varepsilon}{\delta} \right) + c \left(\bar{q}_t^\varepsilon, \frac{\bar{q}_t^\varepsilon}{\delta} \right) \right] - \frac{\lambda(q_t^\varepsilon)}{m} \frac{\varepsilon}{\delta^2} \bar{p}_t^\varepsilon + \frac{1}{\delta} \frac{\sigma \left(q_t^\varepsilon, \frac{q_t^\varepsilon}{\delta} \right)}{\sqrt{m}} u_t \\ &\quad + \frac{\sqrt{\varepsilon}}{\delta} \frac{\sigma \left(q_t^\varepsilon, \frac{q_t^\varepsilon}{\delta} \right)}{\sqrt{m}} \dot{W}_t, \\ \bar{q}_0^\varepsilon &= q_o \in \mathbb{R}^d, \quad \bar{p}_0^\varepsilon = p_o \in \mathbb{R}^d. \end{aligned} \tag{1.3}$$

Let $u_\bullet^\varepsilon \in \mathcal{A}$ and \bar{X}_s^ε solves (1.3) with u^ε in place of u . Let the control space be $\mathcal{Z} = \mathbb{R}^d$, the fast variable space be $\mathcal{Y} = \mathbb{R}^d \times \mathbb{T}^d$. We see that the fast variable is actually $\left(\bar{p}_s^\varepsilon, \frac{\bar{q}_s^\varepsilon}{\delta} \right)$. Let us define now the operator

$$\begin{aligned} \mathcal{L}_q^m \Phi(p, r) &= \frac{1}{\sqrt{m}} [p \cdot \nabla_r \Phi(p, r) + b(q, r) \cdot \nabla_p \Phi(p, r)] \\ &\quad + \frac{1}{m} \left[-\lambda(q) p \cdot \nabla_p \Phi(p, r) + \frac{1}{2} \alpha(q, r) : \nabla_p^2 \Phi(p, r) \right]. \end{aligned}$$

For each fixed q , the operator \mathcal{L}_q^m defines a hypoelliptic diffusion process on $(p, r) \in \mathcal{Y} = \mathbb{R}^d \times \mathbb{T}^d$. Let $\mu(dpdr|q)$ be the unique invariant measure for this process. Notice that \mathcal{L}_q^m is effectively the operator corresponding to the fast motion. The following centering condition is essential for the validity of the results.

Condition 1.3. We assume that for every $q \in \mathbb{R}^d$

$$\int_{\mathcal{Y}} b(q, r) \mu(dpdr|q) = 0.$$

Let us consider the preliminary cell problem

$$\mathcal{L}_q^m \Phi(p, r) = -\frac{1}{\sqrt{m}} p, \quad \int_{\mathcal{Y}} \Phi(p, r) \mu(dr dp|q) = 0. \tag{1.4}$$

It is clear that the solution to (1.4) Φ depends also on q , but we sometimes suppress this in the notation for convenience. By the work of [14], we know that under Condition 1.3, the PDE (1.4) has a unique, smooth solution that does not grow too fast at infinity, see Appendix A for more details. Note that the function Φ is actually a vector valued function $\Phi(p, r) = (\Phi_1(p, r), \dots, \Phi_d(p, r))$.

Let us denote by $\mathcal{AC}([0, T]; \mathbb{R}^d)$ the space of absolutely continuous functions from $[0, T]$ to \mathbb{R}^d . Our first main result reads as follows.

Theorem 1.4. Let $\{(q^\varepsilon, p^\varepsilon), \varepsilon > 0\}$ be the unique solution to (1.2). Under Conditions 1.2 and 1.3, $\{q^\varepsilon, \varepsilon > 0\}$ satisfies the large deviations principle with rate function

$$S_m(\phi) = \begin{cases} \frac{1}{2} \int_0^T (\dot{\phi}_s - r_m(\phi_s))^T Q_m^{-1}(\phi_s) (\dot{\phi}_s - r_m(\phi_s)) ds & \text{if } \phi \in \mathcal{AC}([0, T]; \mathbb{R}^d), \phi_0 = q_o \\ +\infty & \text{otherwise} \end{cases}.$$

where

$$r_m(q) = \frac{1}{\sqrt{m}} \int_{\mathcal{Y}} \nabla_p \Phi(p, r) c(q, r) \mu(dpdr|q) ,$$

$$Q_m(q) = \frac{1}{m} \int_{\mathcal{Y}} \nabla_p \Phi(p, r) \alpha(q, r) (\nabla_p \Phi(p, r))^T \mu(dpdr|q) .$$

To support the claim that the particular parametrization is consistent with the large deviations principle of the overdamped case (0.3), we need to prove that $\lim_{m \rightarrow 0} S_m(\phi) = S_0(\phi)$, where $S_0(\phi)$ is the rate function associated to (0.3). To that end, we recall the corresponding large deviations result from [7].

Let $\mu_0(dr|q)$ be the unique invariant measure corresponding to the operator

$$\mathcal{L}_q^0 = \frac{1}{\lambda(q)} b(q, r) \cdot \nabla_r + \frac{1}{2\lambda(q)} \alpha(q, r) : \nabla_r^2$$

equipped with periodic boundary conditions in r (q is being treated as a parameter here). By Theorem 1.6, Condition 1.3 implies the following centering condition for the drift term b :

$$\int_{\bar{\mathcal{Y}}} b(q, r) \mu_0(dr|q) = 0,$$

where $\bar{\mathcal{Y}} = \mathbb{T}^d$ denotes the d -dimensional torus. Under this centering condition, the cell problem

$$\mathcal{L}_q^0 \chi_\ell(q, r) = -\frac{1}{\lambda(q)} b_\ell(q, r) , \int_{\bar{\mathcal{Y}}} \chi_\ell(q, r) \mu_0(dr|q) = 0 , \ell = 1, 2, \dots, d . \tag{1.5}$$

has a unique bounded and sufficiently smooth solution $\chi = (\chi_1, \dots, \chi_d)$. After these definitions we recall the result from [7] that will be of use to us.

Theorem 1.5 (Theorem 5.3 in [7]). *Let $\{\chi^\varepsilon, \varepsilon > 0\}$ be the unique solution to (0.2). Under Conditions 1.2 and 1.3, $\{\chi^\varepsilon, \varepsilon > 0\}$ satisfies a large deviations principle with rate function*

$$S_0(\phi) = \begin{cases} \frac{1}{2} \int_0^T (\dot{\phi}_s - r_0(\phi_s))^T Q_0^{-1}(\phi_s) (\dot{\phi}_s - r_0(\phi_s)) ds & \text{if } \phi \in \mathcal{AC}([0, T]; \mathbb{R}^d), \phi_0 = q_0 \\ +\infty & \text{otherwise.} \end{cases}$$

where

$$r_0(q) = \frac{1}{\lambda(q)} \int_{\bar{\mathcal{Y}}} \left(I + \frac{\partial \chi}{\partial r}(q, r) \right) c(q, r) \mu_0(dr|q)$$

and

$$Q_0(q) = \frac{1}{\lambda^2(q)} \int_{\bar{\mathcal{Y}}} \left(I + \frac{\partial \chi}{\partial r}(q, r) \right) \alpha(q, r) \left(I + \frac{\partial \chi}{\partial r}(q, r) \right)^T \mu_0(dr|q).$$

For the small mass, i.e., $m \rightarrow 0$, asymptotic that follow, we assume that $\sigma(q, r) = \sqrt{2D\lambda(q)}I, D > 0$, i.e., we assume that the noise is such that we are in fluctuation-dissipation balance. In this case, for a function $f \in \mathcal{C}^2(\mathcal{Y})$, we have

$$\mathcal{L}_q^m f(p, r) = \frac{\lambda(q)}{m} \mathcal{A}f(p, r) + \frac{1}{\sqrt{m}} \mathcal{B}f(p, r) ,$$

where $\mathcal{A}f = -p \cdot \nabla_p f + D\Delta_p f$ and $\mathcal{B}f(p, r) = p \cdot \nabla_r f + b(q, r) \cdot \nabla_p f$. Likewise, we have

$$\mathcal{L}_q^0 f(r) = \frac{1}{\lambda(q)} b(q, r) \cdot \nabla_r f(r) + D\Delta_r f(r) .$$

We denote by $\mu(dpdr|q) = \rho^m(p, r|q) dpdr$ the invariant measure corresponding to the operator \mathcal{L}_q^m . Also, let us write $\mu_0(dr|q) = \rho_0(r|q) dr$ for the invariant measure corresponding to the operator \mathcal{L}_q^0 .

Let us also define $\pi(dp) = \rho^{OU}(p)dp$ to be the invariant measure on \mathbb{R}^d for the Ornstein–Uhlenbeck process with generator \mathcal{A} . With this notation, let us write $\rho^m(p, r) = \tilde{\rho}^m(p, r)\rho^0(p, r)$, where $\rho^0(p, r) = \rho^{OU}(p)\rho_0(r)$, suppressing the dependence on q .

Then, in Sections 5 and 6 respectively we prove the following Theorems which constitute the second main result of our paper.

Theorem 1.6. *Let Condition 1.2 hold and assume that $\sigma(q, r) = \sqrt{2D\lambda(q)}I, D > 0$. Then, for every $q \in \mathbb{R}^d$, we have*

$$\lim_{m \rightarrow 0} \|\tilde{\rho}^m(p, r) - 1\|_{L^2(\mathcal{Y}; \rho^0)} = 0 .$$

Theorem 1.7. *Let Conditions 1.2 and 1.3 hold and assume that $\sigma(q, r) = \sqrt{2D\lambda(q)}I, D > 0$. Then, for every $q \in \mathbb{R}^d$, we have*

$$\lim_{m \rightarrow 0} \left\| \frac{1}{\sqrt{m}} \nabla_p \Phi - \frac{1}{\lambda(q)} (I + \nabla_r \chi) \right\|_{L^2(\mathcal{Y}; \rho^0)} = 0 .$$

Using then Theorems 1.6 and 1.7 we prove in Section 4 that the rate function $S_m(\phi)$ converges $S_0(\phi)$, as $m \downarrow 0$.

Theorem 1.8. *Let Conditions 1.2 and 1.3 hold and assume that $\sigma(q, r) = \sqrt{2D\lambda(q)}I, D > 0$. Then, we have*

$$\lim_{m \rightarrow 0} S_m(\phi) = S_0(\phi) .$$

Remark 1.9. We believe that Theorems 1.6 and 1.7 and as a consequence Theorem 1.8 are true under more general variable diffusion coefficients as long as Condition 1.2 holds. When, the diffusion coefficient σ is not a multiple of the identity matrix, then the operator \mathcal{A} is not the classical Ornstein-Uhlenbeck that has the Gaussian measure $\rho^{OU}(p)dp \sim e^{-\frac{|p|^2}{2D}} dp$ as its invariant measure. Some of our technical lemmas use this explicit structure in order to derive the necessary estimates. However, since the spirit of the proof does not rely on this structure, we believe that this is only a technical problem.

2 Law of large numbers

In this section we study the limiting behavior of the solution to the control problem (1.3). It turns out that we need to consider the solution to (1.3) together with an appropriate occupation measure and then consider the limit of the pair. Let us be more specific now.

Let $u_s^\varepsilon \in \mathcal{A}$ and \bar{X}_s^ε solves (1.3) with u^ε in place of u . Recall that the control space is $\mathcal{Z} = \mathbb{R}^d$, the fast variable space is $\mathcal{Y} = \mathbb{R}^d \times \mathbb{T}^d$ and that the fast variable is actually $\left(\bar{p}_s^\varepsilon, \frac{\bar{q}_s^\varepsilon}{\delta}\right)$. Let $A \subseteq \mathcal{Z}, B_1 \times B_2 \subseteq \mathcal{Y}$ and $\Gamma \subseteq [0, T]$. Let $\Delta = \Delta(\varepsilon) > 0$ be a separation of scales parameter. We introduce the occupation measure

$$P^{\varepsilon, \Delta}(A \times B_1 \times B_2 \times \Gamma) = \int_{\Gamma} \left[\frac{1}{\Delta} \int_t^{t+\Delta} \mathbf{1}_A(u_s^\varepsilon) \mathbf{1}_{B_1}(\bar{p}_s^\varepsilon) \mathbf{1}_{B_2} \left(\frac{\bar{q}_s^\varepsilon}{\delta} \bmod 1 \right) ds \right] dt . \quad (2.1)$$

Let us define the function

$$\gamma(q, (p, r), z) = \frac{1}{\sqrt{m}} [c(q, r) + \sigma(q, r)z] \cdot \nabla_p \Phi(p, r) . \quad (2.2)$$

Definition 2.1 captures the notion of a viable pair as introduced in [7] which characterizes the required law of large numbers.

Definition 2.1. A pair $(\psi, P) \in \mathcal{C}([0, T]; \mathbb{R}^d) \times \mathcal{P}(\mathcal{Z} \times \mathcal{Y} \times [0, T])$ will be called viable with respect to $(\gamma, \mathcal{L}_q^m)$ or simply viable if there is no confusion, if the following are satisfied. The function ψ_t is absolutely continuous, P is square integrable in the sense that

$$\int_{\mathcal{Z} \times \mathcal{Y} \times [0, T]} |z|^2 P(dz, dpdr, ds) < \infty, \tag{2.3}$$

and

1.

$$\psi_t = q_o + \int_0^t \int_{\mathcal{Z} \times \mathcal{Y}} \gamma(\psi_s, (p, r), z) P(dz, dpdr, ds); \tag{2.4}$$

2. For any $g(p, r) \in \mathcal{D}(\mathcal{L}_q^m)$,

$$\int_0^t \int_{\mathcal{Z} \times \mathcal{Y}} \mathcal{L}_{\psi_s}^m g(p, r) P(dz, dpdr, ds) = 0; \tag{2.5}$$

3.

$$P(\mathcal{Z} \times \mathcal{Y} \times [0, t]) = t. \tag{2.6}$$

We write $(\psi, P) \in \mathcal{V}_{(\gamma, \mathcal{L}_q)}$.

Theorem 2.2. Consider any family $\{u^\varepsilon, \varepsilon > 0\}$ of controls in \mathcal{A} satisfying

$$\sup_{\varepsilon > 0} \mathbb{E} \int_0^T |u_t^\varepsilon|^2 dt < \infty.$$

Let Conditions 1.2 and 1.3 be satisfied. Then the family $\{(\bar{q}_\bullet^\varepsilon, P^{\varepsilon, \Delta}), \varepsilon > 0\}$ is tight. Hence, given any subsequence of $\{(\bar{q}_\bullet^\varepsilon, P^{\varepsilon, \Delta}), \varepsilon > 0\}$, there exists a subsubsequence that converges in distribution with limit (\bar{q}_\bullet, P) . With probability 1, the accumulation point (\bar{q}_\bullet, P) is a viable pair with respect to (γ, \mathcal{L}_q) : $(\bar{q}_\bullet, P) \in \mathcal{V}_{(\gamma, \mathcal{L}_q)}$.

Proof of Theorem 2.2. Part 1. [Tightness]. For a smooth function $g\left(\frac{\bar{p}_t^\varepsilon}{\delta}, \frac{\bar{q}_t^\varepsilon}{\delta}\right)$ we can apply Itô's formula and get

$$\begin{aligned} g\left(\frac{\bar{p}_t^\varepsilon}{\delta}, \frac{\bar{q}_t^\varepsilon}{\delta}\right) - g\left(\frac{\bar{p}_o^\varepsilon}{\delta}, \frac{\bar{q}_o^\varepsilon}{\delta}\right) &= \frac{\varepsilon}{\delta^2} \int_0^t \mathcal{L}_{\bar{q}_s^\varepsilon}^m g\left(\frac{\bar{p}_s^\varepsilon}{\delta}, \frac{\bar{q}_s^\varepsilon}{\delta}\right) ds \\ &+ \frac{1}{\delta} \frac{1}{\sqrt{m}} \int_0^t \left[c\left(\frac{\bar{q}_s^\varepsilon}{\delta}, \frac{\bar{q}_s^\varepsilon}{\delta}\right) + \sigma\left(\frac{\bar{q}_s^\varepsilon}{\delta}, \frac{\bar{q}_s^\varepsilon}{\delta}\right) u_s \right] \cdot \nabla_p g\left(\frac{\bar{p}_s^\varepsilon}{\delta}, \frac{\bar{q}_s^\varepsilon}{\delta}\right) ds \\ &+ \frac{\sqrt{\varepsilon}}{\delta} \frac{1}{\sqrt{m}} \int_0^t \nabla_p g\left(\frac{\bar{p}_s^\varepsilon}{\delta}, \frac{\bar{q}_s^\varepsilon}{\delta}\right) \cdot \sigma\left(\frac{\bar{q}_s^\varepsilon}{\delta}, \frac{\bar{q}_s^\varepsilon}{\delta}\right) dW_s. \end{aligned} \tag{2.7}$$

Let us apply Itô's formula to $\Phi\left(\frac{\bar{p}_t^\varepsilon}{\delta}, \frac{\bar{q}_t^\varepsilon}{\delta}\right)$ in (1.4) and use (2.7) to get a representation formula for \bar{q}_t^ε as follows:

$$\begin{aligned} \bar{q}_t^\varepsilon &= q_o + \int_0^t \frac{1}{\sqrt{m}} \left[c\left(\frac{\bar{q}_s^\varepsilon}{\delta}, \frac{\bar{q}_s^\varepsilon}{\delta}\right) + \sigma\left(\frac{\bar{q}_s^\varepsilon}{\delta}, \frac{\bar{q}_s^\varepsilon}{\delta}\right) u_s \right] \cdot \nabla_p \Phi\left(\frac{\bar{p}_s^\varepsilon}{\delta}, \frac{\bar{q}_s^\varepsilon}{\delta}\right) ds \\ &- \delta \left[\Phi\left(\frac{\bar{p}_t^\varepsilon}{\delta}, \frac{\bar{q}_t^\varepsilon}{\delta}\right) - \Phi\left(\frac{\bar{p}_o^\varepsilon}{\delta}, \frac{\bar{q}_o^\varepsilon}{\delta}\right) \right] + \frac{\sqrt{\varepsilon}}{\sqrt{m}} \int_0^t \nabla_p \Phi\left(\frac{\bar{p}_s^\varepsilon}{\delta}, \frac{\bar{q}_s^\varepsilon}{\delta}\right) \cdot \sigma\left(\frac{\bar{q}_s^\varepsilon}{\delta}, \frac{\bar{q}_s^\varepsilon}{\delta}\right) dW_s. \end{aligned}$$

Using this representation formula, Condition 1.2 and Theorem 3.3 of [14] (see also Appendix A), we can then establish that for every $\eta > 0$

$$\lim_{\rho \downarrow 0} \limsup_{\varepsilon \downarrow 0} \mathbb{P} \left[\sup_{|t_1 - t_2| < \rho, 0 \leq t_1 < t_2 \leq 1} |\bar{q}_{t_1}^\varepsilon - \bar{q}_{t_2}^\varepsilon| \geq \eta \right] = 0.$$

This implies the tightness of the family $\{\bar{q}_\bullet^\epsilon\}$. Tightness of the occupation measures $\{P^{\epsilon,\Delta}, \epsilon > 0\}$ follows from the bound

$$\sup_{\epsilon \in (0,1)} \mathbb{E} [g(P^{\epsilon,\Delta})] = \sup_{\epsilon \in (0,1)} \mathbb{E} \int_0^T \frac{1}{\Delta} \int_t^{t+\Delta} |u^\epsilon(s)|^2 ds dt < \infty. \tag{2.8}$$

for the tightness function $g(r) = \int_{\mathcal{Z} \times \mathcal{Y} \times [0,T]} |z|^2 r(dz, dpdr, dt)$, $r \in \mathcal{P}(\mathcal{Z} \times \mathcal{Y} \times [0, T])$, see Theorem A.19 in [6]. Notice that the last inequality in (2.8) follows by the uniform L^2 bound on the family of controls $\{u^\epsilon, \epsilon > 0\}$.

Hence, the family $\{(\bar{q}^\epsilon, P^{\epsilon,\Delta}), \epsilon > 0\}$ is tight. Due to tightness, for any subsequence of $\epsilon > 0$ there exists subsubsequence that converges, in distribution, to some limit (\bar{q}, P) such that

$$(\bar{q}^\epsilon, P^{\epsilon,\Delta}) \rightarrow (\bar{q}, P) .$$

In addition, as in Proposition 3.1 of [7], we can show that the family $\{P^{\epsilon,\Delta}, \epsilon > 0\}$ is uniformly integrable in the sense that

$$\lim_{M \rightarrow \infty} \sup_{\epsilon > 0} \mathbb{E} \int_{\{z: |z| > M\} \times \mathcal{Y} \times [0,T]} |z| P^{\epsilon,\Delta} (dz, dpdr, dt) = 0.$$

Next, we prove that any accumulation point will be a viable pair according to Definition 2.1.

Part 2. [Proof of (2.3)]. By Fatou’s Lemma we have

$$\mathbb{E} \int_{\mathcal{Z} \times \mathcal{Y} \times [0,T]} |z|^2 P(dz, dpdr, dt) < \infty ,$$

which then implies that $\int_{\mathcal{Z} \times \mathcal{Y} \times [0,T]} |z|^2 P(dz, dpdr, dt) < \infty$ w.p.1.

Part 3. [Proof of (2.4)]. Consider a test function $f = f(q)$ on \mathbb{R}^d . Let $\Psi(p, r) = \Phi(p, r) \cdot \nabla_q f(q)$ which satisfies the cell problem

$$\mathcal{L}_q^m \Psi(p, r) = -\frac{1}{\sqrt{m}} p \cdot \nabla_q f(q) .$$

Making use of (1.4) and (2.7) we get

$$\begin{aligned} \Psi \left(\bar{p}_t^\epsilon, \frac{\bar{q}_t^\epsilon}{\delta} \right) - \Psi \left(\bar{p}_o^\epsilon, \frac{\bar{q}_o^\epsilon}{\delta} \right) &= -\frac{\epsilon}{\delta^2} \int_0^t \frac{1}{\sqrt{m}} \bar{p}_s^\epsilon \cdot \nabla_q f(\bar{q}_s^\epsilon) ds \\ &+ \frac{1}{\delta} \int_0^t \frac{1}{\sqrt{m}} \left[c \left(\bar{q}_s^\epsilon, \frac{\bar{q}_s^\epsilon}{\delta} \right) + \sigma \left(\bar{q}_s^\epsilon, \frac{\bar{q}_s^\epsilon}{\delta} \right) u_s \right] \cdot \nabla_p \Psi \left(\bar{p}_s^\epsilon, \frac{\bar{q}_s^\epsilon}{\delta} \right) ds \\ &+ \frac{\sqrt{\epsilon}}{\delta} \frac{1}{\sqrt{m}} \int_0^t \nabla_p \Psi \left(\bar{p}_s^\epsilon, \frac{\bar{q}_s^\epsilon}{\delta} \right) \cdot \sigma \left(\bar{q}_s^\epsilon, \frac{\bar{q}_s^\epsilon}{\delta} \right) dW_s . \end{aligned} \tag{2.9}$$

Let us now choose $S, \tau \geq 0$ such that $S \leq S + \tau \leq T$. We have

$$f(\bar{q}_{S+\tau}^\epsilon) - f(\bar{q}_S^\epsilon) = \int_S^{S+\tau} \frac{1}{\sqrt{m}} \frac{\epsilon}{\delta} \bar{p}_t^\epsilon \cdot \nabla_q f(\bar{q}_t^\epsilon) dt .$$

Combining the latter expression with (2.9) we get

$$\begin{aligned} f(\bar{q}_{S+\tau}^\epsilon) - f(\bar{q}_S^\epsilon) &- \int_S^{S+\tau} \gamma \left(\bar{q}_t^\epsilon, \left(\bar{p}_t^\epsilon, \frac{\bar{q}_t^\epsilon}{\delta} \right), u_t \right) \cdot \nabla_q f(\bar{q}_t^\epsilon) dt \\ &= -\delta \left(\Psi \left(\bar{p}_t^\epsilon, \frac{\bar{q}_t^\epsilon}{\delta} \right) - \Psi \left(\bar{p}_o^\epsilon, \frac{\bar{q}_o^\epsilon}{\delta} \right) \right) + \sqrt{\epsilon} \frac{1}{\sqrt{m}} \int_0^t \nabla_p \Psi \left(\bar{p}_s^\epsilon, \frac{\bar{q}_s^\epsilon}{\delta} \right) \cdot \sigma \left(\bar{q}_s^\epsilon, \frac{\bar{q}_s^\epsilon}{\delta} \right) dW_s . \end{aligned}$$

Due to the a-priori bounds from Appendix A the right hand side of the last display goes to zero in L^2 , which means that

$$\left| f(\bar{q}_{S+\tau}^\varepsilon) - f(\bar{q}_S^\varepsilon) - \int_S^{S+\tau} \gamma\left(\bar{q}_t^\varepsilon, \left(\bar{p}_t^\varepsilon, \frac{\bar{q}_t^\varepsilon}{\delta}\right), u_t\right) \cdot \nabla_q f(\bar{q}_t^\varepsilon) dt \right| \rightarrow 0$$

as $\varepsilon \downarrow 0$ in means square sense. By Condition 1.2, Lemma 3.2 of [7] guarantees that in distribution

$$\begin{aligned} & \int_S^{S+\tau} \gamma\left(\bar{q}_t^\varepsilon, \left(\bar{p}_t^\varepsilon, \frac{\bar{q}_t^\varepsilon}{\delta}\right), u_t\right) \cdot \nabla_q f(\bar{q}_t^\varepsilon) dt \\ & - \int_{\mathcal{Z} \times \mathcal{Y} \times [S, S+\tau]} \gamma(\bar{q}_t^\varepsilon, (p, r), z) \cdot \nabla_q f(\bar{q}_t^\varepsilon) P^{\varepsilon, \Delta}(dz, dpdr, dt) \rightarrow 0 \end{aligned}$$

and

$$\begin{aligned} & \int_{\mathcal{Z} \times \mathcal{Y} \times [S, S+\tau]} \gamma(\bar{q}_t^\varepsilon, (p, r), z) \cdot \nabla_q f(\bar{q}_t^\varepsilon) P^{\varepsilon, \Delta}(dz, dpdr, dt) \\ & - \int_{\mathcal{Z} \times \mathcal{Y} \times [S, S+\tau]} \gamma(\bar{q}_t, (p, r), z) \cdot \nabla_q f(\bar{q}_t) P(dz, dpdr, dt) \rightarrow 0 \end{aligned}$$

as $\varepsilon \downarrow 0$. Therefore, by defining

$$\bar{\mathcal{A}}_t^{\varepsilon, \Delta} f(q) = \int_{\mathcal{Z} \times \mathcal{Y}} \gamma(q, (p, r), z) \nabla f(q) P_t^{\varepsilon, \Delta}(dz, dpdr),$$

where

$$P_t^{\varepsilon, \Delta}(dz, dpdr) = \frac{1}{\Delta} \int_t^{t+\Delta} 1_{dz}(u_s^\varepsilon) 1_{dp}(\bar{p}_s^\varepsilon) 1_{dr}\left(\frac{\bar{q}_s^\varepsilon}{\delta} \bmod 1\right) ds,$$

we get that, as $\varepsilon \downarrow 0$,

$$\mathbb{E} \left[f(\bar{q}_{S+\tau}^\varepsilon) - f(\bar{q}_S^\varepsilon) - \int_S^{S+\tau} \bar{\mathcal{A}}_t^{\varepsilon, \Delta} f(\bar{q}_t^\varepsilon) dt \right] \rightarrow 0, \tag{2.10}$$

and, in probability,

$$\int_S^{S+\tau} \bar{\mathcal{A}}_s^{\varepsilon, \Delta} f(\bar{q}_s^\varepsilon) ds - \int_{\mathcal{Z} \times \mathcal{Y} \times [S, S+\tau]} \gamma(\bar{q}_s, (p, r), z) \nabla f(\bar{q}_s) P(dz, dpdr, ds) \rightarrow 0. \tag{2.11}$$

Relations (2.10) and (2.11) imply that the pair (\bar{q}, P) solves the martingale problem associated with (2.4), which then proves that (2.4) holds.

Part 4. [Proof of (2.5)]. For functions $f \in C^2(\mathcal{Y})$, let us introduce the auxiliary operator

$$\mathcal{A}_{z,q}^\varepsilon f(p, r) = \frac{\varepsilon}{\delta^2} \mathcal{L}_q^m f(p, r) + \frac{1}{\delta} \frac{1}{\sqrt{m}} [c(q, r) + \sigma(q, r)z] \cdot \nabla_p f(p, r),$$

and define the \mathcal{F}_t -martingale

$$\begin{aligned} M_t^\varepsilon &= f\left(\bar{p}_t^\varepsilon, \frac{\bar{q}_t^\varepsilon}{\delta}\right) - f\left(\bar{p}_0^\varepsilon, \frac{\bar{q}_0^\varepsilon}{\delta}\right) - \int_0^t \mathcal{A}_{u_s^\varepsilon, \bar{q}_s^\varepsilon}^\varepsilon f\left(\bar{p}_s^\varepsilon, \frac{\bar{q}_s^\varepsilon}{\delta}\right) ds \\ &= \frac{\sqrt{\varepsilon}}{\delta} \frac{1}{\sqrt{m}} \int_0^t \nabla_p f\left(\bar{p}_s^\varepsilon, \frac{\bar{q}_s^\varepsilon}{\delta}\right) \sigma\left(\bar{q}_s^\varepsilon, \frac{\bar{q}_s^\varepsilon}{\delta}\right) dW_s. \end{aligned}$$

Let us furthermore set $\mathcal{G}_{q,z}^\varepsilon f(p, r) = \frac{1}{\sqrt{m}} [c(q, r) + \sigma(q, r)z] \cdot \nabla_p f(p, r)$ and define $g(\varepsilon) = \frac{\delta^2}{\varepsilon}$. Then, we have that

$$\begin{aligned}
 & g(\varepsilon)M_t^\varepsilon - g(\varepsilon) \left[f \left(\bar{p}_t^\varepsilon, \frac{\bar{q}_t^\varepsilon}{\delta} \right) - f \left(\bar{p}_0^\varepsilon, \frac{\bar{q}_0^\varepsilon}{\delta} \right) \right] \\
 & + g(\varepsilon) \left[\int_0^t \frac{1}{\Delta} \left[\int_s^{s+\Delta} \mathcal{A}_{u_s^\varepsilon, \bar{q}_s^\varepsilon}^\varepsilon f \left(\bar{p}_\rho^\varepsilon, \frac{\bar{q}_\rho^\varepsilon}{\delta} \right) d\rho \right] ds - \int_0^t \mathcal{A}_{u_s^\varepsilon, \bar{q}_s^\varepsilon}^\varepsilon f \left(\bar{p}_s^\varepsilon, \frac{\bar{q}_s^\varepsilon}{\delta} \right) ds \right] \\
 & = -\frac{\delta}{\varepsilon} \left(\int_0^t \frac{1}{\Delta} \left[\int_s^{s+\Delta} \left[\mathcal{G}_{\bar{q}_s^\varepsilon, u_s^\varepsilon}^\varepsilon f \left(\bar{p}_\rho^\varepsilon, \frac{\bar{q}_\rho^\varepsilon}{\delta} \right) - \mathcal{G}_{\bar{q}_s^\varepsilon, u_s^\varepsilon}^\varepsilon f \left(\bar{p}_\rho^\varepsilon, \frac{\bar{q}_\rho^\varepsilon}{\delta} \right) \right] d\rho \right] ds \right) \\
 & - \frac{\delta}{\varepsilon} \left(\int_{\mathcal{Z} \times \mathcal{Y} \times [0, t]} \mathcal{G}_{\bar{q}_s^\varepsilon, z}^\varepsilon f(p, r) \bar{P}^{\varepsilon, \Delta}(dz, dpdr, ds) \right) \\
 & - \int_0^t \frac{1}{\Delta} \int_s^{s+\Delta} \left[\mathcal{L}_{\bar{q}_\rho^\varepsilon}^m f \left(\bar{p}_\rho^\varepsilon, \frac{\bar{q}_\rho^\varepsilon}{\delta} \right) - \mathcal{L}_{\bar{q}_s^\varepsilon}^m f \left(\bar{p}_\rho^\varepsilon, \frac{\bar{q}_\rho^\varepsilon}{\delta} \right) \right] d\rho ds \\
 & - \int_{\mathcal{Z} \times \mathcal{Y} \times [0, t]} \mathcal{L}_{\bar{q}_s^\varepsilon}^m f(p, r) P^{\varepsilon, \Delta}(dz, dpdr, dt).
 \end{aligned} \tag{2.12}$$

Let us now analyze the different terms in (2.12). We start by observing that $\mathbb{E} [M_T^\varepsilon]^2 \leq C_0 \frac{1}{g(\varepsilon)}$, which then implies that $g(\varepsilon)M_t^\varepsilon \downarrow 0$ in probability, as $\varepsilon \downarrow 0$. Moreover, boundedness of f implies that $g(\varepsilon) \left[f \left(\bar{p}_t^\varepsilon, \frac{\bar{q}_t^\varepsilon}{\delta} \right) - f \left(\bar{p}_0^\varepsilon, \frac{\bar{q}_0^\varepsilon}{\delta} \right) \right]$ converges to zero uniformly. Hence, the left hand side of (2.12) converges to zero in probability as $\varepsilon \downarrow 0$.

Let us next study the right hand side of (2.12). We have the following

1. Conditions 1.2, the L^2 uniform bound on the controls and tightness of $\{\bar{q}^\varepsilon, \varepsilon > 0\}$, imply that the first and the third term in the right hand side of (2.12) converge to zero in probability as $\delta/\varepsilon \downarrow 0$.
2. The second term on the right hand side of (2.12) also converges to zero in probability, by the fact that $\delta/\varepsilon \downarrow 0$ and uniform integrability of $P^{\varepsilon, \Delta}$.

Thus, by combining the behavior of the different terms on the left and on the right hand side of (2.12), we obtain that we should necessarily have that

$$\int_{\mathcal{Z} \times \mathcal{Y} \times [0, T]} \mathcal{L}_{\bar{q}_t^\varepsilon}^m f(p, r) P^{\varepsilon, \Delta}(dz, dpdr, dt) \rightarrow 0, \quad \text{in probability.}$$

which by continuity in $t \in [0, T]$ gives (2.5).

Part 5. [Proof of (2.6).] Finally $P(\mathcal{Z} \times \mathcal{Y} \times [0, t]) = t$ follows from the fact that analogous property holds at the prelimit level, $P(\mathcal{Z} \times \mathcal{Y} \times \{t\}) = 0$ and the continuity of $t \rightarrow P(\mathcal{Z} \times \mathcal{Y} \times [0, t])$ and (2.6) follows. \square

3 Laplace principle

The main result of this section is the following Laplace principle. During the proof of Theorem 3.1 we also establish the alternative representation of Theorem 1.4.

Theorem 3.1. *Let $\{q_\bullet^\varepsilon, \varepsilon > 0\}$ be the unique strong solution to (1). Assume Conditions 1.2 and 1.3. Define*

$$S_m(\phi) = \inf_{(\phi, P) \in \mathcal{V}(\gamma, \mathcal{L}_q^m)} \left[\frac{1}{2} \int_{\mathcal{Z} \times \mathcal{Y} \times [0, T]} |z|^2 P(dz, dpdr, dt) \right] \tag{3.1}$$

with the convention that the infimum over the empty set is ∞ . Then for every bounded and continuous function h mapping $\mathcal{C}([0, T]; \mathbb{R}^d)$ into \mathbb{R} we have

$$\lim_{\varepsilon \downarrow 0} -\varepsilon \ln \mathbb{E}_{q_0} \left[\exp \left(-\frac{h(q_\bullet^\varepsilon)}{\varepsilon} \right) \right] = \inf_{\phi \in \mathcal{C}([0, T]; \mathbb{R}^d)} [S_m(\phi) + h(\phi)] .$$

Moreover, for each $s < \infty$, the set

$$\Phi_s = \{ \phi \in \mathcal{C}([0, T]; \mathbb{R}^d) : S_m(\phi) \leq s \}$$

is a compact subset of $\mathcal{C}([0, T]; \mathbb{R}^d)$.

In other words, $\{q_\bullet^\varepsilon, \varepsilon > 0\}$ satisfies the Laplace principle with rate function $S(\bullet)$.

Proof of Theorem 3.1. The proof of this theorem borrows some of the arguments of the related proof of the LDP for the elliptic overdamped case of Theorem 2.10 in [7]. We present here the main arguments, emphasizing the differences.

Part 1. [Laplace principle lower bound]. Theorem 2.2 and Fatou’s lemma, guarantee the validity of the following chain of inequalities.

$$\begin{aligned} \liminf_{\varepsilon \downarrow 0} \left(-\varepsilon \ln \mathbb{E} \left[\exp \left\{ -\frac{h(q^\varepsilon)}{\varepsilon} \right\} \right] \right) &\geq \liminf_{\varepsilon \downarrow 0} \left(\mathbb{E} \left[\frac{1}{2} \int_0^T |u_t^\varepsilon|^2 dt + h(\bar{q}^\varepsilon) \right] - \varepsilon \right) \\ &\geq \liminf_{\varepsilon \downarrow 0} \left(\mathbb{E} \left[\frac{1}{2} \int_0^T \frac{1}{\Delta} \int_t^{t+\Delta} |u_s^\varepsilon|^2 ds dt + h(\bar{q}^\varepsilon) \right] \right) \\ &= \liminf_{\varepsilon \downarrow 0} \left(\mathbb{E} \left[\frac{1}{2} \int_{\mathcal{Z} \times \mathcal{Y} \times [0, T]} |z|^2 P^{\varepsilon, \Delta}(dz, dpdr, dt) + h(\bar{q}^\varepsilon) \right] \right) \\ &\geq \mathbb{E} \left[\frac{1}{2} \int_{\mathcal{Z} \times \mathcal{Y} \times [0, T]} |z|^2 \bar{P}(dz, dpdr, dt) + h(\bar{q}) \right] \\ &\geq \inf_{(\phi, P) \in \mathcal{V}(\lambda, \mathcal{L}_q^m)} \left\{ \frac{1}{2} \int_{\mathcal{Z} \times \mathcal{Y} \times [0, T]} |z|^2 P(dz, dpdr, dt) + h(\phi) \right\} \\ &= \inf_{\phi \in \mathcal{C}([0, T]; \mathbb{R}^d)} [S_m(\phi) + h(\phi)] . \end{aligned}$$

Hence, the lower bound has been established.

Part 2. [Laplace principle upper bound and alternative representation]. We first observe that one can write (3.1) in terms of a local rate function

$$S_m(\phi) = \int_0^T L^r(\phi_s, \dot{\phi}_s) ds .$$

Here we set

$$L^r(x, \nu) = \inf_{P \in \mathcal{A}_{q, \nu}^r} \int_{\mathcal{Z} \times \mathcal{Y}} \frac{1}{2} |z|^2 P(dz, dpdr) ,$$

where

$$\mathcal{A}_{q, \nu}^r = \left\{ P \in \mathcal{P}(\mathcal{Z} \times \mathcal{Y}) : \int_{\mathcal{Z} \times \mathcal{Y}} \mathcal{L}_q^m f(p, r) P(dz, dpdr) = 0, \forall f \in \mathcal{C}_{\text{loc}}^2(\mathcal{Y}) , \right. \\ \left. \int_{\mathcal{Z} \times \mathcal{Y}} |z|^2 P(dz, dpdr) < \infty \text{ and } \nu = \int_{\mathcal{Z} \times \mathcal{Y}} \gamma(q, (p, r), z) P(dz, dpdr) \right\} .$$

We can decompose the measure $P \in \mathcal{P}(\mathcal{Z} \times \mathcal{Y})$ into the form

$$P(dz, dpdr) = \eta(dz|p, r) \mu(dpdr|q) ,$$

where μ is a probability measure on \mathcal{Y} and η is a stochastic kernel on \mathcal{Z} given \mathcal{Y} . This is referred to as the “relaxed” formulation because the control is characterized as a distribution on \mathcal{Z} (given q and (p, r)) rather than as an element of \mathcal{Z} . We now have, for every $f \in C_{\text{loc}}^2(\mathcal{Y})$ and for every $q \in \mathbb{R}^d$, that

$$\int_{\mathcal{Y}} \mathcal{L}_q^m f(p, r) \mu(dpdr) = 0 .$$

Here we have used the independence of \mathcal{L}_q^m on the control variable z to eliminate the stochastic kernel η . Thus $\mu(dpdr)$ is the unique corresponding to the operator \mathcal{L}_q^m , written as $\mu(dpdr|q)$.

Since the cost is convex in z and γ is affine in z , the relaxed control formulation is equivalent to the following ordinary control formulation of the local rate function

$$L^o(q, \nu) = \inf_{(v, \mu) \in \mathcal{A}_{q, \nu}^o} \frac{1}{2} \int_{\mathcal{Y}} |v(p, r)|^2 \mu(dpdr) ,$$

where

$$\mathcal{A}_{q, \nu}^o = \left\{ v(\bullet) : \mathcal{Y} \rightarrow \mathbb{R}^d, \mu \in \mathcal{P}(\mathcal{Y}), (v, \mu) \text{ satisfy } \int_{\mathcal{Y}} \mathcal{L}_q^m f(p, r) \mu(dpdr) = 0, \forall f \in C_{\text{loc}}^2(\mathcal{Y}), \int_{\mathcal{Y}} |v(p, r)|^2 \mu(dpdr) < \infty \text{ and } \nu = \int_{\mathcal{Y}} \gamma(q, (p, r), v(p, r)) \mu(dpdr) \right\} .$$

One can show as in [7, Section 5] that $L^r(q, \nu) = L^o(q, \nu)$. Let us recall now the definitions of $r_m(q)$ and $Q_m(q)$ from Theorem 1.4. For any $v \in \mathcal{A}_{q, \nu}^o$ we can write

$$\begin{aligned} \nu &= \int_{\mathcal{Y}} \gamma(q, (p, r), v(p, r)) \mu(dpdr|q) \\ &= \int_{\mathcal{Y}} \frac{1}{\sqrt{m}} [c(q, r) + \sigma(q, r)v(p, r)] \cdot \nabla_p \Phi(p, r) \mu(dpdr|q) \\ &= r_m(q) + \int_{\mathcal{Y}} \frac{1}{\sqrt{m}} \nabla_p \Phi(p, r) \sigma(q, r) (v(p, r))^T \mu(dpdr|q) . \end{aligned}$$

Then, $\nu - r_m(q)$ can be treated as β , and $\kappa(q, (p, r)) = \frac{1}{\sqrt{m}} (\nabla_p \Phi(p, r))^T (\sigma(q, r))^T$, $u(p, r) = (v(p, r))^T$ in Lemma 5.1 of [7]. We apply this lemma and then we get that for all $v \in \mathcal{A}_{q, \nu}^o$,

$$\int_{\mathcal{Y}} |v(p, r)|^2 \mu(dpdr|q) \geq (\nu - r_m(q))^T Q_m^{-1}(q) (\nu - r_m(q)) .$$

Moreover, if we take

$$v(p, r) = \bar{u}_\nu(q, (p, r)) = \frac{1}{\sqrt{m}} \sigma^T(q, r) (\nabla_p \Phi(p, r))^T Q_m^{-1}(q) (\nu - r_m(q)) , \tag{3.2}$$

we will have

$$\int_{\mathcal{Y}} |\bar{u}_\nu(q, (p, r))|^2 \mu(dpdr|q) = (\nu - r_m(q))^T Q_m^{-1}(q) (\nu - r_m(q)) .$$

This shows that

$$L^o(x, \nu) = \frac{1}{2} (\nu - r_m(q))^T Q_m^{-1}(q) (\nu - r_m(q)) ,$$

and the minimum is achieved with the control given by (3.2).

Now, that we have identified that the action functional can be written in the proceeding form we can proceed in proving the Laplace principle upper bound. We must show that for all bounded, continuous functions h mapping $\mathcal{C}([0, T]; \mathbb{R}^d)$ into \mathbb{R}

$$\limsup_{\varepsilon \downarrow 0} -\varepsilon \ln \mathbb{E} \left[\exp \left\{ -\frac{h(q^\varepsilon)}{\varepsilon} \right\} \right] \leq \inf_{\phi \in \mathcal{C}([0, T]; \mathbb{R}^d)} [S_m(\phi) + h(\phi)].$$

By the variational representation formula, it is enough to prove that

$$\limsup_{\varepsilon \downarrow 0} \inf_{u \in \mathcal{A}} \mathbb{E} \left[\frac{1}{2} \int_0^T |u_s|^2 ds + h(\bar{q}^\varepsilon) \right] \leq \inf_{\phi \in \mathcal{C}([0, T]; \mathbb{R}^d)} [S_m(\phi) + h(\phi)]. \tag{3.3}$$

To be precise, we consider for the limiting variational problem in the Laplace principle a nearly optimal control pair (ψ, P) . In particular, let $\eta > 0$ be given and consider $\psi \in \mathcal{C}([0, T]; \mathbb{R}^d)$ with $\psi_0 = q_o$ such that

$$S_m(\psi) + h(\psi) \leq \inf_{\phi \in \mathcal{C}([0, T]; \mathbb{R}^d)} [S_m(\phi) + h(\phi)] + \eta < \infty.$$

It is clear now that $L^o(x, \nu)$ is continuous and finite at each pair $(x, \nu) \in \mathbb{R}^{2d}$. Hence, a standard mollification argument, allows us to assume that $\dot{\psi}$ is piecewise constant, see Lemmas 6.5.3 and 6.5.5 in Subsection 6.5 of [6]. The control in feedback from used to prove (3.3) is then given by (3.2), i.e,

$$\bar{u}_t = \bar{u}_{\dot{\psi}_t} \left(\bar{q}_t, \left(\bar{p}_t^\varepsilon, \frac{\bar{q}_t^\varepsilon}{\delta} \right) \right).$$

It is easy to see that Condition 1.2 guarantees that \bar{u}_t is continuous in all of its arguments and that (1.3) has a unique strong solution with $u_t = \bar{u}_t$. Then, by Theorem 2.2, we obtain that in distribution $\bar{q}^\varepsilon \xrightarrow{\mathcal{D}} \bar{q}$, where

$$\bar{q}_t = q_o + \int_0^t \int_{\mathcal{Y}} \gamma \left(\bar{q}_s, (p, r), \bar{u}_{\dot{\psi}_s}(\bar{q}_s, (p, r)) \right) \mu(dpdr|\bar{q}_s) ds.$$

Keeping in mind the definition of $\mathcal{A}_{q, \dot{\psi}_t}^o$ and that $\psi_0 = q_o$, we obtain that

$$\bar{q}_t = q_o + \int_0^t \dot{\psi}_s ds = \psi_t \quad \text{for any } t \in [0, T], \text{ with probability } 1.$$

Therefore, we finally obtain that

$$\begin{aligned} \limsup_{\varepsilon \downarrow 0} \left[-\varepsilon \ln \mathbb{E} \left[\exp \left\{ -\frac{h(q^\varepsilon)}{\varepsilon} \right\} \right] \right] &= \limsup_{\varepsilon \downarrow 0} \inf_u \mathbb{E} \left[\frac{1}{2} \int_0^T |u_t|^2 dt + h(\bar{q}^\varepsilon) \right] \\ &\leq \limsup_{\varepsilon \downarrow 0} \mathbb{E} \left[\frac{1}{2} \int_0^T |\bar{u}_t|^2 dt + h(\bar{q}^\varepsilon) \right] \\ &= \mathbb{E} [S_m(\bar{X}) + h(\bar{X})] \\ &\leq \inf_{\phi \in \mathcal{C}([0, T]; \mathbb{R}^d)} [S_m(\phi) + h(\phi)] + \eta. \end{aligned}$$

Since η is arbitrary, we are done with the proof of the Laplace principle upper bound. At the same time get the explicit form of the rate function

$$S_m(\phi) = \begin{cases} \frac{1}{2} \int_0^T (\dot{\phi}_s - r_m(\phi_s))^T Q_m^{-1}(\phi_s) (\dot{\phi}_s - r_m(\phi_s)) ds & \text{if } \phi \in \mathcal{AC}([0, T]; \mathbb{R}^d), \phi_0 = q_o \\ +\infty & \text{otherwise} \end{cases} \tag{3.4}$$

Part 3. [Compactness of level sets]. This follows directly from the alternative representation (3.4), as it is in the standard quadratic form, see for example [7].

This concludes the proof of the theorem as well as of the alternative representation of Theorem 1.4. \square

4 Convergence of the action functional as $m \rightarrow 0$

Let $D > 0$ and set $\sigma(q, r) = \sqrt{2D\lambda(q)}I$. Recall the definitions of the operators \mathcal{L}_q^m and \mathcal{L}_q^0 and of the corresponding invariant measures from Section 1. Theorem 1.5 follows directly from Lemma 4.1 below, whose proof is based on Theorems 1.6 and 1.7.

Lemma 4.1. *Assume that Conditions 1.2 and 1.3 hold. Let $Q_m(q), r_m(q)$ and $Q_0(q), r_0(q)$ be as in Theorems 1.4 and 1.5 respectively. Then, for any $\eta > 0$, there exist some $m_0 > 0$ such that for every $q \in \mathbb{R}^d$ and every $0 < m < m_0$ we have*

$$|Q_m(q) - Q_0(q)| < \eta, |r_m(q) - r_0(q)| < \eta.$$

Proof. For notational convenience and without loss of generality, we shall set $\sigma(q, r) = 2I, D = \lambda(q) = 1$. Since q is viewed as a parameter, we do not mention it explicitly in the formulas. We have

$$\begin{aligned} Q_m - Q_0 &= 2 \int_{\mathcal{Y}} \left| \frac{1}{\sqrt{m}} \partial_p \Phi(p, r) \right|^2 \rho^0(p, r) \tilde{\rho}^m(p, r) dp dr - 2 \int_{\mathcal{Y}} |I + \partial_r \chi(r)|^2 \rho^0(p, r) dp dr \\ &= 2 \int_{\mathcal{Y}} \left[\left| \frac{1}{\sqrt{m}} \partial_p \Phi(p, r) \right|^2 - |I + \partial_r \chi(r)|^2 \right] \rho^0(p, r) \tilde{\rho}^m(p, r) dp dr \\ &\quad + 2 \int_{\mathcal{Y}} |I + \partial_r \chi(r)|^2 (\tilde{\rho}^m(p, r) - 1) \rho^0(p, r) dp dr \\ &= 2 \int_{\mathcal{Y}} \left[\left(\frac{1}{\sqrt{m}} \partial_p \Phi(p, r) - (I + \partial_r \chi(r)) \right) \left(\frac{1}{\sqrt{m}} \partial_p \Phi(p, r) + (I + \partial_r \chi(r)) \right)^T \right] \rho^m(p, r) dp dr \\ &\quad + 2 \int_{\mathcal{Y}} |I + \partial_r \chi(r)|^2 (\tilde{\rho}^m(p, r) - 1) \rho^0(p, r) dp dr. \end{aligned}$$

Taking absolute value and using Cauchy-Schwarz inequality we obtain

$$\begin{aligned} |Q_m - Q_0| &\leq 2 \left\| \frac{1}{\sqrt{m}} \partial_p \Phi - (I + \partial_r \chi) \right\|_{L^2(\mathcal{Y}; \rho^m)} \left\| \frac{1}{\sqrt{m}} \partial_p \Phi + (I + \partial_r \chi) \right\|_{L^2(\mathcal{Y}; \rho^m)} \\ &\quad + 2 \left\| (I + \partial_r \chi)^2 \right\|_{L^2(\mathcal{Y}; \rho^0)} \left\| \tilde{\rho}^m - 1 \right\|_{L^2(\mathcal{Y}; \rho^0)} \end{aligned}$$

By Theorem 1.6 we have

$$\lim_{m \rightarrow 0} \left\| \tilde{\rho}^m - 1 \right\|_{L^2(\mathcal{Y}; \rho^0)} = 0. \tag{4.1}$$

By Theorem 1.7 we have

$$\lim_{m \rightarrow 0} \left\| \frac{1}{\sqrt{m}} \partial_p \Phi - (I + \partial_r \chi) \right\|_{L^2(\mathcal{Y}; \rho^m)} = 0. \tag{4.2}$$

The results (4.1) and (4.2) imply that there exists a uniform constant C such that

$$\sup_{m \in (0, 1)} \left\| \frac{1}{\sqrt{m}} \partial_p \Phi \right\|_{L^2(\mathcal{Y}; \rho^m)} + \sup_{m \in (0, 1)} \left\| |I + \partial_r \chi|^2 \right\|_{L^2(\mathcal{Y}; \rho^m)} \leq C, \tag{4.3}$$

and by classical elliptic regularity theory there exists a uniform constant C , clearly independent of m , such that

$$\left\| (I + \partial_r \chi)^2 \right\|_{L^2(\mathcal{Y}; \rho^0)} \leq C. \tag{4.4}$$

From (4.1)–(4.4) we infer the first inequality of this Lemma. In a similar way from (4.1) and (4.3) we also derive the second estimate of this Lemma. \square

5 L^2 Convergence of the invariant density

In this section we prove Theorem 1.6. For notational convenience and without loss of generality, let us assume in this Section that $D = \lambda = 1$ and as a consequence that $\alpha(q, r) = 2I$ (recall $\sigma(q, r) = \sqrt{2D\lambda(q)}I$). Since $q \in \mathbb{R}^d$ is viewed as a parameter, it will not be mentioned explicitly.

We want to show that

$$\lim_{m \rightarrow 0} \|\tilde{\rho}^m - 1\|_{L^2(\mathcal{Y}; \rho^0)} = 0. \tag{5.1}$$

where we recall that $\rho^0(p, r) = \rho^{OU}(p)\rho_0(r)$.

Notice that in the case of gradient potential, i.e., when $b(q, r) = -\nabla_r V(q, r)$, then (5.1) is immediately true even without the limit. In fact in this case we have that the invariant density is basically $\rho^m(p, r) = \rho^{OU}(p)\rho_0(r)$ for every finite $m \in \mathbb{R}_+$ which implies that $\tilde{\rho}^m(p, r) = 1$ completing the proof of (5.1). Our goal here is to show that this is true in the more general setting where the drift is not necessarily of gradient form.

By Condition 1.2 the drift $b(q, r)$ and its partial derivatives are uniformly bounded with respect to q . For this reason we sometimes suppress the dependence on q and write $b(q, r) = b(r)$. Also, for notational convenience, let us set

$$h(r) = b(r) - \nabla_r \log \rho_0(r).$$

This definition for $h(r)$ will also be used throughout the rest of the paper.

Notice that in the gradient case, i.e. when $b(r) = -\nabla V(r)$, we have that $h(r) = 0$, but in the general case one has $h(r) \neq 0$. Let us next establish some useful relations

Lemma 5.1. *Let f, g be two functions that belong in the domain of definition of \mathcal{L}_q^m . Then, we have the identity*

$$\begin{aligned} & \int_{\mathcal{Y}} [(\mathcal{L}_q^m f(p, r)) g(p, r) + (\mathcal{L}_q^m g(p, r)) f(p, r)] \rho^0(p, r) dp dr = \\ & - \frac{2}{m} \int_{\mathcal{Y}} \nabla_p f(p, r) \nabla_p g(p, r) \rho^0(p, r) dp dr + \frac{1}{\sqrt{m}} \int_{\mathcal{Y}} f(p, r) g(p, r) h(r) p \rho^0(p, r) dp dr. \end{aligned}$$

In particular, we have that

$$\begin{aligned} & \int_{\mathcal{Y}} (\mathcal{L}_q^m f(p, r)) f(p, r) \rho^0(p, r) dp dr = \\ & = - \frac{1}{m} \int_{\mathcal{Y}} |\nabla_p f(p, r)|^2 \rho^0(p, r) dp dr + \frac{1}{2\sqrt{m}} \int_{\mathcal{Y}} |f(p, r)|^2 h(r) p \rho^0(p, r) dp dr. \end{aligned}$$

Lemma 5.2. *Let f, g be two functions that are in $\mathcal{W}_2^{1,0}(\mathcal{Y})$, i.e., the set where functions and their first derivatives with respect to p are in $L^2(\mathcal{Y})$. Then, there exists a finite constant $K < \infty$ that depends only on $\sup_{r \in \mathbb{T}^d} |h(r)|$ such that*

$$\left| \langle h(r)p, fg \rangle_{L^2(\mathcal{Y}; \rho^0)} \right| \leq K \left[\|f\|_{L^2(\mathcal{Y}; \rho^0)} \|\nabla_p g\|_{L^2(\mathcal{Y}; \rho^0)} + \|\nabla_p f\|_{L^2(\mathcal{Y}; \rho^0)} \|g\|_{L^2(\mathcal{Y}; \rho^0)} \right].$$

Lemma 5.3. For every $\eta > 0$, there exists constant $K < \infty$ that depends only on $\sup_{r \in \mathbb{T}^d} |h(r)|$ such that

$$\langle f, \mathcal{B}f \rangle_{L^2(\mathcal{Y}; \rho^0)} = \frac{1}{2} \langle ph(r), |f|^2 \rangle_{L^2(\mathcal{Y}; \rho^0)} \geq -K \left[\eta \|f\|_{L^2(\mathcal{Y}; \rho^0)}^2 + \frac{1}{4\eta} \|\nabla_p f\|_{L^2(\mathcal{Y}; \rho^0)}^2 \right],$$

where we recall that $\mathcal{B}f = p \cdot \nabla_r f + b(q, r) \nabla_p f$.

The proof of Lemmas 5.1–5.3 are in Appendix B. Let us now define

$$\delta^m(p, r) = \tilde{\rho}^m(p, r) - 1.$$

Recall that our goal is to prove Theorem 1.6, i.e. that (5.1) holds. The next lemmas are towards this direction. The proof of Lemmas 5.4–5.6 are in Appendix B.

Lemma 5.4. For every $m > 0$ we have the following equality

$$\|\nabla_p \delta^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 = \frac{\sqrt{m}}{2} \langle h(r)p, |\delta^m|^2 \rangle_{L^2(\mathcal{Y}; \rho^0)} + \sqrt{m} \langle h(r)p, \delta^m \rangle_{L^2(\mathcal{Y}; \rho^0)}.$$

Lemma 5.5. There is a universal constant $K > 0$ that depends on $\sup_{r \in \mathbb{T}^d} |h(r)|$, but not on $m > 0$, such that for all m sufficiently small

$$(1 - \sqrt{m}) \|\nabla_p \nabla_p \delta^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 \leq \sqrt{m} K \left[1 + \|\delta^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 + \|\nabla_p \delta^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 \right].$$

Lemma 5.6. There is a universal constant $K > 0$ that depends on $\sup_{r \in \mathbb{T}^d} \max(|h(r)|, |\nabla_r h(r)|)$, but not on $m > 0$, such that for all m sufficiently small

$$\begin{aligned} (1 - \sqrt{m}) \|\nabla_p \nabla_r \delta^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 &\leq \\ &\leq \sqrt{m} K \left[1 + \|\delta^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 + \|\nabla_p \delta^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 + \|\nabla_r \delta^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 \right]. \end{aligned}$$

Let us define \mathcal{L}^1 to be the operator \mathcal{L}_q^m with $m = 1$. We recall that

$$\mathcal{L}^1 = \mathcal{A} + \mathcal{B},$$

where $\mathcal{A} = -p \cdot \nabla_p + \Delta_p$ and $\mathcal{B} = p \cdot \nabla_r + b(q, r) \cdot \nabla_p$. It is easy to check that, with respect to the measure $\rho^0(p, r) dp dr$ we can actually write that

$$\mathcal{L}^1 = -AA^* + \mathcal{B}$$

where

$$A = \nabla_p, \quad \text{and} \quad A^* = -(\nabla_p - p).$$

One can also check that the adjoint operator of \mathcal{B} is formally given by

$$\mathcal{B}^* = -\mathcal{B} + ph(r).$$

Notice that the latter relation implies that \mathcal{B} is antisymmetric only if $h(r) = 0$ which essentially is the case of gradient drift. However, in the general case $h(r) \neq 0$ which would imply that \mathcal{B} is not antisymmetric. Next, we introduce the operator

$$\mathcal{C} = [\mathcal{A}, \mathcal{B}] = [\nabla_p, p \nabla_r + b(r) \nabla_p] = \nabla_r.$$

A word on notation now. In order to make the notation lighter we will write from now on

$$\|\cdot\| = \|\cdot\|_{L^2(\mathcal{Y}; \rho^0)}, \quad \text{and} \quad \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{L^2(\mathcal{Y}; \rho^0)},$$

for the norm and for the inner product in the space $L^2(\mathcal{Y}; \rho^0)$.

In order to show that (5.1) holds, we use the work of [30]. In particular, as in [30], let a, b, c be constants to be chosen such that $1 > a > b > c > 0$ and let us define the norm

$$((f, f)) = \|f\|^2 + \alpha \|Af\|^2 + 2b\mathcal{R} \langle Af, Cf \rangle + c \|Cf\|^2 .$$

In fact, as it is argued in [30], the norms $((f, f))$ and $\|f\|_{H^1(\mathcal{Y}; \rho^0)}^2$ are equivalent as soon as $b < \sqrt{ac}$, in that

$$\min\{1, a, c\} \left(1 - \frac{b}{\sqrt{ac}}\right) \|f\|_{H^1(\mathcal{Y}; \rho^0)}^2 \leq ((f, f)) \leq \max\{1, a, c\} \left(1 + \frac{b}{\sqrt{ac}}\right) \|f\|_{H^1(\mathcal{Y}; \rho^0)}^2 .$$

Since, we are dealing with a real Hilbert space, all the inner products are real. By polarization we have

$$((f, \mathcal{L}^1 f)) = \langle f, \mathcal{L}^1 f \rangle + a \langle Af, A\mathcal{L}^1 f \rangle + b [\langle A\mathcal{L}^1 f, Cf \rangle + \langle Af, C\mathcal{L}^1 f \rangle] + c \langle Cf, C\mathcal{L}^1 f \rangle .$$

One important difference between the current setup and the setup of [30] is that there $\mathcal{B}^* = -\mathcal{B}$ whereas here that is not the case, as we have $\mathcal{B}^* = -\mathcal{B} + ph(r)$. Keeping that in mind and repeating the argument of the proof of Theorem 18 in [30], we obtain that there are constants a, b, c that are sufficiently small such that $1 \gg a \gg b \gg 2c$ with $b^2 < ac$ (the exact same constants as in [30]) such that

$$\begin{aligned} ((f, \mathcal{L}^1 f)) &\geq K \left[\|Af\|^2 + \|Cf\|^2 \right] \\ &\quad + \{ \langle f, \mathcal{B}f \rangle + a \langle Af, \mathcal{B}Af \rangle + b \langle ph(r)Af, Cf \rangle + c \langle Cf, \mathcal{B}Cf \rangle \} \\ &\geq K \|f\|_{H^1}^2 + \{ \langle f, \mathcal{B}f \rangle + a \langle Af, \mathcal{B}Af \rangle + b \langle ph(r), AfCf \rangle + c \langle Cf, \mathcal{B}Cf \rangle \} . \end{aligned} \tag{5.2}$$

The bracket term of the right hand side of the inequality is due to the fact that in our case $h(r) \neq 0$ and thus \mathcal{B} is not anti-symmetric. The bracket term is equal to zero in [30].

Let us now choose $f = \delta^m$ in (5.2). The strategy of the proof is: (a) bound from below the bracket term on the right hand side of (5.2) using Lemmas 5.2-5.6 and the equation that δ^m satisfies, and (b) bound from above the left hand side of (5.2) using Lemmas 5.2-5.6 and the equation that δ^m satisfies. Putting the two bounds together one will then obtain a bound for $\|\delta^m\|_{H^1}^2$ which will give the convergence to zero of (5.1) that we need, combined with Poincaré inequality for the measure $\rho^0(p, r)dpdr$.

We would like to highlight here that one of the obstacles in putting the lower and upper bounds together, are the order one terms $\langle f, \mathcal{L}^1 f \rangle$ in the definition of $((f, \mathcal{L}^1 f))$ and $\langle f, \mathcal{B}f \rangle$ in the lower bound (5.2). However, as it turns out, see (5.6), for $f = \delta^m$, we actually have that $\langle \mathcal{L}^1 \delta^m, \delta^m \rangle - \langle \mathcal{B} \delta^m, \delta^m \rangle = o(\sqrt{m})$ which then allows us to proceed with the bounds. The rest of the terms are being handled via Lemmas 5.2-5.6.

We start with obtaining a lower bound for the bracket term on the right hand side of (5.2) using Lemmas 5.4-5.6 and the equation that δ^m satisfies. For this purpose, let us define

$$\begin{aligned} R(\delta^m) &= \langle \delta^m, \mathcal{B} \delta^m \rangle + a \langle A \delta^m, \mathcal{B} A \delta^m \rangle + b \langle ph(r) A \delta^m, C \delta^m \rangle + c \langle C \delta^m, \mathcal{B} C \delta^m \rangle \\ &= \langle \delta^m, \mathcal{B} \delta^m \rangle + R_1(\delta^m) . \end{aligned}$$

Let $\eta > 0$ to be chosen. By Lemmas 5.2-5.3, recalling that $A \delta^m = \nabla_p \delta^m$ and $C \delta^m = \nabla_r \delta^m$ and using the generalized Cauchy inequality $ab \leq \eta a^2 + \frac{1}{4\eta} b^2$ we have that

$$\begin{aligned} R_1(\delta^m) &\geq -K \left\{ a \left[\eta \|\nabla_p \delta^m\|^2 + \frac{1}{4\eta} \|\nabla_p \nabla_p \delta^m\|^2 \right] + b \left[\eta \|\nabla_r \delta^m\|^2 + \frac{1}{4\eta} \|\nabla_p \nabla_p \delta^m\|^2 \right] \right. \\ &\quad \left. + b \left[\eta \|\nabla_p \delta^m\|^2 + \frac{1}{4\eta} \|\nabla_p \nabla_r \delta^m\|^2 \right] + c \left[\eta \|\nabla_r \delta^m\|^2 + \frac{1}{4\eta} \|\nabla_p \nabla_r \delta^m\|^2 \right] \right\} . \end{aligned}$$

Next, using Lemmas 5.4, 5.5 and 5.6 we subsequently obtain

$$\begin{aligned}
 R_1(\delta^m) &\geq -K \left\{ a \left[\eta \|\nabla_p \delta^m\|^2 + \frac{\sqrt{m}}{4\eta} (1 + \|\delta^m\|^2 + \|\nabla_p \delta^m\|^2) \right] \right. \\
 &\quad + b \left[\eta \|\nabla_r \delta^m\|^2 + \frac{\sqrt{m}}{4\eta} (1 + \|\delta^m\|^2 + \|\nabla_p \delta^m\|^2) \right] \\
 &\quad + b \left[\eta \|\nabla_p \delta^m\|^2 + \frac{\sqrt{m}}{4\eta} (\|\delta^m\|^2 + \|\nabla_p \delta^m\|^2 + \|\nabla_r \delta^m\|^2) \right] \\
 &\quad \left. + c \left[\eta \|\nabla_r \delta^m\|^2 + \frac{\sqrt{m}}{4\eta} (\|\delta^m\|^2 + \|\nabla_p \delta^m\|^2 + \|\nabla_r \delta^m\|^2) \right] \right\} \\
 &\geq -K \left[\eta (\|\nabla_p \delta^m\|^2 + \|\nabla_r \delta^m\|^2) + \frac{\sqrt{m}}{4\eta} (1 + \|\delta^m\|^2 + \|\nabla_p \delta^m\|^2 + \|\nabla_r \delta^m\|^2) \right] \\
 &\geq -K \left[\eta \|\delta^m\|_{H^1}^2 + \frac{\sqrt{m}}{4\eta} (1 + \|\delta^m\|^2 + \|\delta^m\|_{H^1}^2) \right],
 \end{aligned}$$

where the positive constant $K < \infty$ may change from line to line but it is always independent of m . Choosing now $\eta = \eta(m)$ such that $\lim_{m \downarrow 0} \eta(m) = \lim_{m \downarrow 0} \frac{\sqrt{m}}{\eta(m)} = 0$, we obtain for $\hat{\eta}(m) = \max\{\eta(m), \frac{\sqrt{m}}{\eta(m)}\} \downarrow 0$, that

$$R_1(\delta^m) \geq -K\hat{\eta}(m) [1 + \|\delta^m\|^2 + \|\delta^m\|_{H^1}^2].$$

So, overall we have that for m sufficiently small there is $\hat{\eta}(m) \downarrow 0$ as $m \downarrow 0$ such that

$$R(\delta^m) \geq \langle \delta^m, \mathcal{B}\delta^m \rangle - K\hat{\eta}(m) [1 + \|\delta^m\|^2 + \|\delta^m\|_{H^1}^2],$$

or in other words by (5.2) with $f = \delta^m$ we have that for m sufficiently small there is $\hat{\eta}(m) \downarrow 0$ as $m \downarrow 0$ such that

$$(\langle \delta^m, \mathcal{L}^1 \delta^m \rangle) \geq K(1 - \hat{\eta}(m)) \|\delta^m\|_{H^1}^2 + \langle \delta^m, \mathcal{B}\delta^m \rangle - K\hat{\eta}(m) [1 + \|\delta^m\|^2]. \tag{5.3}$$

Hence, recalling the definition of the inner product $(\langle \cdot, \cdot \rangle)$, using (5.3) and rearranging the expression a little bit we have obtained the following bound

$$\begin{aligned}
 K(1 - \hat{\eta}(m)) \|\delta^m\|_{H^1}^2 &\leq \langle \delta^m, \mathcal{L}^1 \delta^m \rangle - \langle \delta^m, \mathcal{B}\delta^m \rangle + K\hat{\eta}(m) [1 + \|\delta^m\|^2] \\
 &\quad + a \langle A\delta^m, A\mathcal{L}^1 \delta^m \rangle + b [\langle A\mathcal{L}^1 \delta^m, C\delta^m \rangle + \langle A\delta^m, C\mathcal{L}^1 \delta^m \rangle] + c \langle C\delta^m, C\mathcal{L}^1 \delta^m \rangle.
 \end{aligned} \tag{5.4}$$

The next goal is to derive an appropriate upper bound for the left hand side of (5.4). First, we need to obtain the equation that δ^m satisfies. By factoring out $\rho^m(p, r) = \rho^0(p, r)\tilde{\rho}^m(p, r)$ where $\rho^0(p, r) = \rho^{OU}(p)\rho_0(r)$, we obtain the following equation for $\tilde{\rho}^m(p, r)$:

$$\mathcal{L}_q^m \tilde{\rho}^m(p, r) = \frac{2}{\sqrt{m}} \mathcal{B}\tilde{\rho}^m(p, r) - \frac{1}{\sqrt{m}} ph(r)\tilde{\rho}^m(p, r).$$

where we recall that $h(r) = b(r) - \nabla_r \log \rho_0(r)$. Hence, the equation for $\delta^m(p, r) = \tilde{\rho}^m(p, r) - 1$ is

$$\mathcal{L}_q^m \delta^m(p, r) = \frac{2}{\sqrt{m}} \mathcal{B}\delta^m(p, r) - \frac{1}{\sqrt{m}} ph(r) [\delta^m(p, r) + 1],$$

or in terms of the operator $\mathcal{L}^1 = \mathcal{L}_q^{m=1}$ we have

$$\mathcal{L}^1 \delta^m(p, r) = (1 + \sqrt{m}) \mathcal{B}\delta^m(p, r) - \sqrt{m} ph(r) [\delta^m(p, r) + 1]. \tag{5.5}$$

By multiplying both sides of (5.5) by δ^m and integrating over \mathcal{Y} with respect to the measure $\rho^0(p, r)dpdr$ we then obtain that

$$\langle \mathcal{L}^1 \delta^m, \delta^m \rangle - \langle \mathcal{B} \delta^m, \delta^m \rangle = \sqrt{m} \langle \mathcal{B} \delta^m, \delta^m \rangle - \sqrt{m} \langle ph(r), (\delta^m + 1) \delta^m \rangle . \quad (5.6)$$

Hence, using (5.4) and (5.6) we have the following bound

$$\begin{aligned} K(1 - \hat{\eta}(m)) \|\delta^m\|_{H^1}^2 &\leq \sqrt{m} \langle \mathcal{B} \delta^m, \delta^m \rangle - \sqrt{m} \langle ph(r), (\delta^m + 1) \delta^m \rangle + K \hat{\eta}(m) [1 + \|\delta^m\|^2] \\ &\quad + a \langle A \delta^m, A \mathcal{L}^1 \delta^m \rangle + b [\langle A \mathcal{L}^1 \delta^m, \mathcal{C} \delta^m \rangle + \langle A \delta^m, \mathcal{C} \mathcal{L}^1 \delta^m \rangle] + c \langle \mathcal{C} \delta^m, \mathcal{C} \mathcal{L}^1 \delta^m \rangle \\ &\leq K \hat{\eta}(m) [1 + \|\delta^m\|^2] \\ &\quad + \sqrt{m} [\langle \mathcal{B} \delta^m, \delta^m \rangle - \langle ph(r), (\delta^m + 1) \delta^m \rangle] \\ &\quad + a \langle A \delta^m, A \mathcal{L}^1 \delta^m \rangle + b [\langle A \mathcal{L}^1 \delta^m, \mathcal{C} \delta^m \rangle + \langle A \delta^m, \mathcal{C} \mathcal{L}^1 \delta^m \rangle] + c \langle \mathcal{C} \delta^m, \mathcal{C} \mathcal{L}^1 \delta^m \rangle \\ &\leq K \hat{\eta}(m) [1 + \|\delta^m\|^2] \\ &\quad + T_1(\delta^m) + aT_2(\delta^m) + bT_3(\delta^m) + cT_4(\delta^m) . \end{aligned} \quad (5.7)$$

Our next goal is to derive upper bounds for the terms $T_i(\delta^m)$ for $i = 1, 2, 3, 4$. For better readability, we collect the required bounds in the following lemma, which we also prove in Appendix B.

Lemma 5.7. *Let the terms $T_i(\delta^m)$ for $i = 1, 2, 3, 4$ be defined as in (5.7). Then, there exists a constant $K < \infty$ that does not depend on m , and a sequence $\eta(m), \frac{\sqrt{m}}{\eta(m)} \downarrow 0$ as $m \rightarrow 0$ such that for m sufficiently small the following bounds holds*

$$\begin{aligned} |T_1(\delta^m)| &\leq \sqrt{m} \frac{K+2}{2} \|\delta^m\|^2 + \sqrt{m} \frac{K}{2} \|\nabla_p \delta^m\|^2 ; \\ |T_2(\delta^m)| &\leq \left(\eta(m) + \sqrt{m} + \frac{\sqrt{m}}{\eta(m)} \right) K \|\delta^m\|_{H^1}^2 + \left(\frac{\sqrt{m}}{\eta(m)} + \sqrt{m} \right) K (1 + \|\delta^m\|^2) ; \\ |T_3(\delta^m)| &\leq \left(\eta(m) + \sqrt{m} + \frac{\sqrt{m}}{\eta(m)} \right) K \|\delta^m\|_{H^1}^2 + \frac{\sqrt{m}}{\eta(m)} K [1 + \|\delta^m\|^2] + (1 + \sqrt{m}) \|\nabla_r \delta^m\|^2 ; \\ |T_4(\delta^m)| &\leq \left(\eta(m) + \frac{\sqrt{m}}{\eta(m)} \right) K \|\delta^m\|_{H^1}^2 + \frac{\sqrt{m}}{\eta(m)} \|\delta^m\|^2 . \end{aligned}$$

Now that we have obtained the desired bounds for the terms $T_i(\delta^m)$ for $i = 1, 2, 3, 4$ let us put them together. There are some constants $K_1, K_2 < \infty$, and a sequence $\hat{\eta}(m) = \max\{\eta(m), \frac{\sqrt{m}}{\eta(m)}\} \downarrow 0$ such that for m sufficiently small

$$\begin{aligned} (1 - \hat{\eta}(m)) \|\delta^m\|_{H^1}^2 &\leq \hat{\eta}(m) K_1 [1 + \|\delta^m\|^2] + \left[\sqrt{m} K_1 \|\delta^m\|^2 + \sqrt{m} K_1 \|\nabla_p \delta^m\|^2 \right] \\ &\quad + a \left[\hat{\eta}(m) K_1 \|\delta^m\|_{H^1}^2 + \hat{\eta}(m) K_1 (1 + \|\delta^m\|^2) \right] \\ &\quad + b \left[\hat{\eta}(m) K_1 [1 + \|\delta^m\|^2 + \|\delta^m\|_{H^1}^2] + (1 + \sqrt{m}) \|\nabla_r \delta^m\|^2 \right] \\ &\quad + c \left[\hat{\eta}(m) K_1 [\|\delta^m\|_{H^1}^2 + \|\delta^m\|^2] \right] \\ &\leq (\hat{\eta}(m) + \sqrt{m}) K_2 [1 + \|\delta^m\|^2 + \|\delta^m\|_{H^1}^2] + b(1 + \sqrt{m}) \|\nabla_r \delta^m\|^2 . \end{aligned} \quad (5.8)$$

Now we choose m small enough such that $\hat{\eta}(m) < 1, (\hat{\eta}(m) + \sqrt{m}) K_2 < 1/2$. Moreover, we also note that since by construction $b \ll 1$ we can write for m small enough $b(1 + \sqrt{m}) \ll 1/2$. In fact the proof of [30] shows that we can choose a, b, c to be positive but as small as we want, as long we choose the constants a, b, c to be ordered appropriately. Putting these estimates together, we get that there is some constant $K_3 < \infty$ such that

for m small enough, one has

$$\|\delta^m\|_{H^1}^2 \leq K_3 \frac{\hat{\eta}(m) + \sqrt{m}}{1/2 - \hat{\eta}(m)} [1 + \|\delta^m\|^2] . \tag{5.9}$$

In order now to close the estimate we need to use Poincaré inequality. Here we make the assumption that the drift $b(r)$ is such that the invariant measure $\rho^0(p, r)dpdr$ satisfies the Poincaré inequality with constant $\kappa > 0$. In particular, for a function $Q(p, r)$, we have that the Poincaré inequality in the following form holds

$$\left\| Q - \int_{\mathcal{Y}} Q \right\|_{L^2(\mathcal{Y}; \rho^0)}^2 \leq \kappa \|Q\|_{H^1(\mathcal{Y}; \rho^0)}^2 .$$

Let us set now $Q(p, r) = \delta^m(p, r)$. Notice that by definition of $\delta^m(p, r)$ we have

$$\int_{\mathcal{Y}} \delta^m(p, r) \rho^0(p, r) dpdr = 0 .$$

Therefore, we have obtained

$$\|\delta^m\|^2 \leq \kappa \|\delta^m\|_{H^1}^2 . \tag{5.10}$$

Inserting now (5.10) into (5.9), we finally obtain that for m small enough

$$\|\delta^m\|^2 \leq \frac{K_3}{\kappa} \frac{\hat{\eta}(m) + \sqrt{m}}{1/2 - \hat{\eta}(m)} [1 + \|\delta^m\|^2] , \tag{5.11}$$

from which the desired result finally follows:

$$\|\delta^m\|^2 \leq K_4 (\hat{\eta}(m) + \sqrt{m}) \rightarrow 0 .$$

This concludes the $L^2(\mathcal{Y}; \rho^0)$ convergence of the invariant measures.

6 Convergence of the solution to the cell problem

The goal of this section is to analyze the cell problem (1.4) that $\Phi(p, r)$ satisfies and we want to prove Theorem 1.7. As it will become clear from the proof below, we prove even more. We rigorously derive an asymptotic expansion of $\Phi(p, r)$ in terms of powers of \sqrt{m} .

Let us recall our assumption $\alpha(q, r) = 2D\lambda(q)I$. Let $\ell = 1, 2, \dots, d$ be a given direction and let us define

$$\Psi_\ell(p, r) = \Phi_\ell(p, r) - \frac{\sqrt{m}}{\lambda(q)} p \cdot e_\ell ,$$

where e_ℓ is the unit vector in direction ℓ . Then, bearing in mind (1.4) the equation that $\Psi_\ell(p, r)$ satisfies is given by

$$\mathcal{L}_q^m \Psi_\ell(p, r) = -\frac{b_\ell(q, r)}{\lambda(q)} . \tag{6.1}$$

Moreover, by Condition 1.3 we have that that for every $m > 0$

$$\int_{\mathcal{Y}} \Psi_\ell(p, r) \mu(dr dp | q) = 0 .$$

Let us write for notational convenience the hypoelliptic operator

$$\mathcal{L}_q^m = \frac{\lambda(q)}{m} \mathcal{A} + \frac{1}{\sqrt{m}} \mathcal{B} ,$$

where we have already defined $\mathcal{A} = -p \cdot \nabla_p + D\Delta_p$ and $\mathcal{B} = p \cdot \nabla_r + b(q, r) \cdot \nabla_p$.

Let us now write the expansion

$$\Psi_\ell(p, r) = \Psi_{\ell,0}(p, r) + \sqrt{m}\Psi_{\ell,1}(p, r) + m\Psi_{\ell,2}(p, r) + \Psi_{\ell,3}^m(p, r).$$

Assume that the functions $\Psi_{\ell,0}, \Psi_{\ell,1}, \Psi_{\ell,2}$ and $\Psi_{\ell,3}^m$ satisfy the following equations

$$\mathcal{A}\Psi_{\ell,0}(p, r) = 0, \tag{6.2}$$

$$\mathcal{B}\Psi_{\ell,0}(p, r) + \lambda(q)\mathcal{A}\Psi_{\ell,1}(p, r) = 0, \tag{6.3}$$

$$\mathcal{B}\Psi_{\ell,1}(p, r) + \lambda(q)\mathcal{A}\Psi_{\ell,2}(p, r) = -b_\ell(q, r), \tag{6.4}$$

$$\mathcal{L}_q^m \Psi_{\ell,3}^m(p, r) = -\sqrt{m}\mathcal{B}\Psi_{\ell,2}(p, r). \tag{6.5}$$

and that for $i = 0, 1, 2, 3$, we have that $\int_{\mathcal{Y}} \Psi_{\ell,i}(p, r) \rho^m(p, r|q) = 0$. The next step is to analyze the solutions to (6.2)–(6.5). First we notice that (6.2) basically implies that $\Psi_{\ell,0}(p, r) = \Psi_{\ell,0}(r)$, i.e., function $\Psi_{\ell,0}(r)$ is a function of r alone. Then, using this we get by (6.3) that it has to be the case that

$$\Psi_{\ell,1}(p, r) = \frac{1}{\lambda(q)} \nabla_r \Psi_{\ell,0}(r) \cdot p + \hat{\Psi}_{\ell,0}(r)$$

for some function $\hat{\Psi}_{\ell,0}(r)$. From equation (6.1) and (6.2), (6.3), (6.4), (6.5) we see that up to an additive constant we can assume that $\hat{\Psi}_{\ell,0}(r) = 0$. Lastly, we notice that the solvability condition for (6.4) is

$$\begin{aligned} \int_{\mathbb{R}^d} [\mathcal{B}\Psi_{\ell,1}(p, r) + b_\ell(q, r)] \pi(dp) &= 0 \Rightarrow \\ \int_{\mathbb{R}^d} [\mathcal{B}(\nabla_r \Psi_{\ell,0}(r) \cdot p) + \mathcal{B}\hat{\Psi}_{\ell,0}(r) + b_\ell(q, r)] \pi(dp) &= 0 \Rightarrow \\ \int_{\mathbb{R}^d} [\Delta_r \Psi_{\ell,0}(r) |p|^2 + b(q, r) \cdot \nabla_r \Psi_{\ell,0}(r) + p \cdot \nabla_r \hat{\Psi}_{\ell,0}(r) + b_\ell(q, r)] \pi(dp) &= 0 \Rightarrow \\ D\Delta_r \Psi_{\ell,0}(r) + b(q, r) \cdot \nabla_r \Psi_{\ell,0}(r) &= -b_\ell(q, r), \end{aligned}$$

where the Gaussian structure of the invariant measure $\pi(dp) \sim e^{-\frac{|p|^2}{2D}} dp$ and integration by parts were used. Notice that this is exactly the solution to (1.5) with $\alpha = 2D\lambda(q)I$. Thus, by uniqueness of the solution to (1.5) we basically have that for every $\ell = 1, \dots, d$

$$\Psi_{\ell,0}(r) = \chi_\ell(r).$$

Hence, we have that

$$\begin{aligned} \nabla_p \Phi(p, r) &= \nabla_p \left(\frac{\sqrt{m}}{\lambda(q)} p + \Psi(p, r) \right) \\ &= \nabla_p \left(\frac{\sqrt{m}}{\lambda(q)} p + \Psi_0(p, r) + \sqrt{m}\Psi_1(p, r) + m\Psi_2(p, r) + \Psi_3^m(p, r) \right) \\ &= \frac{\sqrt{m}}{\lambda(q)} [I + \nabla_r \chi(r)] + m\nabla_p \Psi_2(p, r) + \nabla_p \Psi_3^m(p, r). \end{aligned}$$

Having established the last display, it is easy to see that in order to show (4.2), we basically need to show that

$$\lim_{m \rightarrow 0} \left\| \sqrt{m} \nabla_p \Psi_2(p, r) + \frac{1}{\sqrt{m}} \nabla_p \Psi_3^m(p, r) \right\|_{L^2(\mathcal{Y}; \rho^m)} = 0,$$

or, in other words, it is sufficient to show

$$\lim_{m \rightarrow 0} \left\| \sqrt{m} \nabla_p \Psi_2(p, r) \right\|_{L^2(\mathcal{Y}; \rho^m)} = 0, \tag{6.6}$$

and

$$\lim_{m \rightarrow 0} \left\| \frac{1}{\sqrt{m}} \nabla_p \Psi_3^m(p, r) \right\|_{L^2(\mathcal{Y}; \rho^m)} = 0. \tag{6.7}$$

Relation (6.6) can be claimed to be true by the fact that $\Psi_2(p, r)$ is solution to the elliptic problem (6.4) and Theorem 1.6.

So, it remains to prove (6.7). At this point let us recall that $\Psi_{\ell,3}^m(p, r)$ is solution to (6.5), i.e., it solves

$$\mathcal{L}_q^m \Psi_{\ell,3}^m(p, r) = -\sqrt{m} \mathcal{B} \Psi_{\ell,2}(p, r). \tag{6.8}$$

Notice that for the purposes of this section q is seen as a fixed parameter by the operators and recall that we have already assumed $\alpha(q, r) = 2D\lambda(q)I$. Namely $D\lambda(q)$ is seen as a fixed constant. Hence, from now on and for notational convenience, we shall assume without loss of generality that $\alpha(q, r) = 2I$, i.e., that $D = \lambda(q) = 1$. Let us first apply Lemma 5.1 and we get

$$\begin{aligned} \int_{\mathcal{Y}} (\mathcal{L}_q^m \Psi_{\ell,3}^m) \Psi_{\ell,3}^m \rho^0(p, r) dp dr &= -\frac{1}{m} \int_{\mathcal{Y}} |\nabla_p \Psi_{\ell,3}^m|^2 \rho^0(p, r) dp dr \\ &+ \frac{1}{2\sqrt{m}} \int_{\mathcal{Y}} (\Psi_{\ell,3}^m)^2 h(r) p \rho^0(p, r) dp dr. \end{aligned} \tag{6.9}$$

Lemmas 6.1-6.4 that follow are proven in Appendix C.

Lemma 6.1. *We have the Poincaré inequality*

$$\left\| f - \int_{\mathcal{Y}} f(p, r) \rho^0(p, r) dp dr \right\|_{L^2(\mathcal{Y}; \rho^0)} \leq \kappa \|\nabla_p f\|_{L^2(\mathcal{Y}; \rho^0)}, \tag{6.10}$$

for some constant $\kappa > 0$ independent of m .

Lemma 6.2. *We have*

$$\limsup_{m \rightarrow 0} \frac{\|\Psi_{\ell,3}^m\|_{L^2(\mathcal{Y}; \rho^0)}}{m^{3/2}} \leq C < \infty, \quad \limsup_{m \rightarrow 0} \frac{\left\| \frac{1}{\sqrt{m}} \nabla_p \Psi_{\ell,3}^m \right\|_{L^2(\mathcal{Y}; \rho^0)}}{m} \leq C < \infty \tag{6.11}$$

for some constant $C > 0$ independent of m .

Lemma 6.3. *We have*

$$\lim_{m \rightarrow 0} \|\Psi_{\ell,3}^m\|_{L^4(\mathcal{Y}; \rho^0)} = 0. \tag{6.12}$$

Lemma 6.4. *We have*

$$\begin{aligned} \int_{\mathcal{Y}} (\mathcal{L}_q^m f(p, r)) g(p, r) \rho^m(p, r) dp dr + \int_{\mathcal{Y}} f(p, r) (\mathcal{L}_q^m g(p, r)) \rho^m(p, r) dp dr &= \\ = -\frac{2}{m} \int_{\mathcal{Y}} [\nabla_p f(p, r) \cdot \alpha(q, r) \nabla_p g(p, r)] \rho^m(p, r) dp dr. \end{aligned} \tag{6.13}$$

We set in particular in (6.9) $f = g = \Psi_{\ell,3}^m$, then we will have that

$$\int_{\mathcal{Y}} (\mathcal{L}_q^m \Psi_{\ell,3}^m(p, r)) \Psi_{\ell,3}^m(p, r) \rho^m(p, r) dp dr = -\frac{2}{m} \int_{\mathcal{Y}} |\nabla_p \Psi_{\ell,3}^m(p, r)|^2 \rho^m(p, r) dp dr .$$

But, we also know that $\Psi_{\ell,3}^m(p, r)$ satisfies (6.8). Therefore, multiplying both sides of (6.8) by $\Psi_{\ell,3}^m(p, r)$ and integrating against the invariant density $\rho^m(p, r)$ gives us the identity

$$\frac{2}{m} \int_{\mathcal{Y}} |\nabla_p \Psi_{\ell,3}^m(p, r)|^2 \rho^m(p, r) dp dr = \sqrt{m} \int_{\mathcal{Y}} (\mathcal{B} \Psi_{\ell,2}(p, r)) \Psi_{\ell,3}^m(p, r) \rho^m(p, r) dp dr ,$$

or, in other words

$$\left\| \frac{1}{\sqrt{m}} \nabla_p \Psi_{\ell,3}^m \right\|_{L^2(\mathcal{Y}; \rho^m)}^2 = \frac{\sqrt{m}}{2} \langle \mathcal{B} \Psi_{\ell,2}, \Psi_{\ell,3}^m \rangle_{L^2(\mathcal{Y}; \rho^m)} . \tag{6.14}$$

We now have the estimate

$$\begin{aligned} \|\Psi_{\ell,3}^m\|_{L^2(\mathcal{Y}; \rho^m)}^2 &= \int_{\mathcal{Y}} (\Psi_{\ell,3}^m)^2 \rho^m(p, r) dp dr \\ &= \int_{\mathcal{Y}} (\Psi_{\ell,3}^m)^2 \delta^m(p, r) \rho^0(p, r) dp dr + \int_{\mathcal{Y}} (\Psi_{\ell,3}^m)^2 \rho^0(p, r) dp dr \\ &\leq \|\Psi_{\ell,3}^m\|_{L^4(\mathcal{Y}; \rho^0)}^4 \|\delta^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 + \|\Psi_{\ell,3}^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 . \end{aligned}$$

Applying Lemma 6.3 and Lemma 6.2 and the fact that $\lim_{m \rightarrow 0} \|\delta^m\|_{L^2(\mathcal{Y}; \rho^0)} = 0$ we see that

$$\lim_{m \rightarrow 0} \|\Psi_{\ell,3}^m\|_{L^2(\mathcal{Y}; \rho^m)} = 0 .$$

Thus we have by (6.14)

$$\left\| \frac{1}{\sqrt{m}} \nabla_p \Psi_{\ell,3}^m \right\|_{L^2(\mathcal{Y}; \rho^m)}^2 \leq \frac{\sqrt{m}}{2} \|\mathcal{B} \Psi_{\ell,2}\|_{L^2(\mathcal{Y}; \rho^m)} \|\Psi_{\ell,3}^m\|_{L^2(\mathcal{Y}; \rho^m)} \rightarrow 0$$

as $m \rightarrow 0$. This is (6.7), completing the proof of Theorem 1.7.

A On properties of the solution to the hypoelliptic cell problem

In this section we recall some results on the solution to the hypoelliptic Poisson equation (1.4) from [14]. Since the set-up of the current paper has some differences from the setup in [14], we formulate the results that we need in the current setup, even though we emphasize that the derivation follows basically from [14].

Under the assumptions made in this paper, Theorem 3.3 from [14] guarantees that, (1.4) has a smooth solution that does not grow too fast at infinity. In particular, we have that for every $\eta > 0$, we can write

$$\Phi(p, r) = e^{\frac{\eta}{2}|p|^2} \tilde{\Phi}(p, r)$$

where $\tilde{\Phi} \in \mathcal{S}$, the Schwartz space of smooth functions with fast decay. Furthermore, as it can be derived from the proof of Theorem 3.3 of [14], if we let $\sigma_{\max} = \max_{i,j=1,\dots,d} \sup_{(q,r)} |\sigma_{i,j}(q, r)|$, then we have that for every $\eta \in (0, 2\sigma_{\max}^{-2})$ the solution Φ is unique (up to additive constants) in the space $L^2(\mathcal{Y}, e^{-\eta|p|^2} dp dr)$.

Moreover, it is clear that for each fixed q , the operator \mathcal{L}_q^m defines a hypoelliptic diffusion process on $(p, r) \in \mathcal{Y} = \mathbb{R}^d \times \mathbb{T}^d$. Let us define this process by (p, \cdot, r) . We recall then the following useful bounds from [14].

1. There exists a constant C such that

$$\mathbb{E} \left[e^{\frac{\sigma_{\max}^{-2}}{2} |p_t|^2} \right] < \mathbb{E} \left[e^{\frac{\sigma_{\max}^{-2}}{2} |p_0|^2 + Ct} \right]$$

2. For every $T > 0$, there exist constants $\eta, C > 0$ such that

$$\mathbb{E} \left[\sup_{t \in [0, T]} e^{\eta |p_t|^2} \right] < C \mathbb{E} \left[e^{\eta |p_0|^2} \right]$$

Based then on these bounds, the computations of [14] reveal that the following bounds for the solution to (1.4) are true. In particular we have that for every $T, p > 0$ there exists a constant $C > 0$ that is independent of ε, δ such that

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} \left| \Phi \left(\bar{p}_t^\varepsilon, \frac{\bar{q}_t^\varepsilon}{\delta} \right) \right|^p \right] &\leq C \delta^{-p/2}, \text{ and} \\ \mathbb{E} \left[\sup_{t \in [0, T]} \left| \nabla_p \Phi \left(\bar{p}_t^\varepsilon, \frac{\bar{q}_t^\varepsilon}{\delta} \right) \right|^p \right] &\leq C. \end{aligned}$$

These bounds are used in the proofs of this paper and more specifically in the derivation of Theorems 2.2 and 3.1.

B Proofs of Lemmas in Section 5.

Proof of Lemma 5.1. The proof goes in a standard way using integration by parts. We present the main steps for completeness.

$$\begin{aligned} \int_{\mathcal{Y}} (\mathcal{L}_q^m f(p, r)) g(p, r) \rho^0(p, r) dp dr &= \int_{\mathcal{Y}} f(p, r) (\mathcal{L}_q^m)^* (g(p, r) \rho^0(p, r)) dp dr \\ &= \int_{\mathcal{Y}} f(p, r) \left[\frac{1}{m} (\mathcal{A})^* (\rho^0(p, r)) g(p, r) + (-\mathcal{L}_q^m g(p, r)) \rho^0(p, r) \right] dp dr \\ &\quad + \frac{2}{m} \int_{\mathcal{Y}} f(p, r) [(\rho^0(p, r) I : \nabla_p^2 g(p, r) + \nabla_p \rho^0(p, r) \cdot I \nabla_p g(p, r))] dp dr \\ &\quad - \frac{1}{\sqrt{m}} \int_{\mathcal{Y}} f(p, r) g(p, r) [b(r) \nabla_p \rho^0(p, r) + p \nabla_r \rho^0(p, r)] dp dr \\ &= \int_{\mathcal{Y}} f(p, r) (-\mathcal{L}_q^m g(p, r)) \rho^0(p, r) dp dr \\ &\quad + \frac{2}{m} \int_{\mathcal{Y}} f(p, r) (\rho^0(p, r) I : \nabla_p^2 g(p, r) + \nabla_p \rho^0(p, r) \cdot I \nabla_p g(p, r)) dp dr \\ &\quad + \frac{1}{\sqrt{m}} \int_{\mathcal{Y}} f(p, r) g(p, r) p \cdot h(r) \rho^0(p, r) dp dr \\ &= \int_{\mathcal{Y}} f(p, r) (-\mathcal{L}_q^m g(p, r)) \rho^m(p, r) dp dr \\ &\quad - \frac{2}{m} \int_{\mathcal{Y}} [\nabla_p f(p, r) \cdot I \nabla_p g(p, r)] \rho^0(p, r) dp dr + \frac{1}{\sqrt{m}} \int_{\mathcal{Y}} f(p, r) g(p, r) p \cdot h(r) \rho^0(p, r) dp dr. \end{aligned}$$

To derive the last line, we used integration by parts as well as the definition $h(r) = b(r) - \nabla_r \log \rho_0(r)$. The statement of the lemma follows. \square

Proof of Lemma 5.2. We start with the following calculation

$$\nabla_p \left(e^{-\frac{1}{2}|p|^2} f(p, r) g(p, r) \right) = -p e^{-\frac{1}{2}|p|^2} f(p, r) g(p, r) + e^{-\frac{1}{2}|p|^2} \nabla_p (f(p, r) g(p, r)).$$

Therefore, we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} pf(p, r)g(p, r)\rho^{\text{OU}}(p)dp &= \int_{\mathbb{R}^d} \nabla_p(f(p, r)g(p, r))\rho^{\text{OU}}(p)dp \\ &= \int_{\mathbb{R}^d} (\nabla_p f(p, r)g(p, r) + f(p, r)\nabla_p g(p, r))\rho^{\text{OU}}(p)dp . \end{aligned}$$

Multiplying both sides by $h(r)\rho_0(r)$ and integrating over $r \in \mathbb{T}^d$ we then obtain after using Hölder inequality

$$\begin{aligned} \langle h(r)p, fg \rangle_{L^2(\mathcal{Y};\rho^0)} &= \int_{\mathcal{Y}} (\nabla_p f(p, r)g(p, r) + f(p, r)\nabla_p g(p, r))\rho^0(p, r)dpr \\ &\leq K \left[\|f\|_{L^2(\mathcal{Y};\rho^0)} \|\nabla_p g\|_{L^2(\mathcal{Y};\rho^0)} + \|\nabla_p f\|_{L^2(\mathcal{Y};\rho^0)} \|g\|_{L^2(\mathcal{Y};\rho^0)} \right] . \end{aligned}$$

This completes the statement of the lemma. □

Proof of Lemma 5.3. We notice that

$$\begin{aligned} \langle f, \mathcal{B}f \rangle_{L^2(\mathcal{Y};\rho^0)} &= \int_{\mathcal{Y}} p\nabla_r f(p, r)f(p, r)\rho^0(p, r)dpr \\ &\quad + \int_{\mathcal{Y}} b(r)\nabla_p f(p, r)f(p, r)\rho^0(p, r)dpr \\ &= \text{Term1}_m + \text{Term2}_m . \end{aligned}$$

By integration by parts, we have

$$\begin{aligned} \text{Term1}_m &= \int_{\mathcal{Y}} p\nabla_r f(p, r)f(p, r)\rho^0(p, r)dpr \\ &= - \int_{\mathcal{Y}} pf(p, r)\nabla_r f(p, r)\rho^0(p, r)dpr - \int_{\mathcal{Y}} p\nabla_r \log \rho_0(r) |f(p, r)|^2 \rho^0(p, r)dpr . \end{aligned}$$

Thus, we get

$$\text{Term1}_m = \int_{\mathcal{Y}} p\nabla_r f(p, r)f(p, r)\rho^0(p, r)dpr = -\frac{1}{2} \int_{\mathcal{Y}} p\nabla_r \log \rho_0(r) |f(p, r)|^2 \rho^0(p, r)dpr .$$

Similarly, we have

$$\begin{aligned} \text{Term2}_m &= \int_{\mathcal{Y}} b(r)\nabla_p f(p, r)f(p, r)\rho^0(p, r)dpr \\ &= - \int_{\mathcal{Y}} b(r)f(p, r)\nabla_p f(p, r)\rho^0(p, r)dpr + \int_{\mathcal{Y}} b(r)p |f(p, r)|^2 \rho^0(p, r)dpr . \end{aligned}$$

Thus, we get

$$\text{Term2}_m = \int_{\mathcal{Y}} b(r)\nabla_p f(p, r)f(p, r)\rho^0(p, r)dpr = \frac{1}{2} \int_{\mathcal{Y}} b(r)p |f(p, r)|^2 \rho^0(p, r)dpr .$$

Putting the representations of Term1_m and Term2_m together, we have in fact obtained

$$\langle f, \mathcal{B}f \rangle_{L^2(\mathcal{Y};\rho^0)} = \frac{1}{2} \int_{\mathcal{Y}} ph(r) |f(p, r)|^2 \rho^0(p, r)dpr .$$

Hence, by Lemma 5.2 we have that there exists a constant $K < \infty$ that depends on $\sup_{r \in \mathbb{T}^d} |h(r)|$ such that

$$\begin{aligned} \langle f, \mathcal{B}f \rangle_{L^2(\mathcal{Y};\rho^0)} &= \frac{1}{2} \langle ph(r), |f|^2 \rangle_{L^2(\mathcal{Y};\rho^0)} \geq -K \|f\|_{L^2(\mathcal{Y};\rho^0)} \|\nabla_p f\|_{L^2(\mathcal{Y};\rho^0)} \\ &\geq -K \left[\eta \|f\|_{L^2(\mathcal{Y};\rho^0)}^2 + \frac{1}{4\eta} \|\nabla_p f\|_{L^2(\mathcal{Y};\rho^0)}^2 \right] . \end{aligned}$$

where we use the generalized Cauchy-Schwarz inequality $ab \leq \eta|a|^2 + \frac{1}{4\eta}|b|^2$ for any $\eta \in (0, \infty)$. This concludes the proof of the lemma. \square

Proof of Lemma 5.4. Recall that by (5), the equation for $\delta^m(p, r) = \tilde{\rho}^m(p, r) - 1$ is

$$\mathcal{L}_q^m \delta^m(p, r) = \frac{2}{\sqrt{m}} \mathcal{B} \delta^m(p, r) - \frac{1}{\sqrt{m}} p h(r) [\delta^m(p, r) + 1] . \tag{B.1}$$

Let us multiply now the last equation by $\delta^m(p, r)$ and integrate over \mathcal{Y} against $\rho^0(p, r)$. Doing so, we get

$$\langle \mathcal{L}_q^m \delta^m, \delta^m \rangle_{L^2(\mathcal{Y}; \rho^0)} = \frac{2}{\sqrt{m}} \langle \mathcal{B} \delta^m, \delta^m \rangle_{L^2(\mathcal{Y}; \rho^0)} - \frac{1}{\sqrt{m}} \langle p h(r) [\delta^m(p, r) + 1], \delta^m \rangle_{L^2(\mathcal{Y}; \rho^0)} .$$

The next step is to rewrite the term $\langle \mathcal{B} \delta^m, \delta^m \rangle_{L^2(\mathcal{Y}; \rho^0(p, r))}$. By Lemma 5.3 we have

$$\langle \mathcal{B} f, f \rangle_{L^2(\mathcal{Y}; \rho^0)} = \frac{1}{2} \int_{\mathcal{Y}} p h(r) |f(p, r)|^2 \rho^0(p, r) dp dr .$$

Inserting the latter expression into (B.2) we obtain

$$\begin{aligned} \langle \mathcal{L}_q^m f, f \rangle_{L^2(\mathcal{Y}; \rho^0)} &= \frac{2}{\sqrt{m}} \langle \mathcal{B} f, f \rangle_{L^2(\mathcal{Y}; \rho^0)} \\ &\quad - \frac{1}{\sqrt{m}} \langle p h(r) [f(p, r) + 1], f \rangle_{L^2(\mathcal{Y}; \rho^0)} \\ &= \frac{1}{\sqrt{m}} \langle p h(r), |f|^2 \rangle_{L^2(\mathcal{Y}; \rho^0)} \\ &\quad - \frac{1}{\sqrt{m}} \langle p h(r) [f(p, r) + 1], f \rangle_{L^2(\mathcal{Y}; \rho^0)} \\ &= -\frac{1}{\sqrt{m}} \langle p h(r), f \rangle_{L^2(\mathcal{Y}; \rho^0)} . \end{aligned}$$

Next step is to apply Lemma 5.1 with $f(p, r) = g(p, r) = \delta^m(p, r)$ to get

$$\langle \mathcal{L}_q^m \delta^m, \delta^m \rangle_{L^2(\mathcal{Y}; \rho^0)} = -\frac{1}{m} \|\nabla_p \delta^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 + \frac{1}{2\sqrt{m}} \langle h(r)p, |\delta^m|^2 \rangle_{L^2(\mathcal{Y}; \rho^0)} .$$

Combining the last two expressions, we obtain

$$-\frac{1}{m} \|\nabla_p \delta^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 + \frac{1}{2\sqrt{m}} \langle h(r)p, |\delta^m|^2 \rangle_{L^2(\mathcal{Y}; \rho^0)} = -\frac{1}{\sqrt{m}} \langle p h(r), \delta^m \rangle_{L^2(\mathcal{Y}; \rho^0)} , \tag{B.2}$$

and after rearranging, we obtain

$$\|\nabla_p \delta^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 = \frac{\sqrt{m}}{2} \langle h(r)p, |\delta^m|^2 \rangle_{L^2(\mathcal{Y}; \rho^0)} + \sqrt{m} \langle p h(r), \delta^m \rangle_{L^2(\mathcal{Y}; \rho^0)} .$$

This concludes the proof of the lemma. \square

Proof of Lemma 5.5. The proof goes along the same lines of Lemma 5.4. We take ∂_{p_i} on both sides of the equation (B.1) and we get the following equation

$$\mathcal{L}_q^m \partial_{p_i} \delta^m = \frac{2}{\sqrt{m}} \mathcal{B} \partial_{p_i} \delta^m - \frac{1}{\sqrt{m}} h_i(r) [\delta^m + 1] - \frac{1}{\sqrt{m}} p \cdot h(r) \partial_{p_i} \delta^m .$$

Multiplying both sides of the above equation by $\partial_{p_i} \delta^m$ and integrate with respect to $L^2(\mathcal{Y}; \rho^0)$ -inner product we get

$$\begin{aligned} \langle \mathcal{L}_q^m \partial_{p_i} \delta^m, \partial_{p_i} \delta^m \rangle_{L^2(\mathcal{Y}; \rho^0)} &= \frac{2}{\sqrt{m}} \langle \mathcal{B} \partial_{p_i} \delta^m, \partial_{p_i} \delta^m \rangle_{L^2(\mathcal{Y}; \rho_0)} \\ &\quad - \frac{1}{\sqrt{m}} \langle h_i(r)(\delta^m + 1), \partial_{p_i} \delta^m \rangle_{L^2(\mathcal{Y}; \rho_0)} \\ &\quad - \frac{1}{\sqrt{m}} \langle p \cdot h(r) \partial_{p_i} \delta^m, \partial_{p_i} \delta^m \rangle_{L^2(\mathcal{Y}; \rho_0)}. \end{aligned}$$

We apply Lemma 5.1 with $f(p, r) = g(p, r) = \partial_{p_i} \delta^m(p, r)$ to get

$$\langle \mathcal{L}_q^m \partial_{p_i} \delta^m, \partial_{p_i} \delta^m \rangle_{L^2(\mathcal{Y}; \rho^0)} = -\frac{1}{m} \|\nabla_p \partial_{p_i} \delta^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 + \frac{1}{2\sqrt{m}} \langle h(r)p, |\partial_{p_i} \delta^m|^2 \rangle_{L^2(\mathcal{Y}; \rho^0)}.$$

We now apply Lemma 5.3 and we have

$$\langle \mathcal{B} \partial_{p_i} \delta^m, \partial_{p_i} \delta^m \rangle_{L^2(\mathcal{Y}; \rho^0)} = \frac{1}{2} \int_{\mathcal{Y}} p \cdot h(r) |\partial_{p_i} \delta^m|^2 \rho^0(p, r) dp dr.$$

Furthermore, we can calculate

$$\begin{aligned} \langle h_i(r)(\delta^m + 1), \partial_{p_i} \delta^m \rangle_{L^2(\mathcal{Y}; \rho^0)} &= \int_{\mathcal{Y}} h_i(r)(\delta^m + 1) \partial_{p_i} \delta^m \rho^0(p, r) dp dr, \\ \langle p \cdot h(r) \partial_{p_i} \delta^m, \partial_{p_i} \delta^m \rangle_{L^2(\mathcal{Y}; \rho^0)} &= \int_{\mathcal{Y}} p \cdot h(r) |\partial_{p_i} \delta^m|^2 \rho^0(p, r) dp dr. \end{aligned}$$

Thus, we get the identity

$$\begin{aligned} -\frac{1}{m} \|\nabla_p \partial_{p_i} \delta^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 &= -\frac{1}{2\sqrt{m}} \langle ph(r), |\partial_{p_i} \delta^m|^2 \rangle_{L^2(\mathcal{Y}; \rho^0)} \\ &\quad - \frac{1}{\sqrt{m}} \langle h_i(r)(\delta^m + 1), \partial_{p_i} \delta^m \rangle_{L^2(\mathcal{Y}; \rho^0)}. \end{aligned}$$

Making use of Lemma 5.2 and Young's inequality we estimate

$$\begin{aligned} \|\nabla_p \partial_{p_i} \delta^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 &\leq K\sqrt{m} \|\nabla_p \delta^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 + \sqrt{m} \|\nabla_p \partial_{p_i} \delta^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 \\ &\quad + K\sqrt{m} (\|\delta^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 + \|\nabla_p \delta^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 + 1), \end{aligned}$$

where $K > 0$ is a constant that depends only on $\sup_{r \in \mathbb{T}^d} |h(r)|$. This implies the lemma. \square

Proof of Lemma 5.6. The proof goes again along the same lines of Lemma 5.4. We take ∂_{r_i} on both sides of the equation (B.1) and we get the following equation

$$\mathcal{L}_q^m \partial_{r_i} \delta^m = \frac{2}{\sqrt{m}} \mathcal{B} \partial_{r_i} \delta^m - \frac{1}{\sqrt{m}} p \cdot \partial_{r_i} h(r) [\delta^m + 1] - \frac{1}{\sqrt{m}} p \cdot h(r) \partial_{r_i} \delta^m.$$

Multiplying both sides of the above equation by $\partial_{r_i} \delta^m$ and integrate with respect to $L^2(\mathcal{Y}; \rho^0)$ -inner product we get

$$\begin{aligned} \langle \mathcal{L}_q^m \partial_{r_i} \delta^m, \partial_{r_i} \delta^m \rangle_{L^2(\mathcal{Y}; \rho^0)} &= \frac{2}{\sqrt{m}} \langle \mathcal{B} \partial_{r_i} \delta^m, \partial_{r_i} \delta^m \rangle_{L^2(\mathcal{Y}; \rho_0)} \\ &\quad - \frac{1}{\sqrt{m}} \langle p \cdot \partial_{r_i} h(r) [\delta^m + 1], \partial_{r_i} \delta^m \rangle_{L^2(\mathcal{Y}; \rho_0)} \\ &\quad - \frac{1}{\sqrt{m}} \langle p \cdot h(r) \partial_{r_i} \delta^m, \partial_{r_i} \delta^m \rangle_{L^2(\mathcal{Y}; \rho_0)}. \end{aligned}$$

We apply Lemma 5.1 with $f(p, r) = g(p, r) = \partial_{r_i} \delta^m(p, r)$ to get

$$\langle \mathcal{L}_q^m \partial_{r_i} \delta^m, \partial_{r_i} \delta^m \rangle_{L^2(\mathcal{Y}; \rho^0)} = -\frac{1}{m} \|\nabla_p \partial_{r_i} \delta^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 + \frac{1}{2\sqrt{m}} \langle h(r)p, |\partial_{r_i} \delta^m|^2 \rangle_{L^2(\mathcal{Y}; \rho^0)} .$$

We now apply Lemma 5.3 and we also have

$$\langle \mathcal{B} \partial_{r_i} \delta^m, \partial_{r_i} \delta^m \rangle_{L^2(\mathcal{Y}; \rho^0)} = \frac{1}{2} \int_{\mathcal{Y}} p \cdot h(r) |\partial_{r_i} \delta^m|^2 \rho^0(p, r) dp dr .$$

Furthermore, we can calculate

$$\langle p \cdot \partial_{r_i} h(r) [\delta^m + 1], \partial_{r_i} \delta^m \rangle_{L^2(\mathcal{Y}; \rho^0)} = \int_{\mathcal{Y}} p \cdot \partial_{r_i} h(r) (\delta^m + 1) \partial_{r_i} \delta^m \rho^0(p, r) dp dr ,$$

We can apply a straightforward generalization of Lemma 5.2 with $h(r)$ replaced by $\partial_{r_i} h(r)$, as well as Young’s inequality, to estimate the right hand side of the above equation by

$$\langle p \cdot \partial_{r_i} h(r) [\delta^m + 1], \partial_{r_i} \delta^m \rangle_{L^2(\mathcal{Y}; \rho^0)} \leq K \left(\|\delta^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 + \|\nabla_p \delta^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 + \|\nabla_r \delta^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 + 1 \right) + \frac{1}{2} \|\nabla_p \partial_{r_i} \delta^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 ,$$

where $K > 0$ is a constant that depends only on $\sup_{r \in \mathbb{T}^d} |\nabla_r h(r)|$. We also have

$$\langle p \cdot h(r) \partial_{r_i} \delta^m, \partial_{r_i} \delta^m \rangle_{L^2(\mathcal{Y}; \rho^0)} = \int_{\mathcal{Y}} p \cdot h(r) |\partial_{r_i} \delta^m|^2 \rho^0(p, r) dp dr .$$

Thus we get the identity

$$-\frac{1}{m} \|\nabla_p \partial_{r_i} \delta^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 = -\frac{1}{2\sqrt{m}} \langle ph(r), |\partial_{r_i} \delta^m|^2 \rangle_{L^2(\mathcal{Y}; \rho^0)} - \frac{1}{\sqrt{m}} \langle p \cdot \partial_{r_i} h(r) [\delta^m + 1], \partial_{r_i} \delta^m \rangle_{L^2(\mathcal{Y}; \rho^0)} .$$

Making use of Lemma 5.2 and Young’s inequality again we estimate

$$\|\nabla_p \partial_{r_i} \delta^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 \leq K \sqrt{m} \|\nabla_p \delta^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 + K \sqrt{m} (\|\delta^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 + \|\nabla_p \delta^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 + \|\nabla_r \delta^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 + 1) + \sqrt{m} \|\nabla_p \partial_{r_i} \delta^m\|_{L^2(\mathcal{Y}; \rho^0)}^2 ,$$

where $K > 0$ is a constant that depends only on $\sup_{r \in \mathbb{T}^d} \max(|h(r)|, |\nabla_r h(r)|)$. This implies the lemma. □

Proof of Lemma 5.7. We start with $T_1(\delta^m)$. By Lemma 5.3 with $f = \delta^m$ we have

$$\begin{aligned} T_1(\delta^m) &= \sqrt{m} [\langle \mathcal{B} \delta^m, \delta^m \rangle - \langle ph(r), (\delta^m + 1) \delta^m \rangle] \\ &= \sqrt{m} \left[\frac{1}{2} \langle ph(r), |\delta^m|^2 \rangle - \langle ph(r), (\delta^m + 1) \delta^m \rangle \right] \\ &= \sqrt{m} \left[-\frac{1}{2} \langle ph(r), |\delta^m|^2 \rangle + \langle ph(r), \delta^m \rangle \right] . \end{aligned}$$

Thus, by Lemma 5.2 with $f = g = \delta^m$ we have the following bound

$$\begin{aligned} |T_1(\delta^m)| &\leq \sqrt{m} K \left[\|\delta^m\| \|\nabla_p \delta^m\| + \|\delta^m\|^2 \right] \\ &\leq \sqrt{m} \frac{K+2}{2} \|\delta^m\|^2 + \sqrt{m} \frac{K}{2} \|\nabla_p \delta^m\|^2 . \end{aligned} \tag{B.3}$$

Next we derive an upper bound for $T_2(\delta^m) = \langle A\delta^m, A\mathcal{L}^1\delta^m \rangle$. For this purpose we first notice that

$$\begin{aligned} \langle A\delta^m, A\mathcal{B}\delta^m \rangle &= \langle \nabla_p\delta^m, \nabla_p\mathcal{B}\delta^m \rangle \\ &= \langle \nabla_p\delta^m, \nabla_p(p\nabla_r\delta^m + b(r)\nabla_p\delta^m) \rangle \\ &= \langle \nabla_p\delta^m, \mathcal{B}\nabla_p\delta^m \rangle + \langle \nabla_p\delta^m, \nabla_r\delta^m \rangle \\ &= \frac{1}{2} \langle ph(r), |\nabla_p\delta^m|^2 \rangle + \langle \nabla_p\delta^m, \nabla_r\delta^m \rangle, \end{aligned}$$

where in the last inequality we used Lemma 5.3. Then, using the equation for δ^m , (5.6) and Lemma 5.2 we have

$$\begin{aligned} |T_2(\delta^m)| &= |\langle A\delta^m, A\mathcal{L}^1\delta^m \rangle| \\ &= \left| \frac{(1 + \sqrt{m})}{2} \langle ph(r), |\nabla_p\delta^m|^2 \rangle + (1 + \sqrt{m}) \langle \nabla_p\delta^m, \nabla_r\delta^m \rangle - \sqrt{m} \langle \nabla_p\delta^m, h(r)(\delta^m + 1) \rangle \right. \\ &\quad \left. - \sqrt{m} \langle ph(r), |\nabla_p\delta^m|^2 \rangle \right| \\ &= \left| \frac{(1 - \sqrt{m})}{2} \langle ph(r), |\nabla_p\delta^m|^2 \rangle + (1 + \sqrt{m}) \langle \nabla_p\delta^m, \nabla_r\delta^m \rangle - \sqrt{m} \langle \nabla_p\delta^m, h(r)(\delta^m + 1) \rangle \right| \\ &\leq \frac{(1 - \sqrt{m})}{2} K \|\nabla_p\delta^m\| \|\nabla_p\nabla_p\delta^m\| + \frac{1 + \sqrt{m}}{4\eta} \|\nabla_p\delta^m\|^2 + \eta(1 + \sqrt{m}) \|\nabla_r\delta^m\|^2 \\ &\quad + \sqrt{m}K (\|\delta^m\|^2 + \|\nabla_p\delta^m\|^2) \\ &\leq \frac{(1 - \sqrt{m})}{2} K \left(\eta \|\nabla_p\delta^m\|^2 + \frac{1}{4\eta} \|\nabla_p\nabla_p\delta^m\| \right) + \frac{1 + \sqrt{m}}{4\eta} \|\nabla_p\delta^m\|^2 \\ &\quad + \eta(1 + \sqrt{m}) \|\nabla_r\delta^m\|^2 + \sqrt{m}K (\|\delta^m\|^2 + \|\nabla_p\delta^m\|^2). \end{aligned}$$

Next step now is to use Lemma 5.5. Doing so we get the bound

$$\begin{aligned} |T_2(\delta^m)| &= |\langle A\delta^m, A\mathcal{L}^1\delta^m \rangle| \\ &\leq \frac{(1 - \sqrt{m})}{2} K \left(\eta \|\nabla_p\delta^m\|^2 + \frac{\sqrt{m}}{4\eta} K [1 + \|\delta^m\|^2 + \|\nabla_p\delta^m\|^2] \right) + \frac{1 + \sqrt{m}}{4\eta} \|\nabla_p\delta^m\|^2 \\ &\quad + \eta(1 + \sqrt{m}) \|\nabla_r\delta^m\|^2 + \sqrt{m}K (\|\delta^m\|^2 + \|\nabla_p\delta^m\|^2) \\ &\leq \left(\frac{K(1 - \sqrt{m})}{2} \eta + K \frac{(1 - \sqrt{m})}{8\eta} \sqrt{m} + K\sqrt{m} + \frac{1 + K\sqrt{m}}{4\eta} \right) \|\nabla_p\delta^m\|^2 + \\ &\quad + \eta(1 + \sqrt{m}) \|\nabla_r\delta^m\|^2 + K \left(\frac{(1 - \sqrt{m})}{4\eta} + 1 \right) \sqrt{m} (1 + \|\delta^m\|^2). \end{aligned}$$

Use now Lemma 5.4 and then Lemma 5.2 to bound the term $\frac{1+K\sqrt{m}}{\eta} \|\nabla_p\delta^m\|^2$ by terms of the form $K \frac{1+\sqrt{m}}{\eta} \sqrt{m} (\|\delta^m\|^2 + \|\nabla_p\delta^m\|^2)$. Choosing then $\eta = \eta(m)$ such that $\eta(m) \rightarrow 0$ and $\frac{\sqrt{m}}{\eta(m)} \rightarrow 0$, we get that for m sufficiently small

$$\begin{aligned} |T_2(\delta^m)| &= |\langle A\delta^m, A\mathcal{L}^1\delta^m \rangle| \\ &\leq \left(\eta(m) + \sqrt{m} + \frac{\sqrt{m}}{\eta(m)} \right) K \|\delta^m\|_{H^1}^2 + \left(\frac{\sqrt{m}}{\eta(m)} + \sqrt{m} \right) K (1 + \|\delta^m\|^2). \quad (\text{B.4}) \end{aligned}$$

Next we derive an upper bound for $T_4(\delta^m) = \langle \mathcal{C}\delta^m, \mathcal{C}\mathcal{L}^1\delta^m \rangle$. For this purpose we first

notice that

$$\begin{aligned} \langle \mathcal{C}\delta^m, \mathcal{C}\mathcal{B}\delta^m \rangle &= \langle \nabla_r \delta^m, \nabla_r \mathcal{B}\delta^m \rangle \\ &= \langle \nabla_r \delta^m, \nabla_r (p\nabla_r \delta^m + b(r)\nabla_p \delta^m) \rangle \\ &= \langle \nabla_r \delta^m, \mathcal{B}\nabla_r \delta^m \rangle + \langle \nabla_r \delta^m, \nabla_r b(r)\nabla_p \delta^m \rangle \\ &= \frac{1}{2} \langle ph(r), |\nabla_r \delta^m|^2 \rangle + \langle \nabla_r \delta^m, \nabla_r b(r)\nabla_p \delta^m \rangle, \end{aligned}$$

where in the last inequality we used Lemma 5.3. Then, using the equation for δ^m

$$\begin{aligned} T_4(\delta^m) &= \langle \mathcal{C}\delta^m, \mathcal{C}\mathcal{L}^1\delta^m \rangle \\ &= \frac{(1 + \sqrt{m})}{2} \langle ph(r), |\nabla_r \delta^m|^2 \rangle + (1 + \sqrt{m}) \langle \nabla_r \delta^m, \nabla_r b(r)\nabla_p \delta^m \rangle \\ &\quad - \sqrt{m} \langle \nabla_r \delta^m, p\nabla_r h(r)(\delta^m + 1) \rangle - \sqrt{m} \langle ph(r), |\nabla_r \delta^m|^2 \rangle \\ &= \frac{(1 - \sqrt{m})}{2} \langle ph(r), |\nabla_r \delta^m|^2 \rangle + (1 + \sqrt{m}) \langle \nabla_r \delta^m, \nabla_r b(r)\nabla_p \delta^m \rangle \\ &\quad - \sqrt{m} \langle \nabla_r \delta^m, p\nabla_r h(r)(\delta^m + 1) \rangle. \end{aligned}$$

Using Lemma 5.2 we subsequently obtain

$$\begin{aligned} |T_4(\delta^m)| &\leq \frac{(1 - \sqrt{m})}{2} K \|\nabla_r \delta^m\| \|\nabla_p \nabla_r \delta^m\| + (1 + \sqrt{m}) |\langle \nabla_r \delta^m, \nabla_r b(r)\nabla_p \delta^m \rangle| \\ &\quad + \sqrt{m} |\langle \nabla_r \delta^m, p\nabla_r h(r)(\delta^m + 1) \rangle| \\ &\leq \frac{(1 - \sqrt{m})}{2} K \left(\eta \|\nabla_r \delta^m\|^2 + \frac{1}{4\eta} \|\nabla_p \nabla_r \delta^m\|^2 \right) \\ &\quad + (1 + \sqrt{m}) \left[\eta \|\nabla_r \delta^m\|^2 + \frac{1}{4\eta} K \|\nabla_p \delta^m\|^2 \right] + \sqrt{m} |\langle \nabla_r \delta^m, p\nabla_r h(r)(\delta^m + 1) \rangle| \\ &\leq \frac{(1 - \sqrt{m})}{2} K \left(\eta \|\nabla_r \delta^m\|^2 + \frac{1}{4\eta} \|\nabla_p \nabla_r \delta^m\|^2 \right) \\ &\quad + (1 + \sqrt{m}) \left[\eta \|\nabla_r \delta^m\|^2 + \frac{1}{4\eta} K \|\nabla_p \delta^m\|^2 \right] \\ &\quad + \sqrt{m} K \left[\|\nabla_r \delta^m\| \|\nabla_p \delta^m\| + \|\nabla_p \nabla_r \delta^m\| \|\delta^m\| + \|\nabla_r \delta^m\|^2 \right] \\ &\leq \frac{(1 - \sqrt{m})}{2} K \left(\eta \|\nabla_r \delta^m\|^2 + \frac{1}{4\eta} \|\nabla_p \nabla_r \delta^m\|^2 \right) \\ &\quad + (1 + \sqrt{m}) \left[\eta \|\nabla_r \delta^m\|^2 + \frac{1}{4\eta} K \|\nabla_p \delta^m\|^2 \right] \\ &\quad + \sqrt{m} K \left[\frac{1}{2} \|\nabla_r \delta^m\|^2 + \frac{1}{2} \|\nabla_p \delta^m\|^2 + \frac{1}{2} \|\nabla_p \nabla_r \delta^m\|^2 + \frac{1}{2} \|\delta^m\|^2 + \|\nabla_r \delta^m\|^2 \right] \\ &\leq \frac{(1 - \sqrt{m})}{2} K \left(\eta \|\nabla_r \delta^m\|^2 + \frac{1}{4\eta} \|\nabla_p \nabla_r \delta^m\|^2 \right) \\ &\quad + (1 + \sqrt{m}) \left[\eta \|\nabla_r \delta^m\|^2 + \frac{1}{4\eta} K \|\nabla_p \delta^m\|^2 \right] \\ &\quad + \sqrt{m} K \left[\|\nabla_r \delta^m\|^2 + \|\nabla_p \delta^m\|^2 + \|\nabla_p \nabla_r \delta^m\|^2 + \|\delta^m\|^2 \right]. \end{aligned}$$

The constant K may change from line to line, but it is always independent of m .

Using Lemma 5.6 and then Lemma 5.4 we subsequently obtain

$$\begin{aligned}
 |T_4(\delta^m)| \leq & \frac{(1 - \sqrt{m})}{2} K \left(\eta \|\nabla_r \delta^m\|^2 + \frac{\sqrt{m}}{4\eta} K \left[\|\delta^m\|^2 + \|\nabla_p \delta^m\|^2 + \|\nabla_r \delta^m\|^2 \right] \right) \\
 & + (1 + \sqrt{m}) \left[\eta \|\nabla_r \delta^m\|^2 + \frac{1}{4\eta} K \|\nabla_p \delta^m\|^2 \right] \\
 & + \sqrt{m} K \left[\|\nabla_r \delta^m\|^2 + \|\nabla_p \delta^m\|^2 + \sqrt{m} \left[\|\delta^m\|^2 + \|\nabla_p \delta^m\|^2 + \|\nabla_r \delta^m\|^2 \right] + \|\delta^m\|^2 \right].
 \end{aligned}$$

Finally, choosing $\eta = \eta(m)$ such that $\eta(m) \rightarrow 0$ and $\frac{\sqrt{m}}{\eta(m)} \rightarrow 0$, we get that for m sufficiently small and for some constant $K < \infty$

$$\begin{aligned}
 |T_4(\delta^m)| &= |\langle \mathcal{C}\delta^m, \mathcal{C}\mathcal{L}^1\delta^m \rangle| \\
 &\leq K \left[\left(\eta(m) + \frac{\sqrt{m}}{\eta(m)} \right) \|\delta^m\|_{H^1}^2 + \frac{\sqrt{m}}{\eta(m)} \|\delta^m\|^2 \right]. \tag{B.5}
 \end{aligned}$$

It remains to consider the cross-term

$$T_3(\delta^m) = \langle A\mathcal{L}^1\delta^m, \mathcal{C}\delta^m \rangle + \langle A\delta^m, \mathcal{C}\mathcal{L}^1\delta^m \rangle.$$

Recalling (5.5) we have the following calculations

$$\begin{aligned}
 T_3(\delta^m) &= \langle A\mathcal{L}^1\delta^m, \mathcal{C}\delta^m \rangle + \langle A\delta^m, \mathcal{C}\mathcal{L}^1\delta^m \rangle \\
 &= \langle \nabla_r \delta^m, (1 + \sqrt{m})\mathcal{B}\nabla_p \delta^m + (1 + \sqrt{m})\nabla_r \delta^m - \sqrt{m}h(r)(\delta^m + 1) - \sqrt{m}h(r)p\nabla_p \delta^m \rangle \\
 &+ \langle \nabla_p \delta^m, (1 + \sqrt{m})\mathcal{B}\nabla_r \delta^m + (1 + \sqrt{m})\nabla_r b(r)\nabla_p \delta^m - \sqrt{m}\nabla_r h(r)p(\delta^m + 1) \\
 &\quad - \sqrt{m}h(r)p\nabla_r \delta^m \rangle \\
 &= (1 + \sqrt{m}) [\langle \nabla_p \delta^m, \mathcal{B}\nabla_r \delta^m \rangle + \langle \nabla_r \delta^m, \mathcal{B}\nabla_p \delta^m \rangle] \\
 &+ (1 + \sqrt{m}) \left[\langle \nabla_p \delta^m, \nabla_r b\nabla_p \delta^m \rangle + \|\nabla_r \delta^m\|^2 \right] \\
 &- \sqrt{m} [\langle \nabla_r \delta^m, ph(r)\nabla_p \delta^m \rangle + \langle \nabla_p \delta^m, ph(r)\nabla_r \delta^m \rangle] \\
 &- \sqrt{m} [\langle \nabla_p \delta^m, \nabla_r h(r)p(\delta^m + 1) \rangle + \langle \nabla_r \delta^m, h(r)(\delta^m + 1) \rangle] \\
 &= (1 + \sqrt{m}) [\langle \nabla_p \delta^m, \mathcal{L}^1\nabla_r \delta^m \rangle + \langle \nabla_r \delta^m, \mathcal{L}^1\nabla_p \delta^m \rangle] \\
 &- (1 + \sqrt{m}) [\langle \nabla_p \delta^m, \nabla_p \nabla_r \delta^m \rangle + \langle \nabla_r \delta^m, \nabla_p \nabla_p \delta^m \rangle] \\
 &+ (1 + \sqrt{m}) \left[\langle \nabla_p \delta^m, \nabla_r b\nabla_p \delta^m \rangle + \|\nabla_r \delta^m\|^2 \right] \\
 &- \sqrt{m} [\langle \nabla_r \delta^m, ph(r)\nabla_p \delta^m \rangle + \langle \nabla_p \delta^m, ph(r)\nabla_r \delta^m \rangle] \\
 &- \sqrt{m} [\langle \nabla_p \delta^m, \nabla_r h(r)p(\delta^m + 1) \rangle + \langle \nabla_r \delta^m, h(r)(\delta^m + 1) \rangle].
 \end{aligned}$$

Using now Lemma 5.1 on the first term of the right hand side of the last display we obtain

$$\begin{aligned}
 T_3(\delta^m) &= -2(1 + \sqrt{m}) \langle \nabla_p \nabla_p \delta^m, \nabla_p \nabla_r \delta^m \rangle \\
 &- (1 + \sqrt{m}) [\langle \nabla_p \delta^m, \nabla_p \nabla_r \delta^m \rangle + \langle \nabla_r \delta^m, \nabla_p \nabla_p \delta^m \rangle] \\
 &+ (1 + \sqrt{m}) \left[\langle \nabla_p \delta^m, \nabla_r b\nabla_p \delta^m \rangle + \|\nabla_r \delta^m\|^2 \right] \\
 &- \sqrt{m} [\langle \nabla_r \delta^m, ph(r)\nabla_p \delta^m \rangle + \langle \nabla_p \delta^m, ph(r)\nabla_r \delta^m \rangle] \\
 &- \sqrt{m} [\langle \nabla_p \delta^m, \nabla_r h(r)p(\delta^m + 1) \rangle + \langle \nabla_r \delta^m, h(r)(\delta^m + 1) \rangle].
 \end{aligned}$$

Next we bound terms from above. Using Lemma 5.2, we have for $\eta > 0$

$$\begin{aligned}
 |T_3(\delta^m)| &\leq (1 + \sqrt{m})K \left[\|\nabla_p \nabla_p \delta^m\|^2 + \|\nabla_p \nabla_r \delta^m\|^2 \right] \\
 &\quad + \sqrt{m} \left[\eta \|\nabla_p \delta^m\|^2 + \frac{1}{4\eta} \|\nabla_p \nabla_r \delta^m\|^2 + \eta \|\nabla_r \delta^m\|^2 + \frac{1}{4\eta} \|\nabla_p \nabla_p \delta^m\|^2 \right] \\
 &\quad + \frac{(1 + \sqrt{m})}{2} K \left[\|\nabla_r \delta^m\| \|\nabla_p \nabla_p \delta^m\| + \|\nabla_p \delta^m\| \|\nabla_p \nabla_r \delta^m\| \right] \\
 &\quad + (1 + \sqrt{m}) \left[K \|\nabla_p \delta^m\|^2 + \|\nabla_r \delta^m\|^2 \right] \\
 &\quad + \sqrt{m}K \left[1 + \|\delta^m\|^2 + \|\nabla_p \delta^m\|^2 + \|\nabla_r \delta^m\|^2 \right] .
 \end{aligned}$$

The constant K may change from line to line. Using Lemmas 5.5 and 5.6 we obtain

$$\begin{aligned}
 |T_3(\delta^m)| &\leq \sqrt{m}(1 + \sqrt{m})K \left[1 + \|\delta^m\|^2 + \|\nabla_p \delta^m\|^2 + \|\nabla_r \delta^m\|^2 \right] \\
 &\quad + \sqrt{m} \left[\eta \|\delta^m\|_{H^1}^2 + \frac{\sqrt{m}}{4\eta} \left[1 + \|\delta^m\|^2 + \|\delta^m\|_{H^1}^2 \right] \right] \\
 &\quad + (1 + \sqrt{m})K \left[\eta \|\delta^m\|_{H^1}^2 + \frac{\sqrt{m}}{4\eta} \left[1 + \|\delta^m\|^2 + \|\delta^m\|_{H^1}^2 \right] \right] \\
 &\quad + (1 + \sqrt{m}) \left[K \|\nabla_p \delta^m\|^2 + \|\nabla_r \delta^m\|^2 \right] \\
 &\quad + \sqrt{m}K \left[\|\delta^m\|^2 + \|\nabla_p \delta^m\|^2 + \|\nabla_r \delta^m\|^2 \right] .
 \end{aligned}$$

Applying then Lemma 5.4 to estimate the term $\|\nabla_p \delta^m\|^2$ on the fourth line of the last display, we obtain the following bound

$$|T_3(\delta^m)| \leq \left(\sqrt{m} + \frac{\eta}{\sqrt{m}} \right) K \left[1 + \|\delta^m\|^2 + \|\delta^m\|_{H^1}^2 \right] + \eta K \|\delta^m\|_{H^1}^2 + (1 + \sqrt{m}) \|\nabla_r \delta^m\|^2 .$$

Finally, choosing $\eta = \eta(m)$ such that $\eta(m) \rightarrow 0$ and $\frac{\sqrt{m}}{\eta(m)} \rightarrow 0$, we get that for m sufficiently small and for some constant $K < \infty$

$$|T_3(\delta^m)| \leq \left(\sqrt{m} + \frac{\sqrt{m}}{\eta(m)} \right) K \left[1 + \|\delta^m\|^2 + \|\delta^m\|_{H^1}^2 \right] + \eta(m) K \|\delta^m\|_{H^1}^2 + (1 + \sqrt{m}) \|\nabla_r \delta^m\|^2 .$$

This concludes the proof of the lemma. □

C Proofs of Lemmas in Section 6.

Proof of Lemma 6.1. This can be shown by using Theorem 4.2.5 in [1]. Let $(P_t)_{t \geq 0}$ be the Markov semigroup corresponding to generator \mathcal{L}^1 on \mathcal{Y} .

By Lemma 5.1 with $m = 1$, we obtain for the first term (recall that $\rho^0(p, r)dpdr$ is the invariant measure corresponding to the operator \mathcal{L}^1) that the Dirichlet form associated with $(P_t)_{t \geq 0}$ can be calculated as follows

$$\mathcal{E}(f) = \langle -\mathcal{L}^1 f, f \rangle_{L^2(\mathcal{Y}; \rho^0)} = \|\nabla_p f\|_{L^2(\mathcal{Y}; \rho^0)}^2 .$$

Thus by Theorem 4.2.5 of [1] the validity of Poincaré inequality is equivalent to exponential convergence to equilibrium of the semigroup $(P_t)_{t \geq 0}$:

$$\int_{\mathcal{Y}} \left(P_t f - \int_{\mathcal{Y}} (P_t f) \rho^0(p, r) dpdr \right)^2 \rho^0(p, r) dpdr \leq c(f) e^{-2t/\kappa} .$$

for some constant $\kappa > 0$. The above inequality is true since \mathcal{L}_q^1 admits a spectral gap (see [10]). □

Proof of Lemma 6.2. We make use of our equation (6.8), (6.9) as well as Lemma 5.2 and we get

$$\begin{aligned} & \left\| \frac{1}{\sqrt{m}} \nabla_p \Psi_{\ell,3}^m \right\|_{L^2(\mathcal{Y};\rho^0)}^2 - K \|\Psi_{\ell,3}^m\|_{L^2(\mathcal{Y};\rho^0)} \left\| \frac{1}{\sqrt{m}} \nabla_p \Psi_{\ell,3}^m \right\|_{L^2(\mathcal{Y};\rho^0)} \\ & \leq \sqrt{m} \|\mathcal{B}\Psi_{\ell,2}\|_{L^2(\mathcal{Y};\rho^0)} \|\Psi_{\ell,3}^m\|_{L^2(\mathcal{Y};\rho^0)}, \end{aligned} \tag{C.1}$$

for some constant $K > 0$ independent of m .

We apply Lemma 6.1, using the fact that $\int_{\mathcal{Y}} \Psi_{\ell,3}^m(p, r) \rho^m(p, r) dpdr = 0$, and we have

$$\begin{aligned} & \|\Psi_{\ell,3}^m\|_{L^2(\mathcal{Y};\rho^0)}^2 \\ & \leq \left\| \Psi_{\ell,3}^m - \int_{\mathcal{Y}} \Psi_{\ell,3}^m(p, r) \rho^0(p, r) dpdr \right\|_{L^2(\mathcal{Y};\rho^0)}^2 + \left(\int_{\mathcal{Y}} \Psi_{\ell,3}^m(p, r) \rho^0(p, r) dpdr \right)^2 \\ & = \left\| \Psi_{\ell,3}^m - \int_{\mathcal{Y}} \Psi_{\ell,3}^m(p, r) \rho^0(p, r) dpdr \right\|_{L^2(\mathcal{Y};\rho^0)}^2 + \\ & \quad + \left(\int_{\mathcal{Y}} \Psi_{\ell,3}^m(p, r) \rho^0(p, r) dpdr - \int_{\mathcal{Y}} \Psi_{\ell,3}^m(p, r) \rho^m(p, r) dpdr \right)^2 \\ & \leq \kappa \|\nabla_p \Psi_{\ell,3}^m\|_{L^2(\mathcal{Y};\rho^0)}^2 + \left(\int_{\mathcal{Y}} \Psi_{\ell,3}^m(p, r) \delta^m(p, r) \rho^0(p, r) dpdr \right)^2 \\ & \leq \kappa \|\nabla_p \Psi_{\ell,3}^m\|_{L^2(\mathcal{Y};\rho^0)}^2 + \|\Psi_{\ell,3}^m\|_{L^2(\mathcal{Y};\rho^0)}^2 \|\delta^m\|_{L^2(\mathcal{Y};\rho^0)}^2. \end{aligned}$$

Since we have $\lim_{m \rightarrow 0} \|\delta^m\|_{L^2(\mathcal{Y};\rho^0)} = 0$, we can choose m small enough so that

$$\|\Psi_{\ell,3}^m\|_{L^2(\mathcal{Y};\rho^0)} \leq 2\kappa \|\nabla_p \Psi_{\ell,3}^m\|_{L^2(\mathcal{Y};\rho^0)}. \tag{C.2}$$

Combining (C.2) and (C.1) we see that we have

$$(1 - 2\kappa K \sqrt{m}) \left\| \frac{1}{\sqrt{m}} \nabla_p \Psi_{\ell,3}^m \right\|_{L^2(\mathcal{Y};\rho^0)}^2 \leq \sqrt{m} \|\mathcal{B}\Psi_{\ell,2}\|_{L^2(\mathcal{Y};\rho^0)} \|\Psi_{\ell,3}^m\|_{L^2(\mathcal{Y};\rho^0)}.$$

Using (C.2) again we see that

$$(1 - 2\kappa K \sqrt{m}) \left\| \frac{1}{\sqrt{m}} \nabla_p \Psi_{\ell,3}^m \right\|_{L^2(\mathcal{Y};\rho^0)}^2 \leq 2\kappa m \|\mathcal{B}\Psi_{\ell,2}\|_{L^2(\mathcal{Y};\rho^0)} \left\| \frac{1}{\sqrt{m}} \nabla_p \Psi_{\ell,3}^m \right\|_{L^2(\mathcal{Y};\rho^0)}.$$

This means that we have the bound

$$\left\| \frac{1}{\sqrt{m}} \nabla_p \Psi_{\ell,3}^m \right\|_{L^2(\mathcal{Y};\rho^0)} \leq \frac{2\kappa m}{1 - 2\kappa K \sqrt{m}} \|\mathcal{B}\Psi_{\ell,2}\|_{L^2(\mathcal{Y};\rho^0)}.$$

Now apply (C.2) again we obtain the bound

$$\|\Psi_{\ell,3}^m\|_{L^2(\mathcal{Y};\rho^0)} \leq \frac{4\kappa^2 m^{3/2}}{1 - 2\kappa K \sqrt{m}} \|\mathcal{B}\Psi_{\ell,2}\|_{L^2(\mathcal{Y};\rho^0)}.$$

This proves the lemma. □

Proof of Lemma 6.3. Let us write Ψ in place of $\Psi_{\ell,3}^m$ for simlicity of notations. We set $f = \Psi^2$ and we look for the equation that f satisfies:

$$\begin{aligned}
 & \mathcal{L}_q^m f \\
 &= \frac{1}{m} \mathcal{A}f + \frac{1}{\sqrt{m}} \mathcal{B}f \\
 &= \frac{1}{m} (-p \cdot \nabla_p f + \Delta_p f) + \frac{1}{\sqrt{m}} (b(r) \cdot \nabla_p f + p \cdot \nabla_r f) \\
 &= \frac{1}{m} (-p \cdot (2\Psi \nabla_p \Psi) + 2|\nabla_p \Psi|^2 + 2\Psi \Delta_p \Psi) + \frac{1}{\sqrt{m}} (b(r) \cdot 2\Psi \nabla_p \Psi + p \cdot 2\Psi \nabla_r \Psi) \\
 &= 2\Psi (\mathcal{L}_q^m \Psi) + \frac{2}{m} |\nabla_p \Psi|^2.
 \end{aligned}$$

Using the equation (6.8) we see that

$$\mathcal{L}_q^m f = -2\sqrt{m} \Psi \mathcal{B} \Psi_{\ell,2} + \frac{2}{m} |\nabla_p \Psi|^2. \tag{C.3}$$

Making use of Lemma 5.1 we have

$$\langle \mathcal{L}_q^m f, f \rangle_{L^2(\mathcal{Y}; \rho^0)} = -\frac{1}{m} \|\nabla_p f\|_{L^2(\mathcal{Y}; \rho^0)}^2 + \frac{1}{2\sqrt{m}} \langle h(r) \cdot p, f^2 \rangle_{L^2(\mathcal{Y}; \rho^0)}.$$

This gives

$$\|\nabla_p f\|_{L^2(\mathcal{Y}; \rho^0)}^2 = \frac{\sqrt{m}}{2} \langle h(r) \cdot p, f^2 \rangle_{L^2(\mathcal{Y}; \rho^0)} - m \langle \mathcal{L}_q^m f, f \rangle_{L^2(\mathcal{Y}; \rho^0)}. \tag{C.4}$$

Making use of (C.3), (C.4), the fact that $f \geq 0$ and Lemma 5.2 we get, for some constant $K > 0$ independent of m that may vary from line to line,

$$\begin{aligned}
 & \|\nabla_p f\|_{L^2(\mathcal{Y}; \rho^0)}^2 \\
 &= \frac{\sqrt{m}}{2} \langle h(r) \cdot p, f^2 \rangle_{L^2(\mathcal{Y}; \rho^0)} + 2m^{3/2} \langle \Psi \mathcal{B} \Psi_{\ell,2}, f \rangle_{L^2(\mathcal{Y}; \rho^0)} - 2 \langle |\nabla_p \Psi|^2, f \rangle_{L^2(\mathcal{Y}; \rho^0)} \\
 &\leq \frac{\sqrt{m}}{2} \langle h(r) \cdot p, f^2 \rangle_{L^2(\mathcal{Y}; \rho^0)} + 2m^{3/2} \langle \Psi \mathcal{B} \Psi_{\ell,2}, f \rangle_{L^2(\mathcal{Y}; \rho^0)} \\
 &\leq \frac{\sqrt{m}}{2} K \|f\|_{L^2(\mathcal{Y}; \rho^0)} \|\nabla_p f\|_{L^2(\mathcal{Y}; \rho^0)} + m^{3/2} (\|\Psi \mathcal{B} \Psi_{\ell,2}\|_{L^2(\mathcal{Y}; \rho^0)}^2 + \|f\|_{L^2(\mathcal{Y}; \rho^0)}^2) \\
 &\leq \frac{\sqrt{m}}{2} K \|f\|_{L^2(\mathcal{Y}; \rho^0)} \|\nabla_p f\|_{L^2(\mathcal{Y}; \rho^0)} + m^{3/2} (\|\Psi^2\|_{L^2(\mathcal{Y}; \rho^0)}^2 \|(\mathcal{B} \Psi_{\ell,2})^2\|_{L^2(\mathcal{Y}; \rho^0)}^2 + \|f\|_{L^2(\mathcal{Y}; \rho^0)}^2) \\
 &\leq K \left[\frac{\sqrt{m}}{2} \|f\|_{L^2(\mathcal{Y}; \rho^0)} \|\nabla_p f\|_{L^2(\mathcal{Y}; \rho^0)} + m^{3/2} \|f\|_{L^2(\mathcal{Y}; \rho^0)}^2 \right]. \tag{C.5}
 \end{aligned}$$

Now we apply Lemma 6.1 and we see that for some $\kappa > 0$ we have

$$\begin{aligned}
 & \|f\|_{L^2(\mathcal{Y}; \rho^0)} \\
 &= \left\| f - \int_{\mathcal{Y}} f(p, r) \rho^0(p, r) dp dr + \int_{\mathcal{Y}} f(p, r) \rho^0(p, r) dp dr \right\|_{L^2(\mathcal{Y}; \rho^0)} \\
 &\leq \left\| f - \int_{\mathcal{Y}} f(p, r) \rho^0(p, r) dp dr \right\|_{L^2(\mathcal{Y}; \rho^0)} + \left| \int_{\mathcal{Y}} f(p, r) \rho^0(p, r) dp dr \right| \\
 &\leq \kappa \|\nabla_p f\|_{L^2(\mathcal{Y}; \rho^0)} + \|\Psi\|_{L^2(\mathcal{Y}; \rho^0)}^2.
 \end{aligned}$$

In the last step we used the fact that $f = \Psi^2$. Now we apply Lemma 6.2 and we see that $\|\Psi\|_{L^2(\mathcal{Y}; \rho^0)}^2 \leq Km^3$ for some constant $K > 0$ independent of m . Thus we see that

$$\|f\|_{L^2(\mathcal{Y}; \rho^0)} \leq K [\|\nabla_p f\|_{L^2(\mathcal{Y}; \rho^0)} + m^3] \tag{C.6}$$

Combining (C.5) and (C.6) we see that

$$\|\nabla_p f\|_{L^2(\mathcal{Y};\rho^0)}^2 \leq K[\sqrt{m}\|\nabla_p f\|_{L^2(\mathcal{Y};\rho^0)}^2 + m^3\|\nabla_p f\|_{L^2(\mathcal{Y};\rho^0)} + m^{3/2}\|\nabla_p f\|_{L^2(\mathcal{Y};\rho^0)}^2 + m^{3/2+6}].$$

This gives $\lim_{m \rightarrow 0} \|\nabla_p f\|_{L^2(\mathcal{Y};\rho^0)}^2 = 0$. Apply (C.6) again we see that the claim of the Lemma follows. \square

Proof of Lemma 6.4. The proof of this lemma is completely analogous to that of Lemma 5.1 and thus it is omitted. \square

References

- [1] Bakery, D., Gentil, I., Ledoux, M.: Analysis and Geometry of Markov diffusion operators, *Grundlehren der mathematischen Wissenschaften, Springer*, **348**, 2014. MR-3155209
- [2] Benabou, G: Homogenization of Ornstein-Uhlenbeck process in random environment, *Communications in Mathematical Physics*, **266**, (2006), 699–714. MR-2238895
- [3] Boué, M., and Dupuis, P: A variational representation for certain functionals of Brownian motion, *Annals of Probability*, **26(4)**, (1998), 1641–1659. MR-1675051
- [4] Chen, Z., and Freidlin, M.: Smoluchowski-Kramers approximation and exit problems, *Stochastics and Dynamics*, **5(4)**, (2005), 569–585. MR-2185506
- [5] Dolbeault, J., Mouhot, C., and Schmeiser, C.: Hypocoercivity for linear kinetic equations conserving mass, *Transactions of the American Mathematical Society*, **367(6)**, (2015), 3807–3828. MR-3324910
- [6] Dupuis, P. and Ellis, R.S.: A Weak Conergence Approach to the Theory of Large Deviations, *John Wiley & Sons, New York*, 1997. MR-1431744
- [7] Dupuis, P., and Spiliopoulos, K.: Large deviations for multiscale problems via weak convergence methods, *Stochastic Processes and their Applications*, **122**, (2012), 1947–1987. MR-2914778
- [8] Dupuis, P., Spiliopoulos, K., and Wang, H.: Importance sampling for multiscale diffusions, *SIAM Multiscale Modeling and Simulation*, **12(1)**, (2012), 1–27. MR-2902596
- [9] Dupuis, P., Spiliopoulos, K., and Wang, H.: Rare Event Simulation in Rough Energy Landscapes, *Winter Simulation Conference*, (IEEE, 2011), 504–515.
- [10] Eckmann, J-P, Hairer, M.: Non-equilibrium statistical mechanics of strongly anharmonic chains of oscillators. *Communications in Mathematical Physics*, **212**, (2000), 105–164. MR-1764365
- [11] Einstein, A., Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen, *Annalen der Physik*, **17(8)**, (1905), 549–560.
- [12] Freidlin, M.: Some Remarks on the Smoluchowski-Kramers Approximation, *Journal of Statistical Physics*, **117(314)**, (2004), 617–634. MR-2099730
- [13] Freidlin, M., and Hu, W.: Smoluchowski-Kramers approximation in the case of variable friction, *Journal of Mathematical Sciences*, **79(1)**, (2011), 184–207. MR-3014105
- [14] Hairer, M., and Pavliotis, G.: Periodic Homogenization for Hypoelliptic Diffusions, *Journal of Statistical Physics*, **117(1/2)**, (2004), 261–279. MR-2098568
- [15] Hottovy, S., McDaniel, A., Volpe, G. and Wehr, J.: The Smoluchowski-Kramers limit of stochastic differential equations with arbitrary state-dependent friction, *Communications in Mathematical Physics*, **336(3)**, (2015), 1259–1283. MR-3324144
- [16] Komorowski, T, Landim, C., and Olla, S.: Fluctuations in Markov Processes: Time Symmetry and Martingale Approximation, *Springer*, 2012. MR-2952852
- [17] Kosygina, E., Rezakhanlou, F., and Varadhan, S. R. S.: Stochastic Homogenization of Hamilton-Jacobi-Bellman Equations, *Communications on Pure and Applied Mathematics*, **LIX**, (2006), 0001–0033. MR-2248897

- [18] Langevin, P.: Sur la théorie du mouvement brownien. *C. R. Acad. Sci. (Paris)*, **146**, (1908), 530–533.
- [19] Lifson, S., and Jackson, J.L.: On the self-diffusion of ions in a polyelectrolyte solution, *Journal of Chemical Physics*, **36**, (1962), 2410–2414.
- [20] Lelièvre, T., Stoltz, G., and Rousset, M.: Free Energy Computations: A Mathematical Perspective, *London College Press*, 2010. MR-2681239
- [21] Lions, P.-L., and Souganidis, P.E.: Homogenization of “viscous” Hamilton-Jacobi-Bellman equations in stationary ergodic media, *Communications in Partial Differential Equations*, **30(3)**, (2006), 335–375. MR-2131058
- [22] Lunardi, A., On the Ornstein–Uhlenbeck operator in L^2 spaces with respect to invariant measure, *Transactions of the AMS*, **349(1)**, (1997), 155–169. MR-1389786
- [23] Øksendal, B.: Stochastic differential equations: an introduction with applications, *Springer*, 5th edition, 2003. MR-2001996
- [24] Papanicolaou, G., and Varadhan, S.R.S.: Ornstein-Uhlenbeck process in random potential, *Communications in Pure and Applied Mathematics*, **38**, (1985), 819–834. MR-0812349
- [25] Pavliotis, G.A., and Stuart, A.M.: Periodic homogenization for inertial particles, *Physica D*, **204**, (2005), 161–187. MR-2148377
- [26] Smoluchowski, M., Drei Vorträge über Diffusion Brownsche Bewegung und Koagulation von Kolloidteilchen, *Phys. Z.*, **17**, (1916), 557–585.
- [27] Spiliopoulos, K.: Large deviations and importance sampling for systems of slow-fast motion, *Applied Mathematics and Optimization*, **67**, (2013), 123–161. MR-3009732
- [28] Spiliopoulos, K.: Quenched Large Deviations for Multiscale Diffusion Processes in Random Environments, *Electronic Journal of Probability*, **20(15)**, (2015), 1–29. MR-3317157
- [29] Spiliopoulos, K.: Rare event simulation for multiscale diffusions in random environments, *SIAM Multiscale Modeling and Simulation*, **13(4)**, (2015), 1290–1311. MR-3504804
- [30] C. Villani: Hypocoercivity, *Memoirs of the American Mathematical Society*, **202(950)**, 2009. MR-2562709
- [31] Zhikov, V.V., Kozlov, S.M., Oleinik, O.A., and Kha T'en Ngoan, Averaging and G-convergence of differential operators, *Russian Math Surveys*, **34(5)**, (1979), 65–133. MR-0562800
- [32] Zwanzig, R., Diffusion in a rough potential, *Proc. Natl. Acad. Sci. USA*, **85**, (1988), 2029–2030. MR-0934660

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