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PLANARITY OF WHITNEY LEVELS

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Abstract. First, we characterize all locally connected continua whose all Whitney levels are planar. Second, we show by example that planarity is not a (strong) Whitney reversible property. This answers a question from Illanes-Nadler book [2].

1. Introduction

In this article we investigate planarity of Whitney levels of continua. First, we prove that there are only four locally connected continua whose all Whitney levels are planar. This shows in particular that planarity is a strong Whitney reversible property in the class of locally connected continua. We show even more: we characterize all locally connected continua whose small Whitney levels are planar. There is infinitely many of them. We also show that in general (for non-locally connected continua) this is not the case, precisely, there is a non-planar continuum whose all Whitney levels are planar. This answers [2, Question 54.3].

Let us start establishing basic terms used in this article. A continuum is a compact, connected metric space. A map (or mapping) is a continuous function. Given a continuum X we denote by $C(X)$ the hyperspace of all subcontinua of X equipped with the Hausdorff metric ([5, (0.1) and (0.13)]). A Whitney map for $C(X)$ is a mapping $\omega : C(X) \to [0, \infty)$ such that, (1) if $A \subset B$ and $A \neq B$, then $\omega(A) < \omega(B)$; and (2) for each $x \in X$, $\omega(\{x\}) = 0$. Whitney maps exist for the hyperspace of any continuum ([2, Section 13]). For each $t \in [0, \omega(X)]$, the

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preimage $\omega^{-1}(t)$ is called a Whitney level. It is known that each Whitney level is a continuum ([2, Theorem 19.9]). A topological property $\mathcal{P}$ is said to be:

- a Whitney property, provided that if a continuum X has property $\mathcal{P}$, then for each Whitney map $\omega : C(X) \to [0, \infty)$ and for each $t \in [0, \omega(X))$, $\omega^{-1}(t)$ has property $\mathcal{P}$ ([2, 27.1 (a)]);

- a Whitney reversible property, provided that whenever X is a continuum such that, for *all* Whitney maps $\omega : C(X) \to [0, \infty)$ and for all $t \in (0, \omega(X))$ we have that $\omega^{-1}(t)$ has property $\mathcal{P}$, then X has property $\mathcal{P}$ ([2, 27.1 (b)]);

- a strong Whitney reversible property, provided that whenever X is a continuum such that, for *some* Whitney map $\omega : C(X) \to [0, \infty)$ and for all $t \in (0, \omega(X))$ we have that $\omega^{-1}(t)$ has property $\mathcal{P}$, then X has property $\mathcal{P}$ ([2, 27.1 (c)]);

- a sequential strong Whitney reversible property, provided that whenever X is a continuum such that, there is a Whitney map $\omega : C(X) \to [0, \infty)$ and a sequence $\{t_n : n \in \mathbb{N}\}$ in $(0, \omega(X))$ such that $\lim t_n = 0$ and, for each $n \in \mathbb{N}$, $\omega^{-1}(t_n)$ has property $\mathcal{P}$, then X has property $\mathcal{P}$ ([2, 27.1 (d)]);

- a weak small Whitney property, provided that if a continuum X has property $\mathcal{P}$, then *there is* a Whitney map $\omega : C(X) \to [0, \infty)$ and a number $s \in (0, \omega(X))$ such that for each $t \in [0, s)$, $\omega^{-1}(t)$ has property $\mathcal{P}$ ([1, (0.9)]).

If $\omega : C(X) \to [0, \infty)$ is a Whitney map for a continuum X and $\mathcal{P}$ is a topological property, we say that small Whitney levels have $\mathcal{P}$ if there is a positive number r such that Whitney levels $\omega^{-1}(t)$ have $\mathcal{P}$ for $t \in (0, r)$.

A continuum X is said to be planar provided that X is homeomorphic to a subcontinuum of the euclidean plane. A graph is a continuum which can be written as the union of finitely many arcs any two of which are either disjoint or intersect only in one or both of their end points. Given an integer $n \geq 3$, a simple n-od is a graph that is the union of n arcs having only one end point in common. A simple 3-od is called a simple triod. An n-od is a continuum X for which there is a subcontinuum Y such that $X \setminus Y$ is the union of n sets mutually separated in X (i.e., $X \setminus Y = \cup_{i=1}^{n} E_i$, $E_i \neq \emptyset$ and $\overline{E_i} \cap E_j = \emptyset$ whenever $i \neq j$).

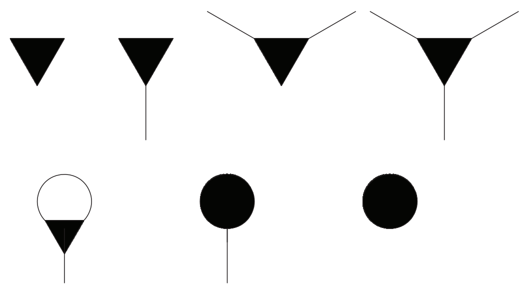

FIGURE 1. Positive Whitney levels of a triod and of L.

2. Locally connected continua

In this section we investigate planarity of Whitney levels of locally connected continua. We will characterize locally connected continua whose all positive Whitney levels are planar (Theorem 2.1) as well as locally connected continua whose small positive Whitney levels are planar (Theorem 2.2). As a consequence we will conclude that being a planar locally connected continuum is a sequential strong Whitney reversible property.

We start with a characterization of locally connected continua whose all Whitney levels are planar. Let L be the one point union of a circle with an arc, where the common point is an end point of the arc.

Theorem 2.1. *The following conditions are equivalent for a locally connected continuum X:*

1. *for each Whitney map $\omega : C(X) \to [0,\infty)$ all Whitney levels of X are planar;*
2. *for some Whitney map $\omega : C(X) \to [0,\infty)$ all Whitney levels of X are planar;*
3. *X contains no 4-od;*
4. *X is one of the following four graphs: an arc, a circle, a simple triod, or the graph L.*

PROOF. The implication (1) $\Longrightarrow$ (2) is obvious, and (2) $\Longrightarrow$ (3) is a consequence of the fact that some Whitney levels of a 4-od contain 3-cells, see [4, Corollary 3.3]. To show (3) $\Longrightarrow$ (4) observe that X contains at most one ramification point, otherwise it would contain a continuum homeomorphic to the letter H, which is a 4-od. If X contains no ramification point, it is an arc or a circle, if it contains one ramification point it is a triod (if it contains no simple closed curve), or it is the continuum L, if it contains a simple closed curve.

To show that (4) $\Longrightarrow$ (1) it is enough to examine all possible Whitney levels of the listed continua. Whitney levels of an arc are arcs, Whitney levels of a circle are circles ([3, 6.4]), and positive Whitney levels of a triod and of the graph L are pictured in Figure 1. The details are left to the reader. In every case all of the Whitney levels are planar, so the proof is complete. □

Now let us investigate continua whose small Whitney levels are planar.

Theorem 2.2. *Let X be a locally connected continuum. Then the following conditions are equivalent:*

(1) *for each Whitney map $\omega : C(X) \to [0, \infty)$ small Whitney levels are planar;*
(2) *for some Whitney map $\omega : C(X) \to [0, \infty)$ small Whitney levels are planar;*
(3) *for some Whitney map $\omega : C(X) \to [0, \infty)$ there is a sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_n > 0$, $\lim_{n=1}^{\infty} t_n = 0$, and $\omega^{-1}(t_n)$ are planar;*
(4) *X is a planar graph containing no simple 4-od.*

Proof. The implications (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (3) are obvious. To show (3) $\Longrightarrow$ (4) suppose (3) and note that X contains no simple 4-od, because small Whitney levels of a simple 4-od contain 3-cells (see [4, Corollary 3.3]). Suppose that X contains infinitely many ramification points, and each ramification point is of order 3. Then X contains an arc A such that $\omega(A) = t_n$ for some $n \in \{1, 2, \dots\}$ and A contains at least two ramification points of X that are not end points of A. Then $\omega^{-1}(t_n)$ contain a 3-cell, contrary to (3), therefore X may contain only finitely many ramification points. A locally connected continuum with finitely many ramification points, all of order 3, is a graph without a simple 4-od.

To prove (4) $\Longrightarrow$ (1), let X be a graph containing no simple 4-od. Define R to be a finite subset of X containing all ramification points of X, all end points of X and such that every simple closed curve in X contains at least two point of R.

Let $r < \min\{\omega(A) : A$ is a continuum containing at least two points of $R\}$. Then any subcontinuum P of X satisfying $\omega(P) < r$ is either an arc or a simple triod. Thus $\omega^{-1}(t)$, for $t < r$ is the union of Whitney levels of arcs $\omega^{-1}(t) \cap C(A)$, where A is a minimal arc in X containing two points of R, and continua of the form $\{P \in C(X) : \omega(P) = t$ and $a \in P\}$ for some $a \in R$. Whitney levels of arcs are arcs ([3, 6.4]), and the later continua are either arcs (if a is an ordinary point), or disks (if a is a point of order 3), so $\omega^{-1}(t)$ is homeomorphic to the graph X

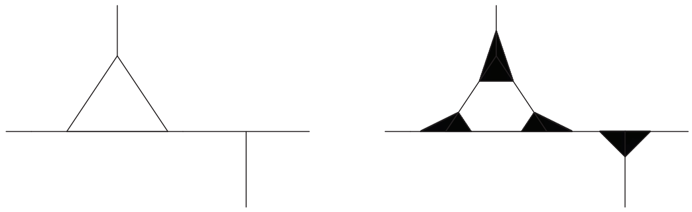

FIGURE 2. A planar graph and its small Whitney level.

whose ramification points are replaced by disks (see Figure 2). If X is planar, then $\omega^{-1}(t)$ is planar as well. □

As a consequence of the equivalence of conditions (3) and (4) of Theorem 2.2 we get the following corollary.

Corollary 2.3. *Planarity is a sequential strong Whitney property in the class of locally connected continua.*

Because local connectedness is a sequential strong Whitney reversible property (see [5, Theorem 14.47]), we get the following corollary.

Corollary 2.4. *Being a planar locally connected continuum is a sequential strong Whitney property.*

3. Non locally connected continua

The main aim of this section is to show an example of a non-planar continuum whose all Whitney levels are planar. By Theorem 2.1 it cannot be locally connected. This shows that the assumptions of local connectedness are necessary in Theorems 2.2 and 2.1, that planarity is not a Whitney reversible property, that answers [2, Question 54.3], and that a non-planarity is not a weak small Whitney reversible property.

Theorem 3.1. *There is a non-planar continuum, whose all positive Whitney levels are planar. Consequently, planarity is not a Whitney reversible property, and non-planarity is not a weak small Whitney property.*

PROOF. Let X be the continuum pictured in Figure 3. It is the union of a sequence of circles C_n having a common center p, the spirals S_n approximating C_n and C_{n+1} respectively, and an arc A having p as its end point.

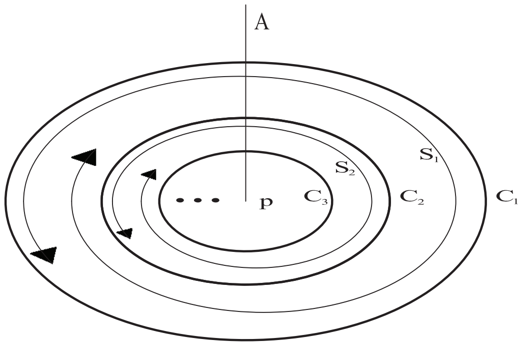

FIGURE 3. The example.

Formally, define

$$\begin{array}{lcl} C_n & = & \{\frac{1}{2^{n-1}}(\cos t, \sin t, 0) : t \in [0, 2\pi)\} \text{ and} \\ S_n & = & \{\frac{r(t)}{2^{n-1}}(\cos t, \sin t, 0) : t \in (-\infty, +\infty)\} \text{ where} \\ r(t) & = & \frac{3}{4} + \frac{1}{2\pi} \arctan t \text{ for } t \in (-\infty, +\infty) \end{array}$$

and put

$$X = \bigcup_{n=1}^{\infty} (C_n \cup S_n) \cup \overline{(0,0,0),(0,0,1)}.$$

The important feature is that the spirals S_n and S_{n+1} approximate C_{n+1} in different directions; this makes the continuum X non-planar. Really, for any potential embedding of X in the plane the spirals S_n and S_{n+1} have to be in different components of the complement $\mathbb{R}^2 \setminus C_{n+1}$.

First we show that all positive Whitney levels of X are planar. To this aim we determine all positive Whitney levels of X. We need to consider two cases.

Case 1: $t \geq \omega(C_n)$ for all $n \in \{1, 2, \dots\}$. If we shrink all circles C_n to different points, the quotient space is an arc, so let $m : X \to [0, 1]$ be a monotone map whose nondegenerate preimages of points are the circles C_n only. Then the mapping $e : \omega^{-1}(t) \to [0, 1]$ defined by $e(P) =$ the left end point of $m(P)$ is an embedding, so $\omega^{-1}(t)$ is an arc.

Case 2: $0 < t < \omega(C_n)$ for some $n \in \{1, 2, \dots\}$. Denote by I the set of all indices that satisfy $\omega(C_i) > t$. Since $\omega(C_i)$ tends to zero, as i tends to infinity, the set I is finite. Note that the continua C_i are terminal in X, so each element E of $\omega^{-1}(t)$ is either contained in some C_i or, if $E \cap C_i \neq \emptyset$, then $C_i \subset E$.

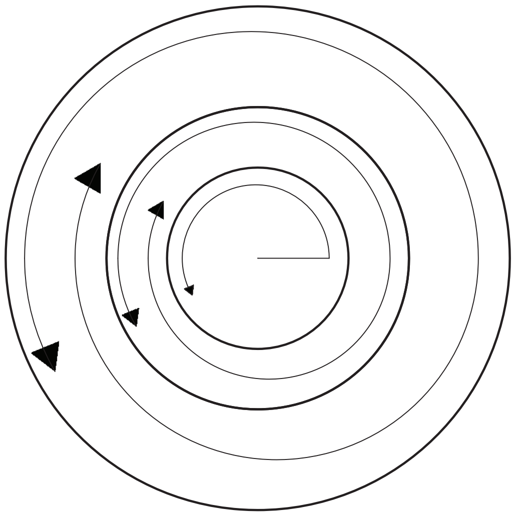

FIGURE 4. A Whitney level of X.

Let $m : X \to Y$ be a map that shrinks every circle C_i, for $i \notin I$, to a point. We will show that $\omega^{-1}(t)$ is homeomorphic to Y. Note that continuum Y is the union of some finite number of circles $m(C_i)$, for $i \in I$, and some spirals. A continuum like that is pictured in Figure 4, they may be different by the number of circles and possibly, if $\omega(C_1) \geq t$, there may be an outer spiral approximating the biggest circle. Clearly the Whitney levels are planar.

Define $e : \omega^{-1}(t) \to Y$ by the following conditions:

- if $P \subset A$, then $e(P)$ is the image under m of the lowest point of P,
- if $P \subset C_i$ for some $i \in I$, then e(P) is the image under m of the most counterclockwise point of P,
- in all other cases $e(P)$ is the image under m of the point of $P \cap (X \setminus A)$ that is farthest away from the origin.

One can verify that e is one-to-one and continuous and that the image of $\omega^{-1}(t)$ under e is Y without an arc in the center, so it is homeomorphic to Y. □

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