Gehring Inequalities on Time Scales

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MARTIN BOHNER AND SAMIR H. SAKER

ABSTRACT. In this paper, we first prove a new dynamic inequality based on an application of the time scales version of a Hardy-type inequality. Second, by employing the obtained inequality, we prove several Gehring-type inequalities on time scales. As an application of our Gehring-type inequalities, we present some interpolation and higher integrability theorems on time scales. The results as special cases, when the time scale is equal to the set of all real numbers, contain some known results, and when the time scale is equal to the set of all integers, the results are essentially new.

1. INTRODUCTION

Let I be a fixed cube with sides parallel to the coordinate axes and let f and g be nonnegative measurable functions defined on I. The classical integral Hölder inequality

$$\int_I |f(x)g(x)|\mathrm{d}x \leq \left[\int_I |f(x)|^p \mathrm{d}x\right]^{\frac{1}{p}} \left[\int_I |g(x)|^q \mathrm{d}x\right]^{\frac{1}{q}},$$

where $1/p + 1/q = 1$, shows that there is a natural scale of inclusion for the $\mathrm{L}^p(I)$-spaces, when the underlying space I has a finite measure $|I|$.

In 1972, Muckenhoupt [14] proved the first simplest reverse integral (mean) inequality, which can be considered as a reverse inclusion, of the form

$$(1.1) \qquad \frac{1}{|I|} \int_I w(x)\mathrm{d}x \leq \kappa \operatorname{essinf}_{x \in I} w(x),$$

where w is a nonnegative measurable function defined on I. A function verifying (1.1) is called an A_1-weight Muckenhoupt function. In [14] (see also [13]), it is proved that any A_1-weight Muckenhoupt function belongs to $\mathrm{L}^r(I)$, for $1 \leq r < s$ and s depending on κ and the dimension of the space.

In 1973, Gehring [8] extended the result of Muckenhoupt for reverse mean inequalities. We say that w satisfies a Gehring condition (or a reverse Hölder inequality) if there exists $p > 1$ and a constant $\kappa > 0$ such that for every cube I with sides parallel to the coordinate axes, we have

$$\left(\frac{1}{|I|} \int_I w^p(x)\mathrm{d}x\right)^{1/p} \leq \frac{\kappa}{|I|} \int_I w(x)\mathrm{d}x.$$

In this case we write $w \in \mathrm{RH}_p$. A well known result obtained by Gehring [8] states that if $w \in \mathrm{RH}_p$, then w satisfies a higher integrability condition, namely

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for sufficiently small $\varepsilon > 0$, $q = p + \varepsilon$, we have for any cube I,

$$\left(\frac{1}{|I|} \int_I w^q(x)\mathrm{d}x \right)^{1/q} \leq \left(\frac{\kappa}{|I|} \int_I w^p(x)\mathrm{d}x \right)^{1/p}.$$

In other words, Gehring's result states that $w \in \mathrm{RH}_p$ implies that there exists $\varepsilon > 0$ such that $w \in \mathrm{RH}_{p+\varepsilon}$. The proof of Gehring's inequality is based on the use of the Calderón–Zygmund decomposition and the scale structure of L^p-spaces. In [12], the author extended Gehring's inequality by means of connecting it to the real method of interpolation by considering maximal operators, and via rearrangements reinterpreted the underlying estimates through the use of K-functionals. This technique allowed to quantify in a precise way, via reiteration, how Calderón–Zygmund decompositions have to be reparameterized in order to characterize different L^p-spaces.

Reverse integral inequalities (cf. [8, 9]) and its many variants and extensions are important in qualitative analysis of nonlinear PDEs, in the study of weighted norm inequalities for classical operators of harmonic analysis, as well as in functional analysis. These inequalities also appear in different fields of analysis such as quasiconformal mappings, weighted Sobolev embedding theorems, and regularity theory of variational problems (see [11]).

In recent years, the study of dynamic inequalities on time scales has received a lot of attention. For details, we refer to the books [2, 3, 5, 6] and the recent paper [1] and the references cited therein. The general idea in studying dynamic inequalities on time scales is to prove a result for an inequality, where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. This idea goes back to its founder Stefan Hilger [10]. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e., when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$, and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ with $q > 1$. The study of dynamic inequalities on time scales helps avoid proving results twice – once for differential inequalities and once again for difference inequalities.

Following this trend and to develop the study of dynamic inequalities on time scales, we aim in this paper to prove Gehring-type inequalities on time scales, which contain the classical integral inequalities of Gehring's type and their discrete versions as special cases. We believe that the reverse dynamic inequalities on time scales will be, just like in the classical case, similarly important for the analysis of dynamic equations on time scales.

The rest of the paper is organized as follows: In Section 2, we recall some definitions and notations related to time scales which will be used throughout the paper. Section 3 features some auxiliary results, in particular, a time scales version of Hardy's inequality. In Section 4, we present the proofs of our Gehring-type inequalities on time scales and give some interpolation results as well as some higher integrability theorems for monotone nonincreasing functions on time scales, see Section 5. As special cases, we offer discrete versions of the Gehring inequalities. To the best of the authors' knowledge, nothing is known regarding the discrete analogues of Gehring inequalities or even their extensions, and thus the presented discrete inequalities are essentially new.

2. Time Scales Preliminaries

We assume that the reader is familiar with time scales as presented in the monographs [5, 6]. For concepts concerning general measure and integration on time scales, see [6, Chapter 5] and [4, 7]. Here, we only state four facts that are essentially used in the proofs of our results. For a function $f : \mathbb{T} \to \mathbb{R}$, where $\mathbb{T}$ is a time scale, we denote the delta derivative by f^{Δ} and the forward shift by $f^{\sigma} = f \circ \sigma$, where σ is the time scales jump operator. The time scales *product rule* says that for two differentiable functions f and g, the product fg is differentiable with

$$(2.1) \qquad (fg)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta}.$$

On the other hand, the time scales *integration by parts rule* says that for two integrable functions $f, g : \mathbb{T} \to \mathbb{R}$ and $a, b \in \mathbb{T}$, we have

$$(2.2) \qquad \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f(t)g^{\Delta}(t)\Delta t.$$

We also need the time scales *chain rule* which says that if $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable, then $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable with

$$(2.3) \qquad (f \circ g)^{\Delta} = g^{\Delta} \int_0^1 f'(hg^{\sigma} + (1-h)g)\mathrm{d}h.$$

Finally, we need the time scales *Hölder inequality* which says that for two nonnegative integrable functions $f, g : \mathbb{T} \to \mathbb{R}$ and $a, b \in \mathbb{T}$ and $p, q > 1$ with $1/p + 1/q = 1$, we have

$$(2.4) \qquad \int_a^b f(t)g(t)\Delta t \leq \left[\int_a^b f^p(t)\Delta t\right]^{1/p} \left[\int_a^b f^q(t)\Delta t\right]^{1/q},$$

and p, q are called the corresponding *exponents*.

Throughout this paper, we assume that the functions in the statements of the theorems are nonnegative and rd-continuous functions, delta differentiable, locally delta integrable, and the integrals considered are assumed to exist (finite, i.e., convergent).

3. Auxiliary Results

In this section, we give some auxiliary results that are used in the proofs of our main results.

Definition 3.1. Throughout this paper, we suppose that $\mathbb{T}$ is a time scale with $0 \in \mathbb{T}$, and we let $T > 0$ with $T \in \mathbb{T}$. For any function $f : (0, T] \to \mathbb{R}$ which is Δ-integrable, nonnegative, and nonincreasing, we define the *average* function $\mathcal{A}f : (0, T] \to \mathbb{R}$ by

$$(3.1) \qquad \mathcal{A}f(t) := \frac{1}{t}\int_0^t f(s)\Delta s \quad \text{for all} \quad t \in (0, T].$$

Some simple facts about $\mathcal{A}f$ are given next.

Lemma 3.2. *If $f : (0, T] \to \mathbb{R}$ is Δ-integrable, nonnegative, and nonincreasing, then*

$$(3.2) \qquad \mathcal{A}f \geq f.$$

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Proof. Due to

$$\mathcal{A}f(t) = \frac{1}{t}\int_0^t f(s)\Delta s \geq \frac{1}{t}\int_0^t f(t)\Delta s = f(t),$$

(3.2) follows immediately. $\qquad\square$

Lemma 3.3. *If* $f : (0,T] \to \mathbb{R}$ *is* Δ*-integrable, nonnegative, and nonincreasing, then so is* $\mathcal{A}f$.

Proof. In this proof, we write $F = \mathcal{A}f$ for brevity. We show that F inherits the nonincreasing nature of f. Let $t_1 < t_2$. Then

$$
\begin{aligned}
F(t_1) - F(t_2) &= \frac{1}{t_1}\int_0^{t_1} f(s)\Delta s - \frac{1}{t_2}\left[\int_0^{t_1} f(s)\Delta s + \int_{t_1}^{t_2} f(s)\Delta s\right]\\
&= \frac{t_2 - t_1}{t_2}\left[\frac{1}{t_1}\int_0^{t_1} f(s)\Delta s - \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} f(s)\Delta s\right]\\
&\geq \frac{t_2 - t_1}{t_2}\left[\frac{1}{t_1}\int_0^{t_1} f(t_1)\Delta s - \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} f(t_1)\Delta s\right] = 0,
\end{aligned}
$$

completing the proof. $\qquad\square$

Now we present a Hardy inequality (see also [3, Corollary 1.5.1]) which, for completeness, we prove in our special setting.

Theorem 3.4. *If* $q > 1$ *and* $f : (0,T] \to \mathbb{R}$ *is* Δ*-integrable, nonnegative, and nonincreasing, then*

(3.3) $$\mathcal{A}\left[(\mathcal{A}f)^\sigma\right]^q \leq \left(\frac{q}{q-1}\right)^q \mathcal{A}f^q.$$

Proof. In this proof, we write $F = \mathcal{A}f$ for brevity. Using Lemma 3.3, the chain rule shows that

(3.4)
$$
\begin{aligned}
(F^q)^\Delta &\overset{(2.3)}{=} qF^\Delta \int_0^1 (hF^\sigma + (1-h)F)^{q-1}\,\mathrm{d}h\\
&\leq qF^\Delta \int_0^1 (hF^\sigma + (1-h)F^\sigma)^{q-1}\,\mathrm{d}h = qF^\Delta (F^\sigma)^{q-1}.
\end{aligned}
$$

Moreover, since

$$tF(t) = \int_0^t f(s)\Delta s,$$

the product rule yields

(3.5) $$f(t) \overset{(2.1)}{=} F(\sigma(t)) + tF^\Delta(t).$$

Now, putting $u(t) = t$ and $v(t) = F^q(t)$, we use integration by parts to find

$$
\begin{aligned}
\int_0^t (F(\sigma(s)))^q \Delta s &= \int_0^t u^\Delta(s)v(\sigma(s))\Delta s\\
&\overset{(2.2)}{=} u(t)v(t) - \lim_{s\to 0^+} u(s)v(s) - \int_0^t u(s)v^\Delta(s)\Delta s\\
&= tF^q(t) - \int_0^t sv^\Delta(s)\Delta s
\end{aligned}
$$

$$\geq \quad -\int_0^t sv^\Delta(s)\Delta s$$

$$\overset{(3.4)}{\geq} \quad -q\int_0^t sF^\Delta(s)F^{q-1}(\sigma(s))\Delta s$$

$$\overset{(3.5)}{=} \quad -q\int_0^t \left[f(s) - F(\sigma(s))\right]F^{q-1}(\sigma(s))\Delta s$$

$$= \quad -q\int_0^t f(s)F^{q-1}(\sigma(s))\Delta s + q\int_0^t (F(\sigma(s)))^q\Delta s$$

so that, by using Hölder's inequality with exponents q and $q/(q-1)$,

$$(q-1)\int_0^t (F(\sigma(s)))^q\Delta s \quad \leq \quad q\int_0^t f(s)(F(\sigma(s)))^{q-1}\Delta s$$

$$\overset{(2.4)}{\leq} \quad q\left[\int_0^t (f(s))^q\Delta s\right]^{1/q}\left[\int_0^t (F(\sigma(s)))^q\Delta s\right]^{(q-1)/q},$$

resulting in (3.3). $\qquad\qquad\square$

In the main results of this paper, we assume that there exists a constant $\lambda \geq 1$ such that

$$(3.6) \qquad\qquad\qquad \sigma(t) \leq \lambda t \quad \text{for all} \quad t \in \mathbb{T}.$$

We now apply the time scales chain rule to obtain some estimates that will be used later.

Lemma 3.5. *Let $x(t) = t$. If $0 < \gamma < 1$, then*

$$(3.7) \qquad\qquad\qquad \left(x^{1-\gamma}\right)^\Delta \geq \frac{1-\gamma}{\sigma^\gamma},$$

and if $\gamma > 1$ and (3.6) holds, then

$$(3.8) \qquad\qquad\qquad \left(x^{1-\gamma}\right)^\Delta \geq \frac{(1-\gamma)\lambda^\gamma}{\sigma^\gamma}.$$

Proof. By the chain rule, we obtain

$$\left(x^{1-\gamma}\right)^\Delta(t) \overset{(2.3)}{=} (1-\gamma)x^\Delta(t)\int_0^1 \frac{dh}{(hx(\sigma(t)) + (1-h)x(t))^\gamma}$$

$$= (1-\gamma)\int_0^1 \frac{dh}{(h\sigma(t) + (1-h)t)^\gamma}.$$

Thus, if $0 < \gamma < 1$, then

$$\left(x^{1-\gamma}\right)^\Delta(t) \geq (1-\gamma)\int_0^1 \frac{dh}{(h\sigma(t) + (1-h)\sigma(t))^\gamma} = \frac{1-\gamma}{(\sigma(t))^\gamma},$$

which is (3.7), and if $\gamma > 1$ and (3.6) holds, then

$$\left(x^{1-\gamma}\right)^\Delta(t) \geq (1-\gamma)\int_0^1 \frac{dh}{(ht + (1-h)t)^\gamma} = \frac{1-\gamma}{t^\gamma} \overset{(3.6)}{\geq} \frac{(1-\gamma)\lambda^\gamma}{(\sigma(t))^\gamma},$$

which is (3.8). $\qquad\qquad\square$

Lemma 3.6. *If F is nonnegative and nondecreasing and $\gamma > 1$, then*

$$(3.9) \qquad\qquad\qquad (F^\gamma)^\Delta \geq \gamma F^\Delta F^{\gamma-1}.$$

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Proof. Again we apply the chain rule to see that

$$(F^\gamma)^\Delta \overset{(2.3)}{=} \gamma F^\Delta \int_0^1 (hF^\sigma + (1-h)F)^{\gamma-1}\mathrm{d}h$$

$$\geq \gamma F^\Delta \int_0^1 (hF + (1-h)F)^{\gamma-1}\mathrm{d}h$$

$$= \gamma F^\Delta F^{\gamma-1},$$

which shows (3.9).   □

4. MAIN RESULTS

We say that $f : (0, T] \to \mathbb{R}$ belongs to $\mathrm{L}_\Delta^p(0, T]$ provided $\int_0^T |f(t)|^p \Delta t < \infty$. The first theorem will be used later in the proof of the Gehring inequality.

Theorem 4.1. *If $f \in \mathrm{L}_\Delta^p(0, T]$ for $p > 1$ is nonnegative and nonincreasing, then, for any $q \in (0, p)$, we have*

(4.1) $$\mathcal{A}f^p \leq \frac{q}{p} [\mathcal{A}f^q]^{p/q} + \frac{(p-q)\lambda^{p/q}}{p} \mathcal{A}\left[(\mathcal{A}f^q)^\sigma\right]^{p/q}.$$

Proof. From the Hardy inequality, see (3.3), we see that the second integral on the right-hand side of (4.1) is finite. Now, we consider this integral. Then, for $0 < q < p$, we put

$$\gamma = \frac{p}{q} > 1 \quad \text{and} \quad F(t) = \int_0^t f^q(s)\Delta s.$$

Using the notation from Lemma 3.5, we have

$$\frac{(p-q)\lambda^{p/q}}{pt} \int_0^t \left[\frac{1}{\sigma(s)} \int_0^{\sigma(s)} f^q(\tau)\Delta\tau\right]^{p/q} \Delta s$$

$$= \frac{(\gamma-1)\lambda^\gamma}{\gamma t} \int_0^t \left[\frac{1}{\sigma(s)} \int_0^{\sigma(s)} f^q(\tau)\Delta\tau\right]^\gamma \Delta s$$

$$= \frac{(\gamma-1)\lambda^\gamma}{\gamma t} \int_0^t \left[\frac{F(\sigma(s))}{\sigma(s)}\right]^\gamma \Delta s$$

$$= -\frac{1}{\gamma t} \int_0^t F^\gamma(\sigma(s)) \frac{(1-\gamma)\lambda^\gamma}{(\sigma(s))^\gamma} \Delta s$$

$$\overset{(3.8)}{\geq} -\frac{1}{\gamma t} \int_0^t F^\gamma(\sigma(s)) \left(x^{1-\gamma}\right)^\Delta (s)\Delta s$$

$$\overset{(2.2)}{=} \lim_{s\to 0^+} \frac{F^\gamma(s)x^{1-\gamma}(s)}{\gamma t} - \frac{F^\gamma(t)x^{1-\gamma}(t)}{\gamma t} + \frac{1}{\gamma t} \int_0^t (F^\gamma)^\Delta (s)x^{1-\gamma}(s)\Delta s$$

$$= \frac{1}{\gamma t} \int_0^t s^{1-\gamma} (F^\gamma)^\Delta (s)\Delta s + \frac{1}{\gamma t} \lim_{s\to 0^+} \left[s\left(\frac{F(s)}{s}\right)^\gamma\right] - \frac{1}{\gamma}\left(\frac{F(t)}{t}\right)^\gamma$$

$$\overset{(3.9)}{\geq} \frac{1}{\gamma t} \int_0^t \frac{\gamma F^\Delta(s)F^{\gamma-1}(s)}{s^{\gamma-1}}\Delta s - \frac{1}{\gamma}\left(\frac{F(t)}{t}\right)^\gamma$$

$$= \frac{1}{t} \int_0^t f^q(s) [\mathcal{A}f^q(s)]^{\gamma-1} \Delta s - \frac{1}{\gamma} [\mathcal{A}f^q(t)]^\gamma$$

$$\overset{(3.2)}{\geq} \frac{1}{t} \int_0^t f^q(s) \, [f^q(s)]^{\gamma-1} \, \Delta s - \frac{1}{\gamma} \, [\mathcal{A}f^q(t)]^\gamma$$

$$= \frac{1}{t} \int_0^t [f^q(s)]^\gamma \, \Delta s - \frac{1}{\gamma} \, [\mathcal{A}f^q(t)]^\gamma$$

$$= \mathcal{A}f^p(t) - \frac{q}{p} \, [\mathcal{A}f^q(t)]^{p/q}$$

from which (4.1) follows. □

Now, we are ready to state and prove our first time scales version of Gehring's mean inequality for monotone functions.

Theorem 4.2 (Gehring Inequality I). *Assume (3.6). If* $f \in \mathrm{L}^q_\Delta(0,T]$ *for* $q > 1$ *is nonnegative and nonincreasing such that*

$$(4.2) \qquad\qquad \mathcal{A}f^q \leq \kappa \, [\mathcal{A}f]^q \qquad \text{for some} \quad \kappa > 0,$$

then $f \in \mathrm{L}^p_\Delta(0,T]$ *for any* $p > q$ *satisfying*

$$(4.3) \qquad\qquad \tilde{\kappa} := \frac{q\kappa^{p/q}}{p - (p-q)(\lambda\kappa)^{p/q}\left(\frac{p}{p-1}\right)^p} > 0,$$

and in this case,

$$(4.4) \qquad\qquad \mathcal{A}f^p \leq \tilde{\kappa} \, [\mathcal{A}f]^p.$$

Proof. Assuming (4.2), we find

$$\frac{1}{t}\int_0^t f^p(s)\Delta s \overset{(4.1)}{\leq} \frac{q}{p}\left[\frac{1}{t}\int_0^t f^q(s)\Delta s\right]^{p/q}$$

$$+ \frac{(p-q)\lambda^{p/q}}{pt}\int_0^t \left[\frac{1}{\sigma(s)}\int_0^{\sigma(s)} f^q(\tau)\Delta\tau\right]^{p/q}\Delta s$$

$$\overset{(4.2)}{\leq} \frac{q}{p}\kappa^{p/q}\left[\frac{1}{t}\int_0^t f(s)\Delta s\right]^p + \frac{(p-q)\lambda^{p/q}}{pt}\int_0^t \kappa^{p/q}\left[\frac{1}{\sigma(s)}\int_0^{\sigma(s)} f(\tau)\Delta\tau\right]^p\Delta s$$

$$\overset{(3.3)}{\leq} \frac{q}{p}\kappa^{p/q}\left[\frac{1}{t}\int_0^t f(s)\Delta s\right]^p + \frac{(p-q)(\lambda\kappa)^{p/q}}{pt}\left(\frac{p}{p-1}\right)^p\int_0^t f^p(s)\Delta s$$

so that, due to (4.3),

$$\frac{1}{t}\int_0^t f^p(s)\Delta s \leq \tilde{\kappa}\left[\frac{1}{t}\int_0^t f(s)\Delta s\right]^p,$$

from which (4.4) follows. □

As a special case of Theorem 4.2 when $\mathbb{T} = \mathbb{R}$, we get the classical Gehring inequality (see Section 1) with $\lambda = 1$. In the case when $\mathbb{T} = \mathbb{N}$, we have the following result with $\lambda = 2$.

Corollary 4.3 (Discrete Gehring Inequality I). *Let* $q > 1$ *and* $\{a_n\}_{n\in\mathbb{N}_0}$ *be a nonnegative and nonincreasing sequence such that*

$$\frac{1}{n}\sum_{i=0}^{n-1} a_i^q \leq \kappa \left(\frac{1}{n}\sum_{i=0}^{n-1} a_i\right)^q.$$

Then, for $p > q$, we have

$$\frac{1}{n}\sum_{i=0}^{n-1} a_i^p \leq \tilde{\kappa}\left(\frac{1}{n}\sum_{i=0}^{n-1} a_i\right)^p,$$

provided

$$\tilde{\kappa} := \frac{q\kappa^{p/q}}{p - (p-q)(2\kappa)^{p/q}\left(\frac{p}{p-1}\right)^p} > 0.$$

It is natural to ask what happens if in (4.4) we fix $p > 1$ and consider the improvement to this inequality that would result from lowering the exponent on the right-hand side. The following result gives an answer.

Theorem 4.4. *Suppose that the assumptions of Theorem 4.2 hold and define $\tilde{\kappa}$ as in (4.3). Then, for all $0 < r < 1$, we have*

$$(4.5) \qquad \mathcal{A}f^p \leq \overline{\kappa}\left[\mathcal{A}f^r\right]^{p/r}, \quad \text{where} \quad \overline{\kappa} := \tilde{\kappa}^{1/\theta} \quad \text{with} \quad \theta := \frac{1 - \frac{1}{p}}{\frac{1}{r} - \frac{1}{p}}.$$

Proof. Note first that $\theta \in (0,1)$ and

$$\frac{1-\theta}{p} + \frac{\theta}{r} = 1.$$

Then, by the Hölder inequality with exponents $p/(1-\theta)$ and r/θ, we have

$$\left[\frac{1}{t}\int_0^t f^p(s)\Delta s\right]^{1/p} \stackrel{(4.4)}{\leq} \frac{\tilde{\kappa}^{1/p}}{t}\int_0^t f(s)\Delta s$$

$$= \frac{\tilde{\kappa}^{1/p}}{t}\int_0^t f^{1-\theta}(s)f^{\theta}(s)\Delta s$$

$$\stackrel{(2.4)}{\leq} \frac{\tilde{\kappa}^{1/p}}{t}\left[\int_0^t f^p(s)\Delta s\right]^{(1-\theta)/p}\left[\int_0^t f^r(s)\Delta s\right]^{\theta/r}$$

$$= \tilde{\kappa}^{1/p}\left[\frac{1}{t}\int_0^t f^p(s)\Delta s\right]^{(1-\theta)/p}\left[\frac{1}{t}\int_0^t f^r(s)\Delta s\right]^{\theta/r}$$

so that, by dividing, we find

$$\left[\frac{1}{t}\int_0^t f^p(s)\Delta s\right]^{\theta/p} \leq \tilde{\kappa}^{1/p}\left[\frac{1}{t}\int_0^t f^r(s)\Delta s\right]^{\theta/r},$$

i.e., (4.5). □

By Theorem 4.4, under the assumptions of Theorem 4.2, if $f \in \mathrm{L}_{\Delta}^r(0,T]$ for $0 < r < 1$, then $f \in \mathrm{L}_{\Delta}^p(0,T]$ for $p > 1$. But in the general case when $p \neq r$, $\mathrm{L}_{\Delta}^p(0,T]$ neither includes nor is included in $\mathrm{L}_{\Delta}^r(0,T]$. The following theorem gives some results for $\mathrm{L}_{\Delta}^p(0,T]$-interpolation.

Theorem 4.5. *Suppose that $0 < p_0 < p_1 < \infty$ and that $0 < \theta < 1$.*

(i) *If $p = (1-\theta)p_0 + \theta p_1$ and $f \in \mathrm{L}_{\Delta}^{p_0}(0,T] \cap \mathrm{L}_{\Delta}^{p_1}(0,T]$, then $f \in \mathrm{L}_{\Delta}^p(0,T]$ and*

$$\mathcal{A}f^p \leq \left[\mathcal{A}f^{p_0}\right]^{1-\theta}\left[\mathcal{A}f^{p_1}\right]^{\theta}.$$

(ii) *If* $p = \frac{1}{\frac{1-\theta}{p_0} + \frac{\theta}{p_1}}$ *and* $f \in L_\Delta^{p_0}(0,T) \cap L_\Delta^{p_1}(0,T)$, *then* $f \in L_\Delta^p(0,T)$ *and*

$$\mathcal{A}f^p \le [\mathcal{A}f^{p_0}]^{(1-\theta)p/p_0}[\mathcal{A}f^{p_1}]^{\theta p/p_1}.$$

Proof. For (i), we apply the Hölder inequality with exponents $1/(1-\theta)$ and $1/\theta$ to see that

$$\frac{1}{t}\int_0^t f^p(s)\Delta s \quad = \quad \frac{1}{t}\int_0^t f^{(1-\theta)p_0}(s)f^{\theta p_1}(s)\Delta s$$

$$\overset{(2.4)}{\le} \quad \left[\frac{1}{t}\int_0^t f^{p_0}(s)\Delta s\right]^{1-\theta}\left[\frac{1}{t}\int_0^t f^{p_1}(s)\Delta s\right]^{\theta},$$

which shows (i). For (ii), we apply the Hölder inequality with exponents $1/(1-\gamma)$ and $1/\gamma$, where

$$\gamma := \frac{\theta p}{p_1} \quad \text{so that} \quad 1-\gamma = \frac{(1-\theta)p}{p_0},$$

to see that

$$\frac{1}{t}\int_0^t f^p(s)\Delta s \quad = \quad \frac{1}{t}\int_0^t f^{(1-\theta)p}(s)f^{\theta p}(s)\Delta s$$

$$\overset{(2.4)}{\le} \quad \left[\frac{1}{t}\int_0^t f^{(1-\theta)p/(1-\gamma)}(s)\Delta s\right]^{1-\gamma}\left[\frac{1}{t}\int_0^t f^{\theta p/\gamma}(s)\Delta s\right]^{\gamma}$$

$$= \quad \left[\frac{1}{t}\int_0^t f^{p_0}(s)\Delta s\right]^{(1-\theta)p/p_0}\left[\frac{1}{t}\int_0^t f^{p_1}(s)\Delta s\right]^{\theta p/p_1},$$

which shows (ii). $\qquad\qquad\square$

In the following, we give a new proof of Gehring's mean inequality on time scales. The inequality will be proved by using a condition similar to the condition (1.1) due to Muckenhoupt. In fact, we do not assume that the reverse Hölder inequality holds.

Theorem 4.6 (Gehring Inequality II). *Assume* (3.6). *If* $f : (0,T] \to \mathbb{T}$ *is nonnegative and nonincreasing such that*

(4.6) $$\mathcal{A}f^\sigma \le \nu f \quad \text{for some} \quad \nu > 1,$$

then $f \in L_\Delta^p(0,T)$ *for* $p \in [1, \alpha/(\alpha - 1))$, *where* $\alpha = \lambda\nu$, *and we have*

(4.7) $$\mathcal{A}(f^p)^\sigma \le \tilde{\nu}[\mathcal{A}f^\sigma]^p, \quad \text{where} \quad \tilde{\nu} := \frac{\alpha}{\alpha - p(\alpha - 1)} > 0.$$

Proof. For this proof, we put

$$F(t) := \int_0^t f^\sigma(s)\Delta s, \quad l(t) = \log(t), \quad L(t) = \log(F(t)).$$

By the chain rule, we get

$$\frac{1}{\alpha}l^\Delta(t) \quad \overset{(2.3)}{=} \quad \frac{1}{\lambda\nu}\int_0^1 \frac{\mathrm{d}h}{h\sigma(t) + (1-h)t}$$

$$\overset{(3.6)}{\le} \quad \frac{1}{\lambda\nu}\int_0^1 \frac{\mathrm{d}h}{h\sigma(t) + (1-h)\frac{\sigma(t)}{\lambda}}$$

$$\le \quad \frac{1}{\lambda\nu}\cdot\frac{\lambda}{\sigma(t)} = \frac{1}{\nu\sigma(t)}$$

$$\overset{(4.6)}{\leq} \quad \frac{f(\sigma(t))}{F(\sigma(t))} = \frac{F^{\Delta}(t)}{F(\sigma(t))}$$

$$= \quad F^{\Delta}(t) \int_0^1 \frac{\mathrm{d}h}{hF(\sigma(t)) + (1-h)F(\sigma(t))}$$

$$\leq \quad F^{\Delta}(t) \int_0^1 \frac{\mathrm{d}h}{hF(\sigma(t)) + (1-h)F(t)}$$

$$\overset{(2.3)}{=} \quad L^{\Delta}(t),$$

and hence, by integrating,

$$\log \left(\frac{t}{\sigma(s)} \right)^{1/\alpha} = \frac{1}{\alpha}l(t) - \frac{1}{\alpha}l(\sigma(s)) \leq L(t) - L(\sigma(s)) = \log \left(\frac{F(t)}{F(\sigma(s))} \right)$$

so that

$$f(\sigma(s)) \leq \frac{1}{\sigma(s)} \int_0^{\sigma(s)} f(\sigma(\tau))\Delta\tau = \frac{F(\sigma(s))}{\sigma(s)} \leq \left(\frac{\sigma(s)}{t} \right)^{1/\alpha} \frac{F(t)}{\sigma(s)},$$

and by integrating again, putting $\gamma := p(1 - 1/\alpha) \in (0, 1)$, and using the notation from Lemma 3.5, we obtain

$$\frac{1}{t} \int_0^t f^p(\sigma(s))\Delta s \quad \leq \quad \frac{F^p(t)}{t^{1+p/\alpha}} \int_0^t \frac{\Delta s}{(\sigma(s))^{p(1-1/\alpha)}}$$

$$\overset{(3.7)}{\leq} \quad \frac{F^p(t)}{(1-\gamma)t^{1+p/\alpha}} \int_0^t \left(x^{1-\gamma} \right)^{\Delta}(s)\Delta s$$

$$= \quad \frac{t^{1-\gamma}F^p(t)}{(1-\gamma)t^{1+p/\alpha}} = \frac{1}{1-\gamma} \left(\frac{F(t)}{t} \right)^p,$$

proving (4.7).   □

As a special case of Theorem 4.6 when $\mathbb{T} = \mathbb{N}$, we have the following result.

Corollary 4.7 (Discrete Gehring Inequality II). *Let $\{a_n\}_{n\in\mathbb{N}_0}$ be a nonnegative and nonincreasing sequence. If there exists a constant $\nu > 1$ such that*

$$\frac{1}{n} \sum_{i=1}^n a_i \leq \nu a_n,$$

then, for $p \in [1, \alpha/(\alpha - 1)]$, where $\alpha = 2\nu$, we have

$$\frac{1}{n} \sum_{i=1}^n a_i^p \leq \tilde{\nu} \left[\frac{1}{n} \sum_{i=1}^n a_i \right]^p, \quad where \quad \tilde{\nu} := \frac{\alpha}{\alpha - p(\alpha - 1)}.$$

5. Higher Integrability

In the following, as an application of Gehring's inequality (4.7), we prove a higher integrability theorem for monotone nonincreasing functions. First notice that for all nonnegative and nonincreasing functions $f \in \mathrm{L}^q_{\Delta}(0, T]$ with $q > 1$, we always have

$$(5.1) \quad \mathcal{A}f^q(t) = \frac{1}{t} \int_0^t f^q(s)\Delta s = \frac{1}{t} \int_0^t f^{q-1}(s)f(s)\Delta s \geq \frac{f^{q-1}(t)}{t} \int_0^t f(s)\Delta s.$$

Let us now consider the class of nonnegative and nonincreasing functions $f \in L_\Delta^q(0, T]$ that satisfy the reverse of (5.1), namely

(5.2)
$$\mathcal{A}f^q \leq \eta f^{q-1}\mathcal{A}f \quad \text{for some} \quad \eta > 1.$$

Theorem 5.1. *Assume* (3.6). *If* $f \in L_\Delta^q(0, T]$ *for* $q > 1$ *is nonnegative and nonincreasing such that* (5.2) *holds, then* $f \in L_\Delta^p(0, T]$ *for* $p \in [q, q+c]$, $c \in (q, \eta)$, *and we have*
(5.3)
$$\mathcal{A}(f^p)^\sigma \leq \tilde{\eta}\left[\mathcal{A}f^q\right]^{p/q}, \quad \text{where} \quad \tilde{\eta} := \frac{\lambda\eta_q^{1+p/q}}{\lambda\eta_q - \frac{p}{q}(\lambda\eta_q - 1)} \quad \text{with} \quad \eta_q = \frac{\eta q}{q-1}.$$

Proof. In this proof, we write $F = \mathcal{A}f^q$ for brevity. By using the Hölder inequality with exponents $q/(q-1)$ and q, we obtain

$$\frac{1}{t}\int_0^t F(\sigma(s))\Delta s \overset{(5.2)}{\leq} \frac{\eta}{t}\int_0^t (f(\sigma(s)))^{q-1} \cdot \frac{1}{\sigma(s)}\int_0^{\sigma(s)} f(\tau)\Delta\tau\Delta s$$

$$\overset{(2.4)}{\leq} \frac{\eta}{t}\left[\int_0^t (f(\sigma(s)))^q\Delta s\right]^{(q-1)/q}\left[\int_0^t \left(\frac{1}{\sigma(s)}\int_0^{\sigma(s)} f(\tau)\Delta\tau\right)^q\Delta s\right]^{1/q}$$

$$\overset{(3.3)}{\leq} \frac{\eta q}{(q-1)t}\left[\int_0^t (f(s))^q\Delta s\right]^{(q-1)/q}\left[\int_0^t (f(s))^q\Delta s\right]^{1/q}$$

$$= \frac{\eta_q}{t}\int_0^t f^q(s)\Delta s = \eta_q F(t),$$

i.e.,

(5.4)
$$\mathcal{A}F^\sigma \leq \eta_q F.$$

Since F is also nonnegative and nonincreasing (see Lemma 3.3), it satisfies the assumptions of Theorem 4.6, and thus

(5.5)
$$\frac{1}{t}\int_0^t [F(\sigma(s))]^r\,\Delta s \leq \tilde{\eta}_q\left[\frac{1}{t}\int_0^t F(\sigma(s))\Delta s\right]^r$$

with

$$\tilde{\eta}_q = \frac{\alpha_q}{\alpha_q - r(\alpha_q - 1)} \quad \text{and} \quad \alpha_q = \lambda\eta_q \quad \text{for} \quad r = \frac{p}{q} \in \left[1, \frac{\alpha_q}{\alpha_q - 1}\right).$$

Noting that

(5.6)
$$F(t) = \frac{1}{t}\int_0^t f^q(s)\Delta s \geq f^q(t),$$

we obtain

$$\frac{1}{t}\int_0^t (f(\sigma(s)))^p\Delta s \quad = \quad \frac{1}{t}\int_0^t (f^q(\sigma(s)))^r\Delta s$$

$$\overset{(5.6)}{\leq} \quad \frac{1}{t}\int_0^t (F(\sigma(s)))^r\Delta s$$

$$\overset{(5.5)}{\leq} \quad \tilde{\eta}_q\left(\frac{1}{t}\int_0^t F^\sigma(s)\Delta s\right)^r$$

$$\overset{(5.4)}{\leq} \quad \tilde{\eta}_q\eta_q^r[F(t)]^r = \tilde{\eta}[F(t)]^r$$

$$= \quad \tilde{\eta} \left[\frac{1}{t} \int_0^t f^q(s) \Delta s \right]^{p/q},$$

proving (5.3). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In Theorem 5.1, if $\mathbb{T} = \mathbb{R}$, then we have that $\sigma(t) = t$, $\alpha_q = \eta_q$, and we get the following result.

Corollary 5.2. *Let $\eta > 1$ and $q > 1$. Then every nonnegative nonincreasing function f satisfying*

$$\int_0^t f^q(x)\mathrm{d}x \leq \eta f^{q-1}(t) \int_0^t f(x)\mathrm{d}x$$

belongs to $\mathrm{L}_\Delta^p(0,T](0,T]$ for $p \in [q, q+c]$ and $c \in (q, \eta)$, and we have

$$\frac{1}{t} \int_0^t f^p(x)\mathrm{d}x \leq \tilde{\eta} \left(\frac{1}{t} \int_0^t f^q(x)\mathrm{d}x \right)^{p/q},$$

where

$$\tilde{\eta} := \frac{\left(\frac{\eta q}{q-1} \right)^{\frac{p}{q}+1}}{\frac{\eta q}{q-1} - \frac{p}{q}\left(\frac{\eta q}{q-1} - 1 \right)}.$$

In Theorem 5.1, if $\mathbb{T} = \mathbb{N}$, then we have that $\sigma(t) = t + 1$, and by choosing $\lambda = 2$, we get the following result.

Corollary 5.3. *Let $\eta > 1$ and $q > 1$. Suppose $\{a_n\}_{n \in \mathbb{N}_0}$ is a nonnegative and nonincreasing sequence satisfying*

$$\sum_{i=0}^{n-1} a_i^q \leq \eta a_n^{q-1} \sum_{i=0}^{n-1} a_i.$$

Then, for $p \in [q, q+c]$, $c \in (q, \eta)$, we have

$$\frac{1}{n} \sum_{i=0}^{n-1} a_i^p \leq \tilde{\eta} \left(\frac{1}{n} \sum_{i=0}^{n-1} a_i^q \right)^{p/q},$$

where

$$\tilde{\eta} := \frac{2\left(\frac{\eta q}{q-1} \right)^{\frac{p}{q}+1}}{2\frac{\eta q}{q-1} - \frac{p}{q}\left(2\frac{\eta q}{q-1} - 1 \right)}.$$

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